Studies on Free Quantum Groups : Analysis, Algebra and Probability

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- Liberation of classical groups,
- Analogue to (duals of) free groups,
- Free probability analogue of classical groups.

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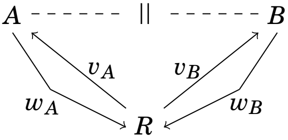
Example : the quantum permutation group S_N^+.

Part I

Analysis

The graph isomorphism game

Two graphs X and Y, two players A and B.



Winning condition

$w_A \sim w_B$ if and only if $v_A \sim v_B$.

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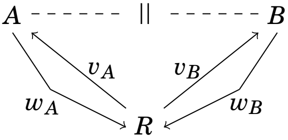

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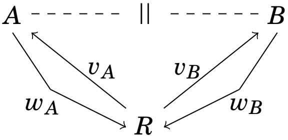

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Perfect Classical Strategy : permutation matrix P_σ such that $P_\sigma A_X = A_Y P_\sigma$.

Perfect Quantum Strategy : matrix $P = (P_{ij})_{1\leqslant i,j\leqslant N} \in M_N(\mathscr{B}(H))$ such that $PA_X = A_Y P$ and P is a *quantum permutation*, i.e.

$$P_{ij}^2 = P_{ij} = P_{ij}^* \text{ and } \sum_{k=1}^{N} P_{ik} = \mathrm{Id}_H = \sum_{k=1}^{N} P_{kj}.$$

The quantum permutation group

Definition (Wang)

Let $C\left(S_N^+\right)$ be the universal C*-algebra generated by N^2 elements P_{ij} such that

1. $P_{ij}^2 = P_{ij} = P_{ij}^*$,
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With the coproduct $\Delta : C\left(S_N^+\right) \to C\left(S_N^+\right) \otimes C\left(S_N^+\right)$

$$\Delta\left(P_{ij}\right) = \sum_{k=1}^{N} P_{ik} \otimes P_{kj}.$$

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Note : Abelianization = $S_N \rightsquigarrow S_N^+$ is a “liberation” of S_N.

Analytical properties of the quantum permutation group

- <u>BANICA :</u> Non-amenable for $N \geqslant 5$,
- <u>BRANNAN :</u> Exact, ICC (for $N \geqslant 8$), C*-simple with unique trace (for $N \geqslant 8$), Haagerup property, Rapid Decay,
- <u>DE COMMER–YAMASHITA–F. :</u> $\Lambda_{cb}\left(C\left(S_N^+\right)\right) = 1$,
- <u>VOIGT :</u> K-amenable, Baum-Connes conjecture,
- <u>BRANNAN–CHIRVASITU–F. :</u> Residually finite, hyperlinear.

Strategy to prove RFD

Assume we have quotients $\pi_i : \Gamma \twoheadrightarrow \Gamma_i$ such that

1. Γ_i is RFD for all i,
2. Any non-trivial element in Γ has a non-trivial image in at least one of the Γ_i's.

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Definition

A compact quantum group $\mathbb{G}$ is *topologically generated by* $\pi_i : C(\mathbb{G}) \to C(\mathbb{G}_i)$ if for any representation V of $\mathbb{G}$,

$$\mathrm{Fix}_{\mathbb{G}}(V) = \bigcap_i \mathrm{Fix}_{\mathbb{G}_i}(\pi_{i*}(V))$$

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- $\pi_1 : C\left(S_N^+\right) \to C\left(S_{N-1}^+\right)$ given by $u_{11} \mapsto 1$,
- $\pi_2 : C\left(S_N^+\right) \to C(S_N)$ given by abelianization.

Theorem (Branna-Chirvasitu-F.)

For all N, S_N^+ is topologically generated by S_{N-1}^+ and S_N.

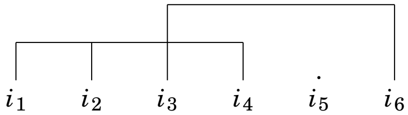
Part II

Algebra

Invariants of classical permutations

Let p be a partition of a finite set, for instance $\{\{1,2,4\},\{3,6\},\{5\}\}$

$p =$



and set

$$f_p\left(e_{i_1}\otimes e_{i_2}\otimes e_{i_3}\otimes e_{i_4}\otimes e_{i_5}\otimes e_{i_6}\right)=\delta_{i_1i_2i_4}\delta_{i_3i_6}.$$

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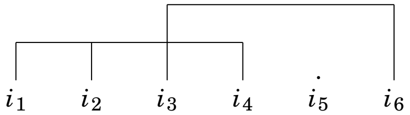

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Theorem (Jones, Martin)

Set $V = \mathbb{C}^N$ with the permutation representation of S_N. Then, for any $k \geqslant 1$,

$$\mathrm{Fix}_{S_N}\left(V^{\otimes k}\right) = \mathrm{Span}\left\{f_p^* \mid p \in P(k)\right\}$$

where $P(k)$ is the set of partitions of $\{1, \cdots, k\}$.

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A partition is *non-crossing* if it can be drawn without letting the strings cross.

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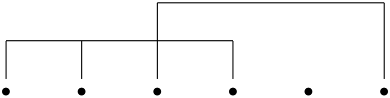

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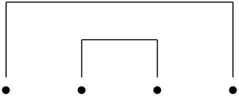

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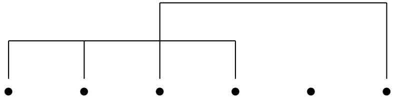
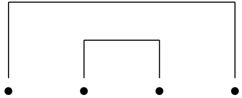
$p =$ 

$q =$ 

Theorem (Banica)

Set $V = \mathbb{C}^N$ with the representation $e_i \mapsto \sum P_{ij} \otimes e_j$ of S_N^+. Then, for any $k \geqslant 1$,

$$\mathrm{Fix}_{S_N}\left(V^{\otimes k}\right) = \mathrm{Span}\left\{f_p^* \mid p \in NC(k)\right\}$$

where $NC(k)$ is the set of non-crossing partitions of $\{1, \cdots, k\}$.

Partition quantum groups

Definition

A *category of coloured partition* $\mathscr{C}$ is a set of partitions with colours on the points and stable under the category operations : concatenations and relfections.

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Examples : $\mathbb{G}_N(P) = S_N$, $\mathbb{G}_N(NC) = S_N^+$.

This parallels the passage from classical to free probability theory.

Free quantum groups

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Recipe of a free fusion semi-ring :

1. A set S with involution $x \mapsto \overline{x}$ and $* : S \times S \to S \cup \{\emptyset\}$,
2. Build the free abelian group $R^+(S)$ over the set of words on S,
3. Set $\overline{w_1 \cdots w_n} = \overline{w}_n \cdots \overline{w}_1$ and $(w_1 \cdots w_n) * (w'_1 \cdots w'_k) = w_1 \cdots w_{n-1}(w_n * w'_1) w'_2 \cdots w'_k$,
4. Define a semi-ring structure on $R^+(S)$ by the formula

$$w \otimes w' = \sum_{w = az, w' = \overline{z}b} ab + a * b.$$

Example : $S = \{x\}$, $x * x = x$, $\overline{x} = x \rightsquigarrow R^+\left(S_N^+\right)$.

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Definition

A compact quantum group is *free* if $R^+(\mathbb{G})$ is free as a fusion semi-ring.

Applications

Motivation : Powers property for free groups.

Theorem (Banica, Brannan, Chirvasitu, Lemeux)

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If $\mathbb{G}$ is a free quantum group, then it is C^*-simple with unique trace.

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Theorem (F.)

Any free partition quantum group is isomorphic to a free product of copies of O_N^+, U_N^+ and

$$\left(\hat{*}^{n}_{S_N^+} \widetilde{H}_N^+\right) \underset{S_N^+}{*} \left(\widehat{\Lambda} \wr_* S_N^+\right).$$

Part III

Probability

Random transpositions

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Theorem (Diaconis-Shahshahani)

Both $\left(\mu_{tr}^{*2k}\right)_{k\in\mathbb{N}}$ *and* $\left(\mu_{tr}^{*2k+1}\right)_{k\in\mathbb{N}}$ *converge in* $N\ln(N)/2$ *steps in total variation distance.*

$$\|\mu-\nu\|_{TV} = \sup_{A\subset S_N} |\mu(A)-\nu(A)|.$$

The cut-off phenomenon

The result is more precise : setting $k(N) = N\ln(N)/2$, then for any $\epsilon > 0$,

$$\lim_{N\to+\infty} \|\mu_{\mathrm{tr}}^{*(1-\epsilon)k(N)} - U\|_{TV} = 1$$

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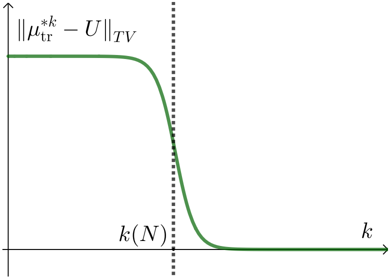
This is the *cut-off phenomenon*



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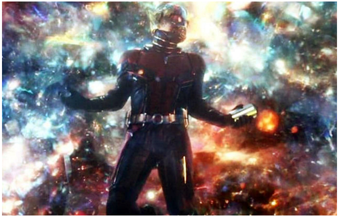

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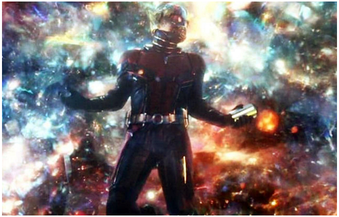

Classical probability	Quantum probability
$\mu \in \mathrm{Prob}(S_N)$	$\varphi \in S\left(C\left(S_N^+\right)\right)$
$\mu * \nu$	$(\varphi * \psi) \circ \Delta$
Uniform measure U	Haar state h
$\\|\mu - \nu\\|_{TV}$	$\sup_p \vert\varphi(p) - \psi(p)\vert,\ p \in \mathrm{Proj}\left(L^\infty\left(S_N^+\right)\right)$

Quantum transpositions

Let C be the set of transpositions, it is a conjugacy class !

$$\int_{S_N} f \mathrm{d}\mu_{\mathrm{tr}} = \int_{S_N} \left(\int_{S_N} f(\sigma \bullet \sigma^{-1}) \mathrm{d}\sigma \right) \mathrm{d}\mu_{\mathrm{tr}} = \mathbb{E}(f)(12).$$

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- $\mathbb{E} : C(S_N^+) \to C^*\left(\sum_{i=1}^{N} P_{ii}\right)$ conditional expectation,
- $\pi : C(S_N^+) \to C(S_N)$ abelianization,
- $\mathrm{ev}_{(12)} : C(S_N) \to \mathbb{C}$ evaluation map.

Definition

The *uniform measure on quantum transpositions* is the state

$$\varphi_{\mathrm{tr}} = \mathrm{ev}_{(12)} \circ \pi \circ \mathbb{E}.$$

Quantum cut-off

More explicitly, for a coefficient of the n-th irreducible representation of S_N^+,

$$\varphi_{\mathrm{tr}}(u^n_{ij}) = \delta_{ij}\frac{Q_n(N-2)}{Q_n(N)},$$

$Q_0(X) = 1$, $Q_1(X) = X - 1$ and

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Theorem (F.)

The random m-cycle walk exhibits a cut-off at $N\ln(N)/m$ steps.

The end

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