MATRICIAL APPROXIMATION OF QUANTUM AUTOMORPHISM GROUPS OF GRAPHS

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Abstract. We report on a joint work [BCF20] with M. Brannan and A. Chirvasitu investigating three approximation properties "by matrices" for quantum permutation groups and quantum reflection groups: the Connes embedding property, residual finite-dimensionality and inner linearity.

1. Introduction

We should start with a disclaimer: this text will only be concerned with the "most simple" examples of quantum automorphism groups of graphs, namely the quantum permutation groups S_N^+ and the quantum reflection groups H_N^{s+}. However, it is our hope that some of the ideas used in this work carry to more general quantum automorphism groups of graphs. These notes come from talks given for audiences with sometimes no operator algebra background, but at least a strong algebra culture. This is why we decided to work in the algebraic setting.

The quantum permutation groups were introduced by S. Wang in [Wan98] and have been studied from then on by operator algebraists. Moreover, they recently attracted attention from other communities due to a connection with quantum information theory (see for instance [LMR20]), based on so-called *graph isomorphism games* and the notion of quantum permutation. To define such a quantum permutation, first note that classical permutations can be seen as specific matrices, with only 1's and 0's, the sum on any row and column being 1. Since measurement in quantum mechanics is based on Hilbert space projections, one may give the following tentative generalization:

Definition 1.1. A *quantum permutation* of size N is a matrix $(p_{ij})_{1\leqslant i,j\leqslant N}$ where $p_{ij} \in \mathcal{B}(H)$ for some fixed Hilbert space H and such that for all $1 \leqslant i,j \leqslant N$,

$$p_{ij}^* = p_{ij} = p_{ij}^2 \text{ and } \sum_{k=1}^N p_{ik} = \mathrm{Id}_H = \sum_{k=1}^N p_{kj}.$$

To investigate these objects, it is natural to introduce a universal algebra of which they are representations. This gives a possible definition of the quantum permutation groups.

Definition 1.2. The *quantum permutation algebra* $\mathcal{O}(S_N^+)$ is the universal $*$-algebra over $\mathbb{C}$ generated by N^2 elements $(u_{ij})_{1\leqslant i,j\leqslant N}$ such that

- $p_{ij}^* = p_{ij} = p_{ij}^2$ for all $1 \leqslant i,j \leqslant N$,
- $\sum_{k=1}^N u_{ik} = 1 = \sum_{k=1}^N u_{kj}$ for all $1 \leqslant i,j \leqslant N$,

- $u_{ij}u_{ij'} = \delta_{j,j'}u_{ij}$ and $u_{ij}u_{i'j} = \delta_{i,i'}u_{ij}$ for all $1 \leqslant i,j \leqslant N$.

Remark 1.3. The third condition is automatic for projections in a Hilbert space whose sum is a again a projection, but not for idempotents in a general $*$-algebra. Indeed, by [BES94], there exists four non-zero idempotents p_1, p_2, p_3, p_4 acting on a vector space V whose sum is 0. Setting $p_5 = 1$ and defining an involution on the algebra generated by these five elements through the formula $p_i^* = p_i$ for all $1 \leqslant i \leqslant 5$, we get a $*$-algebra with five projections adding up to one, but such that $p_5p_1 = p_1 \neq 0$.

The algebra $\mathcal{O}(S_N^+)$ has a natural Hopf $*$-algebra structure where the coproduct is given on the generators by

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}$$

and the counit and antipode are defined by $\varepsilon(u_{ij}) = \delta_{ij}$ and $S(u_{ij}) = u_{ji}$. Last but not least, there is a positive linear form $h : \mathcal{O}(S_N^+) \to \mathbb{C}$ which is *invariant* in the sense that for all $x \in \mathcal{O}(S_N^+)$,

$$(h \otimes \mathrm{id})\Delta(x) = h(x).1 = (\mathrm{id} \otimes h)\Delta(x).$$

This means that we have an underlying *compact quantum group* (in the sense of [Wor98]) denoted by S_N^+ and called the *quantum permutation group on N points*. This definition can be generalized to encompass a graph structure on the N points.

Definition 1.4. Let Γ be a graph with N vertices and let d_Γ be its adjacency matrix. A quantum permutation $P = (p_{ij})_{1\leqslant i,j\leqslant N}$ is said to be a *quantum automorphism of* Γ if

$$Pd_\Gamma = d_\Gamma P.$$

The quotient of $\mathcal{O}(S_N^+)$ by the relations $Pd_\Gamma = d_\Gamma P$ yields a $*$-algebra $\mathcal{O}(\mathrm{QAut}(\Gamma))$ on which the map Δ factors. This defines the *quantum automorphism group* of Γ, denoted by $\mathrm{QAut}(\Gamma)$.

Our primary motivation for this work was the study of the *Connes embedding property* for the *von Neumann algebra* $L^\infty(S_N^+)$, which we now define. The positivity of h means that $h(x^*x) \geqslant 0$ for all x, and this enables to define a pre-Hilbert space structure on $\mathcal{O}(S_N^+)$. Taking the completion[1] yields a Hilbert space $L^2(S_N^+)$ and left multiplication yields an embedding of $\mathcal{O}(S_N^+)$ into $\mathcal{B}(L^2(S_N^+))$. The weak closure of the image is what we denote by $L^\infty(S_N^+)$. In this setting, the Connes embedding property[2] means that given any finite number of self-adjoint elements $a_1, \cdots, a_n \in L^\infty(S_N^+)$, any $\varepsilon > 0$ and any integer $m > 0$, there exists an integer k and matrices $M_1, \cdots, M_n \in M_k(\mathbb{C})$ such that for any $i_1, \cdots, i_p \in \{1, \cdots, n\}$ with $p \leqslant m$,

$$\left| h(a_{i_1} \cdots a_{i_p}) - \frac{1}{k}\mathrm{Tr}(M_{i_1} \cdots M_{i_p}) \right| < \varepsilon.$$

Such a statement can certainly be called a matricial approximation property. Note however that it is analytical in nature since it is really a property of h rather than of the von Neumann algebra

[1] It turns out (see for instance [NT13]) that no separation is needed.

[2] From a free probabilistic point of view, this is the same as saying that any finite set of self-adjoint noncommutative random variables has enough matricial microstates

$L^\infty(S_N^+)$. Here is another, purely algebraic property which also corresponds to realizing relations of the algebra by matrices:

Definition 1.5. A $*$-algebra A is said to be *residually finite-dimensional* (RFD for short) if there exists integers $(n_i)_{i\in I}$ such that there is an embedding of $*$-algebras

$$A \hookrightarrow \prod_{i\in I} M_{n_i}(\mathbb{C}).$$

Note that no assumption is made on the cardinality of the set I.

In other words, finite-dimensional $*$-representations separate the points. If Γ is a *finitely generated* discrete group and $A = \mathbb{C}[\Gamma]$, then this is the same as saying that Γ is a *residually finite* group. It turns out that for compact quantum groups, this is stronger that the Connes embedding property, as proven in [BBCW19, Thm 2.1]):

Theorem 1.6 (Bhattacharya-Brannan-Chirvasitu-Wang). *Let $\mathbb{G}$ be a compact quantum group and let $\mathcal{O}(\mathbb{G})$ be its canonical Hopf $*$-algebra. If $\mathcal{O}(\mathbb{G})$ is RFD, then $L^\infty(\mathbb{G})$ has the Connes embedding property.*

Sketch of proof. Pick any faithful tracial state τ on the product of matrix algebras in which $\mathcal{O}(\mathbb{G})$ embeds and restrict it to a faithful tracial state $\widetilde{\tau}$ on $\mathcal{O}(\mathbb{G})$ which is by construction *amenable* (i.e. a pointwise limit of traces which factor through finite-dimensional algebras). It is then well-known that the sequence $\widetilde{\tau}^{*n}$ converges to the Haar state pointwise and amenability is preserved under such limits so that h is amenable. Eventually E. Kirchberg proved in [Kir94] that the von Neumann algebra coming from the GNS construction of an amenable trace has the Connes embedding property. □

We can therefore focus on residual finite-dimensionality. Here is the main result we want to discuss from now on:

Theorem 1.7 (Brannan-Chirvasitu-F.). *The Hopf $*$-algebras $\mathcal{O}(S_N^+)$ and $\mathcal{O}(H_N^{s+})$ are RFD for any $N \in \mathbb{N}$ and any $s \in \mathbb{N}\cup\{+\infty\}$.*

2. Topological generation

The basic strategy for the proof of Theorem 1.7 for S_N^+ is induction on N, using the key notion of *topological generation*. This idea was first introduced by A. Chirvasitu in [Chi15] (though not under that name). To explain it, let us write $\mathbb{H} < \mathbb{G}$ if $\mathbb{G}$ and $\mathbb{H}$ are compact quantum groups with a surjective $*$-homomorphism $\pi : C(\mathbb{G}) \to C(\mathbb{H})$ intertwining the coproducts. Moreover, recall that of *Hopf $*$-ideal* in $\mathcal{O}(\mathbb{G})$ is a $*$-ideal $I \subset \mathcal{O}(\mathbb{G})$ such that

$$\Delta(I) \subset I \otimes \mathcal{O}(\mathbb{G}) + \mathcal{O}(\mathbb{G}) \otimes I.$$

Definition 2.1. Consider $\mathbb{G}_1, \mathbb{G}_2 < \mathbb{G}$ given by surjections π_1 and π_2. We say that $\mathbb{G}$ is *topologicaly generated* by $\mathbb{G}_1$ and $\mathbb{G}_2$ if the kernel of the map

$$\pi := (\pi_1 \otimes \pi_2) \circ \Delta : \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{G}_1) \otimes \mathcal{O}(\mathbb{G}_2)$$

does not contain any non-trivial Hopf $*$-ideal.

The following result [BCF20, Thm 3.3 and Thm 3.12] is the key tool to prove residual finite-dimensionality. In this statement, we see the inclusion $S_{N-1}^+ < S_N^+$ through the surjection

$$\pi_1 : \mathcal{O}(S_N^+) \to \mathcal{O}(S_{N-1}^+)$$

sending u_{11} to 1.

Theorem 2.2 (Brannan-Chirvasitu-F.)**.** *For any $N \geqslant 6$, the quantum permutation group S_N^+ is topologically generated by S_{N-1}^+ and S_N. The same holds for H_N^{s+} for any finite s.*

Before embarking into the proof, let us comment on this statement. From the point of view of quantum automorphism groups of graphs, S_N^+ is associated to the graph consisting in N disjoint vertices. Then, S_{N-1}^+ is nothing but the stabilizer of one vertex. For quantum reflection groups, H_{N-1}^{s+} is the stabilizer of one of the N disjoint cycles forming the graph. This suggests the following question:

Question 1. *Under which condition on a graph Γ is $\mathrm{QAut}(\Gamma)$ generated by $\mathrm{Aut}(\Gamma)$ and the quantum stabilizer of a subgraph ?*

Unfortunately, the proof of Theorem 2.2 does not give hints into this conjecture because of the lack of a convenient description of the intertwiner spaces of general quantum automorphism groups of graphs. We now prepare for the proof of Theorem 2.2. The main tool is the explicit description of the invariant theory of S_N^+, which comes from the work of T. Banica [Ban99b]. Let us first consider the so-called *fundamental comodule* $V = \mathbb{C}^N$ of $\mathcal{O}(S_N^+)$, which is given on the canonical basis by

$$e_i \mapsto \sum_{j=1}^{N} u_{ij} \otimes e_j.$$

All we need to know for the proof are the following three facts:

- Any irreducible comodule over $\mathcal{O}(S_N^+)$ is isomorphic to a sub-comodule of $V^{\otimes k}$ for some k,
- For any integer k, there is a generating family of the space of invariant linear maps

  $$V^{\otimes k} \to \mathbb{C}$$

  which is indexed by $NC(k)$, the set of all the *non-crossing partitions* of $\{1, \cdots, k\}$,
- If $\dim(V) \geqslant 4$, then the previous generating family $(f_p)_{p\in NC(k)}$ is linearly independent.

Proof of Theorem 2.2. We only do the proof for S_N^+, the case of H_N^{s+} begin similar. There are some preliminary manipulations which reduce topological generation to an equivalent but more tractable description. First, note that any comodule over $\mathcal{O}(\mathbb{G})$ is a comodule over both $\mathcal{O}(\mathbb{G}_1)$ and $\mathcal{O}(\mathbb{G}_2)$ and that $\mathbb{G}$ is topologically generated by $\mathbb{G}_1$ and $\mathbb{G}_2$ if and only if, for any two comodules M and N,

$$\mathrm{Hom}_{\mathcal{O}(\mathbb{G})}(M,N) = \mathrm{Hom}_{\mathcal{O}(\mathbb{G}_1)}(M,N) \cap \mathrm{Hom}_{\mathcal{O}(\mathbb{G}_2)}(M,N).$$

From this, decomposing into irreducible representations and using standard manipulations involving *Frobenius reciprocity*, the problem reduces to proving the following statement: let $V = \mathbb{C}^N$ be the fundamental comodule of $\mathcal{O}(S_N^+)$ and let

$$f : V^{\otimes k} \to \mathbb{C}$$

be a linear map which is both S_{N-1}^+-invariant and S_N-invariant. Then, f is S_N^+-invariant.

So let us consider such a map f and, for $1 \leqslant i \leqslant N$, let $V_i = e_i^\perp$. Because f is S_{N-1}^+ invariant, its restriction to $V_1^{\otimes k}$ is a linear combination of partitions maps: there exist $(\lambda_p)_{p\in NC(k)} \in \mathbb{C}$ such that

$$f_{|V_1^{\otimes k}} = \sum_{p\in NC(k)} \lambda_p f_p.$$

Let us set

$$\widetilde{f} = f - \sum_{p\in NC(k)} \lambda_p f_p.$$

This is still invariant under S_{N-1}^+ and S_N and vanishes on $V_1^{\otimes k}$. Our task is to show that it vanishes on the whole of $V^{\otimes k}$.

For this purpose, let us set $V_i' = \mathbb{C}e_i$, so that

$$V^{\otimes k} = \bigoplus_{\epsilon_1,\cdots,\epsilon_k} V_1^{\epsilon_1} \otimes \cdots \otimes V_1^{\epsilon_k}$$

where ϵ is either prime or nothing. Let us consider one of these summands where V_i appears ℓ times and denote it by W. Since S_{N-1}^+ acts trivially on V_1', there exists a linear S_{N-1}^+-equivariant isomorphism

$$\Phi : W \to V_1^{\otimes \ell}.$$

As a consequence, there exist $(\mu_p)_{p\in NC(\ell)} \in \mathbb{C}$ such that

$$\widetilde{f} \circ \Phi^{-1} = \sum_{p\in NC(\ell)} \mu_p f_p.$$

The idea now is to use the linear independence of the partition maps to conclude that $\mu_p = 0$ for all $p \in NC(\ell)$, hence that $\widetilde{f} \circ \Phi^{-1} = 0$.

To do this, set $V_{1,N} = V_1 \cap V_N$ and observe that

$$\Phi^{-1}\left(V_{1,N}^{\otimes \ell}\right) \subset V_N^{\otimes k}.$$

Now, by S_N-invariance, we can exchange e_1 and e_N without changing the value of $\widetilde{f}$, hence it vanishes on $V_N^{\otimes k}$. Thus, $\widetilde{f} \circ \Phi^{-1}$ vanishes on $V_{1,N}^{\otimes \ell}$. Now, since $N \geqslant 6$, $\dim(V_{1,N}) \geqslant 4$ so that noncrossing partition maps on $V_{1,N}^{\otimes \ell}$ are linearly independent. This forces $\mu_p = 0$ for all $p \in NC(k)$, hence $\widetilde{f} = 0$. $\square$

So far, Theorem 2.2 is useless since we do not know that $\mathcal{O}(S_5^+)$ is RFD so that we cannot start the induction. Fortunately, it was recently showed by T. Banica in [Ban21, Thm 7.10] that S_5^+ enjoys a much stronger property: there is no quantum group sitting in between S_5 and S_5^+. The idea of the proof is that by [Ban99a] and [TW18], any quantum subgroup of S_5^+ yields a subfactor at index 5, with extra properties if it contains S_5. Moreover, this correspondence is injective. Now one has to look at the complete list of subfactors at index 5 satisfying the extra properties and check that none of the corresponding quantum groups contains S_5.

Proof of Theorem 1.7 for S_N^+. First we can extend the statement of Theorem 2.2 to $N = 5$ by noticing that the quantum subgroup generated by S_4^+ and S_5 strictly contains S_5, hence equals S_5^+. Then, it was proven by A. Chirvasitu in [Chi15, Cor 2.12] that if $\mathbb{G}$ is topologically generated by $\mathbb{G}_1$ and $\mathbb{G}_2$, and if $\mathcal{O}(\mathbb{G}_1)$ and $\mathcal{O}(\mathbb{G}_2)$ are RFD, then $\mathcal{O}(\mathbb{G})$ is RFD. It therefore just remains to prove that S_4^+ is RFD (for $N \geqslant 3$, $S_N^+ = S_N$ is finite) and this follows from the existence (see for instance [BC07, Thm 4.1]) of an embedding

$$C(S_4^+) \hookrightarrow C(SU(2), M_4(\mathbb{C})).$$

□

We may here make a more general remark: it is conjectured that for any N, there is no quantum group sitting in between S_N and S_N^+. Such an inclusion is called *maximal*. This again suggests a more general question:

Question 2. *Under which condition on a graph Γ is the inclusion* $\mathrm{Aut}(\Gamma) \subset \mathrm{QAut}(\Gamma)$ *maximal ?*

Let us add that for S_N^+ this is a longstanding, apparently difficult, problem. Solving it in that case would be the first important step.

Let us now turn to quantum reflection groups. Theorem 2.2 is of no use for H_N^{s+} since the argument of [Ban21] does not work here to prove that H_5^{s+} is topologically generated by H_4^{s+} and H_5^s. Indeed, we still do not know whether this is true. We can nevertheless conclude by a trick using the *free wreath product* (see [Bic04]) structure of quantum reflection groups. More precisely, let B_i be the $*$-subalgebra of $C(S_N^+)$ generated by the i-th row and set $A_0 = \mathcal{O}(S_N^+)$. We define a sequence of $*$-algebras inductively by setting

$$A_{i+1} = (\mathcal{O}(\mathbb{Z}_s) * A_i)/\langle[\mathcal{O}(\mathbb{Z}_s), B_i]\rangle.$$

Then, $A_N = \mathcal{O}(H_N^{s+})$ and this is enough to conclude.

Proof of Theorem 1.7 for H_N^{s+}. If s is finite, one has to prove that taking the free product with a finite-dimensional $*$-algebra and quotienting by commutators of this finite-dimensional algebra with a finite-dimensional *abelian* subalgebra preserves the RFD property. This is done by embedding the $*$-algebra into an amalgamated free product of the form $A *_D A$ where A is RFD and D is finite-dimensional and then appealing to a result from [LS12].

If s is infinite, one can prove ([BCF20, Lem 2.13]) that $H_N^{\infty+}$ is topologically generated by all the H_N^{s+}'s for all finite s, and then conclude. □

Remark 2.3. The previous argument in fact gives more: if Γ is a residually finite group, then $\mathcal{O}(\widehat{\Gamma} \wr_* S_N^+)$ is residually finite-dimensional.

3. Flat matrix models

In this final section we will explain a strengthening of Theorem 1.7. We have defined quantum permutation algebras as universal objects with respect to quantum permutations, so let us look at the simplest case, namely when all the projections have rank 1. Then, because their sum is the identity, they act on a space of dimension N, i.e. they must belong to $M_N(\mathbb{C})$ and the quantum permutation is then said to be *flat*. The set of all flat quantum permutations is denoted by X_N

and is naturally a compact space as a closed (defined by polynomial equations) subspace of a grassmanian[3]. Moreover, there is a $*$-homomorphism

$$\pi : \mathcal{O}(S_N^+) \to C(X_N, M_N(\mathbb{C}))$$

sending u_{ij} to the function sending a flat quantum permutation to its (i,j)-th coefficient called the *universal flat matrix model*. A reasonable definition of "being able to recover S_N^+ from flat quantum permutations" would be that π is *inner faithful* in the following sense:

Definition 3.1. A $*$-homomorphism $\pi : \mathcal{O}(\mathbb{G}) \to A$ is said to be *inner faithful* if its kernel does not contain any non-trivial Hopf $*$-ideal.

Let us define by induction maps $\Delta^{(k)} : \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{G})^{\otimes k}$ by setting $\Delta^{(1)} = \Delta$ and

$$\Delta^{(k+1)} = (\Delta \otimes \mathrm{id}) \circ \Delta^{(k)} = (\mathrm{id} \otimes \Delta) \circ \Delta^{(k)}.$$

Then, setting $\pi^{*k} = \pi^{\otimes k} \circ \Delta^{(k)} : \mathcal{O}(\mathbb{G}) \to A^{\otimes k}$, we observe that

$$\bigcap_{k\in\mathbb{N}} \ker(\pi^{*k}) = \{0\}$$

because this intersection is a Hopf $*$-ideal. Thus, if A is residually finite-dimensional then so is $\mathcal{O}(\mathbb{G})$. It was conjectured by T. Banica that the universal flat matrix model is inner faithful for S_N^+. We managed to prove this for almost all values of N in [BCF20, Cor 4.9].

Theorem 3.2 (Brannan-Chirvasitu-F.)**.** *For $N \leqslant 5$ and $N \geqslant 10$, the universal flat matrix model of S_N^+ is inner faithful.*

We will not explain the proof in details here. Part of it relies on elementary manipulations with latin squares. Another part uses a refined topological generation result [BCF20, Prop 3.10] which we now state since it is responsible for the $N \geqslant 10$ assumption in the statement of Theorem 3.2.

Proposition 3.3. *Let $M \geqslant 5$ and $N \geqslant 2M$. Let $\mathbb{G} < S_M^+ * S_{N-M}^+$ be such that the composition*

$$C(S_M^+) \hookrightarrow C(S_M^+) * C(S_{N-M}^+) \to C(\mathbb{G})$$

is faithful. Then, $\mathbb{G}$ and S_N topologically generate S_N^+.

A close look at the proof of Theorem 3.2 in fact shows that we can even do more: we just need (at most) three points of X_N to get an inner faithful map. This means that $\mathcal{O}(S_N^+)$ has an inner faithful map to a finite-dimensional C*-algebra. In other words, S_N^+ is *inner linear*, a property analogous to linearity for discrete groups.

Let us conclude with yet another question coming naturally from this result. If Γ is a finite *vertex-transitive*[4] graph, one can consider the closed subspace $X_\Gamma \subset X_N$ of matrices which commute with the adjacency matrix d_Γ of Γ. This yields a universal flat matrix model for $\mathrm{QAut}(\Gamma)$.

Question 3. *Under which condition on a vertex-transitive graph Γ is the universal flat matrix model of $\mathrm{QAut}(\Gamma)$ inner faithful ?*

[3]It is therefore a projective $*$-algebraic manifold, the geometry of which has not been studied yet.

[4]In the sense that its automorphism group acts transitively on vertices.

Note that even for classical transitive automorphism groups (or more generally transitive subgroups of S_N), the answer is not clear. More precisely, the following criterion was given in [BF17, Prop 5.5]: the matrix model is inner faithful (in fact automatically faithful) if and only if there exist N automorphisms $\sigma_1, \cdots, \sigma_N$ such that for any vertex v and $i \neq j$, $\sigma_i(v) \neq \sigma_j(v)$. Moreover, it may be that the model is inner faithful for the quantum autmorphism group but not for the classical one, or vice-versa.

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