The INTERMEDIATE QUANTUM PERMUTATION PROBLEM

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Dedicated to Roland Speicher on the occasion of his birthday.

These are notes from a talk given on the occasion of Roland Speicher's sixty-fifth birthday. They deal with a problem on which Roland and I have worked in the past, without success. I will introduce and motivate the question, and then explain our failed attempt, in hope that someone can make it successful.

1 The problem

1.1 Intermediate quantum permutation groups

As the title suggests, the problem we are interested in concerns quantum groups, and more precisely *compact quantum groups*. Fortunately for the reader, even though this is the setting in which the question arose, it can be stated with very little quantum group apparatus and there is no need for the general theory, at least for the moment.

In a nutshell, the question is to decide whether there exists quantum groups $\mathbb{G}$ that sit in between the usual permutation group S_N and the quantum permutation group S_N^+. To make sense of this, we therefore have to introduce both S_N^+ as a quantum group, and the notion of quantum subgroup. The story starts with an object introduced by SH. WANG in [Wan98]. Even though the original setting involves C*-algebra, there is a purely algebraic version [Bic08] which will be more practical for our use.

DEFINITION 1.1. The *quantum permutation algebra on N points* is the universal $*$-algebra $\mathcal{O}(S_N^+)$ generated by N^2 elements $(p_{ij})_{1\leqslant i,j\leqslant N}$ such that

- $p_{ij}^2 = p_{ij} = p_{ij}^*$ for all $1 \leqslant i,j \leqslant N$;
- $\displaystyle\sum_{k=1}^N p_{ik} = 1 = \sum_{k=1}^N p_{kj}$ for all $1 \leqslant i,j \leqslant N$;
- $p_{ik}p_{ik'} = \delta_{kk'}p_{ik}$ and $p_{ki}p_{k'i} = \delta_{kk'}p_{ki}$ for all $1 \leqslant i,k,k' \leqslant N$.

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If $P \in M_N(\mathcal{O}(S_N^+))$ is the matrix with coefficients $(p_{ij})_{1\leqslant i,j\leqslant N}$, then the pair

$$\boxed{S_N^+ = \left(\mathcal{O}(S_N^+), P\right)}$$

is an *orthogonal compact matrix quantum group* in the sense of [Wor87]. The reader may refer to [Fre23] for a comprehensive treatment of the theory of orthogonal compact matrix quantum group in the algebraic setting.

To understand what S_N^+ has to do with permutations, the simplest thing to do is to consider its *abelianization*. Consider the classical permutation group S_N, seen as the group of $N \times N$ permutation matrices. If c_{ij} denotes the function sending a matrix to its (i,j)-th coefficient, then the algebra $\mathcal{O}(S_N)$ of all complex-valued functions on S_N is generated by the elements $(c_{ij})_{1\leqslant i,j\leqslant N}$, which moreover satisfy the relations of Definition 1.1. Therefore, there is a surjective $*$-homomorphism

$$\pi_{\mathrm{ab}} : \mathcal{O}(S_N^+) \to \mathcal{O}(S_N)$$

sending p_{ij} to c_{ij}. Because the right-hand side is abelian, any commutator in $\mathcal{O}(S_N^+)$ is in $\ker(\pi_{\mathrm{ab}})$, and it is not difficult to show (see for instance [Fre23, Ex 1.3]) that the converse holds.

This is a justification for thinking of S_N^+ as a *quantum* or – perhaps more accurately – *free* version of S_N. But there is more in that comparison. Indeed, consider the group law on S_N. At the level of the coefficient functions c_{ij}, we have for $\sigma, \tau \in S_N$,

$$\Delta_{S_N} : c_{ij}(\sigma\tau) = \sum_{k=1}^{N} c_{ik}(\sigma)c_{kj}(\tau) = \sum_{k=1}^{N} (c_{ik} \otimes c_{kj})(\sigma, \tau),$$

where we used the canonical $*$-isomorphism

$$\mathcal{O}(S_N \times S_N) \simeq \mathcal{O}(S_N) \otimes \mathcal{O}(S_N).$$

At the level of the function algebra, the group law is therefore encoded through the map

$$c_{ij} \longmapsto \sum_{k=1}^{N} c_{ik} \otimes c_{kj}.$$

As it turns out, a similar map exists for S_N^+, thereby providing a type of group-like structure.

Lemma 1.2. *There exists a unique $*$-homomorphism $\Delta : \mathcal{O}(S_N^+) \to \mathcal{O}(S_N^+) \otimes \mathcal{O}(S_N^+)$ such that for all $1 \leqslant i,j \leqslant N$,*

$$\boxed{\Delta(p_{ij}) = \sum_{k=1}^{N} p_{ik} \otimes p_{kj}}$$

With this in hand, we can make sense of a "quantum subgroup" of S_N^+, which is the core of our problem. This should be the analogue of a subset stable under products, which by contravariance of the $\mathcal{O}$ functor yields the following:

DEFINITION 1.3. A *quantum subgroup* of S_N^+ is a pair $\mathbb{G} = (\mathcal{O}(\mathbb{G}), \Delta_{\mathbb{G}})$, where

- $\mathcal{O}(\mathbb{G})$ is a $*$-algebra;
- There is a surjective $*$-homomorphism $\pi_{\mathbb{G}} : \mathcal{O}(S_N^+) \to \mathcal{O}(\mathbb{G})$;
- $\Delta_{\mathbb{G}} : \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{G}) \otimes \mathcal{O}(\mathbb{G})$ is a $*$-homomorphism such that

$$\Delta_{\mathbb{G}} \circ \pi = (\pi \otimes \pi) \circ \Delta.$$

If $\mathbb{G}$ is a quantum subgroup of S_N^+, then we write $\mathbb{G} \leqslant S_N^+$. There are lots of quantum subgroups of S_N^+:

- S_N itself;
- Hence, all finite groups acting faithfully on N points;
- In particular, all quantum automorphism groups of graphs on N vertices.

Note however that there are finite quantum groups which are not quantum subgroups of S_N^+ (see [BBN12] for examples), and that it is an open problem to characterize these quantum subgroups. But we are not interested in all quantum subgroups here, but only in those which contain S_N as a quantum subgroup. Here is a simple way of defining that notion.

DEFINITION 1.4. A quantum subgroup $\mathbb{G} \leqslant S_N^+$ is said to be an *intermediate quantum permutation group* if the map π factors the map $\pi_{\mathrm{ab}} : \mathcal{O}(S_N^+) \to \mathcal{O}(S_N^+)$, or more plainly, if there exists a $*$-homomorphism $\pi' : \mathcal{O}(\mathbb{G}) \to \mathcal{O}(S_N)$ such that

$$\pi_{\mathrm{ab}} = \pi' \circ \pi_{\mathbb{G}}.$$

Remark 1.5. Let us just check for sanity that in the previous definition, the map π' indeed realizes S_N as a quantum subgroup of $\mathbb{G}$. We have to prove that $(\pi' \otimes \pi') \circ \Delta_{\mathbb{G}} = \Delta_{S_N} \circ \pi'$, and by surjectivity it is enough to check that equality on the elements $\pi_{\mathbb{G}}(p_{ij})$ for all $1 \leqslant i, j \leqslant N$,

$$\begin{aligned}
(\pi' \otimes \pi') \circ \Delta_{\mathbb{G}}\left(\pi_{\mathbb{G}}(p_{ij})\right) &= (\pi' \otimes \pi') \circ (\pi_{\mathbb{G}} \otimes \pi_{\mathbb{G}}) \circ \Delta(p_{ij}) \\
&= (\pi_{\mathrm{ab}} \otimes \pi_{\mathrm{ab}}) \circ \Delta(p_{ij}) \\
&= \Delta_{S_N} \circ \pi_{\mathrm{ab}}(p_{ij}) \\
&= \Delta_{S_N} \circ \pi' \circ \pi_{\mathbb{G}}(p_{ij}).
\end{aligned}$$

We will write $S_N \leqslant \mathbb{G} \leqslant S_N^+$ if $\mathbb{G}$ is an intermediate quantum permutation group. We can now state the question this text is about.

Problem. *Is there an integer N and a compact quantum group $\mathbb{G}$ such that $S_N \leqslant \mathbb{G} \leqslant S_N^+$ but $\mathbb{G} \notin \{S_N, S_N^+\}$, in the sense that π and π' are not injective ? If yes, then we write*

$$\boxed{S_N < \mathbb{G} < S_N^+}$$

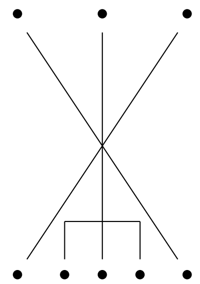
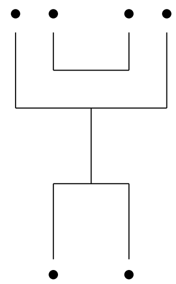
Before going further, let us summarize what is known about the problem for small values of N.

- For $1 \leqslant N \leqslant 3$, the map π_{ab} is injective (see for instance [Fre23, Ex 1.9]), hence there is no intermediate quantum permutation group;
- For $N = 4$, browsing through the list of all quantum subgroups of S_4^+ shows that there is no intermediate quantum permutation group, and this result from [BB09] led to the statement of the problem;
- For $N = 5$, it was shown in [Ban21] that there is no intermediate quantum permutation group, and the proof relies on the classification of subfactors at index 5 (see for instance the survey [JMS14]).

1.2 A COMBINATORIAL FORMULATION

My personal history with the intermediate quantum permutation problem started in 2014, when I arrived as a post-doctoral researcher in the UNIVERSITÄT DES SAARLANDES in the team of ROLAND SPEICHER, though I first heard it from M. WEBER. Anyways, the question is stated as a quantum group question, but concerns a specific type of quantum groups, namely *easy quantum groups*. These have a rich combinatorial structure, which was formalised by T. BANICA and ROLAND SPEICHER in a seminal work [BS09]. That formalisation is based on the framework of partitions of finite sets. Let us therefore give some definitions.

A *partition* is given by two integers k and ℓ and a partition p of the set $\{1,\ldots,k+\ell\}$. It is useful to represent such partitions as diagrams, in particular for computational purposes. A diagram consists in an upper row of k points, a lower row of ℓ points and some strings connecting these points if and only if they belong to the same set of the partition. Let us consider for instance the partitions $p_1 = \{\{1,8\},\{2,6\},\{3,4\},\{5,7\}\}$ and $p_2 = \{\{1,4,5,6\},\{2,3\}\}$. Their diagram representations are:

$p_1 =$  $p_2 =$ 

The key notion is that of a non-crossing partition.

DEFINITION 1.6. Let p be a partition. A *crossing* in p is a tuple $k_1 < k_2 < k_3 < k_4$ of integers such that:

- k_1 and k_3 are in the same block,
- k_2 and k_4 are in the same block,
- The four points are *not* in the same block.

If there is no crossing in p, then it is said to be a *non-crossing* partition. The set of non-crossing partitions will be denoted by NC. In the example above, p_1 is crossing whereas p_2 is not.

There are several operations on the set $\mathscr{P}$ of all partitions. To describe them, let us denote by $\mathscr{P}(k,\ell)$ the set of all partitions with k upper points and ℓ lower points.

- The *tensor product* of two partitions $p \in \mathscr{P}(k,\ell)$ and $q \in \mathscr{P}(k',\ell')$ is the partition $p \otimes q \in \mathscr{P}(k+k',\ell+\ell')$ obtained by *horizontal concatenation*, i.e. the first k of the $k+k'$ upper points are connected by p to the first l of the $\ell+\ell'$ lower points, whereas q connects the remaining k' upper points with the remaining l' lower points.

- The *composition* of two partitions $p \in \mathscr{P}(k,\ell)$ and $q \in \mathscr{P}(\ell,m)$ is the partition $qp \in \mathscr{P}(k,m)$ obtained by *vertical concatenation*. Connect k upper points by p to l middle points and then continue the lines by q to m lower points. This yields a partition, connecting k upper points with m lower points. By the composition procedure, certain loops might appear resulting from blocks around the middle points. These are simply removed.

- The *involution* of a partition $p \in \mathscr{P}(k,\ell)$ is the partition $p^* \in \mathscr{P}(\ell,k)$ obtained by turning p upside down.
- We also have a *rotation* on partitions. Let $p \in \mathscr{P}(k,\ell)$ be a partition connecting k upper points with ℓ lower points. Shifting the very left upper point to the left of the lower points (or the converse) – without changing the strings connecting the points – gives rise to a partition in $\mathscr{P}(k-\ell,\ell+1)$ (or in $\mathscr{P}(k+\ell,\ell-1)$), called a *rotated version* of p. This procedure may also be performed on the right-hand side of the k upper and l lower points. In particular, for a partition $p \in \mathscr{P}(0,\ell)$, we might rotate the very left point to the very right and vice-versa.

These operations (tensor product, composition, involution and rotation) are called the *category operations*. The name comes from the fact that they enable the construction of tensor categories, and then of compact quantum groups through *Tannaka-Krein duality* (see [Fre23, Chap 3] for a detailed exposition). The following definition summarizes the properties needed for such a reconstruction procedure to work.

DEFINITION 1.7. A collection $\mathscr{C}$ of subsets $\mathscr{C}(k,\ell) \subseteq \mathscr{P}(k,\ell)$ (for every $k,\ell \in \mathbb{N}_0$) is a *category of partitions* if it is invariant under the category operations and if the *identity partition* $| \in \mathscr{P}(1,1)$ is in $\mathscr{C}(1,1)$.

The fundamental idea of [BS09] is that to any category of partitions $\mathscr{C}$ and any integer N, one can associate a quantum group. In particular, S_N^+ corresponds to the category NC of all non-crossing partitions, while S_N corresponds to the category $\mathscr{P}$ of all partitions. It is not true however that conversely, any quantum subgroup of S_N^+ is given by a category of partitions, but it is not difficult to slightly enlarge the setting for that purpose.

DEFINITION 1.8. A *linear category of partitions* is a collection of *vector spaces* $\mathscr{D}(k,\ell)$ of *complex linear combinations of partitions* (for every $k,\ell \in \mathbb{N}_0$) which is invariant under the obvious (bi)linear extensions of the category operations, and such that $| \in \mathscr{D}(1,1)$.

Setting $\mathscr{D}(k,\ell) = \mathrm{Span}\,\mathscr{C}(k,\ell)$ associates to any category of partitions a linear category of partitions, but they do not all come from that construction. What is nevertheless true, is the following:

Proposition 1.9. *Any quantum group $\mathbb{G}$ containing S_N as a quantum subgroup is given by a linear category of partitions. Moreover, for any quantum subgroup $\mathbb{G} \leqslant S_N^+$ which is given by a linear category of partitions $\mathscr{D}$, one has*

$$\mathrm{Span}\,NC(k,\ell) \subset \mathscr{D}(k,\ell).$$

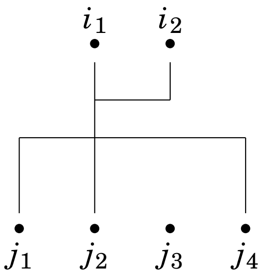
There is a subtlety in the above statement. As we said, the data of a linear category of partitions does not determine a compact quantum group, since for instance S_N^+ always corresponds to NC. The way the integer N enters the picture is at the core of [BS09] and consists in a way of associating to any partition $p \in \mathscr{P}(k,\ell)$ a linear map

$$T_p : \left(\mathbb{C}^N\right)^{\otimes k} \to \left(\mathbb{C}^N\right)^{\otimes \ell}.$$

To describe it, let $(e_i)_{1\leqslant i\leqslant N}$ be the canonical basis of $\mathbb{C}^N$. Then, we set

$$\boxed{T_p\left(e_{i_1}\otimes\cdots\otimes e_{i_k}\right) = \sum_{j_1,\cdots,j_\ell=1}^{N} \delta_p(\mathbf{i},\mathbf{j}) e_{j_1}\otimes\cdots\otimes e_{j_\ell}}$$

where $\delta_p(\mathbf{i},\mathbf{j})$ is computed as follows: we write the indices $i_1,\cdots,i_k$ on the upper row of p and the indices $j_1,\cdots,j_\ell$ on the lower row. If whenever two indices are connected, they are equal, then we set $\delta_p(\mathbf{i},\mathbf{j}) = 1$, and otherwise we set $\delta_p(\mathbf{i},\mathbf{j}) = 0$. For instance, for the partition

$p =$ 

we have

$$\delta_p((i_1, i_2), (j_1, j_2, j_3, j_4)) = \delta_{i_1 i_2 j_2} \delta_{j_1 j_4}.$$

The association $p \mapsto T_p$ is extended to linear categories of partitions in the obvious way:

$$T_{\sum_q \lambda_q q} \mapsto \sum_q \lambda_q T_q.$$

With this in hand, we can give a more combinatorial reformulation of the problem.

Problem. *Is there an integer N, a linear category of partitions $\mathscr{D}$ and integers k_1, k_2, ℓ_1, ℓ_2 such that*

- $\mathrm{Span}\{T_p \mid p \in \mathscr{D}(k_1, \ell_1)\} \neq \mathrm{Span}\{T_p \mid p \in \mathscr{P}(k_1, \ell_1)\}$;
- $\mathrm{Span}\{T_p \mid p \in \mathscr{D}(k_2, \ell_2)\} \neq \mathrm{Span}\{T_p \mid p \in NC(k_2, \ell_2)\}$?

This might not seem helpful at first sight, but provides a basic strategy to try to produce a genuine intermediate quantum permutation group. That is, take a formal linear combination of partitions

$$\gamma = \sum_{p \in \mathscr{P}(n)} \lambda_p p$$

and consider the linear category of partitions $\mathscr{D}(\gamma)$ generated by this and all non-crossing partitions. Then, the corresponding compact quantum groups sits in between S_N and S_N^+ by construction.

To ensure that it is not equal to S_N^+, there must be at least one crossing partition appearing with non-zero coefficient in γ. Moreover, because we can make arbitrary linear combinations with non-crossing partitions, we can even assume that all partitions appearing in γ have crossings. The problem now is that it might be possible that when turning these linear combinations into linear maps, on can combine them in order to obtain a map of the form T_p for some crossing partition p. But then, $\mathbb{G} = S_N$ and we loose. We can safely say that for the moment, we do not have an efficient way of ensuring that this does not happen. One last remark one can make is that there are no non-crossing partitions on at most three points, and only one on four points. As a consequence, one has to look for linear combinations of partitions on at least five points.

2 A PROBABILISTIC APPROACH

Before trying to answer the question, we should decide whether we believe or not that there exist intermediate quantum permutation groups. The results up to $N = 5$ suggest that there are none, but that could be a specificity of low dimension. We therefore need a better intuition, and there is one which is given by free probability theory. To explain it, we first have to recall some of the fundamental works of ROLAND SPEICHER on the subject, and more precisely two fundamental results: the *classification of universal independences*, and the *free de Finetti theorem*. Let us work counter-chronologically and explain these two theorems and their connection to our problem.

2.1 De Finetti theorems

2.1.1 The classical case

The de Finetti theorem is a deep result which in a sense relates probability theory and group theory. To explain this, let us consider an infinite family $(X_i)_{i\in\mathbb{N}}$ of random variables, all defined on a common probability space $(\Omega, \mathbb{P})$. We want to translate the fact that these variables are somewhat not distinguishable from one another by empirical means. Assuming that we empirically have access at least to the joint moments, the next definition is very natural.

DEFINITION 2.1. The family of random variables $(X_i)_{i\in\mathbb{N}}$ is said to be *exchangeable* if its joint distribution is invariant under permutation. Concretely, this means that for any integer $N \in \mathbb{N}$ and any permutation $\sigma \in S_N$, the tuples $(X_1, \cdots, X_N)$ and $(X_{\sigma(1)}, \cdots, X_{\sigma(N)})$ have the same joint distribution. Otherwise said, for any $1 \leqslant n \leqslant N$ and $1 \leqslant i_1, \cdots, i_n \leqslant N$,

$$\boxed{\mathbb{E}\left(X_{i_1}\cdots X_{i_n}\right) = \mathbb{E}\left(X_{\sigma(i_1)}\cdots X_{\sigma(i_n)}\right)}$$

This is really a group-theoretical property: it means that the natural action of the family of permutation groups $(S_N)_{N\in\mathbb{N}}$ given, for $\sigma \in S_N$, by

$$\sigma.X_i = \begin{cases} X_{\sigma^{-1}(i)} & \text{if } i \leqslant N \\ X_i & \text{if } i > N \end{cases}$$

preserves the probability distribution. One therefore sometimes terms this kind of property a *distributional symmetry*.

Example 2.2. Assume that the variables are all independent and identically distributed. Then, there exist integers $n_1, \cdots, n_N$ such that

$$\begin{aligned}
\mathbb{E}\left(X_{i_1}\cdots X_{i_n}\right) &= \mathbb{E}\left(X_1^{n_1}\cdots X_N^{n_N}\right) \\
&= \mathbb{E}\left(X_1^{n_1}\right)\cdots\mathbb{E}\left(X_N^{n_N}\right) \\
&= \mathbb{E}\left(X_{\sigma(1)}^{n_{\sigma(1)}}\right)\cdots\mathbb{E}\left(X_{\sigma(N)}^{n_{\sigma(N)}}\right) \\
&= \mathbb{E}\left(X_{\sigma(i_1)}\cdots X_{\sigma(i_n)}\right)
\end{aligned}$$

where independence was used in the second and fourth lines and identical distribution in the third one. Thus, $(X_i)_{i\in\mathbb{N}}$ is exchangeable.

Example 2.3. Independence is far from necessary, as the following trivial example shows: if $X_i = X_1$ for all $i \in \mathbb{N}$, then the family $(X_i)_{i\in\mathbb{N}}$ is exchangeable, simply because all joint moments are just ordinary moments of X_1.

To get a better understanding of the general situation, observe that both examples above satisfy the following general criterion: if the variables are conditionnally independent and identically distributed, then they are exchangeable. The contents of the de Finetti theorem is a converse to this observation. The first problem in general is to find a σ-algebra with respect to which these conditions are achieved, and there is a natural candidate for that.

DEFINITION 2.4. Let $(X_i)_{i\in\mathbb{N}}$ be a family of random variables. Their *tail algebra* is the σ-algebra

$$\mathcal{T} = \bigcap_{N\in\mathbb{N}} \sigma(X_i \mid i \geqslant N).$$

Intuitively, an event is measurable with respect to $\mathscr{T}$ if it does not depend upon the outcome of any fixed finite number of the variables $(X_i)_{i\in\mathbb{N}}$. We can now state the motivating theorem of this section, which relates permutational symetry with conditional independence. Even though the name of B. DE FINETTI stays attached to it, he only proved it for Bernoulli random variables in [dF37, Chap III], and the general statement hereafter is due to E. HEWITT and L. SAVAGE in [HS55].

THEOREM 2.5 (DE FINETTI, HEWITT-SAVAGE) The family of random variables $(X_i)_{i\in\mathbb{N}}$ is exchangeable if and only if the variables are independent and identically distributed *conditionnally to their tail algebra*.

2.1.2 The free case

We will now explain how this result can be generalized to a non-commutative setting. More precisely, if S_N^+ plays the rôle of a free analogue of S_N, then what does invariance under its action entail ? To answer that question, one first needs to make sense of the words “non-commutative setting”. Basically, this means that random variables should be replaced by operators on a Hilbert space. It is nevertheless convenient to assume a bit more structure on the set of random variables.

DEFINITION 2.6. A *non-commutative measure space* is given by a *von Neumann algebra* $\mathscr{N}$ acting on a Hilbert space H that is, a subalgebra $\mathscr{N}$ of the algebra $\mathscr{B}(H)$ of all bounded operators on H, which is stable under taking adjoints and closed under the *strong operator topology*[1].

We will moreover assume that $\mathscr{N}$ admits a *tracial state*, which is a linear map $\tau : \mathscr{N} \to \mathbb{C}$ such that

- $\tau(xy) = \tau(yx)$ for all $x, y \in \mathscr{N}$;
- $\tau(x^*x) \geqslant 0$ for all $x \in \mathscr{N}$;
- τ is continuous with respect to the strict operator topology.

Then, one may think of τ as the integration functional with respect to a probability measure, and we will accordingly call $(\mathscr{N}, \tau)$ a *non-commutative probability space*. Given an element $X \in \mathscr{N}$, which we may think of as a non-commutative random variable, its *moments* are the numbers

$$\boxed{m_k(X) = \tau\left(X^k\right)}$$

Classically, there are many instances where the moments of a random variable enable to recover the whole distribution, for instance if its support is compact. In the non-commutative setting, if we assume X to be self-adjoint (i.e. $X^* = X$), then X has a compact spectrum[2] $\mathrm{Sp}(X)$ and its moments therefore define a unique probability measure on $\mathrm{Sp}(X)$ which is called the *spectral measure of* X. Even though we will not use that measure later on, we will restrict ourselves to the case of self-adjoint random variables.

Exchangeability is, as we have explained above, an invariance property of the joint distribution under the action of the symmetric groups. Based on this, we should look for some kind of action[3] of S_N^+ on a family of non-commutative random variables to define a quantum version of exchangeability. Note that classically, we have

$$X_{\sigma(i)} = \sum_{j=1}^{N} X_j \delta_{j\sigma(i)} = \sum_{j=1}^{N} X_j c_{ji}(\sigma),$$

so that it would be natural for an action of S_N^+ to send X_i for some $1 \leqslant i \leqslant N$ to

$$\sum_{j=1}^{N} X_j \otimes p_{ji}.$$

This can be straightforwardly extended to monomials in the X_i's,

$$X_{i_1} \cdots X_{i_n} \mapsto \sum_{j_1,\cdots,j_n} X_{j_1} \cdots X_{j_n} \otimes p_{j_1 i_1} \cdots p_{j_n i_n}.$$

To obtain a quantum notion of exchangeability we simply have to translate the fact that τ should be invariant under the above map.

DEFINITION 2.7. A family of non-commutative random variables $(X_i)_{i\in\mathbb{N}}$ in a non-commutative probability space $(\mathcal{N}, \tau)$ is said to be *quantum exchangeable* if for any $N \in \mathbb{N}$ and $1 \leqslant i_1, \cdots, i_n \leqslant N$,

$$\boxed{\tau\left(X_{i_1} \cdots X_{i_n}\right) = \sum_{j_1,\cdots,j_n=1}^{N} \tau\left(X_{j_1} \cdots X_{j_n}\right) p_{j_1 i_1} \cdots p_{j_n i_n}} \tag{1}$$

To convince the reader that this is the correct invariance condition, let us make a sanity check. If Equation (1) is satisfied, then it also holds when p_{ij} is replaced by the coefficients of any quantum permutation matrix. In particular, it should hold for the coefficients of a permutation matrix P_σ for $\sigma \in S_N$. In that case, $p_{ij} = \delta_{i\sigma(j)}$, so that Equation (1) becomes

$$\tau(X_{i_1} \cdots X_{i_n}) = \tau(X_{\sigma(i_1)} \cdots X_{\sigma(i_n)}).$$

Therefore, quantum exchangeability implies classical exchangeability. In order to make sense of the statement of Theorem 2.5, we now have to define the tail algebra of a family of non-commutative random variables, and this is rather straightforward.

DEFINITION 2.8. Let $(X_i)_{i\in\mathbb{N}}$ be non-commutative random variables in a non-commutative probability space $(\mathcal{N}, \tau)$. Their *tail algebra* is the von Neumann subalgebra

$$\boxed{\mathcal{N}_{\text{tail}} = \overline{\bigcap_{i\in\mathbb{N}} \langle X_j \mid j \geqslant i \rangle}}$$

where the closure is taken in the strong operator topology.

It follows from the general theory of von Neumann algebras that there exists a linear map $\mathbb{E}_{\text{tail}} : \mathcal{N} \to \mathcal{N}_{\text{tail}}$ which is a *conditional expectation* in the sense that it satisfies the following properties:

- $\tau \circ \mathbb{E}_{\text{tail}} = \tau$;
- $\mathbb{E}_{\text{tail}}(axb) = a\mathbb{E}_{\text{tail}}(x)b$ for all $x \in \mathcal{N}$ and $a, b \in \mathcal{N}_{\text{tail}}$;
- $\mathbb{E}_{\text{tail}}(x^*x)$ is a positive operator for all $x \in \mathcal{N}$;
- $\mathbb{E}_{\text{tail}}(1) = 1$;
- $\|\mathbb{E}_{\text{tail}}\| = 1$.

Remark 2.9. If τ is not assumed to be tracial, then the existence of a conditional expectation can still be proven as soon as one assumes the variables to be classically exchangeable, see [KS09, Prop 4.2].

The last ingredient needed is a non-commutative analogue of independence. Note that because X_i and X_j are not assumed to commute, independence is of no use: how can we compute for instance $\mathbb{E}_{\text{tail}}(X_1X_2X_1X_2)$? What we are looking for is a rule which enables us to compute any mixed moment of the variables in terms of their individual moments in a coherent way. It turns out that they are only two ways of doing this (a precise meaning for that sentence is given in Subsection 2.2 below): classical independence and *free independence*.

The problem is that we have to introduce free independence not only in the scalar setting, but also in the conditional one and that entails subtleties. Plainly, we are going to explain what it means for non-commutative random variables to be freely independent with respect to the conditional expectation $\mathbb{E}_{\text{tail}}$. This definition should be a statement about the conditional joint moments of the variables and these would classically be numbers of the form $\mathbb{E}_{\text{tail}}(P(X_1)Q(X_2))$ for polynomials $P,Q\in\mathbb{C}[X]$. Here, first note that we should take an alternating product of polynomials since X_1 and X_2 do not commute. But this is not sufficient, since the polynomials should have coefficients in $\mathcal{N}_{\text{tail}}$, and the latter need not commute with X_1 nor with X_2. Hence, the following definition is necessary.

DEFINITION 2.10. Given a von Neumann algebra $\mathcal{N}$ and a von Neumann subalgebra $\mathcal{M}$, a *non-commutative $\mathcal{M}$-valued polynomial* is a linear combination of elements of the form

$$b_0Xb_1X\cdots b_{n-1}Xb_n$$

for some $b_0,\cdots,b_n\in\mathcal{M}$. The space of such polynomials is denoted by $\mathcal{M}\langle X\rangle$.

We are now ready for the definition.

DEFINITION 2.11. The variables $(X_i)_{i\in\mathbb{N}}$ are said to be *conditionally freely independent* with respect to $\mathcal{N}_{\text{tail}}$ if for any $i_1\neq i_2\neq\cdots\neq i_n$ and any $p_1,\cdots,p_m\in\mathcal{N}_{\text{tail}}\langle X\rangle$ such that

$$\mathbb{E}_{\text{tail}}\left(p_k(X_{i_k})\right)=0$$

for all $1\leqslant k\leqslant n$, we have

$$\boxed{\mathbb{E}_{\text{tail}}\left(p_1(X_{i_1})\cdots p_n(X_{i_n})\right)=0}$$

The definition may seem complicated at first sight, but the main point is that it enables to compute by induction any mixed moment. Let us do this in a simple case. Setting $Y_i=X_i-\mathbb{E}(X_i)$, we have for $i\neq j$

$$\begin{aligned}
\mathbb{E}_{\text{tail}}(X_iX_j) &= \mathbb{E}_{\text{tail}}\left((Y_i+\mathbb{E}_{\text{tail}}(X_i))(Y_j+\mathbb{E}_{\text{tail}}(X_j))\right)\\
&= \mathbb{E}_{\text{tail}}(Y_iY_j)-\mathbb{E}_{\text{tail}}\left(\mathbb{E}_{\text{tail}}(X_i)Y_j\right)-\mathbb{E}_{\text{tail}}\left(Y_i\mathbb{E}_{\text{tail}}(X_j)\right)\\
&+\mathbb{E}_{\text{tail}}\left(\mathbb{E}_{\text{tail}}(X_i)\mathbb{E}_{\text{tail}}(X_j)\right)\\
&= \mathbb{E}_{\text{tail}}(Y_iY_j)-\mathbb{E}_{\text{tail}}(X_i)\mathbb{E}_{\text{tail}}\left(Y_j\right)-\mathbb{E}_{\text{tail}}\left(Y_i\right)\mathbb{E}_{\text{tail}}(X_j)\\
&+\mathbb{E}_{\text{tail}}(X_i)\mathbb{E}_{\text{tail}}(X_j)\\
&= 0-0-0+\mathbb{E}_{\text{tail}}(X_i)\mathbb{E}(X_j)\\
&= \mathbb{E}_{\text{tail}}(X_i)\mathbb{E}_{\text{tail}}(X_j)
\end{aligned}$$

Remark 2.12. For products of two variables, the result in the free case in the same as in the classical one. To see the difference, one should use non-commutativity. An indeed, assuming for instance $\mathbb{E}_{\text{tail}}(X_1)=0=\mathbb{E}_{\text{tail}}(X_2)$ for simplicity, free independence yields

$$\mathbb{E}_{\text{tail}}(X_1X_2X_1X_2)=0,$$

while for classically independent random variables,

$$\mathbb{E}_{\text{tail}}(X_1X_2X_1X_2)=\mathbb{E}_{\text{tail}}(X_1^2X_2^2)=\mathbb{E}_{\text{tail}}(X_1^2)\mathbb{E}_{\text{tail}}(X_2^2).$$

We are now in position to state the main result of [KS09], which provides an analogue of the de Finetti theorem in the context of free probability theory, using S_N^+ as the symmetries.

THEOREM 2.13 (KÖSTLER-SPEICHER) A family of non-commutative random variables $(X_i)_{i\in\mathbb{N}}$ is quantum exchangeable if and only if the variables are freely independent and identically distributed with respect to their tail algebra.

2.2 UNIVERSAL INDEPENDENCES

Comparing the classical and free de Finetti theorems, one might be led to the following reasoning: since S_N and S_N^+ are exactly the symmetries characterizing classical and free independence, families of intermediate quantum permutation groups $S_N < \mathbb{G}_N < S_N^+$ should encode intermediate notions of independence. This is where another result of ROLAND SPEICHER comes into play: *there is no other notion of independence*. The goal of this section is to give a precise meaning to that statement.

Classical and free independence give general methods to compute the mixed moments of families of (non-commutative) random variables. This immediately raises the question of the existence of more notions of independence, and of a classification thereof. One issue is of course to properly axiomatize this concept, and we will work in the original framework of [Spe97]. First, we consider unital algebras equipped with a unital linear form, which plays the role of integration with respect to a fixed measure. Given two such object $(\mathscr{A}_1, \varphi_1)$ and $(\mathscr{A}_2, \varphi_2)$, one can form the *unital free product*[4]

$$\mathscr{A} = \mathscr{A}_1 * \mathscr{A}_2,$$

and the question is to find a universal way of producing a unital linear map $\varphi = \varphi_1 \bullet \varphi_2$ on $\mathscr{A}$.

Second, we need a few combinatorial definitions. The main idea is that we want a rule to compute $\varphi(a_1 \cdots a_n)$ recursively by grouping terms which belong to the same factor. For classical independence, this is done by simply gathering all terms belonging $\mathscr{A}_1$ on one side, all terms belonging to $\mathscr{A}_2$ on the other side and then multiplying the values of the original linear forms on these two products. In general, things are trickier and require heavier notations. Given a partition $p \in \mathscr{P}(k)$, we will say that $a_1 \cdots a_k \in \mathscr{A}_p$ if whenever m and n are connected in p, a_m and a_n belong to the same factor. In that situation, we can gather terms which are connected and the following definition is practical to formulate this:

DEFINITION 2.14. A *block* of a partition p is a maximal set of points which are connected in p.

If $a_1 \cdots a_k \in \mathscr{A}_p$, and if $B_1, \cdots, B_b$ denote the blocks of p ordered according to their leftmost point, then we may set

$$\boxed{\varphi_p(a_1 \cdots a_k) = \varphi_{k(1)}\left(\overrightarrow{\prod}_{i\in B_1} a_i\right) \times \cdots \times \varphi_{k(b)}\left(\overrightarrow{\prod}_{i\in B_1} a_i\right)}$$

where the arrow indicates that the product is taken in the same order as in the original product, and $k(\cdot)$ denotes the index of the factor to which all the elements belong. The previous formula in fact makes sense more generally for any partition q which is compatible with p in the sense that any block of q is wholly contained in a single block of p. We will write $q \prec p$ in that case. With this in hand, we can give the definition of a universal product.

DEFINITION 2.15. A *universal product* is a construction associating to (φ_1, φ_2) a new linear form $\varphi_1 \bullet \varphi_2$ such that

- $\varphi_1 \bullet (\varphi_2 \bullet \varphi_3) = (\varphi_1 \bullet \varphi_2) \bullet \varphi_3$;
- For any $k \in \mathbb{N}$ and $p, q \in \mathscr{P}(k)$ with $q \prec p$, there exists $\mathbf{t}(p;q) \in \mathbb{C}$ such that for all $a_1 \cdots a_k \in \mathscr{A}_p$,
$$\varphi_p(a_1 \cdots a_k) = \sum_{q \prec p} \mathbf{t}(p;q)\varphi_q(a_1 \cdots a_k).$$

With this in hand, the main result of [Spe97] can be easily stated.

THEOREM 2.16 (SPEICHER) There exist exactly two universal independences, namely classical independence and free independence.

Before turning to the core of this text, let us make a remark. The previous discussion might be a motivation to believe that there is no intermediate quantum permutation group, but it really points to a more precise and weaker problem.

Problem: Assume we have a sequence $S_N < \mathbb{G}_N < S_N^+$ of compact quantum groups such that $\mathbb{G}_N \neq S_N$ for N large enough. If a family of non-commutative random variables is invariant under the action of all the $\mathbb{G}_N$'s, is it automatically free and identically distributed conditionally to the tail algebra ?

This statement is closer to the idea that if there is no notion of independence between classical and free, then there should be no "quantum symmetries" implementing it. Moreover, the problem above comes quite close to deep results of W. LIU, who proved such generalized de Finetti theorems in [Liu15].

3 MIXING PROBABILITY AND COMBINATORICS

A few years ago, ROLAND SPEICHER and I had a try at the intermediate quantum permutation group problem. Even though it did not lead us to a solution, the method proved useful in other works involving quantum permutation groups (see for instance [FSW25]). Besides, our strategy might still be useful for the original problem, since it does not incorporate an important ingredient yet, as we will see at the end.

The starting point is the quantum analogue of the Haar measure, which is called the *Haar state* and exists for any compact quantum group. Let us gather the essential features of that object in one statement (see for instance [Fre23, Sec 5.2] for details).

THEOREM 3.1 (WORONOWICZ) Let $(\mathbb{G}, u)$ be an intermdiate quantum permutation group. Then, there exists a unique state $h_{\mathbb{G}} : \mathcal{O}(\mathbb{G}) \to \mathbb{C}$ such that

$$(h_{\mathbb{G}} \otimes \mathrm{id}) \circ \Delta_{\mathbb{G}} = h_{\mathbb{G}}.1 = (\mathrm{id} \otimes h_{\mathbb{G}}) \circ \Delta_{\mathbb{G}}.$$

Moreoveor, h is tracial and faithful[5].

So let $S_N < \mathbb{G} < S_N^+$ and let φ be the state on $\mathcal{O}(S_N^+)$ defined by

$$\varphi = h_{\mathbb{G}} \circ \pi.$$

We will investigate the *moments* of φ in the sense of non-commutative probability theory, that is to say its values on monomials in the coefficients of u.

DEFINITION 3.2. For an integer k, a *moment of order k* of φ is a number of the form

$$\boxed{\varphi\left(p_{i_1 j_1} \cdots p_{i_k j_k}\right)}$$

The key point is that the definition of φ forces its moments to satisfy several symmetry properties. To express these properties, we will need some combinatorial notations. For a tuple of integers $I = (i_1, \cdots, i_k)$ let $\ker(I)$ denote the partition of the set $\{1, \cdots, k\}$ where a and b belong to the same subset if and only if $i_a = i_b$. Here is the first important property.

Lemma 3.3. *The moment $\varphi(p_{i_1j_1}\cdots p_{i_kj_k})$ only depends on the partitions $p = \ker(i_1, \cdots, i_k)$ and $q = \ker(j_1, \cdots, j_k)$. We will denote by $\varphi(p, q)$ that common value.*

Proof. This comes from the fact that since $S_N < \mathbb{G}$, $h_{\mathbb{G}}$ must in particular be invariant under permutations. More precisely, we observe that

$$\begin{aligned}(\varphi \otimes \pi) \circ \Delta &= (h_{\mathbb{G}} \otimes \mathrm{id}) \circ (\pi \otimes \pi) \circ \Delta \\ &= (h_{\mathbb{G}} \otimes \mathrm{id}) \circ \Delta \circ \pi \\ &= (h_{\mathbb{G}} \circ \pi).1 \\ &= \varphi.1\end{aligned}$$

and applying π' to both sides of that equality yields

$$\varphi\left(p_{i_1j_1}\cdots p_{i_kj_k}\right) = \sum_{n_1\cdots n_k} \varphi\left(p_{i_1n_1}\cdots p_{i_kn_k}\right) c_{n_1j_1}\cdots c_{n_kj_k}.$$

Evaluating at a permutation $\sigma \in S_N$ and using the fact that $c_{ij}(\sigma) = \delta_{i\sigma(j)}$, this becomes

$$\varphi\left(p_{i_1j_1}\cdots p_{i_kj_k}\right) = \varphi\left(p_{i_1\sigma(j_1)}\cdots p_{i_k\sigma(j_k)}\right).$$

As a consequence, we can permute arbitrarily the j-indices without changing the value of φ. Using $(\mathrm{id} \otimes \varphi) \circ \Delta$ instead yields the same result for the i-indices. This means that the moment only depends on which indices are equal and which are different but not on their precise value. This is exactly what the partitions $\ker(i_1, \cdots, i_k)$ and $\ker(j_1, \cdots, j_k)$ encode. ■

We can now work directly on the partitions and exploiting the various definitions yields a list of constraints on the moments of φ.

Lemma 3.4. *The numbers $\varphi(p, q)$ enjoy the following properties:*

1. *Rotating an endpoint of p and q to the other end does not change the value of the moment;*
2. $\varphi(p, q) = \overline{\varphi(q, p)}$;
3. *If two neighbouring points are connected in p but not in q (or the converse) then the corresponding moment vanishes;*
4. *If two neighbouring points are connected both in p and in q then $\varphi(p, q) = \varphi(\widetilde{p}, \widetilde{q})$, where $\sim$ corresponds to contracting these two points,*
5. *If p or q contains a singleton, then $\varphi(p, q)$ is a linear combination of moments of order at most k with no singletons in the partitions,*
6. *We have*
$$\varphi(p, q) = \sum_{r \in P(k)} C(N, r)\varphi(p, r)\varphi(r, s),$$
where $C_N(r)$ is the number of tuples K of $\{1, \cdots, N\}$ such that $\ker(K) = r$.

Proof. Let us work itemwise.

1. Let us for instance rotate the rightmost point to the left. This is equivalent to considering the moment
$$\varphi\left(p_{i_kj_k}p_{i_1j_1}\cdots p_{i_{k-1}j_{k-1}}\right)$$
which is the same as the original one by traciality of φ. The same holds if the first point is rotated to the end.

2. The quantum group structure of S_N^+ provides an anti-homomorphism S of $\mathcal{O}(S_N^+)$ called the *antipode* and satisfying $S(p_{ij}) = p_{ji}$. Moreover, it follows from the general theory that $\varphi\circ S = \varphi$. Applying S exchanges the i and j indices, but also reverses the order of the factors. Nevertheless, one can further apply the involution of $\mathcal{O}(S_N^+)$ to put the factors back in the original order, at the price of taking complex conjugates[6].

3. Assume that the points a and $a+1$ are connected in p but not in q. This means that the monomial on which we evaluate φ has a factor of the form $p_{i_aj_a}p_{i_aj_{a+1}}$ with $j_a \neq j_{a+1}$. By the defining relations of $\mathcal{O}(S_N^+)$, this product must vanish, hence also the corresponding moment.

4. Assume that points a and $a+1$ are connected both in p and in q. This means that the monomial on which we evaluate φ has a factor of the form $p_{i_aj_a}^2 = p_{i_aj_a}$, hence the moment can be obtained by contraction.

5. Up to rotating, we may assume that the first point of p is a singleton. Using the second defining relations of $\mathcal{O}(S_N^+)$, we then have
$$\varphi\left(p_{i_2j_2}\cdots p_{i_kj_k}\right) = \sum_{i=1}^{N}\varphi\left(p_{ij_1}p_{i_2j_2}\cdots p_{i_kj_k}\right).$$
Consider a term in this sum. If $i = i_a$ for some $a \in \{1,\cdots,k\}$, then the moment is of the form $\varphi(p',q)$ where p' has one singleton less than p since its first point is now connected to at least one other point. If $i = i_1$ then this is just $\varphi(p,q)$. As a consequence, the latter can be written as a linear combination of a moment of lower order and moments where the first partition has one singleton less than p. Applying the same procedure again, and doing the same for q, we end up with no singletons at all.

6. Noticing that $(\pi\otimes\pi)\circ\Delta = \Delta_{\mathbb{G}}\circ\pi$, we have
$$\begin{aligned}\varphi\left(p_{i_1j_1}\cdots p_{i_kj_k}\right) &= h_{\mathbb{G}}\circ\pi\left(p_{i_1j_1}\cdots p_{i_kj_k}\right)\\ &= (h_{\mathbb{G}}\otimes h_{\mathbb{G}})\circ\Delta_{\mathbb{G}}\circ\pi\left(p_{i_1j_1}\cdots p_{i_kj_k}\right)\\ &= (\varphi\otimes\varphi)\circ\Delta\left(p_{i_1j_1}\cdots p_{i_kj_k}\right)\\ &= \sum_{n_1,\cdots,n_k}\varphi\left(p_{i_1n_1}\cdots p_{i_kn_k}\right)\varphi\left(p_{n_1j_1}\cdots p_{n_kj_k}\right)\end{aligned}$$

and the result follows. ■

With this in hand, we can elucidate all moments of φ up to order four. Note that we already have two such available families of moments : those of $h_{S_N^+}$ and those of h_{S_N}.

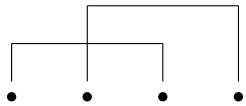
Proposition 3.5. *Let $(\mathbb{G},u)$ be an intermediate quantum permutation group. Then, either all moments of φ up to order four equal those of $h_{S_N^+}$, or they all equal those of h_{S_N}.*

Proof. Let us start by observing that a partition of length at most three with no singleton must be the one-block partition. Thus, denoting by $\mathbf{1}_k$ the one-block partition of $\{1,\cdots,k\}$, all moments of length $k \leqslant 3$ are determined by

$$\varphi(\mathbf{1}_k, \mathbf{1}_k) = \varphi(p_{ij}) = \varphi(\mathbf{1}_1).$$

Summing over i shows that this number equals $1/N$. In particular, moments up to order 3 do not depend on $\mathbb{G}$.

We now consider moments of order four. By Lemma 3.4, these are linear combinations of moments of lower order (which are independent from $\mathbb{G}$) and moments of the form $\varphi(p,q)$ where p and q do not contain singletons and do not have two neighbouring points connected up to a rotation. The only partition satisfying these two conditions is

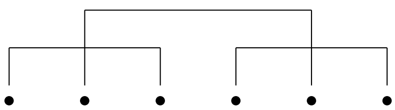
$p_c = \{\{1,3\},\{2,4\}\} =$ 

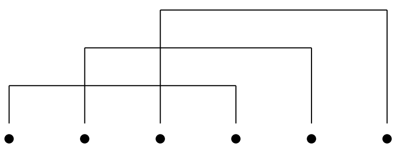
Thus, any moment of order at most 4 is of the form $\lambda + \mu\varphi(p_c, p_c)$ for some $\lambda, \mu \in \mathbb{C}$. But then, point (6) of Lemma 3.4 shows that $\varphi(p_c, p_c)$ satisfies a polynomial equation of degree two. As a consequence, there are at most two possible values for it. For $N \geqslant 4$, we already know two distinct solutions for this : $h_{S_N^+}(p_c, p_c)$ and $h_{S_N}(p_c, p_c)$. Since they determine the values of all the other moments of order at most four, the result follows. ∎

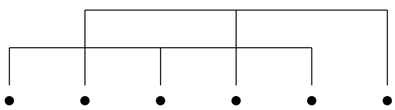
Remark 3.6. One can of course write down explicitly the quadratic relation alluded to above, and deduce from it the two possible values:

$$h_{S_N}(p_c, p_c) = \frac{1}{N(N-1)}$$
$$h_{S_N^+}(p_c, p_c) = \frac{2N-5}{N(N-1)(N^2+5N+1)}$$

The argument can be pushed slightly further, by observing that for a partition on a odd number of points, there will always be either a singleton or a point connected to one of its neighbours up to rotation. As a consequence, moments up to order five are in fact completely determined by the value of $\varphi(p_c, p_c)$. Things break down however at $k = 6$. The reason for that is that there are now three partitions that have no singleton and no connected neighbours, namely (up to rotations)

$p_1 = \{\{1,3\},\{2,5\},\{4,6\}\} =$ 

$p_2 = \{\{1,4\},\{2,5\},\{3,6\}\} =$ 

$p_3 = \{\{1,3,5\},\{2,4,6\}\} =$ 

This gives us a system of quadratic equations with six unknowns, which might have more than two triples of solutions. To conclude, let us simply point out an interesting fact: the Haar state is – as its name suggests – a state. In other words, its values of elements of the form x^*x is always a positive real number. This gives a set of additional constraints on the values, which might help rule out solutions of the systems of quadratic equations until there are only two left. We did not find an efficient way of doing this so far unfortunately, but perhaps some reader of this document will do !

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NOTES

1. The strong operator topology on $\mathscr{B}(H)$ is the topology for which a sequence $(T_n)_{n\in\mathbb{N}}$ of bounded operators converges to T if and only if $\|T_n(\xi)-\xi\| \to 0$ for all $\xi \in H$.

2. The spectrum of an element X in a complex unital algebra A is the set of $\lambda \in \mathbb{C}$ such that $X-\lambda.1$ is not invertible.

3. There is a well-defined notion of action of a compact quantum group on a von Neumann algebra (see for instance [De 17] for a comprehensive treatment) but we do not need it here, since we are only interested in the notion of invariance of a state under the action.

4. The reader may refer to [Fre23, Sec 6.4.1] for the definition of the unital free product of unital algebras.

5. A state φ on a $*$-algebra is *faithful* if $\varphi(x^*x) = 0$ implies $x = 0$.

6. It follows from the fact that $h_{\mathbb{G}}(x^*x) \geqslant 0$ for all $x \in \mathscr{O}(\mathbb{G})$ (that is the definition of a *state*) that $h_{\mathbb{G}}(x^*) = \overline{h_{\mathbb{G}}(x)}$.