Apollonian carpets and the boundary of the Kähler cone of a hyperkähler manifold

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Abstract

The ample cone of a compact Kähler n-manifold M is the intersection of its Kähler cone and the real subspace generated by integer (1,1)-classes. Its isotropic boundary is the set of all points η on its boundary such that $\int_M \eta^n = 0$. We are interested in the relation between the shape of the isotropic boundary of the ample cone of a hyperkähler manifold and the dynamics of its holomorphic automorphism group G. In this case, the projectivization of the ample cone is realized as an open, locally polyhedral subset in a hyperbolic space $\mathbb{H}$. The isotropic boundary S is realized as a subset of the hyperbolic boundary (the absolute) A of $\mathbb{H}$, which is naturally identified with a Euclidean sphere. It is clear that the isotropic boundary S contains the limit set of G acting on its ample cone. We prove that, conversely, all irrational points on S belong to the limit set. Using a result of N. Shah about limiting distributions of curves under geodesic flow on hyperbolic manifolds, we prove that every real analytic curve in S is contained in a geodesic sphere in S, and in presence of such curves the limit set is the closure of the union of these geodesic spheres. We study the geometry of such fractal sets, called Apollonian carpets, and establish a link between the geometry of the Apollonian carpet and the structure of the automorphism group.

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1 Introduction

The abyss and its veil thus came to stand for elements of inner and outer experience, respectively, both of which were part of the necessary synthesis. In Belyj's later writings the covering over the abyss does not merely represent the deception of logical form, but rather **"the golden Apollonian carpet"***, which is essential to the creative art as is the spirit of Dionysus which dwells in the abyss.*

LAURA GOERING, "THE ABYSS OF LANGUAGE AND THE LANGUAGE OF THE ABYSS: BELY'S SYMBOLIST ESSAYS", THE ANDREJ BELYJ SOCIETY NEWSLETTER, 8, 1989

1.1 Boundary of the ample cone of a hyperkähler manifold

Throughout this paper, we treat a hyperkähler manifold as a complex manifold of Kähler type, admitting a holomorphic symplectic form. Let M be a compact hyperkähler manifold of maximal holonomy (Definition 2.2; see Subsection 2.1 for more details and references). Then the second cohomology of M is equipped with a non-degenerate, rational, bilinear symmetric form q of signature $(3, b_2 - 3)$, called **the Bogomolov–Beauville–Fujiki** form, or the **BBF form** (Subsection 2.1). Complex automorphisms of M act on $H^2(M)$ preserving q and the Hodge decomposition. Denote by $\mathcal{K}_M$ the Kähler cone, i.e. the set of all Kähler classes in $H^{1,1}(M)$, and by $\mathcal{A}_M$ the set of all ample classes in $\mathrm{NS}_{\mathbb{R}}(M) = (H^{1,1}(M) \cap H^2(M, \mathbb{Q})) \otimes_{\mathbb{Q}} \mathbb{R}$. This set is called **the ample cone of** M; it is obtained as the intersection of $\mathrm{NS}_{\mathbb{R}}$ and the Kähler cone of M.

This paper grew out from an attempt to understand the geometry of the boundary of $\mathcal{K}_M$ and $\mathcal{A}_M$. These cones were described as follows in [AV1]. Let $\mathcal{C}_M \subset H^{1,1}(M, \mathbb{R})$ be the one of two connected components of the set of all vectors with positive square that contains the Kähler cone, and consider the set of so-called "MBM classes" $H_2(M, \mathbb{Z})$(Subsection 2.2). Denote by R_I the set of all effective MBM classes of type (1,1) (that is,

MBM classes represented by a linear combination of complex curves with positive coefficients). Then

$$\mathcal{K}_M = \{\eta \in \mathcal{C}_M \quad | \quad \forall u \in R_I, \quad q(\eta, u) > 0\} \tag{1.1}$$

In other words, the Kähler cone is a subset of $\mathcal{C}_M$ given by a system of linear inequalities. Its boundary inside $\mathcal{C}_M$ is locally polyhedral ([AV2]). In this paper, we are interested in the **isotropic boundary** of $\mathcal{K}_M$ and $\mathcal{A}_M$, that is, the part of the boundary contained in the set $\{\eta \in H^{1,1}(M, \mathbb{R}) \ | \ q(\eta, \eta) = 0\}$. We denote the isotropic boundary by $\partial\mathcal{K}_M$ and $\partial\mathcal{A}_M$.

When the Picard rank of M is not maximal, that is, $\operatorname{rk}(\mathrm{NS}) < b_2 - 2$, the isotropic boundary contains an open subset in the cone of isotropic elements $\eta \in H^{1,1}(M)$, as follows from [V2, Proposition 3.4]. However, $\partial\mathcal{A}_M = \partial\mathcal{K}_M \cap \mathrm{NS}_{\mathbb{R}}$ is a complicated fractal set, if R_I is non-empty and the Picard number of M is at least 3. Further in this paper we relate the geometry of this fractal with the geometric properties of the manifold M and its automorphism group.

1.2 The limit set of the automorphism group

Dynamical properties of the group $\mathrm{Aut}(M)$ of holomorphic automorphisms of a hyperkähler manifold M is closely related to the dynamics of the group of isometries of the hyperbolic space. We always use one of the standard models of the hyperbolic space, identifying it with the projectivisation of the cone of positive vectors in a quadratic vector space of signature $(1, n)$. We are mostly interested in the case when the hyperbolic space is $\mathbb{H} := \mathbb{P}(\mathcal{C}_M \cap \mathrm{NS}_{\mathbb{R}})$, when M is projective of Picard number at least 3.

The action of $\mathrm{Aut}(M)$ on $H^{1,1}(M, \mathbb{R})$ preserves the BBF form, giving a homomorphism $\mathrm{Aut}(M) \longrightarrow O(1, b_2 - 3)$.

Any holomorphic automorphism of M which preserves a Kähler class $[\omega] \in \mathcal{K}_M$, acts on M by isometries of the corresponding Calabi-Yau metric. Since the automorphism group of M is discrete, the group of automorphisms preserving $[\omega]$ is finite, and $\mathrm{Aut}(M)$ is mapped to $O(1, b_2 - 3)$ with finite kernel. We are interested in the dynamics of its image acting on the corresponding hyperbolic space $\mathbb{P}\mathcal{C}_M$, and, even more, on a smaller hyperbolic space, $\mathbb{P}(\mathcal{C}_M \cap NS_{\mathbb{R}})$.

Consider a quadratic vector space (V, q) of signature $(1, n)$, and let $\mathbb{H} := \mathbb{P}(V^+)$ be the corresponding hyperbolic space. The **absolute** of $\mathbb{H}$ is the set $\mathrm{Abs} := \mathbb{P}(\{\eta \in V \ | \ q(\eta, \eta) = 0\})$, identified with the Euclidean sphere. Given a group $\Gamma \subset SO(1, n)$ acting on $\mathbb{H}$, its **limit set** is the set of all $x \in \mathrm{Abs}$ obtained as limit points of an orbit of Γ.

It turns out that $\partial\mathcal{A}_M$ is closely related to the limit set of the automorphism group. We call a cohomology class $\eta \in \mathrm{NS}_{\mathbb{R}}$ **irrational** if it is not proportional to a rational cohomology class.

Theorem 1.1: Let M be a hyperkähler manifold, and $\mathrm{Aut}(M)$ the group of its holomorphic automorphisms. Then all irrational points on the isotropic boundary $\partial(\mathbb{P}\,\mathrm{Amp})$ belong to the limit set of $\mathrm{Aut}(M)$ acting on $\mathbb{P}\,\mathrm{Amp}$. Conversely, any point in the limit set of $\mathrm{Aut}(M)$-action belongs to $\partial(\mathbb{P}\,\mathrm{Amp})$.
Proof: Theorem 5.20. ■

By [CNS, Theorem 1.2.24], all points of the limit set $\Lambda(\Gamma)$ have dense orbits in $\Lambda(\Gamma)$, hence this set is fractal. Its fractal geometry is the main subject of this paper.

1.3 CHOPBs and Apollonian carpets

The group of automorphisms of a hyperkähler manifold can be described explicitly in terms of its MBM classes and Hodge structure, as shown in [Ma], [AV2] and [AV4]. Let $\mathsf{Mon} \subset O(H^2(M,\mathbb{Z}),q)$ be the group generated by the monodromy of the Gauss-Manin local systems for all complex deformations of M. This group is called **the monodromy group** (Section 4). It has finite index in $O(H^2(M,\mathbb{Z}),q)$ ([V1]). The **Hodge monodromy group** Mon_I is its subgroup which preserves the Hodge decomposition associated to a complex structure I of hyperkähler type. It is mapped to $O(H^2(M,\mathbb{Z})\cap H^{1,1}(M))$ with finite kernel and finite index (Proposition 4.3).

Let Γ be the image of Mon_I in $O(H^2(M,\mathbb{Z})\cap H^{1,1}(M))$, and Γ_0 a subgroup of Γ which preserves the Kähler cone. As shown by Markman [Ma], the group $\mathrm{Aut}(M)$ of automorphisms of a hyperkähler manifold is a subgroup of Mon_I which preserves the Kähler cone. Then the natural map $\mathrm{Aut}(M) \longrightarrow \Gamma_0$ has finite kernel and finite index.

Let $\mathbb{H}$ be the hyperbolic space obtained as the projectivisation of the positive cone Pos in $\mathrm{NS}_{\mathbb{R}}$. Since Γ has finite index in $O(H^2(M,\mathbb{Z})\cap H^{1,1}(M))$, the quotient $\mathcal{M} := \mathbb{H}/\Gamma$ is a hyperbolic orbifold of finite volume. The ample cone can be described as

$$\mathcal{A}_M = \{\eta \in \mathrm{Pos} \quad | \quad q(\eta,u) > 0 \ \ \forall u \in R_I\}$$

as follows from (1.1). In fact, $\mathcal{A}_M$ is a connected component of the complement in Pos to the union of hyperplanes $z^\perp$, where z belongs to the set R of MBM classes of type $(1,1)$. The group Γ preserves the set R and acts on it with finitely many orbits ([AV2]). Therefore, the image of the

union of the orthogonal complements to R is a finite union of closed hyperbolic orbifolds $\mathcal{M}_1, ..., \mathcal{M}_k$ immersed in $\mathcal{M} = \mathbb{H}/\Gamma$ (Theorem 4.11). The complement $\mathcal{M} \setminus \bigcup_i \mathcal{M}_i$ is a union of finitely many locally convex orbifold polyhedral spaces. Each of these orbifolds is obtained as the quotient of a connected component U of $\mathbb{H} \setminus \bigcup_{z \in R} z^\perp$, with U being a convex, open subset of $\mathbb{H}$ called **an ample chamber**. Every ample chamber P is the ample cone of an appropriate birational model (M, I_P) of M ([Ma], [AV4]). The automorphism group of (M, I_P) is mapped to the stabilizer of P in Mon_I with finite kernel and finite index (Proposition 4.3).

This way, we reduce the study of the automorphism group of a hyperkähler manifold to the study of the orbifold fundamental group of a certain type of a polyhedral space, called CHOPB.

Definition 1.2: Let $\mathcal{H} = \mathbb{H}^n/\Gamma$ be a hyperbolic orbifold, and $S_1, ..., S_n \subset \mathcal{H}$ a collection of closed locally geodesic immersed hypersurfaces. A **convex hyperbolic orbifold with polyhedral boundary** (CHOPB) is a connected component of the complement $\mathcal{H} \setminus \bigcup_i S_i$.

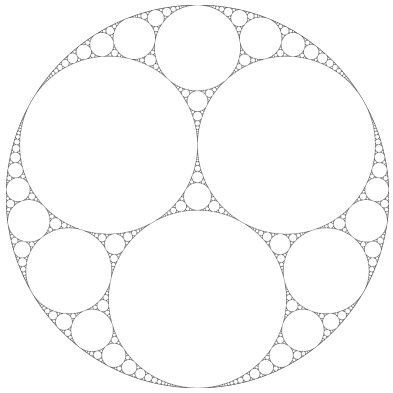
Let $\mathcal{P}$ be a CHOPB, and $P \subset \mathbb{H}^n$ a connected component of its preimage in $\mathbb{H}^n$. **The isotropic boundary** of the CHOPB is the set $\overline{P} \cap \mathrm{Abs}$, where $\mathrm{Abs} = \partial\mathbb{H}^n$. It turns out that the isotropic boundary of a CHOPB is either a countable set or a fractal (Theorem 5.20); in particular, its interior is empty, unless $P = \mathbb{H}^n$

As mentioned in Subsection 1.2, the limit set Λ of $\Gamma_0 = \pi_1(\mathcal{P})$ is in the isotropic boundary of P, and, conversely, all non-cusp points of the isotropic boundary belong to Λ (Theorem 5.20).

The fractal nature of this set is explored in some detail in Section 7. We define **the Apollonian carpet** of the isotropic boundary of a CHOPB as the union of all positive dimensional irreducible real analytic subvarieties in its isotropic boundary, and prove that each component of the Apollonian carpet is a geodesic sphere S in Abs (Theorem 6.2). Moreover, the convex hull of each S is a hyperbolic subspace in the closure of P, and its image in the quotient $\mathbb{H}^n/\Gamma$ is a closed hyperbolic immersed submanifold $\mathcal{M}$ in the closure of $\mathcal{P}$. This implies, in particular, that the lattice $\pi_1(\mathcal{M})$ is a subgroup of $\pi_1(\mathcal{P})$. (Theorem 7.3; here this result is stated for CHOPBs associated with the hyperkähler manifolds, but the proof works generally).

1.4 Indra's pearls, Apollonian gaskets and geometry of K3 surfaces (after A. Baragar et al)

The "Apollonian gasket" is a classical concept, known since Apollonius of Perga, c. 240 BC - c. 190 BC. ([MSW]). Apollonian gasket is obtained by taking three kissing circles, inscribing a fourth circle kissing these three, then inscribing a circle inside a triangle formed by the segments of three circles, and so on, ad infinitum:



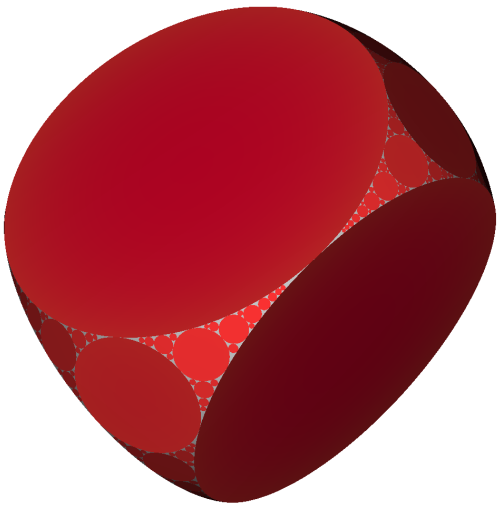
This construction can (and should) be performed on a 2-sphere. Since the Moebius group acts transitively on triples of pairwise kissing circles, the Apollonian gasket on $\mathbb{C}P^1$ is unique up to a conformal equivalence ([MSW, p. 201]).

Interestingly enough, the Apollonian gasket is a limit set of a discrete subgroup of the Moebius group $PSL(2,\mathbb{C})$ acting on $\mathbb{C}P^1$. This is why the Apollonian gasket is one of the central subjects of "Indra's Pearls" [MSW], by D. Mumford, C. Series and D. Wright. In [MSW] this was shown by an explicit argument with matrices. Using Theorem 6.2 together with Theorem 5.20, we obtain a conceptual argument which implies that the Apollonian gasket is a limit set (Subsection 7.4.1).

The Apollonian gasket is ubiquitous in physics ([Sa]) and even in art: the largest artwork in history, "Earth Drawings" by Jim Denevan, drawn in the sand of Black Rock Desert (Nevada), depicts the Apollonian gasket.



This fractal can be obtained as the isotropic boundary of a universal covering of a CHOPB. Consider a circle S in $\mathbb{C}P^1 = \mathrm{Abs}(\mathbb{H}^3)$. The convex hull of S in $\mathbb{H}^3$ is a hyperbolic 2-plane, denoted by W, and S is its isotropic boundary. The 2-plane W can be obtained as a projectivization of the positive cone in a subspace $\widetilde{W} = \mathbb{R}^{1,2}$ in $R^{1,3}$. Choose an integer lattice in $R^{1,3}$, and consider the corresponding lattice group $\Gamma := SO_{\mathbb{Z}}(1,3)$. If $\widetilde{W}$ was chosen rational, the stabilizer $\mathrm{St}_\Gamma(S)$ of S in $\Gamma := SO_{\mathbb{Z}}(1,3)$ is a group which is commensurable with $SO_{\mathbb{Z}}(1,2)$. We can choose $\widetilde{W}$ in such a way that its images under Γ can be tangent, but never intersect transversally (Subsection 7.4.1).[1] Then its image in the hyperbolic manifold $\mathcal{M} = \mathbb{H}^3/\Gamma$ is $\mathbb{H}^2/\mathrm{St}_\Gamma(S)$, a hyperbolic manifold of finite volume. The universal cover of the corresponding CHOPB $\mathcal{P}$ is a locally finite hyperbolic polyhedron, which looks as follows:



(for a more clear visual, in this picture we used the Kleinian model of a

[1] A similar example was obtained earlier in [Ba1].

hyperbolic space; in Klein's model, completely geodesic hypersurfaces are represented by plane discs).

From this picture, it is visually clear that the isotropic boundary of $\mathcal{P}$ is the Apollonian gasket. A formal proof of this observation can be obtained by application of Theorem 7.3, which describes the Apollonian carpet of a CHOPB for an arithmetic hyperbolic orbifold (the statement is made for CHOPBs of hyperkähler origin, but the proof works for this general situation).

This construction was realized in an ample cone of a K3 surface by A. Baragar in [Ba3]. In a serie of papers ([Ba1, Ba2, Ba3, Ba4, Ba5]), Baragar constructed many examples illustrating the fractal nature of the isotropic boundary of the ample cone of a K3 surface.

For K3 surfaces, an MBM class is an integer class of square -2, usually abbreviated as "(-2)-class". The BBF form q on a K3 surface is the intersection form. The corresponding hyperbolic manifold is $\mathbb{P}(\mathrm{Pos}\cap \mathrm{NS}_{\mathbb{R}})/\Gamma$, where Γ is a sublattice of finite rank in $SO(\mathrm{NS}_{\mathbb{Z}}, q)$, and the corresponding hyperbolic polyhedron $P \subset \mathbb{P}(\mathrm{Pos}\cap \mathrm{NS}_{\mathbb{R}})$ is one of the connected components of $\mathbb{P}(\mathrm{Pos}\cap \mathrm{NS}_{\mathbb{R}})\setminus \bigcup_i S_i^{\perp}$, where $\{S_i\} \subset \mathrm{NS}_{\mathbb{Z}}$ is the set of all MBM classes.

In [Ba1], Baragar constructed an arithmetic lattice Λ of rank 4, such that its isometry group acts on the set of (-2)-classes transitively. This means that $O(\Lambda)$ acts on the set of faces of P transitively. Moreover, for any pair of (-2)-classes $a, b \in \Lambda$, the lattice $\langle a, b\rangle$ generated by a and b has signature $(1,1)$ (this means that their orthogonal complements don't intersect in the absolute) or $(0,1)$ (and this means that the orthogonal circles are tangent). This gives a fractal formed by kissing circles, which is conformally equivalent to the Apollonian gasket by [MSW, p. 201], as we indicated earlier.

In his earliest work on the subject ([Ba1]), Baragar established the fractal nature of the isotropic boundary of the ample cone of a K3 surface, and computed the Hausdorff dimension of this fractal, identified with the limit set of the automorphism group. This computation is based on a result of D. Sullivan, relating the Hausdorff dimension of the limit set with the growth rate of the number of orbit points in a ball of radius r.

This observation was later used by I. Dolgachev [Do], who used the bounds on the Hausdorff dimension to estimate the growth rate of the number of curves on a K3 surface in a given orbit of the automorphism group. Note that the framework built in the present paper can be used to extend these estimates to an arbitrary hyperkähler manifold. Indeed, the Hausdorff dimension of the isotropic boundary of an ample cone is bounded from below by the dimension of the components of its Apollonian carpet.

1.5 The structure of this paper

The paper is organized as follows. In Section 2 we recall some basic definitions and facts about the hyperkähler manifolds. We also introduce the convex cones in $H^{1,1}(M,\mathbb{R})$ which are central for our further investigation. In Section 3 we prove that whenever ample, movable or effective cone does not coincide with the positve cone in the real Neron-Severi group, its isotropic boundary is nowhere dense (the cone has no "round part"); on the contrary, when the Picard number is not maximal, the Kähler cone always has a "round part", which is open in the boundary of the positive cone. The statement for the ample cone is mentioned in [Mat], with a partial proof, and for the effective cone it is proved in [De]. Our starting point was an earlier version of [De] where the proof for the effective cone was given for the known families of IHS manifolds only, by adapting Kovacs' argument [Ko1] on K3 surfaces. It seems that the simple general proof of this result relying only on hyperbolic geometry was not widely known. We have therefore decided to give a self-contained and complete account here.

In Sections 4 and 5, we study the boundary of the ample cone using hyperbolic geometry. In Section 6, we recall a theorem of N. Shah which permits us to prove that the "analytic" part of the absolute of the ample cone is a union of spheres. In Section 7, we describe these spheres more explicitly and give a number of examples of their possible behavior. We also make a connection to the structure of the automorphism group of the hyperkähler manifold.

2 Hyperkähler manifolds

We briefly recall the basic definitions of hyperkähler geometry. We refer to [Bea], [Hu1] and [Ma] for a more detailed discussion.

2.1 The basic definitions

Let M be a complex manifold with a Riemannian metric g. Denote by I the complex structure on M, and recall that g is Hermitian if for any pair of tangent vectors u and v we have $g(u,v)=g(Iu,Iv)$. In this case one defines $\omega(u,v)=g(Iu,v)$. Then ω is a two-form on M, and M is called Kähler if $d\omega=0$. If M is Kähler then ω is a symplectic form on M.

Definition 2.1: A **hyperkähler structure** on a manifold M is a Riemannian metric g and a triple of complex structures I, J, K, satisfying

quaternionic relations $IJ = -JI = K$, such that g is Kähler for I, J and K. A **hyperkähler manifold** is a manifold with a hyperkähler structure.

A hyperkähler manifold carries symplectic forms ω_I, ω_J and ω_K associated to g and the complex structures I, J and K, as above. Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega_I = \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I). The holomorphic symplectic forms on (M, J) and (M, K) are be defined analogously.

Recall that for a Riemannian metric g and a point $x \in M$ the subgroup of $GL(T_xM)$ generated by the operators of parallel transport (along all loops with endpoints at x, with respect to the Levi–Civita connection) is called the holonomy group of g. A hyperkähler manifold can be equivalently defined as a Riemannian manifold whose holonomy is contained in $Sp(n)$ (the group of all orthogonal isometries of a quaternionic vector space preserving I, J, K).

In this paper we will consider only one special class of hyperkähler manifolds.

Definition 2.2: A compact hyperkähler manifold M is **of maximal holonomy**, or **IHS**, if $\pi_1(M) = 1$ and the holonomy group of the Levi–Civita connection equals $Sp(n)$, where $2n = \dim_{\mathbb{C}}(M)$.

For the rest of this paper, we abuse the language slightly, and "a hyperkähler manifold" for us always means a **compact hyperkähler manifold of maximal holonomy.**

On any hyperkähler manifold of maximal holonomy, the space of holomorphic two-forms is one-dimensional, spanned by the holomorphic symplectic form described above.

Fix a hyperkähler manifold M and consider its singular cohomology group $H^2(X, \mathbb{Q})$. One of the main properties of hyperkähler manifolds of maximal holonomy is the existence of a canonical quadratic form $q \in \mathrm{Sym}^2 H^2(X, \mathbb{Q})^*$, called the **Bogomolov–Beauville–Fujiki** form or the **BBF form**. This form can be characterized by the following property (see [So] for a more detailed discussion):

Theorem 2.3: (Fujiki, [F]) For any hyperkähler manifold M of complex dimension $2n$ there exists a constant $c_M \in \mathbb{Q}$ and a quadratic form q on $H^2(X, \mathbb{Q})$ such that for any $a \in H^2(M, \mathbb{Q})$ we have

$$\int_M a^{2n} = c_M q(a)^n.$$

We may assume, after rescaling, that the BBF form q is integral and primitive on $H^2(M,\mathbb{Z})$. In the above theorem the integral on the left hand side denotes the pairing of the cohomology class $a^{2n} \in H^{4n}(M,\mathbb{Q})$ with the fundamental class of M determined by the canonical orientation of M, see [So] for details.

The form q has signature $(3, b_2(M) - 3)$. If I is one of the complex structures that appears as part of a hyperkähler structure on M, then q is negative definite on the primitive forms of I-type $(1,1)$ and positive definite on the subspace of $H^2(M,\mathbb{R})$ spanned by ω_I, $\mathrm{Re}(\Omega_I)$ and $\mathrm{Im}(\Omega_I)$, where ω_I is the Kähler form and Ω_I is the holomorphic symplectic form.

2.2 Cones in the cohomology of a hyperkähler manifold

Let us fix a hyperkähler manifold M of maximal holonomy with a hyperkähler structure (g, I, J, K). We will consider the Hodge decomposition of the cohomology of M with respect to I if not stated otherwise.

The main goal of the present paper is to study the shape of several natural convex cones in the real vector space $H^{1,1}(M,\mathbb{R})$. The latter means the intersection of $H^{1,1}(M)$ with $H^2(M,\mathbb{R})$ inside $H^2(M,\mathbb{C})$. As was explained above, the vector space V carries the quadratic form q, the restriction of the BBF form to the $(1,1)$-part of the cohomology. The form q has signature $(1, b_2(M) - 3)$ on $H^{1,1}(M,\mathbb{R})$.

Recall that the Néron–Severi group $\mathrm{NS}(M)$ is the image of the Picard group $\mathrm{Pic}(M)$ in $H^2(M,\mathbb{Z})$ under the map that computes the first Chern class of a line bundle. By Hodge theory, the group $\mathrm{NS}(M)$ is contained in $H^{1,1}(M,\mathbb{R})$, and we will denote by $\mathrm{NS}_{\mathbb{R}}$ the $\mathbb{R}$-subspace of $H^{1,1}(M,\mathbb{R})$ spanned by $\mathrm{NS}(M)$. The condition that a hyperkähler manifold M is projective is equivalent to the condition that $q|_{\mathrm{NS}_{\mathbb{R}}}$ has signature $(1, \rho(M) - 1)$, where $\rho(M)$ is the Picard number of M, see [Hu1].

Let us recall the definitions of the cones that we are going to study.

1. The positive cone $\mathcal{C}_M \subset H^{1,1}(M,\mathbb{R})$ is the connected component of the cone $\{v \in H^{1,1}(M,\mathbb{R}) \mid q(v) > 0\}$ that contains the class of the Kähler form ω_I.

2. The Kähler cone $\mathcal{K}_M \subset \mathcal{C}_M$ is the cone consisting of the cohomology classes of all Kähler forms on M. The cone $\mathcal{K}_M$ is open and its closure is called the nef cone.

3. The birational Kähler cone $\mathcal{BK}_M \subset \mathcal{C}_M$ is the cone consisting of the cohomology classes of the form $f^*[\omega]$, where $f\colon M \dashrightarrow M'$ is an arbitrary bimeromorphic map to a hyperkähler manifold M' and ω is an arbitrary Kähler form on M'. Note that the cone $\mathcal{BK}_M$ is not convex (but its closure is, as we shall see from its description below).

4. The pseudoeffective cone $\mathcal{P}_M \subset H^{1,1}(M,\mathbb{R})$ consists of the cohomology classes of all closed positive $(1,1)$-currents on M. It is known that $\mathcal{P}_M$ is closed and contains $\mathcal{C}_M$.

5. The ample cone $\mathcal{A}_M \subset \mathrm{NS}_{\mathbb{R}}(M)$ is spanned by the classes of ample line bundles. The ample cone is open in $\mathrm{NS}_{\mathbb{R}}(M)$.

6. The movable cone $\mathcal{M}_M \subset \mathrm{NS}_{\mathbb{R}}(M)$ is spanned by the classes of movable line bundles, i.e. the line bundles whose complete linear systems have base loci of codimension at least two.

We summarize the relations between the cones introduced above. We have the following inclusions:

$$\mathcal{K}_M \subset \mathcal{BK}_M \subset \mathcal{C}_M \subset \mathcal{P}_M.$$

Moreover, it is known that $\overline{\mathcal{BK}_M}$ is dual to $\mathcal{P}_M$ with respect to q, see [Hu2, Corollary 4.6].

The ample and the movable cones are related to the Kähler cone and the birational Kähler cone as follows. For the ample cone we have $\mathcal{A}_M = \mathcal{K}_M \cap \mathrm{NS}_{\mathbb{R}}(M)$. For the movable cone $\mathcal{M}_M^{in} = (\overline{\mathcal{BK}_M})^{in} \cap \mathrm{NS}_{\mathbb{R}}(M)$, where the superscript *in* means the interior of the cone, see [Ma, Proposition 5.6 and Lemma 6.22].

The Kähler cone and the birational Kähler cone admit an important description in terms of wall and chamber decomposition of the positive cone. To give this description we need to recall the notions of an MBM class and of a prime exceptional divisor.

A cohomology class $z \in H^2(M,\mathbb{Z})$, $q(z) < 0$ on a hyperkähler manifold M with a complex structure I is called **an MBM class** if some (equivalently, any) deformation (M,I') of (M,I) with $\mathrm{NS}_{\mathbb{R}}$ generated by z contains a rational curve; see [AV5]. We will denote by $\mathrm{MBM}(M) \subset H^2(M,\mathbb{Z})$ the set of primitive MBM classes. The set $\mathrm{MBM}(M)$ is a deformation invariant of a hyperkähler manifold M.

Example 2.4: Recall that a (-2)-class on a K3 surface M is a cohomology class $\eta \in H^2(M, \mathbb{Z})$ with self-intersection -2. The MBM classes on a K3 surface are exactly the (-2)-classes ([AV1]).

Recall from [AV2] that the set of MBM classes on a hyperkähler manifold M is q-bounded: there exists a constant $\alpha > 0$ such that for all $v \in \mathrm{MBM}(M)$ we have $-\alpha \leq q(v) < 0$. Now consider the intersection $\mathcal{S} = \mathrm{MBM}(M) \cap \mathrm{NS}(M)$. Then the collection of hyperplanes $v^\perp$, $v \in \mathcal{S}$ is locally finite in $\mathcal{C}_M$ (see e.g. [SSV, Proposition 5.12]) and the complement $\mathcal{C}_M \setminus \cup_{v\in\mathcal{S}} v^\perp$ is a disjoint union of open chambers, the Kähler cone $\mathcal{K}_M$ being one of the chambers, see [AV1]. In other words, the MBM classes are orthogonal to the faces of the Kähler chambers, and all faces of the Kähler cones of deformations and birational models of M are realized in this way ([AV4]).

Now we recall an analogous description of the closure of the birational Kähler cone $\overline{\mathcal{BK}(M)}$. Recall that a **prime exceptional divisor** is a prime divisor D on M whose cohomology class is negative: $q([D]) < 0$. Prime exceptional divisors are uniruled, see [Bou]. We will denote by $\mathrm{PE}(M) \subset \mathrm{NS}(M)$ the set of cohomology classes of prime exceptional divisors on M.

It follows from the uniruledness (see [AV5, Lemma 2.11]) that the cohomology class $[D]$ of a prime exceptional divisor D is MBM. We will call an MBM class of type $(1,1)$ **divisorial** if its orbit under the action of the Hodge monodromy group (see [Ma]) contains a class that is proportional to the class of a prime exceptional divisor. Let us now denote by $\mathcal{S}$ the set of divisorial primitive MBM classes on a hyperkähler manifold M. The complement $\mathcal{C}_M \setminus \cup_{v\in\mathcal{S}} v^\perp$ is again a disjoint union of open chambers and the closure of one of these chambers is $\overline{\mathcal{BK}(M)}$.

Let us finally recall the following description of the pseudoeffective cone given in [Bou]:

$$\mathcal{P}_M = \overline{\mathcal{C}_M + \sum_{v\in\mathrm{PE}(M)} \mathbb{R}_+ \cdot v}, \tag{2.1}$$

which means that the boundary of the cone $\mathcal{P}_M$ is locally polyhedral away from $\mathcal{C}_M$, with extremal rays spanned by the classes of the prime exceptional divisors.

3 Hyperbolic spaces and arithmetic lattices

In this section we recall some standard notions of hyperbolic geometry in the form convenient for applications to hyperkähler manifolds. As a general reference on hyperbolic geometry one may use, for example, [Ka1], where most of the facts that we recall are proven in greater generality. In order to streamline the exposition, we choose not to deduce what we need from those general theorems, but to give self-contained elementary proofs, claiming no originality whatsoever. We also immediately apply the statements about the hyperbolic spaces to hyperkähler manifolds, reformulating them in the terminology introduced in the previous section.

3.1 The hyperbolic space

Let V be a vector space over $\mathbb{R}$ and $q \in \mathrm{Sym}^2 V^*$ a quadratic form of signature $(1,d)$. Let $V^+ = \{x \in V \mid q(x) > 0\} \subset V$ be the cone of positive vectors and $\mathbb{P}(V^+)$ its image in the real projective space $\mathbb{P}(V)$. Analogously, set $V^- = \{v \in V \mid q(v) < 0\}$. Denote by $Q = \{x \in \mathbb{P}(V) \mid q(x) = 0\} \subset \mathbb{P}(V)$ the real projective quadric defined by the quadratic form q. The quadric Q is the boundary sphere of the hyperbolic space $\mathbb{P}(V^+)$. For a non-zero vector $v \in V$ we will denote by $[v]$ its image in $\mathbb{P}(V)$.

The group $O(V,q)$ has 4 connected components and we denote the connected component of the identity by $SO^+(V,q)$. It is clear that $\mathbb{P}(V^+) \simeq SO^+(V,q)/SO(d)$. The hyperbolic space $\mathbb{P}(V^+)$ carries an $SO^+(V,q)$-invariant Riemannian metric g. It is easy to see that this metric is unique up to a constant multiplier. One can also check that $SO^+(V,q)$ is the group of all oriented isometries of $\mathbb{H}^d = (\mathbb{P}(V^+), g)$. We recall the following classification of isometries of a hyperbolic space.

Theorem-definition: Let $\alpha \in SO^+(1,d)$ be an isometry acting on $\mathbb{H}^d$. Then one and only one of the following cases occurs:

1. α has an eigenvector x with $q(x) > 0$ (then α is called **elliptic**);
2. α has an eigenvector x with $q(x) = 0$ and eigenvalue λ satisfying $|\lambda| > 1$ (then α is called **hyperbolic**, or **loxodromic**);
3. α has a unique eigenvector x with $q(x) = 0$ (then α is called **parabolic**).

3.2 Wall and chamber decompositions of the hyperbolic space

For the applications to hyperkähler geometry, we need to study wall and chamber decompositions of the hyperbolic space $\mathbb{P}(V^+)$. Such decompositions are determined by a subset $\mathcal{S} \subset V^-$ of elements orthogonal to the walls. We will always assume the following:

$$\mathcal{S} \neq \emptyset, \quad -\mathcal{S} = \mathcal{S}, \quad \mathcal{S} \text{ is discrete in } V \text{ and } q\text{-bounded}. \tag{3.1}$$

The latter condition means that there exists some constant $\alpha > 0$ such that $-\alpha \leq q(v) < 0$ for any $v \in \mathcal{S}$.

For $v \in V^-$ denote $H_v = v^\perp \subset V$ and

$$V^\diamondsuit = V^+ \setminus \cup_{v \in \mathcal{S}} H_v. \tag{3.2}$$

It is well-known that the collection of hyperplanes H_v, $v \in \mathcal{S}$ is locally finite in V^+, see e.g. [SSV, Proposition 5.12]. Therefore $V^\diamondsuit$ is a union of disjoint open connected components which we will call the **$\mathcal{S}$-chambers** of V^+.

Given an $\mathcal{S}$-chamber $\mathcal{C}$ we will denote by $\mathbb{P}(\mathcal{C})$ its image in $\mathbb{P}(V)$. We are interested in the structure of the boundary of $\mathbb{P}(\mathcal{C})$ at infinity, or, more precisely, in the structure of the set $\mathbb{P}(\overline{\mathcal{C}}) \cap Q$. The properties of this set depend on the geometry of $\mathcal{S}$, and there are two substantially different cases to consider: either the set $\mathcal{S}$ is "big", i.e. spans V and is invariant under the action of an arithmetic lattice in $SO(V, q)$, or $\mathcal{S}$ is contained in a proper subspace $N \subset V$. In the applications, the first case arises when V is the space spanned by the Néron–Severi lattice and $\mathcal{C}$ is the ample or the movable cone, and the second case arises when $V = H^{1,1}(X, \mathbb{R})$ and $\mathcal{C}$ is the Kähler cone (the subspace N is then spanned by the Néron-Severi lattice). We will treat these two cases separately below.

3.2.1 The case when $\mathcal{S}$ spans V

In the above setting, assume moreover that $V = V_\mathbb{Z} \otimes \mathbb{R}$ for some lattice $V_\mathbb{Z}$. Denote $V_\mathbb{Q} = V_\mathbb{Z} \otimes \mathbb{Q}$ and assume that q is rational, i.e. $q \in \mathrm{Sym}^2 V_\mathbb{Q}^*$. Let $\Gamma \subset O(V_\mathbb{Z}, q)$ be a subgroup of finite index and assume that $\mathcal{S} \subset V_\mathbb{Z}$ and $\mathcal{S}$ is Γ-invariant. The following proposition is well known. We sketch an elementary proof for the sake of completeness.

Proposition 3.1: In the above setting assume that $d \geq 2$. For any $x \in Q$ the orbit Γx is dense in Q in the analytic topology.

Proof: Let $U \subset Q$ be a non-empty open subset. We have to prove that $\Gamma x \cap U \neq \emptyset$. We will consider separately two cases: when Q contains a rational point and when it doesn't. We will construct some $\gamma \in \Gamma$ with $\gamma^n x \in U$ for sufficiently big n. In the first case γ will be parabolic, and in the second case hyperbolic.

Case 1. Assume that Q contains a rational point: let $u \in V_{\mathbb{Q}}$ be a non-zero vector with $q(u) = 0$. In this case, intersecting Q with lines passing through $[u] \in Q$ we see that rational points are dense in Q. Therefore we may assume that $[u] \in U$.

Let $G \subset O(V, q)$ be the stabilizer of u. The group G is unipotent and its Lie algebra is $u \wedge u^\perp \subset \Lambda^2 V$, where we identify the Lie algebra of the orthogonal group $O(V, q)$ with $\Lambda^2 V$ via the form q. For any $v \in u^\perp$, $w \in V$ and $t \in \mathbb{R}$ the action of $e^{tu\wedge v} \in G$ on w is given by the following formula:

$$e^{tu\wedge v} w = w + tq(v, w)u - tq(u, w)v - \frac{t^2}{2} q(u, w)q(v, v)u. \tag{3.3}$$

Rescaling u, we may assume that $u \in V_{\mathbb{Z}}$. The assumption $d \geq 2$ implies that we may find $v \in u^\perp \cap V_{\mathbb{Z}}$ with $q(v, v) \neq 0$. Replacing v by its multiple, we may assume that the operator $\gamma = e^{u\wedge v}$ preserves $V_{\mathbb{Z}}$ and, furthermore, that $\gamma \in \Gamma$. Choose $w \in V$ so that $x = [w]$. Then $q(u, w) \neq 0$ unless u and w are proportional. In any case, we see from the equation (3.3) that $\gamma^n x \to [u]$ for $n \to +\infty$, therefore $\gamma^n x \in U$ for n big enough.

Case 2. Assume that Q does not contain rational points. We may still find a two-dimensional rational subspace $W_{\mathbb{Q}} \subset V_{\mathbb{Q}}$ such that $q|_{W_{\mathbb{Q}}}$ is non-degenerate, $x \notin \mathbb{P}(W)$ and $\mathbb{P}(W) \cap U \neq \emptyset$, where $W = W_{\mathbb{Q}} \otimes \mathbb{R} \subset V$. Then $Q \cap \mathbb{P}(W)$ consists of two points $[u]$ and $[u']$, where $u, u' \in W$ and we may assume that $[u] \in U$.

By our assumption, the quadratic form $q|_W$ does not admit non-zero rational isotropic vectors. After possibly rescaling q by a rational factor, we may represent it in an appropriate basis by a diagonal matrix with entries 1 and $-d$, where $d > 1$ is a square-free integer. We may identify the field $K = \mathbb{Q}[\sqrt{d}]$ with W as a rational vector space, then q is identified with the norm of an algebraic number. It is well known that the group of units in the ring of algebraic integers $\mathcal{O}_K$ is infinite. Choose an element $a \in \mathcal{O}_K^*$ of infinite order. Multiplication by a preserves the lattice $\mathcal{O}_K \subset W$. Replacing a by its power we may assume that there exists an element $\gamma \in \Gamma$ such that $\gamma|_{W^\perp} = \mathrm{Id}$ and $\gamma|_W$ acts as multiplication by a. Then u and u' are eigenvectors of $\gamma|_W$ with eigenvalues λ and λ^{-1}, and possibly replacing γ by its inverse we may assume that $\lambda > 1$. Recall that $x \notin \mathbb{P}(W)$ by construction and note that $x \notin \mathbb{P}(W^\perp)$, since $q|_{W^\perp}$ is negative-definite. Since λ is the

unique eigenvalue of γ with absolute value greater than one, it is clear that $\gamma^n x \to [u]$ as $n \to +\infty$ and so $\gamma^n x \in U$ for n big enough. ■

Recall our notation: $Q = \{x \in \mathbb{P}(V) \mid q(x) = 0\}$ is the boundary of $\mathbb{P}(V^+)$ and $\Gamma \subset O(V_{\mathbb{Z}}, q)$ is a subgroup of finite index.

Proposition 3.2: Assume that $d \geq 2$. For any $x \in \mathbb{P}(V) \setminus Q$ we have $Q \subset \overline{\Gamma x}$, where the overline denotes the closure in $\mathbb{P}(V)$ in the analytic topology.

Proof: The closed subset $\overline{\Gamma x} \subset \mathbb{P}(V)$ is clearly Γ-invariant, so by Proposition 3.1 it is enough to show that $\overline{\Gamma x}$ contains at least one point of Q.

Fix an arbitrary Euclidean norm $\|\cdot\|$ on V and some $v \in V$ with $[v] = x$. Since Γ is an arithmetic lattice in a non-compact orthogonal group and the orbit of v under the action of this group is unbounded in V, we can find a sequence $\gamma_i \in \Gamma$, such that for $y_i = \gamma_i v$ we have $\|y_i\| \to +\infty$ for $i \to +\infty$. Since the unit $\|\cdot\|$-sphere in V is compact, we may also assume that the unit length vectors $y_i/\|y_i\|$ converge to a point $z \in V$. Since $q(y_i) = q(x)$, we see that $q(z) = \lim q(y_i)/\|y_i\| = 0$, and the image of z in $\mathbb{P}(V)$ lies in Q. ■

Theorem 3.3: In the above setting let $\mathcal{C}$ be one of the $\mathcal{S}$-chambers for some $\mathcal{S}$ satisfying (3.1) and $\overline{\mathcal{C}} \subset V$ be the closure of $\mathcal{C}$ in V. Then the set $\mathbb{P}(\overline{\mathcal{C}}) \cap Q$ is nowhere dense in Q.

Proof: Let $U \subset Q$ be an open subset. The set of positive vectors V^+ consists of two connected components and $\mathcal{C}$ lies in one of them. We can therefore find $u \in V$ such that its image $[u] \in \mathbb{P}(V)$ lies in U and $q(w, u) > 0$ for any $w \in \mathcal{C}$.

By Proposition 3.2 for any $v \in \mathcal{S}$ there exists a sequence $\gamma_i \in \Gamma$ such that $[\gamma_i v] \to [u]$ for $i \to +\infty$. We may choose a sequence of vectors $v_i \in V^-$ such that v_i is proportional to $\gamma_i v$ and $v_i \to u$ for $i \to +\infty$. Let $H_i = v_i^\perp$, $H_i^- = \{w \in V \mid q(w, v_i) < 0\}$ and $H_i^+ = \{w \in V \mid q(w, v_i) > 0\}$. Since the vectors v_i converge to u, the hyperplanes H_i converge to $u^\perp$. Since $\mathbb{P}(u^\perp)$ is tangent to Q at the point $[u] \in U$, it follows that infinitely many of the hyperplanes $\mathbb{P}(H_i)$ have non-empty intersection with U. Therefore, passing to a subsequence, we may assume that $\mathbb{P}(H_i^-) \cap U \neq \emptyset$ for all i.

Observe that for any $w \in \cap_i H_i^-$ we have $q(w, v_i) < 0$, so $q(w, u) \leq 0$, because v_i converge to u. It follows that there must exist some i_0 such that $\mathcal{C} \not\subseteq H_{i_0}^-$: otherwise for $w \in \mathcal{C}$ we would have $q(w, u) \leq 0$ which contradicts

the choice of u. Since $\mathcal{S}$ is Γ-invariant, $\gamma_i v \in \mathcal{S}$ for every i, so the chamber $\mathcal{C}$ is either contained in H_i^- or in H_i^+. Therefore $\overline{\mathcal{C}} \cap H_{i_0}^- = \emptyset$ and we conclude that the non-empty open subset $U \cap \mathbb{P}(H_{i_0}^-) \subset U$ does not intersect $\mathbb{P}(\overline{\mathcal{C}})$. ■

Recall that by our assumptions (3.1) we have $-\mathcal{S} = \mathcal{S}$. Define

$$\mathcal{S}' = \{v \in \mathcal{S} \mid q(v,w) > 0 \ \forall w \in \mathcal{C}\} \tag{3.4}$$

and note that $\mathcal{S} = \mathcal{S}' \coprod (-\mathcal{S}')$. For $v \in \mathcal{S}'$ let $H_v^+ = \{w \in V \mid q(v,w) > 0\}$.

Corollary 3.4: In the above setting we have $\overline{\mathcal{C}} = \overline{\cap_{v\in\mathcal{S}'} H_v^+} = \cap_{v\in\mathcal{S}'} \overline{H_v^+}$.

Proof: Denote $\mathcal{C}' = \cap_{v\in\mathcal{S}'} H_v^+$. Note that the set $\mathcal{C}' \cap V^+$ is convex and open, because the collection of hyperplanes H_v, $v \in \mathcal{S}'$ is locally finite in V^+ by [SSV, Proposition 5.12]. It follows that $\mathcal{C} = \mathcal{C}' \cap V^+$, because $\mathcal{C}$ is a connected component of the complement to this collection of hyperplanes, and clearly $\mathcal{C} \subset \mathcal{C}' \cap V^+$.

Let us show that $\mathcal{C}' \cap V^- = \emptyset$. Assume that $v \in \mathcal{C}' \cap V^-$. Since $\mathcal{C}'$ is convex, it contains the segments connecting v with the points of $\mathcal{C}$, and the intersections of these segments with the boundary of V^+ form a subset of the boundary of $\mathcal{C}$ with non-empty interior. This implies that the boundary of $\mathcal{C}$ contains an open subset of the boundary of V^+, contradicting Theorem 3.3. It follows that $\mathcal{C}' \cap V^- = \emptyset$.

We see that $\mathcal{C}' \subset \overline{V^+}$. The convexity of $\mathcal{C}'$ now implies that $\mathcal{C}' \subset \overline{\mathcal{C}}$. Since $\mathcal{C} \subset \mathcal{C}'$, we also have $\overline{\mathcal{C}} \subset \overline{\mathcal{C}'}$ and the first equality in the statement of the corollary follows.

To prove the second equality, note that any closed convex cone with non-empty interior is the closure of its interior. Hence it suffices to prove that the cones $\overline{\mathcal{C}'}$ and $\cap_{v\in\mathcal{S}'} \overline{H_v^+}$ have the same non-empty interior. It is clear that the interior of $\overline{\mathcal{C}'}$ is $\mathcal{C}$, which is non-empty and is contained in $\cap_{v\in\mathcal{S}'} \overline{H_v^+}$. To prove the inclusion in the opposite direction, note that a point in the interior of $\cap_{v\in\mathcal{S}'} \overline{H_v^+}$ has an open neighbourhood that lies in H_v^+ for all $v \in \mathcal{S}'$, therefore lies in the interior of $\mathcal{C}'$. This completes the proof. ■

Applying the above statements to the ample cone of a projective hyperkähler manifold we recover the following theorem proved by Kovács in the K3 case, [Ko1], [Ko2] (see also [Mat] and [De]). Let M be a hyperkähler manifold. Denote $V_{\mathbb{Z}} = \mathrm{NS}(M)$ and let $\Gamma \subset O(V_{\mathbb{Z}}, q)$ be the image of the Hodge monodromy group under the natural representation, where q is the

BBF form. By [Ma, Lemma 6.23] Γ is of finite index in $O(V_{\mathbb{Z}}, q)$. Recall the definitions of the cones in the cohomology of M given in section 2.2.

Theorem 3.5: Assume that M is a projective hyperkähler manifold with Picard number $\rho(M) > 2$.

1. If $\mathrm{NS}(M)$ does not contain MBM classes, then $\mathcal{A}_M = \mathcal{C}_M \cap \mathrm{NS}_{\mathbb{R}}(M)$. Otherwise the boundary of $\mathcal{A}_M$ is nowhere dense in the boundary of $\mathcal{C}_M \cap \mathrm{NS}_{\mathbb{R}}(M)$.

2. If M does not contain any prime exceptional divisors then $\overline{\mathcal{M}_M} = \overline{\mathcal{C}_M} \cap \mathrm{NS}_{\mathbb{R}}(M)$. Otherwise the boundary of $\overline{\mathcal{M}_M}$ is nowhere dense in the boundary of $\mathcal{C}_M \cap \mathrm{NS}_{\mathbb{R}}(M)$.

Proof: *Part 1.* Let $\mathcal{S} = \mathrm{MBM}(M) \cap \mathrm{NS}(M)$. Since $\mathcal{A}_M = \mathcal{K}_M \cap \mathrm{NS}_{\mathbb{R}}(M)$, we deduce the following from the description of $\mathcal{K}_M$ given in section 2.2: $\mathcal{A}_M$ is a connected component of the set $(\mathcal{C}_M \cap \mathrm{NS}_{\mathbb{R}}(M)) \setminus \cup_{v \in \mathcal{S}} v^{\perp}$. Since $\mathcal{S}$ is Γ-invariant, we may apply Theorem 3.3.

Part 2. Analogously to part 1, but with $\mathcal{S}$ defined to be the set of divisorial MBM classes. Theorem 6.17 in [Ma] shows that the interior of $\mathcal{M}_M$ is one of the chambers in the decomposition of $\mathcal{C}_M \cap \mathrm{NS}_{\mathbb{R}}(M)$ defined by $\mathcal{S}$. We again conclude by Theorem 3.3. ■

We recall that $\mathrm{PE}(M) \subset \mathrm{NS}(M)$ denotes the set of classes of prime exceptional divisors.

Theorem 3.6: Assume that M is a projective hyperkähler manifold with $\rho(M) > 2$. If $\mathrm{PE}(M)$ is empty, then $\mathcal{P}_M = \overline{\mathcal{C}_M}$, otherwise

$$\mathcal{P}_M \cap \mathrm{NS}_{\mathbb{R}}(M) = \overline{\sum_{v \in \mathrm{PE}(M)} \mathbb{R}_+ \cdot v}.$$

Proof: We use the description of $\mathcal{P}_M$ given in (2.1). The case when $\mathrm{PE}(M) = \emptyset$ is clear, so we may assume the contrary.

Denote by $\mathcal{S}$ the set of divisorial MBM classes and for $v \in \mathcal{S}$ denote $H_v^+ = \{u \in \mathrm{NS}_{\mathbb{R}}(M) \mid q(u, v) > 0\}$. Then $\mathcal{S}$ defines the wall and chamber decomposition of $\mathcal{C}_M \cap \mathrm{NS}_{\mathbb{R}}(M)$ and we denote by $\mathcal{C}$ the chamber whose closure is $\overline{\mathcal{M}_M}$. By Corollary 3.4 we have $\overline{\mathcal{C}} = \cap_{v \in \mathcal{S}'} \overline{H_v^+}$, where $\mathcal{S}'$ is defined in (3.4). On the other hand, $\overline{\mathcal{C}}$ is dual to $\mathcal{P}_M \cap \mathrm{NS}_{\mathbb{R}}(M)$ and so we have $\mathrm{PE}(M) \subset \mathcal{S}'$, hence $\overline{\mathcal{C}} \subset \cap_{v \in \mathrm{PE}(M)} \overline{H_v^+}$. The latter inclusion must be equality,

because the set of faces of the cone $\mathcal{C}$ inside the positive cone is given by the orthogonal complements to the negative extremal rays of $\mathcal{P}_M \cap \mathrm{NS}_{\mathbb{R}}(M)$, i.e. by the vectors $v \in \mathrm{PE}(M)$.

We have shown that $\overline{\mathcal{M}_M} = \overline{\mathcal{C}} = \cap_{v \in \mathrm{PE}(M)} \overline{H_v^+}$. The claim now follows from duality between $\overline{\mathcal{M}_M}$ and $\mathcal{P}_M \cap \mathrm{NS}_{\mathbb{R}}(M)$. ■

3.2.2 The case when $\mathcal{S}$ is contained in a proper subspace of V

We go back to the setting introduced in the beginning of section 3.1. We additionally assume that there exists a proper subspace $N \subset V$ such that $\mathcal{S} \subset N$. Then $Q^\circ = Q \setminus \mathbb{P}(N)$ is an open subset of Q. For $v \in \mathcal{S}$ let $Q_v = Q \cap \mathbb{P}(H_v)$, where as above $H_v = v^\perp$.

Lemma 3.7: The collection of subvarieties Q_v, $v \in \mathcal{S}$ is locally finite in Q°: for any $x \in Q^\circ$ there exists an open subset $U \subset \mathbb{P}(V)$ such that $x \in U$ and U intersects Q_v only for finitely many $v \in \mathcal{S}$.

Proof: Let $u \in V$ be such that $[u] = x$. The idea is to translate the vector u in the direction given by some $w \in N^\perp$, so that the translated vector is contained in the positive cone. Local finiteness of the hyperplanes H_v, $v \in \mathcal{S}$ inside the positive cone is known, so we may find a ball around the translated vector that intersects only finite number of the hyperplanes. Translating this ball back, we get a neighbourhood of u that we need. Note that $N^\perp \subset H_v$ for all $v \in \mathcal{S}$, so that all hyperplanes are invariant under translation in the direction of any $w \in N^\perp$.

More precisely, we note that since $x \in Q^\circ$, we have $u \notin N$. The latter implies that there exists $w \in N^\perp$ with $q(u, w) \neq 0$. It follows that $u' = u + t_0 w \in V^+$ for some $t_0 \in \mathbb{R}$. By [SSV, Proposition 5.12] the collection of hyperplanes H_v, $v \in \mathcal{S}$ is locally finite in V^+, so there exists an open ball $B' \subset V^+$ containing u' that intersects only finitely many hyperplanes H_v. Set $B = B' - t_0 w$ and note that since $w \in N^\perp$ the ball B also intersects only finitely many hyperplanes H_v, $v \in \mathcal{S}$. Defining U to be the image of B in $\mathbb{P}(V)$ we conclude the proof of the claim. ■

Assume that $\mathcal{C} \subset V^+$ is one of the $\mathcal{S}$-chambers and let $\mathscr{B} = \mathbb{P}(\overline{\mathcal{C}}) \cap Q$ be the isotropic part of the boundary of $\mathbb{P}(\mathcal{C})$. Denote by $\mathscr{B}^{in}$ the interior of $\mathscr{B}$ as a subset of Q.

Proposition 3.8: In the above setting the set $\mathscr{B}$ has the following properties.

(1) The intersection $\mathscr{B}^{in} \cap Q^\circ$ is non-empty and dense in $\mathscr{B} \cap Q^\circ$;

(2) If there exists $w \in N^\perp$ with $q(w) < 0$ then $\mathscr{B}^{in}$ is dense in $\mathscr{B}$;

(3) If there exists a subspace $W \subset N^\perp$ such that $\dim(W) \geq 2$ and $q|_W$ is negative definite, then $\mathscr{B}^{in}$ and $\mathscr{B}$ are arcwise connected.

Proof: *Proof of (1).* Note that when $q|_N$ is negative definite, then $\mathcal{S}$ is finite, because it is q-bounded and discrete. In this case the collection of hyperplanes H_v, $v \in \mathcal{S}$ is finite and this easily implies part (1). It remains to prove (1) when $q|_N$ is either degenerate or has mixed signature. Note that in both cases $q|_{N^\perp}$ is negative semidefinite.

Since $\mathcal{C} \subset V$ is open and $N \subset V$ is a proper subspace, there exists $u \in \mathcal{C} \setminus N$. Since $u \notin N$, there exists $w \in N^\perp$ with $q(u, w) \neq 0$. By the remark above, we may assume that $q(w) \leq 0$. It follows that there exists $t_0 \in \mathbb{R}$ such that $u' = u + t_0 w$ is isotropic. Since for any $v \in \mathcal{S}$ and $t \in \mathbb{R}$ we have $q(u + tw, v) = q(u, v) \neq 0$, the affine line $u + \mathbb{R}w$ does not intersect any of the hyperplanes H_v, $v \in \mathcal{S}$. In follows that $\overline{\mathcal{C}}$ contains the segment joining u and u', so $[u'] \in \mathscr{B} \cap Q^\circ$. Since u lies in the interior of $\overline{\mathcal{C}}$, we have $[u'] \in \mathscr{B}^{in} \cap Q^\circ$, so $\mathscr{B}^{in} \cap Q^\circ$ is non-empty. The fact that it is dense in $D \cap Q^\circ$ follows from Lemma 3.7: any point $x \in \mathscr{B} \cap Q^\circ$ has an open neighbourhood $U \subset \mathbb{P}(V)$ that intersects only finitely many hyperplanes H_v, $v \in \mathcal{S}$. This finite collection of hyperplanes and the quadric Q cut U into finitely many connected components, and $\mathbb{P}(\mathcal{C}) \cap U$ is one of these components. The claim of part (1) follows easily from this.

Proof of (2). Let us fix a subspace W such that $q|_{W^\perp}$ is negative definite. By assumption we may find such a W with $W^\perp$ at least one-dimensional. Then $N \subset W$ and therefore $\mathcal{S} \subset W$.

Let $W^+ = W \cap V^+$ and $\mathcal{C}_W = \mathcal{C} \cap W$. Note that the linear projection of V onto W along $W^\perp$ induces a fibration of $p\colon \mathbb{P}(\overline{V^+}) \to \mathbb{P}(\overline{W^+})$. The fibre $F_p = p^{-1}[v]$ over a point $[v] \in \mathbb{P}(\overline{W^+})$ is isomorphic to the projectivization of the cone

$$\{av + w \mid a \in \mathbb{R},\, w \in W^\perp,\, q(av + w) \geq 0\} \subset W^\perp + \mathbb{R}v. \tag{3.5}$$

This shows that F_p is either a closed ball of dimension $\dim(V) - \dim(W)$ or a single point. Since $\mathcal{S} \subset W$, each fibre F_p is either contained in $\mathbb{P}(H_v)$ or does not intersect it, for any given $v \in \mathcal{S}$. It follows that p induces a fibration $\pi\colon \mathbb{P}(\overline{\mathcal{C}}) \to \mathbb{P}(\overline{\mathcal{C}_W})$ with the same fibres as p. Note the following property of the fibration p: for any $x \in \mathbb{P}(W) \cap Q$ it is clear from (3.5) that

$p^{-1}(x) = \{x\}$. Moreover, if $y_i \in \mathbb{P}(\overline{V^+})$ is an arbitrary sequence of points such that the sequence $\{p(y_i)\}$ converges to x, then the sequence $\{y_i\}$ also converges to x. Analogous statements also hold true for the fibration π.

If for a point $x \in \mathscr{B} \setminus \mathbb{P}(N)$, we see from part (1) that x lies in the closure of $\mathscr{B}^{in}$. To conclude the proof of part (2), take a point $x \in \mathscr{B} \cap \mathbb{P}(N)$ and a sequence of points $y_i \in \mathbb{P}(\mathcal{C})$ converging to x. Since $x \in \mathbb{P}(W)$, we have $\pi(x) = x$ and we may choose arbitrary $z_i \in \mathscr{B}^{in}$ such that $\pi(z_i) = \pi(y_i) \in \mathbb{P}(\mathcal{C})$. Then $\{z_i\}$ converges to x by the remark in the end of the previous paragraph.

Proof of (3). We may now additionally assume that $\dim(W^\perp) \geq 2$. We use the fibration π constructed above. Let $x_1, x_2 \in \mathscr{B}^{in}$. Then $y_1 = \pi(x_1)$ and $y_2 = \pi(x_2)$ lie in $\mathbb{P}(W^+)$. Connect y_1 to y_2 by a segment and lift this segment to $\mathscr{B}^{in}$ by taking a section of π. This gives an arc connecting x_1 to some point $x_2' \in \mathscr{B}^{in}$, where $\pi(x_2') = \pi(x_2)$. The condition on the dimension of $W^\perp$ implies that the boundary of the fibre $\pi^{-1}(y_2)$ is a sphere of positive dimension, in particular it is connected. Therefore we may connect x_2 and x_2' by an arc lying in that sphere. This proves the claim for $\mathscr{B}^{in}$, and the proof for $\mathscr{B}$ is analogous. ■

We now apply the above proposition to the Kähler cone of a hyperkähler manifold.

Theorem 3.9: Let M be a hyperkähler manifold, $\mathcal{C}_M \subset H^{1,1}_{\mathbb{R}}(M)$ its positive cone and $\mathcal{K}_M$ its Kähler cone. Let $\mathscr{B}_M = (\overline{\mathcal{K}_M} \cap \partial\mathcal{C}_M) \setminus \{0\}$ be the isotropic part of the boundary of $\mathcal{K}_M$ and $\mathscr{B}_M^\circ = \mathscr{B}_M \setminus \mathrm{NS}_{\mathbb{R}}(M)$. Denote by $\rho(M)$ the Picard rank of M.

1. If $\rho(M) \leq h^{1,1}(M) - 1$ then the interior of $\mathscr{B}_M^\circ$ in $\partial\mathcal{C}_M$ is non-empty and dense in $\mathscr{B}_M^\circ$;
2. If $\mathrm{NS}_{\mathbb{R}}(M)^\perp \subset H^{1,1}_{\mathbb{R}}(M)$ contains a q-negative vector (for instance, if $\rho(M) \leq h^{1,1}(M) - 2$) then the interior of $\mathscr{B}_M$ in $\partial\mathcal{C}_M$ is non-empty and dense in $\mathscr{B}_M$;
3. If $\mathrm{NS}_{\mathbb{R}}(M)^\perp \subset H^{1,1}_{\mathbb{R}}(M)$ contains a q-negative subspace of dimension two (for instance, if $\rho(M) \leq h^{1,1}(M) - 3$) then the both $\mathscr{B}_M$ and its interior in $\partial\mathcal{C}_M$ are arcwise connected.

Proof: This follows directly from Proposition 3.8 applied to $V = H^{1,1}_{\mathbb{R}}(M)$, $N = \mathrm{NS}_{\mathbb{R}}(M)$, $\mathcal{S} = \mathrm{MBM}(M)$, $\mathcal{C} = \mathcal{K}_M$. ■

4 The monodromy group and automorphisms of hyperkähler manifolds

Here we present some basic results about the rational curves, automorphisms and the Kähler cone of a hyperkähler manifold. We follow [V1, AV1, AV3, AV4].

4.1 The monodromy orbifold of a hyperkähler manifold

We recall that an **orbifold** is a topological space with a maximal atlas consisting of quotients of open subsets of $\mathbb{R}^n$ by finite groups, with smooth transition functions respecting the group actions, see [MP] for a precise definition and [ALR] for a detailed discussion. It is sometimes useful to think of orbifolds in terms of topological groupoids, see loc. cit. for this point of view. The basic example (and the only one that we will need) is the **action groupoid**: if M is a smooth manifold and Γ is a discrete group acting on M properly discontinuously, then the orbifold groupoid is given by the two maps $s, t\colon \Gamma \times M \to M$, where s is the projection to M and t is the map defining the action, see [ALR, section 1] for details. The orbifold represented by the action groupoid is the quotient orbifold M/Γ.

A **covering** of an orbifold is a map $\pi\colon X_1 \to X_2$ which can be locally represented as $B/\Gamma_1 \longrightarrow B/\Gamma_2$, where $B \subset \mathbb{R}^n$ is an open ball and $\Gamma_1 \subset \Gamma_2$, and such that any point in X_2 has a neighbourhood U_2 such that its preimage is a union of open subsets in X_1 mapping to U_2 in this fashion. An orbifold X is called **simply connected** if any connected covering of X is isomorphic to X. A **universal covering** of X is a connected covering which is simply connected. One can show that any orbifold admits an universal covering which is unique up to an isomorphism. The **fundamental group** of a connected orbifold X is the group of automorphisms of its universal covering compatible with the projection to X. See [ALR, section 2.2] for the definition of the orbifold fundamental group in terms of groupoids.

Going back to the example of the quotient orbifold, let Γ be a discrete group acting properly discontinuously on a contractible manifold Z. Then Z is the universal covering of the orbifold Z/Γ, and $\pi_1(Z/\Gamma) = \Gamma$.

Let $\Gamma \subset SO^+(1, n)$ be a lattice, i.e. a discrete subgroup of finite covolume. A **hyperbolic orbifold** is the quotient $\mathbb{H}^m/\Gamma$ of the hyperbolic space. It is called a **hyperbolic manifold** if Γ has no elements of finite order (then, by the classification of automorphisms of the hyperbolic space, Γ acts freely on $\mathbb{H}$).

Let G be a semisimple algebraic group defined over $\mathbb{Q}$. An **arithmetic**

lattice is a subgroup $\Gamma \subset G$ which is commensurable with $G_{\mathbb{Z}}$. Recall that by a theorem of Borel and Harish–Chandra [Mo], for an algebraic Lie group G defined over $\mathbb{Q}$ and without non-trivial rational characters (such as a semisimple group), $G_{\mathbb{Z}}$ has finite covolume. This theorem implies that for an arithmetic lattice $\Gamma \subset SO^+(1,n)$ the quotient $\mathbb{H}^m/\Gamma$ is a hyperbolic orbifold.

We will consider hyperbolic orbifolds that arise from the action of the automorphism group of a hyperkähler manifold on its cohomology. For a hyperkähler manifold M we will denote by $\mathsf{Diff}_0(M)$ the connected component of its diffeomorphism group containing the identity (the group of isotopies). We will denote by $\mathsf{Comp}(M)$ the (infinite-dimensional) space of all complex structures of hyperkähler type on M, and we define the **Teichmüller space** as $\mathsf{Teich}(M) = \mathsf{Comp}(M)/\mathsf{Diff}_0(M)$. It is known that $\mathsf{Teich}(M)$ is a finite-dimensional non-Hausdorff complex analytic space (see e.g. [C], [V1]).

Let $\Gamma_0 := \mathsf{Diff}(M)/\mathsf{Diff}_0(M)$ denote the mapping class group of M. For a complex structure $I \in \mathsf{Comp}(M)$ let Γ^I be the subgroup of Γ_0 preserving the connected component $\mathsf{Teich}^I \subset \mathsf{Teich}$. The image of Γ^I in $O(H^2(M,\mathbb{Z}))$ is called **the monodromy group** of (M,I), denoted by $\mathsf{Mon}(M,I)$ or $\mathsf{Mon}(M)$ when I is clear from the context.

Remark 4.1: This notion is due to E. Markman ([Ma]). Monodromy group can be alternatively defined as the subgroup of $O(H^2(M,\mathbb{Z}))$ generated by the monodromy operators for all Gauss–Manin local systems obtained from deformations of (M,I).

Let $I \in \mathsf{Comp}(M)$ be a complex structure on M. The **Hodge monodromy group** is the group Mon_I of all $s \in \mathsf{Mon}(M,I) \subset O(H^2(M,\mathbb{Z}))$ that preserve the Hodge decomposition $H^2(M,\mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$.

We recall (see [Ma]) that the automorphism group of (M,I) is discrete, the restriction map $\mathrm{Aut}(M,I) \to \mathrm{Aut}(M,I)|_{H^2(M,\mathbb{Z})}$ has finite kernel, and its image is the set of all $\tau \in \mathsf{Mon}_I$ that preserve the Kähler cone. Slightly more generally, the following claim is true.

Claim 4.2: The orthogonal hyperplanes to the MBM classes partition the positive cone $\mathcal{C}_M$ in chambers, one of which is the Kähler cone $\mathcal{K}_M$. The group Mon_I acts on the set of the chambers (called **Kähler chambers**), and its elements which lift to automorphisms of M are exactly the ones which preserve $\mathcal{K}_M$. The Kähler chambers are the Kähler cones of the birational models of M and their monodromy images.

We sometimes denote by $H^{1,1}(M,\mathbb{Z})$ the group $H^2(M,\mathbb{Z}) \cap H^{1,1}(M) = \mathrm{NS}(M)$. Clearly, Mon_I acts on $H^{1,1}(M,\mathbb{Z})$.

Proposition 4.3: Let M be a projective hyperkähler manifold with complex structure I. Then the natural homomorphism $\mathsf{Mon}_I \to O(H^{1,1}(M,\mathbb{Z}))$ has finite kernel, and it image is of finite index in $O(H^{1,1}(M,\mathbb{Z}))$.

Proof. Step 1: As shown in [V1], the group Mon has finite index in $O(H^2(M,\mathbb{Z}))$. Therefore, Mon_I has finite index in the group $O_I(H^2(M,\mathbb{Z}))$ of all $s \in O(H^2(M,\mathbb{Z}))$ preserving the Hodge decomposition. Let $\Lambda \subset H^2(M,\mathbb{Z})$ be the orthogonal complement of $H^{1,1}(M,\mathbb{Z})$ in $H^2(M,\mathbb{Z})$. Since we assume that M is projective, the lattice $H^{1,1}(M,\mathbb{Z}) \oplus \Lambda$ has finite index in $H^2(M,\mathbb{Z})$. It follows that the image of $O_I(H^2(M,\mathbb{Z}))$ in $O(H^{1,1}(M,\mathbb{Z}))$ has finite index. This implies that the image of Mon_I in $O(H^{1,1}(M,\mathbb{Z}))$ has finite index.

Step 2: It remains to show that the kernel of the natural map $\mathsf{Mon}_I \to O(H^{1,1}(M,\mathbb{Z}))$ is finite. This would follow if we prove that the kernel K of the natural map $O_I(H^2(M,\mathbb{Z})) \longrightarrow O(H^{1,1}(M,\mathbb{Z}))$ is finite. Since M is projective and K acts trivially on $H^{1,1}(M,\mathbb{Z})$, the group K is contained in the group of automorphisms of the integral polarized Hodge structure $H^2(M,\mathbb{Z})$. The latter group is finite. ■

The intersection

$$\mathrm{Pos}_M = \mathcal{C}_M \cap \left(H^{1,1}(M,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}\right)$$

will be called called **the rational positive cone**. Recall that the intersection $\mathrm{Pos}_M \cap \mathcal{K}_M$ is the ample cone $\mathcal{A}_M$. Clearly, we have an exact analogue of Claim 4.2 in this setting. The resulting chambers, one of which is $\mathcal{A}_M$, are called **ample chambers**. The projectivization $\mathbb{P}\,\mathrm{Pos}_M$ is a hyperbolic space; assuming that M is projective, it follows from Proposition 4.3 that the group Mon_I acts on $\mathbb{P}\,\mathrm{Pos}_M$ as an arithmetic lattice.

Definition 4.4: The quotient orbifold $\mathbb{P}\,\mathrm{Pos}_M \,/\, \mathsf{Mon}_I$ will be called **the monodromy orbifold** of the hyperkähler manifold (M,I). Define **the hyperbolic ample polyhedron** as the image of $\mathbb{P}\mathcal{A}_M$ in the monodromy orbifold.

Remark 4.5: The monodromy orbifold has a finite covering which is a manifold, indeed by Selberg lemma the image of Mon_I in $O(\mathrm{NS}(M))$ has a torsion-free subgroup of finite index, and elements with fixed points on $\mathbb{P}\,\mathrm{Pos}_M$ are torsion.

4.2 Shape of the Kähler cone

Consider the Kähler cone $\mathcal{K}_M$ of a hyperkähler manifold M. The Kähler cone is open and convex in $H^{1,1}(M,\mathbb{R})$. A **wall** of $\mathcal{K}_M$ is the intersection of the boundary $\partial\mathcal{K}_M$ and a hyperplane $W \subset H^{1,1}(M,\mathbb{R})$ that has a non-empty interior (in both). A **face** of $\mathcal{K}_M$ is a nontrivial intersection of walls. A hyperplane W supporting a wall of $\mathcal{K}_M$ is the orthogonal hyperplane to an MBM class. So a face of $\mathcal{K}_M$ is obtained by intersecting the boundary of $\mathcal{K}_M$ and a finite number of orthogonal hyperplanes to MBM classes.

The walls and faces are defined in the same way for the ample cone $\mathcal{A}_M$ of a projective hyperkähler manifold M, by considering $\mathrm{NS}_{\mathbb{R}}(M)$ instead of $H^{1,1}(M,\mathbb{R})$.

In [AV2], it has been proved that the Hodge monodromy group acts with finitely many orbits on the set of MBM classes of type (1,1). As a consequence, the following version of the Kawamata–Morrison conjecture has been obtained.

Theorem 4.6: [AV4] Let (M,I) be a hyperkähler manifold. Then the set F of walls of the Kähler cone $\mathcal{K}_M$ is locally finite outside of the boundary of the positive cone, and the group of complex automorphisms $\mathrm{Aut}(M,I)$ acts on F with finite number of orbits. If M is projective, the same holds for the ample cone.

Remark 4.7: It follows from the above theorem that the boundary $\partial\mathcal{K}_M$ of the Kähler cone is the union of the faces and "the parabolic part", i.e. the set $\{\eta \in \partial\mathcal{K}_M \mid q(\eta) = 0\}$.

Definition 4.8: Let V be a quadratic vector space of signature $(1,n)$, and $\mathbb{H}^n = \mathbb{P}V^+$ the corresponding hyperbolic space. **A hyperbolic hyperplane** in the hyperbolic space $\mathbb{H}^n = \mathbb{P}V^+$ is the image of $\mathbb{P}W^+$, where $W \subset V$ is a hyperplane of signature $(1,n-1)$. Note that the set of hyperbolic hyperplanes is in bijective correspondence with $\mathbb{P}V^-$, where V^- is the set of negative vectors. **A locally geodesic hypersurface** in a hyperbolic orbifold $\mathbb{H}^n/\Gamma$ is the image of a hyperplane $\mathbb{P}W^+ \subset \mathbb{H}^n$. Note that a locally geodesic hypersurface, if it is closed, is also a hyperbolic orbifold.

Remark 4.9: A locally geodesic hypersurface is closed if and only if $\Gamma \cap SO(1,n-1)$ is a lattice in $SO(1,n-1)$ (the stabilizer of W). This is a consequence of Ratner theory when $n>2$, and elementary when $n=2$. In presence of an integral structure, that is, when $\mathbb{H}^n = \mathbb{P}V^+$ with $V = V_{\mathbb{Z}} \otimes \mathbb{R}$, and Γ is arithmetic, this amounts to saying that W is rational. Of course the same applies to the case when $W \subset V$ is a subspace of higher codimension k (but still of signature $(1,n-k)$). One calls $\mathbb{P}W^+$ **a hyperbolic subspace**, and its image in $\mathbb{H}^n/\Gamma$ **a locally geodesic (immersed) submanifold**.

Definition 4.10: Let $\mathcal{H} = \mathbb{H}^n/\Gamma$ be a hyperbolic orbifold, and $S_1, ..., S_n \subset \mathcal{H}$ be a collection of closed locally geodesic hypersurfaces. A **convex hyperbolic orbifold with polyhedral boundary** (CHOPB) is a connected component of the complement $\mathcal{H} \setminus \bigcup_i S_i$.

Theorem 4.11: Let (M,I) be a projective hyperkähler manifold, and $\mathcal{P} := \mathbb{P}\mathcal{A}_M / \operatorname{Aut}(M,I)$ be its hyperbolic ample polyhedron. Then $\mathcal{P}$ is a convex hyperbolic orbifold with polyhedral boundary, and $\pi_1(\mathcal{P})$ is equal to the image of $\operatorname{Aut}(M,I)$ in $O(H^{1,1}(M,\mathbb{Z}))$. Moreover, the natural surjection $\operatorname{Aut}(M,I) \to \pi_1(\mathcal{P})$ has finite kernel.

Proof: The orthogonal hyperplanes to the MBM classes of type $(1,1)$ partition $\mathbb{P}\operatorname{Pos}_M$ in chambers, one of which is $\mathbb{P}\mathcal{A}_M$. Let $\mathcal{M} = \mathbb{P}\operatorname{Pos}_M / \mathsf{Mon}_I$ be the monodromy orbifold of M. The preimage of $\mathcal{P} \subset \mathcal{M}$ in the universal cover $\mathbb{P}\operatorname{Pos}_M$ of $\mathcal{M}$ is the union of $\mathbb{P}\mathcal{A}_M$ and some other chambers which are its Mon_I-translates, and $\mathbb{P}\mathcal{A}_M$ is clearly the universal covering of $\mathcal{P}$. Let $S_i \subset \mathcal{M}$ be the closed hypersurfaces obtained as images of orthogonal complements to the MBM classes of type $(1,1)$. Then $\mathcal{P}$ is a connected component of the set $\mathcal{M} \setminus \bigcup S_i$. As follows from the solution of Morrison-Kawamata conjecture ([AV2]), the set $\{S_i\}$ is finite. Finally, $\pi_1(\mathcal{P})$ is the subgroup of $\mathsf{Mon}_I = \pi_1(\mathbb{P}\operatorname{Pos}_M / \mathsf{Mon}_I)$ which preserves the ample cone $\mathcal{A}_M \subset \operatorname{Pos}_M$. By [Ma], this is precisely the image of $\operatorname{Aut}(M,I)$ in $O(H^{1,1}(M,\mathbb{Z}))$, and the kernel is finite by Proposition 4.3. ■

5 Limit points of the automorphism group

5.1 The absolute and the limit set of a group acting by hyperbolic isometries

Definition 5.1: Let V be a real vector space equipped with a scalar product

q of signature $(1,n)$, and V^+ the set of all vectors with positive square. Then $\mathbb{H}^n = \mathbb{P}V^+$ is the hyperbolic space of constant negative curvature. **The absolute** Abs is the projectivization of the set of all vectors with square 0; it is identified with the boundary of $\mathbb{H}^n$. Clearly, Abs is diffeomorphic to the sphere S^{n-1}, and the union $\mathbb{H}^n \cup \text{Abs}$ is compact.

Remark 5.2: Let $W \subset V$ be a 2-dimensional subspace of signature (1,1). We set $W^+ = V^+ \cap W$. Then $\mathbb{P}W^+ \subset \mathbb{H}$ is a geodesic in the hyperbolic space, and all geodesics are obtained this way.

Definition 5.3: The endpoints of a geodesic $l = \mathbb{P}W^+$ are two points $\text{Abs} \cap \mathbb{P}(W)$.

Remark 5.4: Clearly, any geodesic is uniquely determined by its two endpoints. This defines a bijection between the set of geodesic lines in $\mathbb{H}^n$ and the set of pairs of distinct points in Abs.

Claim 5.5: Let l_1, l_2 be two geodesics on a hyperbolic space, and $\delta : l_1 \longrightarrow \mathbb{R}$ the distance from a point of l_1 to l_2. Let ∞_+, ∞_- be the endpoints of l_1, considered as points in Abs. Then $\lim_{x \longrightarrow \infty_+} \delta(x) = \infty$ if ∞_+ is not an endpoint of l_2, and $\lim_{x \longrightarrow \infty_+} \delta(x) = 0$ otherwise.

Proof: Left as an exercise. ■

Remark 5.6: Claim 5.5 is equivalent to the following geometric observation. Given a constant $R > 0$, denote by $l(R)$ the R-neighbourhood of l in $\mathbb{H}$. Then the ends of l are the only accumulation points of $l(R)$ on Abs.

Claim 5.7: Let $\Gamma \subset SO(1,n)$ be a group of hyperbolic isometries acting on $\mathbb{H}^n$, and $a, b \in \mathbb{H}^n$ any two points. Let $\Lambda_a, \Lambda_b \subset \text{Abs}$ be the set of all points in Abs obtained as accumulation points for $\Gamma \cdot a, \Gamma \cdot b$. Then $\Lambda_a = \Lambda_b$.

Proof: Let $x \in \Lambda_a$, and $\{y_i = \gamma_i a\} \in \Gamma \cdot a$ be a sequence converging to x. Since γ_i are isometries, $d(\gamma_i b, \gamma_i a) < \infty$, and the sequence $\gamma_i b$ converges to x as well. Indeed, since $\mathbb{H}^n \cup \text{Abs}$ is compact, otherwise we would have a point $y \neq x$ which is a limit of a subsequence $\gamma_i b$. However, for any two sequences of points $\{x_i\}$ converging to x and $\{y_i\}$ converging to y, we have $\lim_i d(x_i, y_i) = \infty$, giving a contradiction. ■

Definition 5.8: In these assumptions, **the limit set** of the action of Γ on $\mathbb{H}$ is Λ_a, for some (and, therefore, any) $a \in \mathbb{H}$.

Remark 5.9: Let Λ be the limit set of Γ. Clearly, Γ acts on $\mathrm{Abs} \setminus \Lambda$ properly discontinuously.

5.2 Convex hyperbolic orbifolds with polyhedral boundary in arithmetic hyperbolic manifolds

Definition 5.10: Let $V_{\mathbb{Z}}$ be a lattice equipped with a scalar product of signature $(1,n)$, $V = V_{\mathbb{Z}} \otimes \mathbb{R}$, and $\Gamma \subset SO^+(V)$ a subgroup commensurable with $SO(V_{\mathbb{Z}})$. Then $\mathbb{P}V^+/\Gamma$ is called **an arithmetic hyperbolic orbifold of simplest type**, or **of first type** (for an explanation of this terminology, see [BBKS]). Throughout the paper, we omit the "simplest type", using "arithmetic hyperbolic orbifold" instead. We sometimes use the notation $\mathrm{Pos}(V)$ for V^+.

Remark 5.11: Let $\Gamma \subset SO^+(1,n)$ be a group of isometries of finite covolume (that is, a lattice), and $\mathbb{P}W^+ \subset \mathbb{P}V^+$) a hyperbolic subspace. Clearly, the image of $\mathbb{P}W^+$ in $\mathbb{P}V^+/\Gamma$ has the same volume as $\mathbb{P}W^+/\Gamma_W$, where Γ_W is $\{\gamma \in \Gamma \mid \gamma(W) = W\}$. Any integer lattice in $O(1,n)$ has finite covolume. By Ratner's theorem, any geodesic immersed submanifold of $\mathbb{P}V^+/\Gamma$ is equal to $\mathbb{P}W^+/\Gamma_W$, for some rational subspace $W \subset V$. This implies the following description of a convex hyperbolic orbifold with polyhedral boundary (CHOPB) in arithmetic hyperbolic manifolfds.

Claim 5.12: Let $V_{\mathbb{Z}}$ be a lattice equipped with a scalar product of signature $(1,n)$, and $\Gamma \subset SO^+(V)$ a subgroup commensurable with $SO(V_{\mathbb{Z}})$. Consider a finite collection of rational hyperplanes $W_i \subset V$ of signature $(1,n-1)$, let $\mathfrak{S}$ be $\bigcup_i \Gamma W_i$, and P a connected component of $\mathbb{P}V^+ \setminus \mathbb{P}\mathfrak{S}$. Denote by Γ_P the group $\{\gamma \in \Gamma \mid \gamma(P) = P\}$. Then P/Γ_P is a convex hyperbolic orbifold with polyhedral boundary, and all convex hyperbolic orbifolds with polyhedral boundary in $\mathbb{P}V^+/\Gamma$ are obtained this way. ■

The following question is crucial for this paper.

Question 5.13: What is the limit set of Γ_P acting on P?

The reason is the following theorem, true for all hyperbolic manifolds, but it is much easier to state and prove it when $\mathcal{H}$ is compact. In Subsection

5.4 we will explain how to generalize it to all hyperbolic $\mathcal{H}$.

Theorem 5.14: Let $\mathcal{H} := \mathbb{P}V^+/\Gamma$ be a hyperbolic orbifold, and $\mathcal{P} \subset \mathcal{H}$ a convex hyperbolic orbifold with polyhedral boundary. Let P be a connected component of the preimage of $\mathcal{P}$ in $\mathbb{P}V^+ = \mathbb{H}$; clearly, P is a convex polyhedron in $\mathbb{H}$ with hyperbolic faces. Denote by $\operatorname{Abs} P$ the set $\operatorname{Abs} \cap \overline{P}$. Assume that $\mathcal{H}$ is compact. Then $\operatorname{Abs} P$ is the limit set of Γ_P acting on P.

Proof. Step 1: Clearly, for any $x \in P$, its orbit belongs to P, hence its limit set belongs to $\operatorname{Abs} P$.

Step 2: Conversely, let $x \in \operatorname{Abs} P$, and let l be a geodesic with an end in x. We may choose l so that contains a point in P. Since P is convex, a ray $l_+ \subset l$ converging to x also belongs to P. Choose a fundamental domain D of Γ_P-action on P. Since $\overline{P}/\Gamma_P$ is a closed subset of $\mathcal{H}$, this space is compact, hence D can be chosen relatively compact. Let $R := \mathsf{diam} D$.

Step 3: Choose $z \in D$. Since $\mathsf{diam} D = R$, an R-neighbourhood of any point $y \in l_+$ contains γz for some $\gamma \in \Gamma_P$. Choose a sequence of points $y_i \in l_+$ converging to $x \in \operatorname{Abs} P$, and let $\gamma_i z$ be points which satisfy $d(\gamma_i z, y_i) \leqslant R$. Then $\lim \gamma_i z$ is a point in an R-neighbourhood $l_+(R)$ of l_+. By Claim 5.5, any unbounded sequence in $l_+(R)$ converges to x, hence x belongs to the limit set of Γ_P. ■

5.3 The thick-thin decomposition

Here we give a brief introduction to cusp points on hyperbolic manifolds. For more details, see [Th, Ka3].

Definition 5.15: A horosphere on a hyperbolic space is a sphere which is everywhere orthogonal to the pencil of geodesics passing through a given point at infinity, and **a horoball** is a ball bounded by a horosphere. **A cusp point** for an n-dimensional hyperbolic manifold $\mathbb{H}/\Gamma$ is a point on the boundary $\partial\mathbb{H}$ such that its stabilizer in Γ contains a free abelian group of rank $n-1$. Such subgroups are called **maximal parabolic**. Horospheres are orbits of maximal parabolic subgroups, and all orbits of maximal parabolic subgroups are horospheres.

Claim 5.16: Let V be a quadratic vector space of signature $(1, n)$, and $\Gamma \subset SO^+(V)$ a discrete subroup of finite covolume. Consider a point $p \in \partial\mathbb{H}$,

and let $\Gamma_p \subset \Gamma$ be its stabilizer. Clearly, Γ_p acts on any horosphere S tangent to the boundary in p by isometries. In such a situation, p is a cusp point if and only if $(S\backslash p)/\Gamma_p$ is compact. ■

Remark 5.17: A cusp point p yields a **cusp** in the quotient $\mathbb{H}/\Gamma$, that is, a geometric end of $\mathbb{H}/\Gamma$ of the form $B/\mathbb{Z}^{n-1}$, where $B \subset \mathbb{H}$ is a horoball tangent to the boundary at p.

Theorem 5.18: (Thick-thin decomposition)
Any n-dimensional complete hyperbolic manifold $\mathbb{H}/\Gamma$ of finite volume can be represented as a union of a "thick part", which is a compact manifold with a boundary, and a "thin part", which is a finite union of quotients isometric to B/Γ_p, where B is a horoball tangent to the boundary at a cusp point p, and Γ_p its stabilizer in Γ. Moreover, Γ_p is isomorphic to $\mathbb{Z}^{n-1}$, and B/Γ_p is diffeomorphic to a product of a torus T^{n-1} and an interval.

Proof: See [Th, Section 5.10] or [Ka3, page 491]. ■

When the group $\Gamma \subset SO^+(V)$ is arithmetic, that is, commensurable with a lattice $SO(V_{\mathbb{Z}})$, the cusp points can be identified explicitly. The following theorem seems to be well known.

Theorem 5.19: Let $V_{\mathbb{Z}}$ be a lattice equipped with a scalar product of signature $(1,n)$, and $\Gamma \subset SO^+(V)$ a subgroup commensurable with $SO(V_{\mathbb{Z}})$. Then the cusp points of the hyperbolic manifold $\mathbb{H}/\Gamma$ are in bijective correspondence with Γ-orbits on the set $\mathrm{Abs} \cap \mathbb{P}(V_{\mathbb{Q}})$ of all rational points on Abs.

Proof: From [BJ, Proposition I.4.9] (see also Theorem 5.18 or [Ka3]) we obtain that the cusps of a hyperbolic manifold M correspond to maximal parabolic subgroups $\Gamma_0 \subset \pi_1(M) = \Gamma$.[1] The fixed point α of Γ_0-action on the absolute is the corresponding cusp point of $\mathbb{H}/\Gamma$. This implies that for a general element $\gamma \in \Gamma_0$, the line $\mathbb{R}\alpha$ coincides with $\ker(\mathrm{Id} - \gamma)$, which is rational, because γ is rational.

Conversely, consider a rational point α in the absolute, and let $G_0 \subset SO(1,n)$ be the maximal parabolic group fixing α. Since G_0 is a rational subgroup in $SO(1,n)$, its intersection with the lattice Γ, commensurable to

[1] Let Γ be a lattice in $SO(1,n)$. A subgroup $\Gamma_0 \subset \Gamma$ is **maximal parabolic** if it is a lattice in a maximal parabolic Lie subgroup in $SO(1,n)$.

$SO(V_{\mathbb{Z}})$, is a lattice in G_0. ■

5.4 The absolute of a universal covering of CHOPB

After we defined the cusp points and introduced the thick-thin decomposition, we can state and prove the general version of Theorem 5.14.

Theorem 5.20: Let $\mathcal{H} := \mathbb{P}V^+/\Gamma$ be a hyperbolic manifold, and $\mathcal{P} \subset \mathcal{H}$ a convex hyperbolic orbifold with polyhedral boundary. Let P be a connected component of the preimage of $\mathcal{P}$ in $\mathbb{P}V^+ = \mathbb{H}$; clearly, P is a convex polyhedron in $\mathbb{H}$ with hyperbolic faces. Denote by $\operatorname{Abs} P$ the set $\overline{P} \cap \partial\mathbb{H}$. Then $\operatorname{Abs} P$ is the union of the set of all cusp points of $\mathcal{H}$ which belong to $\operatorname{Abs} P$ and the limit set of Γ_P acting on P.

Proof. Step 1: Clearly, for any $x \in P$, its Γ_P orbit belongs to P, hence its limit set belongs to $\operatorname{Abs} P$.

Step 2: Consider the thick-thin decomposition of $\mathcal{H}$, with the thick part $\mathcal{H}_0$, and let P_0 be the intersection of P and the preimage of $\mathcal{H}_0$. Then P_0 is obtained by removing from P a countable number of horoballs; this set is Γ_P-invariant. The fundamental domain D of Γ_P-action on P_0 has finite diameter R.

Step 3: Fix a non-cusp point $x \in \operatorname{Abs} P$, and fix $z \in P$. We need to show that $\gamma_i z$ converges to x for some sequence $\{\gamma_i\} \in \Gamma_P$. Let $l_+ \subset P$ be a geodesic ray with an end in x. The intersection of a horoball and a geodesic ray is bounded, unless its end coincides with the boundary point of the horoball. So l_+ contains a family of points $\{y_i\} \subset l_+ \cap P_0$ converging to x. This implies that an R-neighbourhood of y_i contains $\gamma_i D$, hence it contains $\gamma_i z$, and $\lim \gamma_i z = x$ by another application of Claim 5.5. ■

Remark 5.21: Note that the cusp points of $\mathcal{H}$ sometimes (but not always) also belong to the limit set of Γ_p.

5.5 Cusp points outside of the limit set

Let $\mathcal{P} \subset \mathcal{H}$ be a CHOPB, and $P \subset \mathbb{H}^n$ a connected component of its preimage in the hyperbolic space, equipped with $\Gamma = \pi_1(\mathcal{P})$-action. Denote by $\Lambda \subset \operatorname{Abs} P$ the limit set of Γ-action on P. In Subsection 5.4, we have

shown that $S := \operatorname{Abs} P \backslash \Lambda$ is a collection of cusp points. We don't have much control over S; the following example implies that S can be infinite.

Proposition 5.22: Let $\mathcal{H}$ be a punctured 2-torus equipped with a complete hyperbolic metric, and $R \subset \mathcal{H}$ a simple geodesic connecting a cusp point to itself. Then $\mathcal{P} := \mathcal{H} \backslash R$ is a CHOPB with $\pi_1(\mathcal{P}) = \mathbb{Z}$. It has an infinite set of cusp points and only two limit points.

Proof: Let $P \subset \mathbb{H}^2$ be a connected component of the preimage of $\mathcal{P}$ in the universal cover of $\mathcal{H}$. Clearly, $\mathcal{P}$ is diffeomorphic to a cylinder, hence $\pi_1(\mathcal{P}) = \mathbb{Z}$. The corresponding free geodesic γ in $\mathcal{P}$ has finite length, hence it corresponds to a hyperbolic automorphism of $\mathbb{H}^2$ preserving P. This implies that $\pi_1(\mathcal{P})$ has only two limit points, which are the ends of γ. The group $\pi_1(\mathcal{P})$ freely acts on the set of cusp points on P, which all lie in S, hence S is infinite. ■

6 The Apollonian carpet

6.1 Real analytic intervals in the limit set

Definition 6.1: Let $V_{\mathbb{Z}}$ be a lattice equipped with a scalar product of signature $(1, n)$, and $\Gamma \subset SO^+(V)$ a subgroup commensurable with $SO(V_{\mathbb{Z}})$. Consider a convex hyperbolic orbifold $\mathcal{P} \subset \mathcal{H}$ with polyhedral boundary. Let P be a connected component of the preimage of $\mathcal{P}$ in $\mathbb{P}V^+ = \mathbb{H}$, and $\operatorname{Abs} P$ the set of accumulation points of P on Abs. **The Apollonian carpet** associated to $\mathcal{P}$ is the union of all positive-dimensional real analytic subvarieties in $\operatorname{Abs} P$.

We start by proving that the Apollonian carpet is a union of geodesic spheres in Abs; this result follows from a theorem of N. Shah.

Theorem 6.2: Let $\mathcal{H}$ be an arithmetic hyperbolic manifold, $\mathcal{H} = \mathbb{P}V^+/\Gamma$, where Γ is commensurable to $SO(V_{\mathbb{Z}})$, and $\mathcal{P} \subset \mathcal{H}$ a convex hyperbolic orbifold with polyhedral boundary. Denote by $P \subset \mathbb{H}$ a connected component of its preimage, and let $\Gamma_P \subset \Gamma$ be its stabilizer. Then the Apollonian carpet is a union of spheres $\operatorname{Abs} \cap \mathbb{P}W_i$, where $W_i \subset V$ is a rational subspace of signature $(1, k)$. Moreover, the group Γ_P acts on the set $\{W_i\}$ with finitely many orbits.

Proof: See Subsection 6.3. ■

Definition 6.3: A **component** of an Apollonian carpet A is a sphere $\mathrm{Abs}\cap\mathbb{P}W_1 \subset A$ which is not contained in another sphere $\mathrm{Abs}\cap\mathbb{P}W_2 \subset A$ of greater dimension.

Remark 6.4: From Theorem 6.2 it follows that an Apollonian carpet is a union of its components.

Remark 6.5: Note that the components of an Apollonian carpet can be disjoint, tangent or intersect each other; see examples in Subsection 7.3.

6.2 Shah's theorem on ergodic properties of the geodesic flow

Let $ST\mathbb{H}^n$ be the manifold of unit tangent vectors to the hyperbolic space. We use $\mathrm{Vis} : ST\mathbb{H}^n \longrightarrow \mathrm{Abs}$ to denote the map taking a tangent vector to the limit point of the corresponding geodesic. If $\mathcal{H} = \mathbb{H}^n/\Gamma$ is a hyperbolic manifold and $(x, v) \in ST\mathcal{H}$, we define $\mathrm{Vis}(x, v)$ by fixing a point in the preimage of x in $\mathbb{H}^n$ and lifting the geodesic to $\mathbb{H}^n$, so that Vis is defined up to Γ-action. The definition also makes sense for orbifolds for a suitable definition of ST at the singular points. In our situation every orbifold has a finite cover which is a manifold (Remark 4.5), and for our purposes we may assume it is already a manifold.

Definition 6.6: Let $(x, v) \in ST\mathbb{H}^n$, with $v \in T_x\mathbb{H}^n$ being a unit tangent vector. Denote by $l_{x,v}(t)$ the geodesic starting to x and tangent to v. The **geodesic flow** is a measure-preserving flow of diffeomorphisms $G_t : ST\mathbb{H}^n \times \mathbb{R} \longrightarrow ST\mathbb{H}^n$ taking $((x, v), t)$ to the tangent vector $l'_{x,v}(t)$. Similarly, one defines the geodesic flow g_t on $\mathcal{H}$.

Remark 6.7: By E. Hopf theorem, the geodesic flow on a hyperbolic manifold is ergodic. The closure of a given geodesic on a hyperbolic manifold or orbifold might be very pathological, such as a product of a Cantor set and an interval.

By Ratner theorem on homogeneous flow, this pathology does not happen when instead of a geodesic, one considers the image in $\mathcal{H}$ of a hyperbolic subspace of dimension $k > 1$ in $\mathbb{H}^n$: the closure is a hyperbolic submanifold, that is, the image of a rational subspace in $\mathbb{H}^n$. The following theorem by N. Shah goes further in this direction.

Theorem 6.8: [Sh, Theorem 1.2] Let $\mathcal{H} = \mathbb{H}^n/\Gamma$ be a hyperbolic manifold, and $C \subset ST\mathcal{H}$ a real analytic interval, such that $\mathrm{Vis}(C)$ is not a singleton. Then there is a hyperbolic subspace $\mathbb{H}^m \subset \mathbb{H}^n$ with $\mathrm{Vis}(C) \subset \partial\mathbb{H}^m$ such that the image $\mathcal{H}_1$ of $\mathbb{H}^m$ in $\mathcal{H}$ is a closed submanifold, and for any function $f \in C_c(ST\mathcal{H})$

$$\lim_{t\to\infty} \frac{1}{|I|} \int_I f(g_t\psi(s))ds = \int_{ST\mathcal{H}_1} f|_{ST\mathcal{H}_1},$$

where $\psi : I \to C$ is a parametrization of C by the unit interval and the integration is taken with respect to the normalized volume form.■

In particular, taking f which is zero outside of a small neighbourhood of some point $x \in ST\mathcal{H}_1$, one obtains that x is a limit point of geodesic rays starting at points of C, cf. [Sh, Remark 1.1]. In the same remark, it is explained how to obtain $\mathcal{H}_1$: choose the smallest subsphere of $\partial\mathbb{H}^n$ containing $\mathrm{Vis}(C)$, it is the absolute of a hyperbolic subspace $\mathbb{H}^k$, and $\mathcal{H}_1$ is the closure of the image of $\mathbb{H}^k$ in $\mathcal{H}$.

6.3 Apollonian carpet is a union of geodesic spheres

We are now in a position to prove Theorem 6.2.

Proof. Step 1: Let $C \subset \mathrm{Abs}\, P$ be a real analytic curve. Connecting a point in P with C by a real analytic family of geodesics, we obtain a real analytic path $C_1 \subset ST\mathcal{P}$ satisfying the assumptions of Shah's theorem. By Shah's theorem, the closure of the image of a general geodesic on closure of the CHOPB $\mathcal{P}$ contains a closed hyperbolic submanifold $\mathcal{H}_1$. It follows that a component $\mathbb{H}^m$ of the preimage of $\mathcal{H}_1$ in $\mathbb{H}^n$ is contained in the closure of P, and its boundary is a sphere S, $C \subset S \subset \mathrm{Abs}\, P$.

Step 2: Consider a vector subspace $W \subset V$, such that $S = \mathbb{P}W \cap \mathrm{Abs}$. The convex hull of S is the hyperbolic subspace $\mathbb{P}W^+ \subset \mathbb{P}V^+$. By Ratner theorem, its image in $\mathcal{H} = \mathbb{P}V^+/\Gamma$ is closed if and only if $\Gamma_W := \{\gamma \in \Gamma \mid \gamma(W) = W\}$ is a lattice in $O(W)$, which happens if and only if W is rational. So W must be rational.

Step 3: For a component $S = \mathbb{P}W \cap \mathrm{Abs}$ of the Apollonian carpet, denote by $H_S \subset \mathcal{H}$ the image of $\mathbb{P}W^+$ in $\mathcal{H}$. We claim that there is only a finite number of H_S. By Mozes-Shah and Dani-Margulis theorems ([MS], Corollaries 1.1, 1.3, 1.4), the closure of the union of countably many such

subsets is a finite union of subsets of the same shape (i.e. images of $\mathbb{P}U^+$ for possibly larger rational U). If there is an infinite number of H_S, we thus obtain a closed hyperbolic submanifold $\mathcal{H}'$ of greater dimension as the closure of some union of them. By maximality of the H_S, $\mathcal{H}'$ cannot coincide with any of them. But the H_S are contained in the the CHOPB $\mathcal{P}$, hence $\mathcal{H}'$ belongs to the closure $\mathcal{P}$. It follows that there is an extra sphere $\mathrm{Vis}(ST\mathcal{H}')$ in $\mathrm{Abs}\, P$, a contradiction. Hence, up to the Γ_P-action, the number of components of the Apollonian carpet is finite. ■

6.4 Closure of the Apollonian carpet

Recall that a group $\Gamma \subset SO^+(1,n)$ is called **elementary** is its limit set has at most 2 points.

Theorem 6.9: Let $\Gamma \subset SO^+(1,n)$ be a non-elementary subgroup, and $\Lambda \subset \mathrm{Abs}$ its limit set. Then the closure of an orbit $\Gamma \cdot x$ for any $x \in \Lambda$ is the whole of Λ.
Proof: [CNS, Theorem 1.2.24]. ■

This result immediately brings the following corollary.

Corollary 6.10: Let $\mathcal{H}$ be a hyperbolic manifold, and $\mathcal{P} \subset \mathcal{H}$ a CHOPB. Denote by $\Lambda(\mathcal{P}) \subset \mathrm{Abs}$ the limit set of $\Gamma \subset \mathrm{Iso}(\mathbb{H}^n)$, and $A \subset \Lambda$ its Apollonian carpet. Assume that A is non-empty. Then Λ is the closure of A. ■

7 Apollonian carpet of a hyperkähler manifold

7.1 Apollonian carpet and MBM classes

Definition 7.1: The Apollonian carpet of a hyperkähler manifold is the union of all geodesic spheres of positive dimension which belong to the absolute of its ample cone.

Remark 7.2: Clearly, the Apollonian carpet of a hyperkähler manifold is the Apollonian carpet of the corresponding convex hyperbolic orbifold with polyhedral boundary $\overline{\mathcal{P}} \subset \mathcal{H}$, obtained as the image of $\overline{\mathbb{P}\,\mathrm{Amp}}$ in $\mathcal{H} = \mathbb{P}\,\mathrm{Pos}_M \,/\, \mathsf{Mon}_I$ (Theorem 4.11).

The Apollonian carpet of a hyperkähler manifold can be described explicitly in terms of its BBF form and MBM classes, as follows.

Theorem 7.3: Let M be a projective hyperkähler manifold, and $NS(M) = H^{1,1}(M,\mathbb{R}) \cap H^2(M,\mathbb{Z})$ its Néron–Severi lattice. Consider the set of all maximal rational subspaces $W_i \subset NS_{\mathbb{R}}$ of signature $(1,k)$, $k \geqslant 2$ such that for any MBM class $\eta \in NS(M)$ with $\eta \not\perp W_i$, we have that $W_i^\perp + \eta$ is not negative definite. Then the sphere $S_i = \operatorname{Abs} \cap \mathbb{P}W_i$ lies in the closure of some ample chamber (which can be seen as the ample cone of (M, I_i) for a complex structure I_i). Moreover S_i belongs to the Apollonian carpet of (M, I_i), and all components of the Apollonian carpet of (M, I_i) are obtained this way. Finally, the group of automorphisms of (M, I_i) contains a lattice in $SO(1,k) = \operatorname{Aut} W_i$, which is commensurable to $SO(W_i, \mathbb{Z})$.

Remark 7.4: The subspace W_i as above can lie in $\eta^\perp$ for some MBM class η. The space generated by all such η for a given W_i is negative definite. If the addition of an extra MBM class η' destroys negativity, this means that W_i supports a face of an ample chamber which does not contain other faces, in other words, W_i^+ itself is a minimal face of the ample chamber. Changing the complex structure if necessary, we may assume that the chamber is the ample cone.

Proof of the theorem, step 1:

When W_i^+ is a face of the ample cone as in Remark 7.4, clearly $\operatorname{Abs} \mathbb{P}W_i^+$ is a sphere on the boundary of the ample cone and a component of Apollonian carpet.

In general, let $U_1, U_2 \subset V$ be spaces of negative signature of a space of signature $(1,n)$. Then $\mathbb{P}(U_1^+)^\perp$ intersects $\mathbb{P}(U_2^+)^\perp$ if and only if $U_1 + U_2$ is negative definite. If $U_1 + U_2$ is degenerate, the spheres $\operatorname{Abs} \mathbb{P}(U_1^+)^\perp$ and $\operatorname{Abs} \mathbb{P}(U_2^+)^\perp$ are tangent.

Take a W_i which is not contained in $\eta^\perp$ with η MBM. The condition of the theorem: $W_i^\perp + \eta$ is not negative definite means that S_i does not meet any orthogonal hyperpane to an MBM class except maybe tangentially. This means that S_i is contained in a chamber which, as before, we assume to be the ample cone. Therefore S_i is again a component of the Apollonian carpet.

Step 2:

Let S_i be a component of the Apollonian carpet of (M, I). Then $S_i = \operatorname{Abs} \cap \mathbb{P}W_i$, where W_i is a rational subspace. Since W_i intersects the positive cone, it must have signature $(1,k)$. Also, S_i is contained in the closure of the ample cone. The ample cone is cut inside the positive cone by the

orthogonal hyperplanes to MBM classes. We derive that for η MBM, S_i cannot intersect $\eta^\perp$ transversally (it is disjoint, tangent to or contained in $\eta^\perp$). This amounts to saying that $W_i^\perp + \eta$ is not negative definite when W_i is not orthogonal to η.

Step 3:

Denote by Γ_I the group of all $\nu \in O^+(H^2(M,\mathbb{Z}))$ preserving the Hodge decomposition. Then $\mathcal{H} := \mathbb{P}\operatorname{Pos}_M / \Gamma_I$ is a hyperbolic manifold, containing the closure of the image of the projectivization of the ample cone as a convex hyperbolic orbifold with polyhedral boundary. Denote by $\Gamma \subset \Gamma_I$ its fundamental group; by Theorem 4.11, Γ is commensurable with $\operatorname{Aut}(M,I)$. Recall that the elements of Γ_I permute the chambers of the decomposition given by MBM classes, and Γ is the subgroup of those elements in Γ_I which preserve one of the chambers, namely the ample cone.

Let S_i be a component of the Apollonian carpet and W_i the associated linear subspace as above. Set $\Gamma_{W_i} := \{\gamma \in \Gamma_I \mid \Gamma(W_i) = W_i\}$. Then Γ_{W_i} is a lattice in $O(W_i)$. If S_i is not contained in the orthogonal to an MBM class, then S_i intersects all $\eta^\perp$, where η is MBM, at most tangentially, and therefore an element γ of Γ_{W_i}, preserving S_i, must also preserve its chamber, i.e. the ample cone. Thus γ lifts to an automorphism of (M,I). If S_i is contained in the orthogonal to an MBM class, then $\mathbb{P}Pos(W_i)$ is a face of the ample cone, and $\gamma \in \Gamma_{W_i}$ can also take the ample cone to another chamber adjacent to $\mathbb{P}Pos(W_i)$, but these are finitely many. So a finite index subgroup of Γ_{W_i} lifts to a subgroup in $\operatorname{Aut}(M,I)$, q.e.d..■

7.2 The Baragar gasket

In the proof of Theorem 7.3 we have seen that the Apollonian carpet has two types of components: the ones which come from the faces of the ample cone, and the ones which do not.

More precisely, let F, $\dim F > 2$, be a face of the ample cone which is minimal with respect to inclusion, that is, L is the intersection of orthogonals to MBM classes which does not meet other orthogonals to MBM classes except possibly tangentially. Then $F = L^+$ for a vector subspace L, and $\mathbb{P}L^+ \cap \operatorname{Abs}$ is a component of the Apollonian carpet.

In the case of K3 surfaces, the union of such components has been studied by A. Baragar, see e.g. [Ba1]. Hence the following definition.

Definition 7.5: The union of all components of the Apollonian carpet obtained this way is called **the Baragar gasket**.

Remark 7.6: The Baragar components do not intersect transversally, but they might be tangential. Other components of the Apollonian gasket can intersect (Example 7.16), but they cannot intersect the Baragar components transversally. In the CHOPB setting, the Baragar components correspond to the hyperspheres which are obtained as isotropic boundaries of faces of the CHOPB which don't contain faces of smaller dimension, and hence are given by a hyperbolic subspace which lies fully on a boundary of the CHOPB.

Remark 7.7: This gives an effective and easy way to determine the Apollonian carpet for particular manifolds when we have good control over the BBF form, the Néron–Severi lattice and the MBM classes. Note that by Nikulin's theorem ([Ni]), any even quadratic lattice of signature $(1, n)$, $n \leqslant 10$ can be realized as the Néron–Severi lattice for an appropriate K3 surface. Note also that for a K3 surface, MBM classes are integer classes with square -2.

7.3 Examples

We give several examples illustrating the shape of the Apollonian carpet and Baragar gasket. For simplicity, we restrict ourselves to K3 surfaces. As we have already observed, a sphere on the boundary of the projectivized positive cone is the intersection of this boundary with a linear subspace W, and the BBF form q (the intersection form in this case) is negative definite on $W^\perp$. This sphere belongs to an ample chamber if it is either disjoint from the walls, or intersects them tangentially. This condition means that adding a (-2)-class z to $W^\perp$ destroys negativity: $z^\perp$ is disjoint from the sphere iff q is of signature $(1, \dim W^\perp)$ on $W^\perp + z$, and tangent to the sphere iff q is negative semi-definite on $W^\perp + z$. Finally, such a sphere is a Baragar component when $W^\perp$ is generated by integral classes of square -2, and not a Baragar component otherwise.

To calculate the signature of q on the relevant subspaces of the real Neron-Severi group, we prove some auxiliary results.

Lemma 7.8: Assume that $L \subset \Lambda$ is a primitive sublattice, and denote by A the quotient Λ/L. Then we have the following exact triples:

$$0 \to A \to \Lambda^\vee/L \to D(\Lambda) \to 0 \tag{7.1}$$

$$0 \to A^\vee \to \Lambda^\vee/L \to D(L) \to 0 \tag{7.2}$$

Proof: By the definition of $D(\Lambda)$ we have an exact triple

$$0 \to \Lambda \to \Lambda^\vee \to D(\Lambda) \to 0.$$

Taking the quotient of the first two terms by L, we obtain (7.1).

Since the sublattice L is primitive, the quotient $A = \Lambda/L$ is free, so we have an exact triple

$$0 \to A^\vee \to \Lambda^\vee \to L^\vee \to 0.$$

Taking the quotient of the last two terms by L, we obtain (7.2). ■

Proposition 7.9: Let (Λ, q) be a quadratic lattice of rank r for some $r \geq 3$. Assume that the $\mathbb{F}_p$-vector space $D(\Lambda) \otimes \mathbb{F}_p$ has dimension $r-k$. Then for any primitive sublattice L of rank $l > k$, p divides $|D(L)|$.

Proof: Applying Lemma 7.8 and using our assumption on $D(\Lambda)$, we deduce from the sequence (7.1) by tensoring it with $\mathbb{F}_p$ that the dimension of $(\Lambda^\vee/L) \otimes \mathbb{F}_p$ is at least $r-k$. On the other hand, $A^\vee$ is a free abelian group of rank $r-l$, and we deduce from the sequence (7.2) by tensoring it with $\mathbb{F}_p$ that $D(L) \otimes \mathbb{F}_p$ is at least one-dimensional. It follows that the group $D(L)$ must contain p^k-torsion for some k, in particular p divides $|D(L)|$.■

The proof of the following two corollaries is an easy exercise, left to the reader.

Corollary 7.10: Let (Λ, q) be a quadratic lattice of signature $(1, r-1)$ for some $r \geq 3$, with $D(\Lambda) \otimes \mathbb{F}_p$ of dimension $r-1$. Let L be a rank-two sublattice generated by q-negative vectors v_1, v_2 with $q(v_1)q(v_2) < p$. Then L is either degenerate and negative semi-definite, or has signature $(1,1)$.

Corollary 7.11: Let (Λ, q) be a quadratic lattice of signature $(1, r-1)$ for some $r \geq 3$, with $D(\Lambda) \otimes \mathbb{F}_p$ of dimension $r-2$, where $p \geqslant 5$. Let L be a rank-three sublattice generated by three elements of square -2, two of which are orthogonal. Then L is of signature $(1,2)$ or $(0,2)$ (i.e. degenerate negative semi-definite).

Example 7.12: Consider the lattice Λ of rank four with the quadratic form $q(x,y,z,w) = 2px^2 - 2y^2 - 2pz^2 - 2pw^2$ for some prime number $p \geqslant 5$. This is an even lattice of signature $(1,3)$. By the classical results about lattice embeddings ([Ni], see also [AV3, section 3.3]), Λ can be primitively embedded into the K3 lattice Λ_{K3}, therefore Λ can be realized as the Néron–Severi lattice of an algebraic K3 surface. Set $V = \Lambda \otimes \mathbb{R}$ and consider $\mathbb{H}^3 = \mathbb{P}V^+$. Then $\mathrm{Abs} = \partial\mathbb{H}^3 = S^2$. It is clear that the set of classes of square -2 is non-empty. By Corollary 7.10, the circles on Abs cut out by the

orthogonal hyperplanes to (-2)-classes v_1 and v_2 do not intersect, except possibly tangentially. Therefore we obtain a circle packing on the boundary of the positive cone of the K3 surface and a non-trivial Baragar gasket (see Definition 7.5) in the closure of its ample cone.

Example 7.13: Note that in the example above, some spheres of the Baragar gasket may be tangent. Let us construct an example where all of the spheres are disjoint. Note that the spheres defined by two vectors $v_1, v_2 \in \Lambda$ (as $v_i^\perp \cap \mathrm{Abs}$) are tangent if and only if the restriction of q to $\langle v_1, v_2\rangle$ is degenerate. In particular, this implies that Λ contains non-trivial isotropic vectors (they correspond to the points of tangency of the spheres). Therefore to construct our example it is sufficient to produce an even lattice without non-zero isotropic elements such that it contains classes of square -2 and satisfies the conditions of Proposition 7.9 for some prime $p \geq 5$.

Start from the rank four lattice Λ_0 with the quadratic form $q_0(x,y,z,w) = 3x^2 - y^2 - 7z^2 - 7w^2$. Let us observe that Λ_0 does not contain non-zero isotropic elements. To see this, assume that x, y, z, w are coprime integers satisfying $3x^2 - y^2 - 7z^2 - 7w^2 = 0$. This implies that $3x^2 \equiv y^2 \,(\mathrm{mod}\ 7)$ and since 3 is not a square mod 7, we deduce that $x \equiv y \equiv 0 \,(\mathrm{mod}\ 7)$. Writing $x = 7x_0$ and $y = 7y_0$ we see that $21x_0^2 - 7y_0^2 - z^2 - w^2 = 0$. It follows that $z^2 + w^2 \equiv 0 \,(\mathrm{mod}\ 7)$, and since -1 is not a square mod 7, we deduce that $z \equiv w \equiv 0 \,(\mathrm{mod}\ 7)$, arriving at a contradiction.

Now let Λ be the rank four lattice with the quadratic form $q(x,y,z,w) = 6p^2x^2 - 2y^2 - 14p^2z^2 - 14p^2w^2$, i.e. q is obtained from $2q_0$ by multiplying the first, third and fourth summands by p^2, for some prime $p \geq 5$. Then (Λ, q) is clearly a sublattice of $(\Lambda_0, 2q_0)$, therefore it does not contain non-trivial isotropic elements. It is also an even lattice of signature $(1,3)$ representing -2 and satisfying the assumptions of Proposition 7.9 for $d = 2$. By the same argument as above, this lattice can be realized as the Néron–Severi lattice of an algebraic K3 surface, giving an example of the Baragar gasket with all spheres disjoint.

Example 7.14: Let us now give an example of an Apollonian carpet which also has non-Baragar components. Let Λ be the rank four lattice with the quadratic form $q(x,y,z,w) = 46x^2 - 2y^2 - 46z^2 - 46w^2$ (i.e. the form from the first example with $p = 23$). This form obviously represents both -2 and -4. Moreover, by Corollary 7.10, if $q(v_1) = -2$ and $q(v_2) = -4$, the circles cut out on the absolute by $v_1^\perp$ and $v_2^\perp$ do not intersect except maybe tangentially. This means that the latter circle lies entirely on the isotropic boundary of the projectivization of the ample cone, i.e. is a component of

the Apollonian carpet. Notice that this is not a Baragar component, because $q(v_2) = -4$ and all MBM classes are of square -2.

Example 7.15: The components of the Apollonian carpet can be of codimension more than one. Consider a K3 surface with Picard lattice Λ of rank 5, with $q(x, y, z, w, u) = 2px^2 - y^2 - z^2 - 2pw^2 - 2pu^2$, then by Corollary 7.11 the orthogonal to $\langle e_2, e_3\rangle$ does not intersect the other walls of the Kähler chambers except possibly tangentially, therefore gives a codimension-two Baragar component of the Apollonian carpet.

Example 7.16: The non-Baragar components can also intersect transversally. As an explicit example, we can take a rank-four lattice Λ with the quadratic form $2q$, where $q = 5x^2 - y^2 - 5z^2 - 5w^2$. The orthogonal circles to vectors $v_1 = (1, 2, 1, 0)$ and $v_2 = (1, 2, 0, 1)$ (here $q(v_1) = q(v_2) = -4$) do intersect, since the restriction of q to the plane they generate is negative. We deduce from Corollary 7.10 applied to q that both circles are components of the Apollonian carpet. See Figure 5 in the next subsection for the picture.

Remark 7.17: The reason why we have only considered K3 surfaces is that the description of MBM classes is particularly simple in this case: indeed these are precisely the elements of square -2 in the lattice. But it is clear from the discussion above that the same method should yield analogous examples for a hyperkähler manifold as soon as one has a description of its MBM classes. Indeed it is known ([AV2]) that MBM classes have bounded Beauville-Bogomolov square, so setting the prime p as above large enough allows to produce lattices with desired properties. Strictly speaking, there is an additional difficulty to prove that a given lattice embeds into the second cohomology lattice of a hyperkähler manifold, as Nikulin's theorem requires unimodularity of this larger lattice. But there exist some ways to get around this difficulty, see e.g. [AV3].

7.4 Illustrations

In this section we provide some examples of computer generated images of the sphere packings on the boundary of the hyperbolic 3-space. All of the presented configurations of spheres may appear in the Néron–Severi groups of algebraic K3 surfaces (some of the lattices considered below are odd; one has to multiply them by two before embedding into the K3 lattice). We hope that out illustrations will give an idea of what the isotropic boundary of the ample cone of a hyperkäher manifold may look like, more vividly than

the mere formal description.

We start by describing the procedure that was used to produce all the figures below. We fix a quadratic lattice (Λ, q) of rank 4 and signature $(1,3)$. We fix an integer $d<0$ and let $\mathcal{S}=\{v\in\Lambda \mid q(v)=d\}$. The collection $\mathcal{S}$ defines an arrangement of hyperplanes $H_v=v^\perp\subset V=\Lambda\otimes\mathbb{R}$, where $v\in\mathcal{S}$.

We fix an isomorphism $\Lambda\simeq\mathbb{Z}^4$, identifying q with a symmetric matrix A. We denote by ν_0, ν_1, ν_3 and ν_3 the eigenvectors of A with eigenvalues $\lambda_0>0>\lambda_1\geq\lambda_2\geq\lambda_3$, normalized by the condition $|q(\nu_i)|=1$. We identify V with $\mathbb{R}^{1,3}$ using the basis $\nu_0,\dots,\nu_3$, so that for $x=(x_0,x_1,x_2,x_3)\in\mathbb{R}^{1,3}$ we have $q(x)=x_0^2-x_1^2-x_2^2-x_3^2$.

The hyperbolic 3-space $\mathbb{H}^3=\mathbb{P}(V^+)$ is identified with the open unit ball in $\mathbb{R}^3$ via the stereographic projection from the point $(-1,0,0,0)$. The inverse of this projection is given by

$$(y_1,y_2,y_3)\mapsto(1+|y|^2:2y_1:2y_2:2y_3),$$

where $|y|$ denotes the standard Euclidan norm in $\mathbb{R}^3$. Under the stereographic projection the boundary Q of $\mathbb{H}^3$ is identified with the unit sphere $S^2\subset\mathbb{R}^3$ and we have

$$Q\cap H_v=\left\{y\in S^2 \;\middle|\; v_0=\sum_{i=1}^3 v_iy_i\right\},$$

where $v=(v_0,v_1,v_2,v_3)\in\mathbb{R}^{1,3}$.

As we have discussed above, the collection of hyperplanes H_v, $v\in\mathcal{S}$ is locally finite in $\mathbb{H}^3$, and it splits $\mathbb{H}^3$ into a countable set of open chambers. We will study the boundary of the $\mathcal{S}$-chamber in $\mathbb{H}^3$ that contains the origin, which corresponds to the point $(1:0:0:0)\in\mathbb{P}(V^+)$. More precisely, since the origin might lie on one of the walls H_v, we will study the boundary of the union of all $\mathcal{S}$-chambers that contain the origin in their closure. The number of such chambers is finite and we denote their union by $\mathcal{C}_0\subset\mathbb{H}^3$. We also define $\mathcal{S}'=\{v\in\mathcal{S}\mid v_0>0\}$, so that $\overline{\mathcal{C}_0}=\cap_{v\in\mathcal{S}'}H_v^+$. We let $Q_v=H_v^-\cap Q$ for $v\in\mathcal{S}'$. Then the isotropic part of the boundary of $\mathcal{C}_0$ is

$$\overline{\mathcal{C}_0}\cap Q=S^2\setminus\cup_{v\in\mathcal{S}'}Q_v.$$

We further consider the stereographic projection from the point $(1,0,0)\in S^2$ onto $\mathbb{R}^2$. The inverse of that projection identifies $\mathbb{R}^2$ with the complement of $(1,0,0)\in S^2$:

$$(z_1,z_2)\mapsto(y_1,y_2,y_3)=\left(\frac{|z|^2-1}{|z|^2+1},\frac{2z_1}{|z|^2+1},\frac{2z_2}{|z|^2+1}\right).$$

Under this projection Q_v is mapped to the subset

$$D_v = \{z \in \mathbb{R}^2 \mid |z|^2(v_1 - v_0) + 2v_2z_1 + 2v_3z_2 > v_0 + v_1\},$$

which is either the interior or the exterior of the circle with centre at the point $(v_2/(v_0-v_1), v_3/(v_0-v_1))$ of radius $|q(v)|^{1/2}|v_1-v_0|^{-1}$, or a half-plane, depending on the non-vanishing and the sign of v_1-v_0. Slightly abusing the terminology, we will call D_v a v-disc (although it might not be really a disc, but a "hole" in $\mathbb{R}^2$ or a half-plane). The isotropic boundary of $\mathcal{C}_0$ under our projection is identified with the complement $\mathcal{B}_0$ to the union of v-discs for all $v \in \mathcal{S}'$.

The v-discs are partially ordered by inclusion as subsets of $\mathbb{R}^2$, and this defines a partial order on $\mathcal{S}'$. It is clear that any increasing chain with respect to this partial order stabilizes (because of the local finiteness of the hyperplanes H_v in $\mathbb{H}^3$). Hence to describe the isotropic boundary $\mathcal{B}_0$ it is sufficient to consider only the maximal v-discs. The maximal v-discs correspond to the faces of $\overline{\mathcal{C}_0}$.

Listing all elements of the set $\mathcal{S}'$, we obtain increasing finite collections of maximal v-discs, and the complements to such finite collections give approximations to $\mathcal{B}_0$. In the pictures below, we show such finite collections of v-discs inside the unit disc in $\mathbb{R}^2$ (which corresponds to the hemisphere in $S^2 \subset \mathbb{R}^3$ given by the inequality $y_1 < 0$). The complement to the v-discs (black points in the pictures) is the approximation of the isotropic boundary of $\mathcal{C}_0$.

7.4.1 Example 0.

The classical Apollonian gasket is a configuration of circles in the plane that is obtained iteratively by the following procedure. One starts from a triple of pairwise tangent circles having three distinct points of tangency. As the first step one adds the two circles that are tangent to the given three. Then one repeats the process for all triples of pairwise tangent circles in the configuration obtained in the previous step. One can obtain the Apollonian configuration starting from the lattice with the intersection matrix (see [Ba1])

$$A = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

We take $d = -1$. The result is shown in Figure 1.

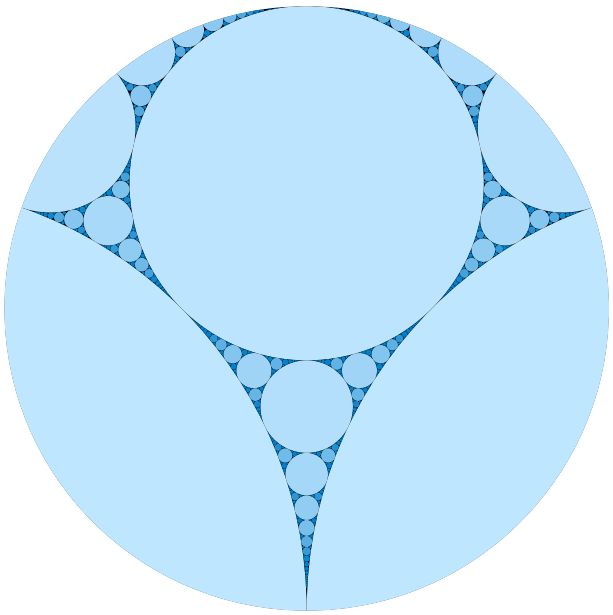


Figure 1: The classical Apollonian gasket

7.4.2 Example 1.

In a typical situation, the maximal v-discs may overlap, and the isotropic boundary (when non-empty) is a fractal. Figure 2 shows such an example, where $d = -1$ and the quadratic form q is given by the matrix

$$A = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

In Figure 3 we show the overlapping v-discs that cut out the isotropic boundary. One may guess from Figure 2 that in this example the isotropic boundary contains infinitely many circles (which form what we call the Apollonian carpet), but since the v-discs overlap, neither of their boundary circles is contained in the isotropic boundary (so the Baragar gasket is empty).

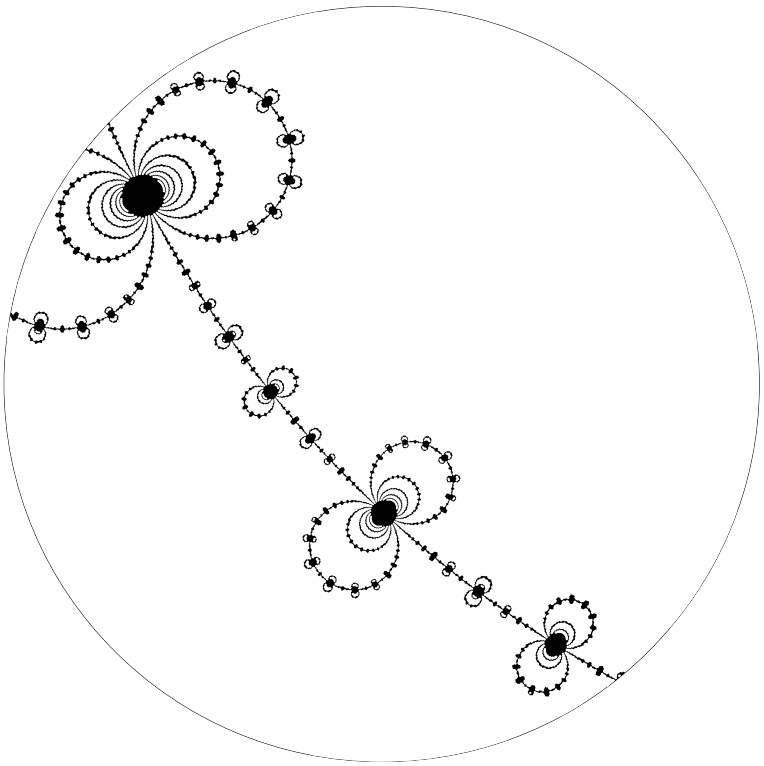


Figure 2: The isotropic boundary is a fractal

Note that Figure 2 shows (part of) the isotropic boundary of only one chamber. For the other chambers the isotropic boundary is obtained by a composition of reflections in some (-1)-vectors. Figure 4 shows the isotropic boundary for two different chambers.

7.4.3 Example 2

Here is the pictural representation of the example Example 7.16.

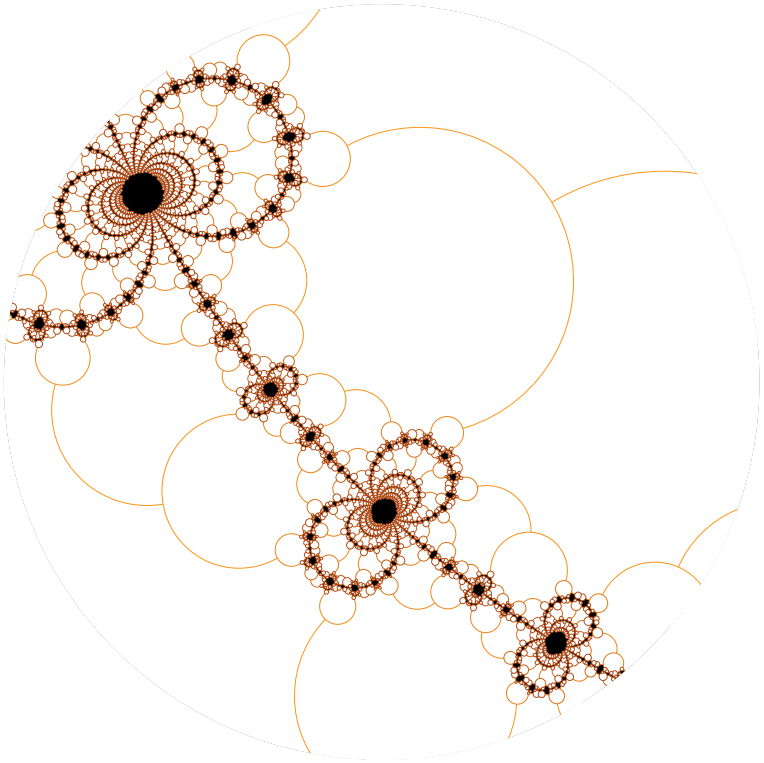


Figure 3: The discs that cut out the fractal

The maximal v-discs are disjoint, although their boundary circles may be tangent. These circles form the Baragar gasket. The circles $C_w = Q \cap H_w$ for primitive elements $w \in \Lambda$ with $q(w) = -4$ are part of the Apollonian carpet, but not of the Baragar gasket, and some of them intersect transversally (this is shown in red on Figure 5).

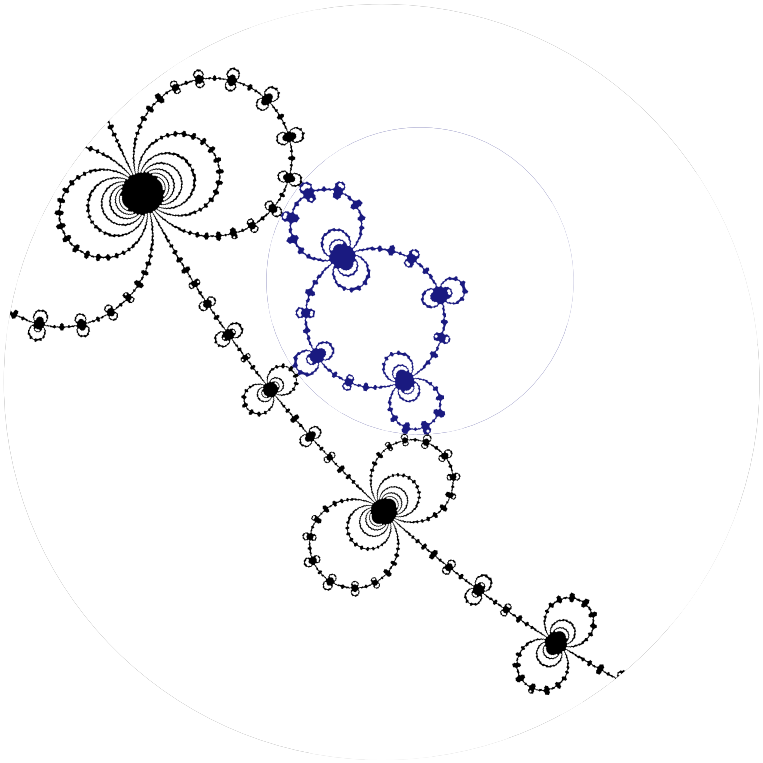


Figure 4: The isotropic boundaries of two different chambers

7.4.4 Example 3.

The quadratic form is given by

$$A = \begin{pmatrix} p^3 & 0 & 0 & 0 \\ 0 & -p^2 & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where $p = 5$. Here we take $d = -1$ as before. The quadratic form satisfies the conditions of Proposition 7.9, therefore the maximal v-discs are disjoint. The result is shown in Figure 6.

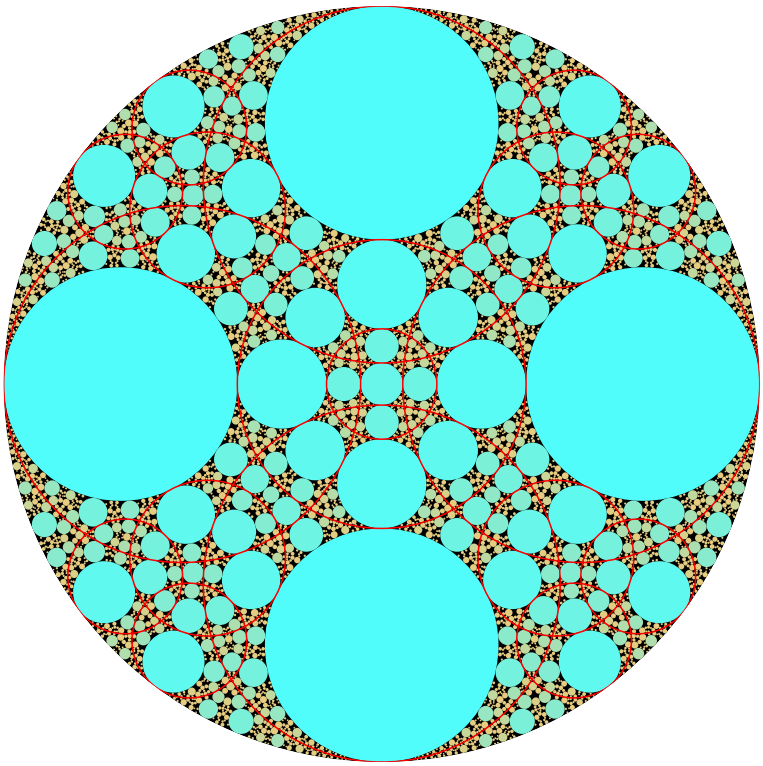


Figure 5: A configuration with intersecting Apollonian circles

7.4.5 Example 4.

We take $d = -1$ and

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -63 & 0 \\ 0 & 0 & 0 & -63 \end{pmatrix}$$

Note, that the above quadratic form does not represent zero, as we have discussed in Example 7.13. Moreover, it satisfies the conditions of Proposition 7.9 (taking $p = 3$), and it follows that the closures of the maximal v-discs are disjoint. Note that due to the large discriminant of q the vectors $v \in \Lambda$ with $q(v) = -1$ are "rare", so that in the picture most of the v-discs have very small radii. They are, nevertheless, everywhere dense in the plane. The big v-disc in the centre of Figure 7 is the D_v with $v = (1, -2, 0, 0)$.

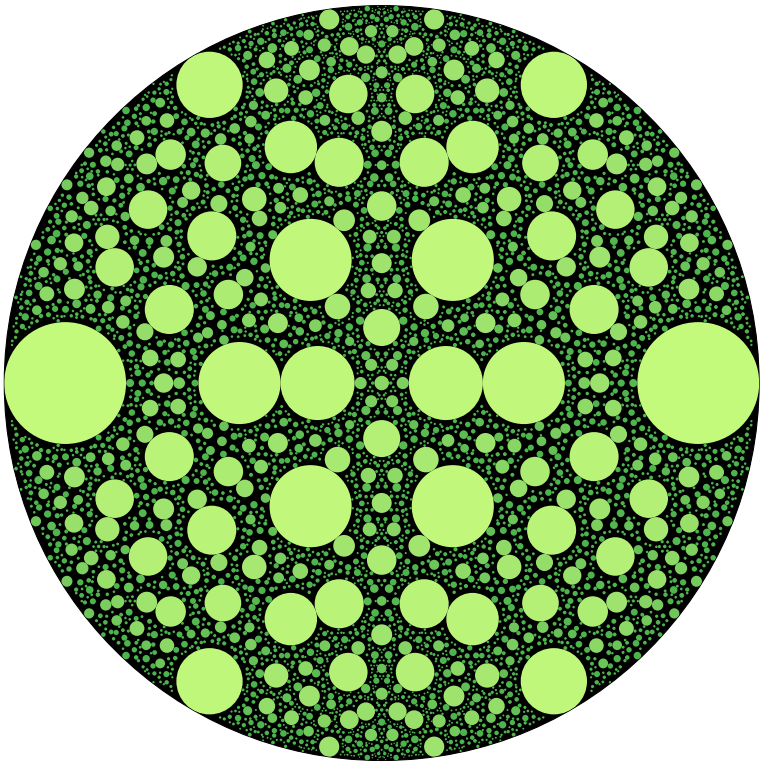


Figure 6: An example of a packing

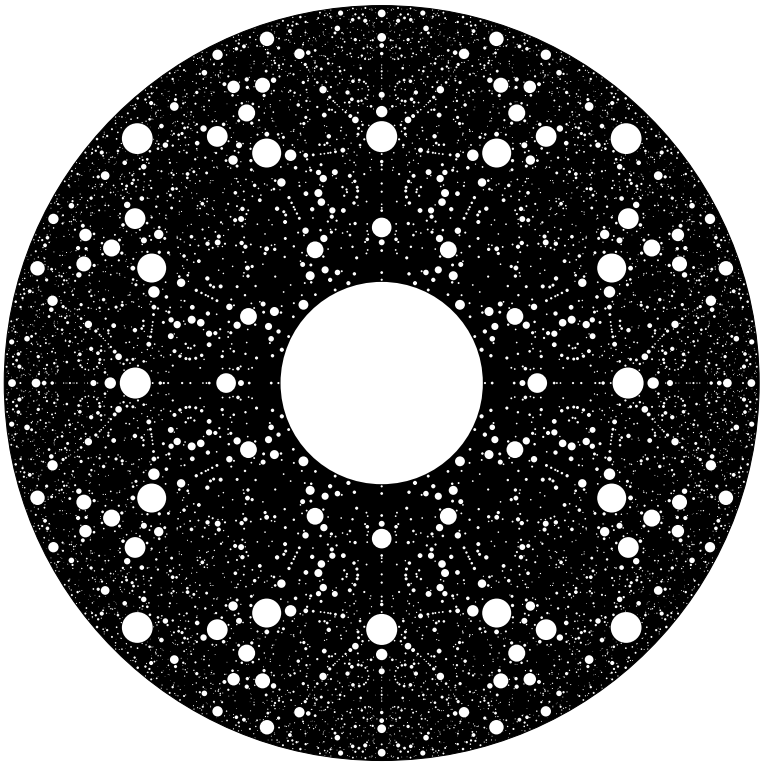


Figure 7: An example where all closures of the maximal v-discs are disjoint

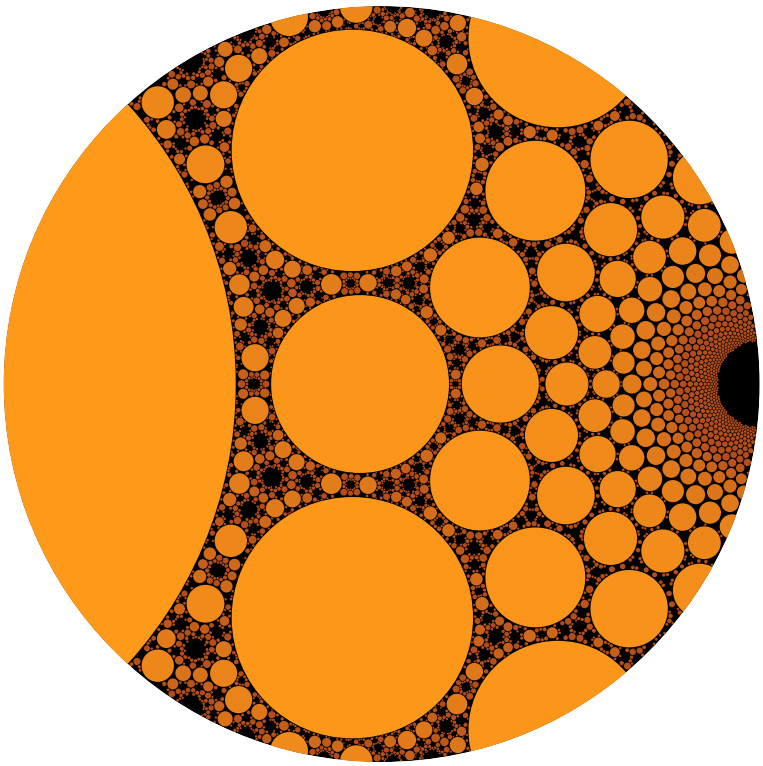


Figure 8: An example where all closures of the maximal v-discs are disjoint

7.4.6 Example 5.

We take $d = -2$ and

$$A = \begin{pmatrix} -2 & 5 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & 5 & -10 \end{pmatrix}$$

The result appears in Figure 8. The conditions of Proposition 7.9 are satisfied for $p = 5$. One may guess from the picture that in this case the closures of the v-discs are also disjoint, like in the previous example.

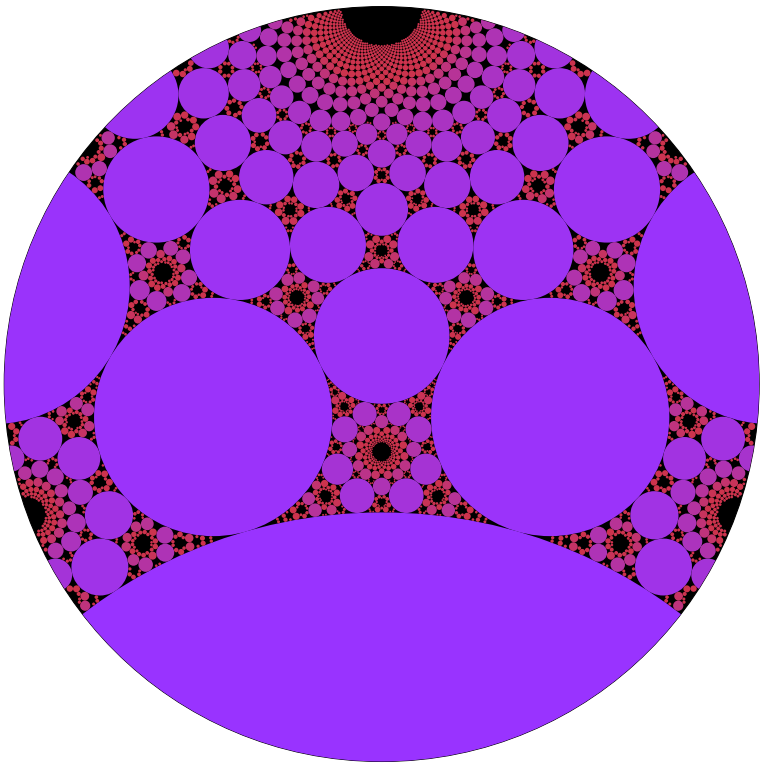


Figure 9: One more example

7.4.7 Example 6.

We take $d = -1$ and

$$A = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

The conditions of Proposition 7.9 are satisfied for $p = 2$. The result appears in Figure 9.

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