Contraction centers in families of hyperkähler manifolds

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Abstract

We study the exceptional loci of birational (bimeromorphic) contractions of a hyperkähler manifold M. Such a contraction locus is the union of all minimal rational curves in a collection of cohomology classes which are orthogonal to a wall of the Kähler cone. Homology classes which can possibly be orthogonal to a wall of the Kähler cone of some deformation of M are called **MBM classes**. We prove that all MBM classes of type (1,1) can be represented by rational curves, called **MBM curves**. Any MBM curve can be contracted on an appropriate birational model of M, unless $b_2(M) \leqslant 5$. When $b_2(M) > 5$, this property can be used as an alternative definition of an MBM class and an MBM curve. Using the results of Bakker and Lehn, we prove that the stratified diffeomorphism type of a contraction locus remains stable under all deformations for which these classes remains of type (1,1), unless the contracted variety has $b_2 \leqslant 4$. Moreover, these diffeomorphisms preserve the MBM curves, and induce biholomorphic maps on the contraction fibers, if they are normal.

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1 Introduction

1.1 Teichmüller spaces in hyperkähler geometry

Let M be a complex manifold. Recall that **the Teichmüller space** Teich of complex structures on M is the quotient $\mathsf{Teich} := \mathsf{Comp} / \mathsf{Diff}_0$, where Comp is the space of complex structures (with the topology of uniform convergence of all derivatives) and Diff_0 the connected component of the unity of the diffeomorphism group. The mapping class group $\mathsf{Diff} / \mathsf{Diff}_0$ on Teich.

In our case M is a compact Kähler irreducible holomorphically symplectic manifold [1] (IHS). We consider the Teichmüller space Teich of all complex structures of hyperkähler type (Subsection 2.2). Up to the action of the subgroup K of the mapping class group acting trivially on cohomology, this is the same space as the moduli space of marked hyperkähler manifolds often considered in algebraic geometry. The subgroup K permutes components of Teich, so that we shall often make no difference between the relevant component of Teich and of the moduli space of marked hyperkähler manifolds[2].

We always consider the component containing the parameter point of our given complex structure, i.e. parametrizing hyperkähler deformations

[1] By the Calabi-Yau theorem, this is the same as a hyperkähler manifold.
[2] The latter has only finitely many connected components, see [H3]

of M. By abuse of notation, this space is also denoted Teich. The action of the mapping class group Γ (i.e. the subgroup of $\mathsf{Diff}\,/\,\mathsf{Diff}_0$ preserving our connected component) on Teich is ergodic, and its orbits are classified using Ratner's orbit classification theorem (Theorem 6.1).

In [Ma2], Theorem 1.1, E. Markman has constructed the universal family on the marked moduli space. Let

$$u : \mathcal{U} \longrightarrow \mathsf{Teich}$$

be its pullback. The map u is a smooth complex analytic submersion with fiber (M, I) at the point $I \in \mathsf{Teich}$ (throughout the paper, (M, I) denotes a manifold M equipped with a complex structure I). In this paper we use the action of the mapping class group Γ on this fibration to study the geometry of families of rational curves on (M, I).

Fix a cohomology class $z \in H^2(M, \mathbb{Z})$. Let $\Gamma_z \subset \Gamma$ be the stabilizer of z in Γ, and Teich_z the Teichmüller space of all complex structures $I \in \mathsf{Teich}$ such that z is of Hodge type (1,1) on (M, I).

Recall that the second cohomology group of a hyperkähler manifold with maximal holonomy is equipped with a canonical bilinear symmetric pairing q, called Bogomolov-Beauville-Fujiki (BBF) form. This form is integral but in general not unimodular, so that it embeds $H_2(M, \mathbb{Z})$ into $H^2(M, \mathbb{Q})$ as an overlattice of $H^2(M, \mathbb{Z})$. It is often convenient to consider the homology classes of curves as second cohomology classes with rational coefficients, and we do this throughout the paper. Assume that $q(z, z) < 0$ and z is represented by a rational curve on some (M, I). It turns out that there is a subspace $\mathsf{Teich}_z^{\mathsf{min}}$, which is the same as Teich_z up to inseparability issues, such that for all $I \in \mathsf{Teich}_z^{\mathsf{min}}$, rational curves with cohomology class proportional to z exist on $X = (M, I)$ and can be contracted birationally (Definition 3.7 and Section 4). For I such that the complex manifold X is projective, this is a consequence of Kawamata base point free theorem (Theorem 4.2). For non-algebraic deformations, it follows from the work by Bakker and Lehn ([BL1]) provided that $b_2(M) > 5$ (Theorem 4.6).

We are interested in the behaviour of the contraction loci (that is, the exceptional loci of the contraction maps) $Z_I \subset (M, I)$ as I varies in $\mathsf{Teich}_z^{\mathsf{min}}$. These loci are also obtained as the unions of rational (or all) curves of class proportional to z. The crucial fact is that for any integral class $z \in H_2(M, \mathbb{Z})$ with $q(z, z) < 0$, the action of the group Γ_z on Teich_z is also ergodic on each connected component. Moreover we can classify, in the same way as for Γ acting on Teich, the orbits of the Γ_z-action on the space $\mathsf{Teich}_z^{\mathsf{min}}$ (Theorem 6.2).

The subvarieties Z_I are exchanged by the action of Γ_z on $\mathcal{U}$ and are thus isomorphic along an orbit, which is often dense. However, when not in the same orbit, they can be very different as complex varieties. Our main purpose in this paper is to show, under some restrictions, that the Z_I form a trivial family in the real analytic category (Theorem 5.4).

Note that the real analytic manifolds do not have continuous moduli: indeed their deformations are controlled by the first cohomology of the tangent bundle, and higher cohomologies of a coherent sheaf in real analytic category are always zero (see [Car], Théorème 3, for submanifolds of $\mathbb{R}^N$, and [GMT], Theorem 2.7 p. 116, [Nar], p.931, for the reduction to this case). However, singular real analytic varieties might have continuous moduli. The easiest way to see this is to look at configuration C of 4 real lines in $\mathbb{R}P^2$. If these lines intersect in one point, the corresponding tangent cone (which is determined intrinsically by the real analytic geometry of the pair $(\mathbb{R}P^2, C)$) is 4 lines in a vector space. The cross-ratio of these 4 lines gives a real analytic invariant of this pair.

Those phenomena are dealt with by Thom-Mather theory. This theory defines stratified diffeomorphism of real analytic varieties as a homeomorphism inducing a diffeomorphism on open strata of a stratification of a manifold by singularities. Thom and Mather proved that in this category real analytic varieties have no continuous moduli (see for example [M]). Later, T. Mostowski and A. Parusiński proved that this diffeomorphism is a bi-Lipschitz equivalence ([Pa3, Theorem 1.6]). We shall see that the deformation of Z_I and related spaces, such as the corresponding component of the Barlet space and the incidence variety, are trivial in stratified diffeomorphism and in the bi-Lipschitz category (Theorem 1.9).

The fact that the family of Z_I, as I varies, is locally trivial in the real analytic category even though Z_I can be singular, is related to the fact that Z_I are contraction loci and follows from results of [BL1]. Bakker and Lehn refer to a concept of "locally trivial deformation" introduced by H. Flenner and S. Kosarew in [FK] (see also Section 4.2). Unlike its name would suggest, a "locally trivial deformation" is not a deformation $\pi : \mathcal{X} \longrightarrow B$ which is equivalent to the product $F \times U \longrightarrow U$ locally on B. Instead, it is a deformation which is locally trivial *locally in* $\mathcal{X}$.

We show that a locally trivial deformation induces a trivial deformation in the real analytic category (Proposition 5.1). Thus the family of contracted IHS manifolds constructed in [BL1] is trivialized real analytically, along with the family of the contraction loci. For other related families, such as the Barlet spaces and incidence spaces associated with minimal rational curves, our techniques (that is, combining an ergodicity theorem with a

result of Thom-Mather type) give bi-Lipschitz and stratified diffeomorphic trivializations.

1.2 Teichmüller spaces, MBM classes and locally trivial deformations

The Teichmüler space Teich_z is a smooth, non-Hausdorff manifold, equipped with a local diffeomorphism to the corresponding period space $\mathbb{P}\mathsf{er}_z := \frac{SO^+(3,b_2-4)}{SO^+(1,b_2-4)\times SO(2)}$ (alternatively, this is just the orthogonal of z in the usual period space $\mathbb{P}\mathsf{er}$, seen as a subset of a quadric in the projective space $\mathbb{P}H^2(M,\mathbb{C})$) , which becomes one-to-one if we glue together the inseparable points. Following E. Markman [Ma1], the set of preimages of a point in $\mathbb{P}\mathsf{er}_z$ (that is, the set of complex structures inseparable from a given I) is identified with the set of the Kähler chambers in the positive cone of $H^{1,1}(M,I)$, so that each Kähler chamber can be seen as the Kähler cone of the corresponding complex structure. The classes relevant for us, those of negative square and represented by a rational curve on some (M,I), are the so-called MBM classes (Subsection 1.3 and Section 3), i.e. such that the orthogonal complement $z^\perp$ contains one of the walls of these Kähler chambers (Theorem 3.4). Restricting ourselves to the Kähler chambers adjacent to the hyperplane $z^\perp$, we obtain the space $\mathsf{Teich}^{\mathsf{min}}_{\pm z} \subset \mathsf{Teich}_z$. Note that both spaces are non-Hausdorff even at their general points, since there are always at least two chambers adjacent to a given wall. Once z is fixed, $z^\perp$ is co-oriented, and we take the set of the chambers adjacent to $z^\perp$ on the positive side (that is, z must be positive on the Kähler cone). This last space, separated at its general point, is denoted $\mathsf{Teich}^{\mathsf{min}}_z \subset \mathsf{Teich}_z$. This is precisely the space of complex structures $I \in \mathsf{Teich}_z$ such that a positive multiple of z is represented by an extremal rational curve: indeed, by a result of Huybrechts and Boucksom, the Kähler cone is characterized as the set of $(1,1)$-classes of positive Beauville-Bogomolov square which are positive on all rational curves ([H1, H2], [Bou]).

It follows that the boundary of the Kähler cone is a union of a "round part" (the boundary of the cone of positive-square classes) and locally polyhedral walls (orthogonals to rational curves) which intersect in locally polyhedral faces of higher codimension.[3] More generally, if F is a face of the Kähler cone of $X = (M,I)$ of codimension k in $H^{1,1}_{\mathbb{R}}(X)$, then F is contained in (and has a common open part with) the intersection of several hyperplanes orthogonal to MBM classes $z_1,\dots,z_k$, where z_i are non-negative on

[3] See e.g. [HT], Proposition 13, for local finiteness issues.

the Kähler cone. We set $\mathsf{Teich}_F = \bigcap_{i=1}^k \mathsf{Teich}^{\mathsf{min}}_{z_i}$: this is the part of Teich where all z_i remain, up to a positive multiple, classes of extremal rational curves.

Thanks to Kawamata base-point-free theorem, it is well-known that if $X = (M, I)$ is projective, the faces of the Kähler cone can be contracted: there is a projective birational morphism $\varphi_F : X \to X'$ sending a curve C to a point iff its class belongs to the subspace $F^\perp$. Conversely, a projective birational contraction contracts some extremal face F.

Let $X \longrightarrow X_1$ be a birational contraction of F as above. In [BL1], Proposition 4.5, Bakker and Lehn prove that any, possibly non-projective, small deformation X_t of X such that all z_i remain of type $(1,1)$ on X_t contracts onto a "locally trivial" deformation of X_1 (in the sense of [FK]), and that all *locally trivial* small deformations of X_1 appear in this way[4].

Bakker and Lehn's result has many applications. The first application, implicit in [BL1], is the existence of bimeromorphic contractions for non-algebraic hyperkähler manifolds, see Theorem 4.6.

Next, we use the locally trivial deformation of the contracted manifold to produce a real analytic trivialization of the universal family over Teich_F preserving the corresponding contraction locus.

As the simplest examples show, the deformation equivalent contraction loci need not be biholomorphic or bimeromorphic. Our last aim is to show that the fibers of their rational quotient fibrations do. To do this, we prove that the diffeomorphisms of contraction loci as above preserve the rational curves. This is done by establishing similar triviality results for Barlet spaces and incidence varieties, which we only get in the stratified diffeomorphism category. We use the following observation. Let $E \stackrel{\varphi}{\longrightarrow} B$ be a proper holomorphic (or even real analytic) map, and assume that B is obtained as a union of dense subsets, $B = \bigcup_{\alpha\in\mathfrak{I}} B_\alpha$, such that for any index α, all fibers of φ over $b \in B_\alpha$ are isomorphic. Then all fibers of φ are homeomorphic, stratified diffeomorphic and bi-Lipschitz equivalent.

This observation is based on the classical results by Thom and Mather (we use the version by Verdier [Ver], particularly well-adapted to our purposes; the bi-Lipschitz case is due to Parusiński[5]) These affirm that for any proper real analytic fibration $E \stackrel{\varphi}{\longrightarrow} B$, there exists a stratification of B such that the restriction of φ to open strata is locally trivial in the category of topological spaces (or in bi-Lipschitz category). Since each B_α in the

[4] We shall use the term "birational contraction" in the non-algebraic setting too, meaning "bimeromorphic contraction".

[5] [Pa3, Theorem 1.6]; see also [Pa1], [Pa2].

decomposition $B = \bigcup_{\alpha\in\mathfrak{I}} B_\alpha$ intersects the open stratum, this implies that all fibers of $E \xrightarrow{\varphi} B$ are homeomorphic and bi-Lipschitz equivalent.

The dense subsets B_α are in our case provided by the ergodicity of the mapping class group action. Unfortunately for this argument we have to exclude from consideration the complex structures with maximal Picard number, since their mapping class group orbits are closed.

We state our main results precisely in the subsection 1.4, after a brief digression on rational curves in the next subsection.

1.3 MBM loci on hyperkähler manifolds

Let $C \subset M$ be a rational curve on a holomorphic symplectic manifold of dimension $2n$. According to a theorem of Ran [R], the irreducible components of the deformation space of C in M have dimension at least $2n-2$.

Definition 1.1: A rational curve C in a holomorphic symplectic manifold M is called **minimal** if every component of its deformation space in M has dimension $2n-2$ at C.

Remark 1.2: In [AV1], Section 4, we have defined and studied minimal rational curves in a maximal irreducible uniruled subvariety $Z \subset X$ as curves of minimal degree with respect to a given Kähler class. From the proof of [AV1], Theorem 4.4, Corollary 4.6, one sees that this is equivalent to saying that C deforms in a family of dimension exactly $2n-2$ within Z. Indeed the fibers of the rational quotient (also called MRC fibration) of Z are k-dimensional, where k is the codimension of Z in X, and a simple dimension count shows that if C deforms in a family of dimension greater than $2n-2$, then C deforms with two fixed points, splitting into a union of lower-degree rational curves ("bend-and-break lemma"). Therefore all the results of [AV1] about minimal rational curves also apply here.

The dimension of Z can take any value between n and $2n-1$. Such a subvariety is always coisotropic, and the rational quotient fibration is equal to the coisotropic one (i.e. the kernel of the restriction of the symplectic form is tangent to the fibers) ([AV1], Theorem 4.4).

The key property of a minimal curve is that such a curve deforms together with its cohomology class $[C]$. More precisely, any small deformation of M on which $[C]$ is still of type $(1,1)$, contains a deformation of $[C]$ ([AV1], Corollary 4.8). Taking closures in the universal family over Teich_z gives a

submanifold of Teich_z of maximal dimension (which does not have to coincide with Teich_z, as it is not Hausdorff) such that every complex structure in this submanifold carries a deformation of C; this curve, however, can degenerate to a reducible curve, and one cannot in general say much about the cohomology classes of its components (Markman's example on K3 surfaces is already enlightening, see [Ma3], Example 5.3).

In [AV1], we have defined and studied the MBM classes: these are classes $z \in H^2(M, \mathbb{Z})$ such that, up to monodromy and birational equivalence, $z^\perp$ contains a wall of the Kähler cone[6]. In other words, $z^\perp$ contains a wall of some Kähler chamber (see [Ma1] for the definition of the latter, but it amounts to say that those are monodromy transforms of Kähler cones of the birational models of M). It is clear that the Beauville-Bogomolov square $q(z)$ is then negative; on the other hand, one can characterize MBM classes as negative classes such that some rational multiple λz is represented by a rational curve on a deformation of M ([AV1], Theorem 5.11, Corollary 5.14; more precisely, on a deformation with Picard group generated by z, λz is represented by a rational curve, and on specializations with larger Picard number this rational curve can break up into a reducible one). For our purposes, it is convenient to extend the notion of MBM on the rational cohomology (or integral homology) classes in an obvious way.

Note that it is apriori possible (though we don't have any examples) that the same rational curve C is contained in two maximal irreducible uniruled subvarieties Z_1 and Z_2 of M, in such a way that the deformations of C lying in Z_1 form a $2n-2$-parameter family whereas those lying in Z_2 need more parameters. Such a C is, by our definition, not minimal, but its generic deformation in Z_1 is.

Definition 1.3: An MBM curve is a minimal curve C such that its class $[C]$ is MBM.

Definition 1.4: Let C be an MBM curve on a hyperkähler manifold (M, I), and B an irreducible component of its deformation space (Chow-Barlet space, well-known to be compact when the ambient manifold is compact Kähler) in M containing the parameter point for C. An **MBM locus** of C is the union of all curves parameterized by B.

As mentioned in the beginning of this subsection, the MBM loci are

[6] A "wall" shall always mean a face of maximal dimension, that is $h^{1,1}-1$.

coisotropic subvarieties which can have any dimension between n and $2n-1$, but the family of minimal rational curves in an MBM locus always has $2n-2$ parameters (see [AV1], section 4, Theorem 4.4, Corollary 4.6).

Definition 1.5: Let z be an MBM class in $H^2(M,\mathbb{Q})$. The **full MBM locus** of z is the union of all MBM curves of cohomology class proportional to z and their degenerations (in other words, the union of all MBM loci for MBM curves of cohomology class proportional to z). Similarly, if F is a codimension k face of the Kähler cone of M, orthogonal to a k-dimensional subspace N in $H^2(M,\mathbb{Q})$, we define the full MBM locus for F as the union of MBM loci of MBM classes in N.

Remark 1.6: If the complex structure on M is in $\mathsf{Teich}_z^{\mathsf{min}}$, the full MBM locus has only finitely many irreducible components and is simply the union of all rational curves of cohomology class proportional to z, and similarly for F. This is because (by Kawamata base-point-free theorem in the projective case and by Bakker and Lehn's work in general with as assumption on b_2[7], see section 5) there exists a birational morphism contracting exactly the curves which have cohomology class in N (that is, orthogonal to F)[8] . The number of irreducible components of an exceptional set of a contraction is finite. One knows that these are **uniruled** ([K2]) and by bend-and-break lemma one finds a minimal rational curve in each. In fact the bend-and-break lemma gives minimal curves in the fibers of the rational quotient and this assures that they are contracted to points, see e.g. [BL1], Prop. 4.11, together with [AV1], proof of Theorem 4.4. These results also show that on a holomorphic symplectic variety the fibers of a contraction map coincide with fibers of the rational quotient of the exceptional locus, in particular the fibers of the contraction map are **rationally connected**.

Remark 1.7: Answering a question of the referee, let us mention without giving details of the calculation (for which we refer to our paper [AV4]) that on a given manifold there can exist rational curves with proportional cohomology classes which are both minimal (and MBM). Indeed, let $\tau : S \to \mathbb{P}^2$ be the double covering of the projective plane ramified along a sextic, and $X = S^{[6]}$ the Hilbert scheme of length-six subschemes of S. It is a classical fact that $H^2(X,\mathbb{Z}) = H^2(S,\mathbb{Z}) \oplus \mathbb{Z}E$ where E is half of the big

[7] In the non-projective case without assumptions on b_2,this statement can be shown using the density of complex structures corresponding to projective manifolds, but we shall not need it.

[8] By a slight abuse of terminology, we say that "F can be contracted".

diagonal class. Also let us denote by $h \in H^2(S, \mathbb{Z})$ the class of the inverse image under τ of a line in $\mathbb{P}^2$.

The direct sum above is orthogonal with respect to the Beauville-Bogomolov intersection form. Using this form, view the classes of curves on X as elements of $H^2(X, \mathbb{Q})$. Any line bundle of degree 6 on $\tau^{-1}C$ where C is a smooth conic in $\mathbb{P}^2$ has at least a pencil of sections, giving a rational curve R on X. In other words, one obtains a rational curve by varying 6-tuples of points on the inverse image of a conic. Moreover this is a minimal rational curve of cohomology class $2h - E$ whose deformations cover a divisor, say D, the MBM locus of R.

Likewise, since the space of sections of a line bundle of degree 4 on a genus two curve is three-dimensional, we obtain a projective plane in $X = S^{[6]}$ from the inverse image of a line (by taking those 6-tuples of points on S of which four are on the inverse image and two remaining points are fixed). A line l in this plane has cohomology class $h - E/2$. It is likewise minimal and its deformations cover a codimension-two subvariety Z, its MBM locus.

In this particular example Z is a part of D. Indeed "4 points on a line" is a special case of "6 points on a conic": just draw a line through the two remaining points. Nevertheless both R and l deform in families of dimension 10, so, for example, $2l$ is not always a degeneration of R as a cycle.

1.4 Main results of this paper

We concentrate on the space $\mathsf{Teich}_z^{\mathsf{min}} \subset \mathsf{Teich}_{\pm z}^{\mathsf{min}} \subset \mathsf{Teich}_z$ described in the first subsection. Recall that to construct $\mathsf{Teich}_z^{\mathsf{min}} \subset \mathsf{Teich}_z$, we first take the complex structures where $z^\perp$ actually contains a wall of the Kähler cone obtaining $\mathsf{Teich}_{\pm z}$, then take the "positive half" (the complex structures such that z is positive on the Kähler classes) of it. On the space $\mathsf{Teich}_z^{\mathsf{min}}$, there is an action of the subgroup of the monodromy group preserving z, and it turns out, thanks to the negativity of z, that almost all orbits of this action are dense. This allows us to make conclusions such as the uniform behaviour of subvarieties swept out by curves of class z on the manifolds represented by the points of $\mathsf{Teich}_z^{\mathsf{min}}$.

Theorem 1.8: Let M be a hyperkähler manifold of maximal holonomy, $b_2(M) > 5$, and $z \in H_2(M, \mathbb{Z}) \subset H^2(M, \mathbb{Q})$ a class of negative Beauville-Bogomolov square. Assume that z is represented by a minimal rational curve in some complex structure I on M (this means that z and the curve are MBM, see [AV1], Section 5; deform to the structure where z generates Picard group and use the deformation invariance). Let $Z = Z_I \subset (M, I)$

be the full MBM locus of z. Then for all $I, I' \in \mathsf{Teich}_z^{\mathsf{min}}$ there exists a real analytic isomorphism $h:\ (M,I) \longrightarrow (M,I')$ identifying Z_I and $Z_{I'}$. The same holds for the full MBM locus of any face F of dimension $\geqslant 3$, as the complex structure varies in Teich_F.

For the proof, see Theorem 5.4. In Subsection 6.3 we also prove the following variant of Theorem 1.8.

Theorem 1.9: In the assumptions of Theorem 1.8, let B_I be the Barlet space of all rational curves of cohomology class proportional to z. Then the map $h:\ Z_I \longrightarrow Z_{I'}$ can be chosen to send any rational curve $C \in B_I$ to some rational curve $h(C) \in B_{I'}$, inducing a homeomorphism from B_I to $B_{I'}$, for all complex structures $I, I' \in \mathsf{Teich}_z^{\mathsf{min}}$ except possibly those with maximal Picard number.

This in turn yields another version/strengthening of the theorem. Recall that a uniruled compact Kähler manifold has a so-called rational quotient, or MRC fibration ([Cam], [KMM]) whose fiber at a general point x consists of all the points which can be reached from x by a chain of rational curves. In particular, considering such a fibration on a desingularization of a component of Z_I gives a rational map $Q:\ Z_I \dashrightarrow Q_I$. Due to the fact that Z_I are contractible the map Q is actually regular and coincides with the contraction itself (cf. Section 4 of [AV1] or Proposition 4.11 of [BL1], and also Remark 1.6).

Theorem 1.10: In the assumptions of Theorem 1.8, Theorem 1.9, consider the contraction maps with exceptional loci Z_I and Z'_I, and let $Q:\ Z_I \to Q_I$, $Q':\ Z_{I'} \to Q_{I'}$ denote the restriction of the contraction maps to Z_I, $Z_{I'}$. Then $h:\ Z_I \longrightarrow Z_{I'}$ induces a bimeromorphism between the fibers of Q and Q'; it is an isomorphism when these fibers are normal.

Proof: We deduce Theorem 1.10 from Theorem 1.9 as follows. By Remark 1.6, the fibers are rationally connected. Any continuous map of rationally connected varieties mapping rational curves family to rational curves is automatically birational. Indeed the tangent spaces to rational curves span the holomorphic tangent space of the rationally connected variety at a general point, so that such a map sends holomorphic tangent space to the holomorphic tangent space. However, a homeomorphism between normal complex analytic spaces which is holomorphic on a dense open set is holomorphic everywhere. This result follows from a version of Riemann removable sin-

gularities theorem, see e. g. [Mag, Theorem 1.10.3]. ■

Restricting the diffeomorphism h to the irreducible components of the full MBM locus, we obtain the same statements for MBM loci of curves.

See Subsection 6.3 for some other variants of the main theorem.

Remark 1.11: One cannot affirm that the same statements hold along the whole of Teich_z, and this is false already for K3 surfaces. Indeed a (-2)-curve on a K3 surface X can become reducible on a suitable deformation X'. What we do affirm is that in Teich_z there is another point, nonseparable from the one corresponding to X', such that on the corresponding K3 surface X'' our curve remains irreducible. In this two-dimensional case, this easily follows from the description of the decomposition into the Kähler chambers in [Ma1]; Theorem 1.8 allows us to go further in the higher-dimensional case.

2 Hyperkähler manifolds

2.1 Hyperkähler manifolds

To save space, we omit most of the standard preliminaries on hyperkähler and holomorphically symplectic geometry (see [Bes] and [Bea]). By a **(simple) hyperkähler**, or irreducible holomorphically symplectic (IHS) manifold we mean a simply-connected compact Kähler manifold M such that $H^{2,0}(M)$ is generated by a nowhere degenerate form Ω. When the context requires, we shall also write (M, I) denoting by M the inderlying differentiable manifold and by I a complex structure on M.

On the second cohomology of a hyperkähler manifold there is an integral quadratic form q, called **Beauville-Bogomolov-Fujiki (BBF) form**. It has signature $(3, b_2 - 3)$ and is positive definite on $\langle \Omega, \overline{\Omega}, \omega \rangle$, where ω is a Kähler form. It is of topological origin and can be defined as follows.

Theorem 2.1: (Fujiki, [F]) Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta, \eta)^n$, for some primitive integer quadratic form q on $H^2(M, \mathbb{Z})$, and $c > 0$ a rational number.

2.2 Teichmüller spaces and the mapping class group

Definition 2.2: Let M be a hyperkähler manifold, and $\mathsf{Diff}_0(M)$ the connected component of the unity of its diffeomorphism group (**the group of**

isotopies). Denote by Comp the space of complex structures of Kähler type on M, and let $\mathsf{Teich} := \mathsf{Comp} / \mathsf{Diff}_0(M)$. We call it **the Teichmüller space** of complex structures on M. It is a complex manifold, possibly non-Hausdorff (more generally for Calabi-Yau manifolds, this statement is essentially contained in [Bo2]; see also [Cat]).

Definition 2.3: Let $\mathsf{Diff}(M)$ be the group of diffeomorphisms of M. We call $\Gamma = \mathsf{Diff}(M)/\mathsf{Diff}_0(M)$ **the mapping class group**.

If M is IHS, Teich modulo the subgroup $K \subset \Gamma$ acting trivially on cohomologies is identified with the marked moduli space, which has finitely many connected components by a result of Huybrechts ([H3]). We consider the subgroup Γ_0 of the mapping class group which preserves the one containing the parameter point for our chosen complex structure.

Definition 2.4: We call the image of Γ_0 in $\mathrm{Aut}\, H^2(M, \mathbb{Z})$ the **monodromy group**, denoted by $\mathsf{Mon}(M)$.

Theorem 2.5: ([V1]) $\mathsf{Mon}(M)$ is a finite index subgroup of the orthogonal lattice $O(H^2(M, \mathbb{Z}), q)$.

Remark 2.6: From now on, to avoid heavy notations, we denote by Teich the connected component of the Teichmüller space containing the parameter point for our given complex structure, and accordingly write Γ instead of Γ_0.

2.3 The period map

Definition 2.7: The map $\mathsf{Per} : \mathsf{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ sending I to the line $H^{2,0}(M, I)$ is called **the period map**.

Remark 2.8: Per maps Teich into an open subset of a quadric, defined by

$$\mathbb{P}\mathsf{er} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \mid \ q(l, l) = 0, q(l, \bar{l}) > 0\}.$$

It is called **the period space** of M.

Remark 2.9: One has

$$\mathbb{P}\mathsf{er} = \frac{SO^+(b_2 - 3, 3)}{SO(2) \times SO^+(b_2 - 3, 1)} = \mathrm{Gr}_{++}(H^2(M, \mathbb{R})),$$

the grassmannian of positive planes in $H^2(M,\mathbb{R})$ (the sign + in SO^+ standing for the connected component of the unity). Indeed, the group $SO^+(H^2(M,\mathbb{R}),q) = SO^+(b_2-3,3)$ acts transitively on $\mathbb{P}\mathsf{er}$, and $SO(2)\times SO^+(b_2-3,1)$ is the stabilizer of a point. From a complex line l one obtains a real oriented plane by taking its real and imaginary part (in that order).

Bogomolov in [Bo2] proved that the period map is a local diffeomorphism, and Huybrechts has shown the surjectivity in [H1]. The second-named author has obtained the following more precise result in [V1].

Theorem 2.10: The points of each connected component of Teich which have the same image in $\mathbb{P}\mathsf{er}$ are exactly the non-separable points (so that the period map is the "Hausdorff reduction" of a component of Teich, i.e. becomes an isomorphism once the non-separable points are identified).

Definition 2.11: Let z be a class of negative square in $H^2(M,\mathbb{Z})$. We call Teich_z the part of Teich consisting of all complex structures on M where z is of type $(1,1)$.

The following proposition is well-known (see e. g. [H1], 1.14).

Proposition 2.12: $\mathsf{Teich}_z = \mathsf{Per}^{-1}(z^\perp)$, where $z^\perp$ is the set of points corresponding to lines orthogonal to z in $\mathbb{P}\mathsf{er} \subset \mathbb{P}H^2(M,\mathbb{C})$.

On Teich_z, we have a natural action of the stabilizer of z in Γ, denoted by $\Gamma_z \subset \Gamma$.

2.4 Ergodicity of the mapping class group action

Definition 2.13: Let (M,μ) be a space with a measure, and G a group acting on M preserving the measure. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M')=0$ or $\mu(M\backslash M')=0$.

It is easy to see that most of the orbits of an ergodic action are dense (the union of non-dense ones has measure zero). A theorem of Moore (see [Mo]) states that a lattice in a non-compact simple Lie group G with finite center acts ergodically on G/H, if H is a non-compact subgroup. Taking $G = SO^+(H^2(M,\mathbb{R}))$ and H the stabilizer of a positive two-plane, we deduce that our mapping class group Γ acts ergodically on $\mathbb{P}\mathsf{er}$: indeed the image of Γ is of finite index in the orthogonal group of $H^2(M,\mathbb{Z})$, so it is a lattice in G.

In [V2] and [V2bis], Theorem 2.5, a more precise result has been established using Ratner theory.

Theorem 2.14: Let L be a integral lattice of signature (a,b) with $a \geqslant 3, b \geqslant 1$ and $a+b \geqslant 5$, $V = L \otimes \mathbb{R}$, Γ a finite index subgroup in $O(L)$. Then there are three types of orbits of Γ-action on $\mathrm{Gr}_{++}(V)$:

1) the orbits of rational planes are closed;

2) the orbits of planes containing no non-zero rational vectors are dense;

3) the orbits of planes containing a single rational line $\langle v \rangle$ are "intermediate": each irreducible component of the orbit closure consists of all planes containing a given rational vector, and this vector is γv for some $\gamma \in \Gamma$.

The theorem applies to the action of the mapping class group Γ on $\mathbb{P}\mathsf{er}$ for an IHS manifold M as soon as $b_2(M) \geqslant 5$, but also to the following situation:

Corollary 2.15: Let M is an IHS manifold with $b_2(M) \geqslant 5+k$, $z_1, \ldots, z_k \in H^2(M, \mathbb{Z})$ span a negative subspace, and $\mathbb{P}\mathsf{er}_z = \bigcap z_i^\perp \subset \mathbb{P}\mathsf{er}$ be the locus of period points of complex structures where each z_i is of type $(1,1)$ (the index z here is the multivector $(z_1, \ldots z_k)$). Let Γ_z be the subgroup of Γ fixing all the z_i. Then there are three types of orbits of Γ_z on $\mathbb{P}\mathsf{er}_z$:

1) the orbits of complex structures with maximal Picard number are closed;

2) the orbits of complex structures with no rational vector in the period plane are dense;

3) the orbit closure of a complex structure whose period plane contains a single rational line $\langle v \rangle$ is a union of irreducible components, each of them consisting of all planes containing a Γ_z-translate of v.

The following observation from [V2bis] (Proposition 2.7) shall be useful.

Proposition 2.16: In the third case of the above orbit classification, each irreducible component of the orbit closure is a fixed point set of an antiholomorphic involution (with respect to the natural complex structure obtained by identifying $\mathrm{Gr}_{++}(V)$ with an open subset of a quadric in $\mathbb{P}V_{\mathbb{C}}$, as in the last section). In particular, it is not contained in any complex submanifold nor contains any positive-dimensional complex submanifold (even locally).

3 MBM curves and the Kähler cone

The notion of MBM classes was introduced in [AV1] and studied further in [AV2]. We recall the setting and some results and definitions.

First of all, the BBF form on $H^{1,1}(M,\mathbb{R})$ has signature $(1, b_2 - 3)$. This means that the set $\{\eta \in H^{1,1}(M,\mathbb{R}) \mid (\eta,\eta) > 0\}$ has two connected components. The component which contains the Kähler cone $\mathrm{Kah}(M)$ is called **the positive cone**, denoted $\mathrm{Pos}(M)$.

The starting point is the following theorem.

Theorem 3.1: (Huybrechts [H1, H2], Boucksom [Bou]) The Kähler cone $\mathrm{Kah}(M)$ is the set of all $\eta \in \mathrm{Pos}(M)$ such that $(\eta, C) > 0$ for all rational curves C.

Observe that it is sufficient to consider the curves of negative square (as only these have orthogonals passing through the interior of the positive cone) and which are moreover extremal, i.e. such that their cohomology class cannot be decomposed as a sum of classes of other curves. An extremal curve is minimal in the sense of our Definition 1.1, though apriori the converse needs not be true.

The Kähler cone is locally polyhedral in the interior of the positive cone (see e.g. [HT], Prop.13), with some round pieces in the boundary, and its walls (that is, codimension-one faces) are supported on the orthogonal complements to the extremal curves.

The notion of an extremal curve is however not adapted to the deformation-invariant context. In order to put the theory in this context we have defined the MBM (monodromy birationally minimal) classes in [AV1][1] Here we recall several equivalent definitions (we refer to section 5 and more specifically to Theorem 5.16 of [AV1] for the equivalence).

Definition 3.2: A negative class z in the image of $H_2(M,\mathbb{Z})$ in $H^2(M,\mathbb{Q})$ is called MBM if Teich_z contains no twistor curves.

Equivalent definitions: A negative class z is MBM iff a rational multiple of z is represented by a rational curve in some complex structure where the Picard group is generated by z over the rationals.

Also, z is MBM iff in some complex structure $X = (M, I)$ where z is of

[1] G. Mongardi has introduced the notion of wall divisors in [Mon], the two notions turned out to be equivalent.

type $(1,1)$, the orthogonal complement $\gamma(z)^\perp$ contains a wall of the Kähler cone of a birational model $X' = (M, I')$ of X (this is the original definition from [AV1] which was at the origin of the terminology).

Moreover in these two equivalent definitions, "some" may be replaced by "all" without changing the content.

Remark 3.3: A **wall** always means a face of maximal dimension (that is, $h^{1,1} - 1$), a change of terminology from [AV1] where already "face" referred only to faces of maximal dimension unless otherwise specified.

Theorem 3.4: ([AV1], Theorem 6.2) The Kähler cone is a connected component of the complement, in $\mathrm{Pos}(M)$, of the union of hyperplanes $z^\perp$ where z ranges over MBM classes of type $(1,1)$.

Definition 3.5: (cf. [Ma1]) **The Kähler chambers** are the other connected components of this complement.

Moreover we have the following connection between the Kähler chambers and the inseparable points of the Teichmüller space (note that for a fixed deformation type the decomposition of Pos into the Kähler chambers is an invariant of a period point rather than of the complex structure itself, since it is determined by the position of $H^{1,1}$ in $H^2(M, \mathbb{R})$):

Theorem 3.6: ([Ma1]), theorem 5.16) The points of a fiber of $\mathbb{P}\mathsf{er}$ over a period point are in bijective correspondence with the Kähler chambers of the decomposition of the positive cone of the corresponding Hodge structure. Each chamber is the Kähler cone of the corresponding complex structure.

Definition 3.7: The space $\mathsf{Teich}^{\mathsf{min}}_{\pm z} \subset \mathsf{Teich}_z$ is obtained by removing the complex structures where $z^\perp$ does not support a wall of the Kähler cone.

In other words, at a general point of Teich_z, where the Picard group is generated by z over the rationals, $\mathsf{Teich}^{\mathsf{min}}_{\pm z}$ coincides with Teich_z, whereas at special points of Teich_z where we have other MBM classes as well, we remove those complex structures where e.g. z becomes a sum of two effective classes, and rational curves representing z thus cease to be extremal.

Notice that the space $\mathsf{Teich}^{\mathsf{min}}_{\pm z}$ is not separated even at its general point, since $z^\perp$ divides the positive cone in at least two chambers. In order to avoid

working with such generically non-separated spaces we divide $\mathsf{Teich}^{\mathsf{min}}_{\pm z}$ in two halves:

Definition 3.8: The space $\mathsf{Teich}^{\mathsf{min}}_{z}$ is the part of $\mathsf{Teich}^{\mathsf{min}}_{\pm z}$ where z has non-negative intersection with Kähler classes (that is, z is pseudo-effective).

Now at a general point $\mathsf{Teich}^{\mathsf{min}}_{z}$ coincides with $\mathbb{P}\mathsf{er}_z$ (but at special points it is still non-separated).

4 MBM loci and birational contractions

4.1 Projective case

Remark 4.1: Let z be an MBM class in some complex structure $I \in \mathsf{Teich}^{min}_{z}$. We have defined the full MBM locus Z of z as the union of subvarieties swept out by minimal rational curves of cohomology class proportional to z. By bend-and-break lemma we can find a minimal rational curve through the general point of any component of Z, so Z is the union of all rational curves C such that $[C]$ is proportional to z.

These loci are interesting since these are centers of elementary birational contractions (Mori contractions). In the projective case this is well-known and follows from Kawamata base-point-freeness theorem.

Theorem 4.2: (Kawamata BPF theorem, [K1]) Let L be a nef line bundle on a projective manifold M such that $L^{\otimes a} \otimes \mathcal{O}(-K_M)$ is big for some a. Then L is semiample.

Recall that a holomorphic line bundle L is **nef** if $c_1(L)$ is in the closure of the Kähler cone, and **big** if the dimension of the space of global sections of its tensor powers has maximal possible growth. For the nef line bundles this last condition is equivalent to $c_1(L)^{\dim M} > 0$. A **semiample** line bundle is a line bundle L such that $L^{\otimes n}$ is base point free for some n; then the linear system of sections of $L^{\otimes n}$ defines a projective morphism with connected fibers $\varphi : M \to M_0$. The bigness of L implies that φ is birational. Clearly, for a curve C, $\varphi(C)$ is a point if and only if $L \cdot C = 0$.

Corollary 4.3: Let M be a projective hyperkähler manifold. Then faces F of the Kähler cone of M, except for the rays contained in the boundary

of $\mathrm{Pos}(M)$, are in bijective correspondence with birational contractions $\pi : M \longrightarrow M_1$, and the exceptional set of π is exactly the full MBM locus of F.

Proof: First of all, note that the result M_1 of a birational contraction is itself projective by the singular version of Huybrechts' criterion ([BL2], Theorem 6.9). The face F of the Kähler cone is a subset with non-empty interior of the orthogonal complement to some rational cohomogy classes (those of extremal rational curves $[C_1], ..., [C_k]$, by Huybrechts-Boucksom description [Bou]), hence it contains an integral point in its interior when M is projective. This point is the Chern class of a nef and big line bundle L. The bundle L is semiample since K_M is zero, and hence defines a contraction $\pi : M \longrightarrow M_1$. Conversely, let $\pi : M \longrightarrow M_1$ be a birational contraction and let L_1 be an ample bundle on M_1. Then $L := \pi^* L_1$ is a big and nef line bundle with $c_1(L) \in \bigcap_i [C_i]^\perp$, where C_i are the extremal rational curves contracted by π (note that the contraction loci are uniruled, as one deduces for instance from [K2], Theorem 1). Hence $\bigcap_i [C_i]^\perp$ is a non-empty face. ■

4.2 Non-projective case: locally trivial deformations

The notion of locally trivial deformations was developed in [FK] and applied to hyperkähler geometry in [BL1].

Definition 4.4: Let $\pi : \mathcal{X} \longrightarrow B$ be a family of complex varieties. Assume that any point $x \in \mathcal{X}$ has a neighbourhood W which is biholomorphic to a product $F \times U$ such that $\pi\Big|_{F\times U}$ is a projection to U (F stands for a neighbourhood of x in the fiber and U for the neighbourhood of its image on the base). Then π is called **a locally trivial deformation**, or **locally trivial deformation in the sense of Flenner-Kosarew**.

Let M be a hyperkähler manifold and $f : M \to M_1$ a birational contraction which contracts precisely the curves whose classes are in the subspace $N \subset H^2(M, \mathbb{Q})$. Let $\mathrm{Def}(M)$, $\mathrm{Def}(M_1)$ are local deformation spaces and $\mathcal{X}$, $\mathcal{X}_1$ the universal families. According to Namikawa [N], there is a natural commutative diagram extending f:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathcal{X}_1 \\ \downarrow & & \downarrow \\ \mathrm{Def}(M) & \xrightarrow{G} & \mathrm{Def}(M_1) \end{array}$$

The fiber of $\mathcal{X}_1$ over a general point of $\mathrm{Def}(M_1)$ is smooth and the restriction of Φ to a general fiber X_t of $\mathcal{X}$ is an isomorphism, so this diagram in itself does not carry information on contractions. An important advance has been recently made by Bakker and Lehn.

Theorem 4.5: ([BL1], Proposition 4.5) Let $f : M \longrightarrow M_1$ be a birational contraction of a projective hyperkähler manifold, with $b_2(M_1) \geqslant 5$. Let $\mathrm{Def}^{lt}(M_1) \subset \mathrm{Def}(M_1)$ be the subspace parametrizing locally trivial deformations of M_1 and $\mathrm{Def}(M, f) \subset \mathrm{Def}(M)$ be the subspace of deformations of M on which the classes contracted by f remain of type $(1,1)$. Then the contraction induces an isomorphism between $\mathrm{Def}(M, f)$ and $\mathrm{Def}^{lt}(M_1)$, so that the small deformations of M preserving the Hodge type of the classes contracted by f contract onto locally trivial small deformations of M_1. ■

Let $M \longrightarrow M_1$ be a birational contraction of a projective hyperkähler manifold obtained from a face F of its Kähler cone $\mathrm{Kah}(M)$ (Corollary 4.3). Assume that F is supported on (i.e. is a subset with non-empty interior of) the intersection of orthogonal complements to linearly independent MBM classes $h_1, ..., h_n \in H^{1,1}(M)$. By analogy with $\mathsf{Teich}^{\mathsf{min}}_z$, we define the space Teich_F. Namely Teich_F is the part of the Teichmüller space of M such that for all $I \in \mathsf{Teich}_F$ the orthogonal complement $\langle h_1, ...h_n\rangle^\perp$ intersects the closure of the Kähler cone of (M, I) in a face F_I of the same codimension n, and all h_i are positive on the Kähler classes. In other words Teich_F is the intersection of $\mathsf{Teich}^{\mathsf{min}}_{h_i}$. Proposition 2.12 easily implies that Teich_F is a generically Hausdorff manifold equipped with the period map $\mathsf{Per} : \mathsf{Teich}_F \longrightarrow \mathrm{Gr}_{+,+}(W_F)$, where $W = \langle h_1, ...h_n\rangle^\perp$. By Torelli theorem $\mathsf{Per} : \mathsf{Teich}_F \longrightarrow \mathrm{Gr}_{+,+}(W_F)$ is locally a diffeomorphism.

The following theorem is essentially due to Bakker and Lehn, though not stated in [BL1] explicitely.[1]

Theorem 4.6: Let $I \in \mathsf{Teich}_F$ be a complex structure on a hyperkähler manifold M. Assume that $\dim_{\mathbb{C}} \mathsf{Teich}_F \neq 1, 2$; this is equivalent to $\dim_{\mathbb{R}} F \neq 1, 2$. Then there exists a birational map $(M, I) \longrightarrow M_1$ which contracts all curves with cohomology classes orthogonal to F and only those curves. Moreover, such a map is uniquely determined by the space $\langle h_1, ..., h_n\rangle$ used to defined the face F.

[1] Note added in proof: it is in the last version, with a reference to the present paper: Corollary 5.9.

Remark 4.7: If (M, I) is algebraic this is true without the extra assumption on $\dim_{\mathbb{R}} F$ (Corollary 4.3). It would be rather suprising if it were necessary in the non-projective case but we don't know how to avoid it. The assumption in the version of Bakker and Lehn's paper available to us is $b_2(M_1) \geqslant 4$, so that $\dim \mathsf{Teich}_F = 2$ is allowed. This seems to be a misprint as their method of proof needs Verbitsky's description of monodromy orbits, available from dimension three on.

Proof of Theorem 4.6. For any algebraic (M, I), $I \in \mathsf{Teich}_F$, the face F is contractible by a morphism $f : M \to M_1$. By Theorem 4.5, F remains contractible on small deformations of M, say over a small open neighbourhood $U_I \subset \mathsf{Teich}_F$ of I. Let now (M, I') be non-algebraic. At this point Bakker and Lehn use the ergodicity of the mapping class group action as follows. Let Γ_F be the subgroup of the mapping class group preserving the h_i. Standard arguments imply that Γ_F is a lattice in the Lie group $O(W)$, where $W = \langle h_1, ..., h_n \rangle^\perp \subset H^2(M, \mathbb{R})$ (see [AV1]). Then one applies the description of orbit closures in Theorem 2.14 and Corollary 2.15 to obtain that the mapping class group orbit of any non-algebraic complex structure contains an algebraic one in its closure. Hence such an orbit has a representative in $U_I \subset \mathsf{Teich}_F$ for I algebraic. For such a representative, all relevant MBM curves can be contracted by Theorem 4.5. However, all complex structures in the same orbit are isomorphic and the isomorphism preserves the classes h_i. ■

Remark 4.8: The key ingredient of the orbit closures description is the application of Ratner theory to Γ_F action on the $O(W)$-homogeneous space $\mathrm{Gr}_{++}(W)$. In order for Ratner theory to be applicable, the connected component H of the stabilizer of a point $v \in \mathrm{Gr}_{++}(W)$ needs to be generated by unipotents. The Lie algebra of H is isomorphic to $\mathfrak{so}(1, \dim F - 1)$, and H is generated by unipotents if and only if $\dim_{\mathbb{R}} F > 2$, whence the restrictions on $b_2(M)$. ■

5 Locally trivial deformations and real analytic geometry

Any deformation of a smooth complex manifold is trivial (locally on the base) in real analytic category. This is most easy to see by constructing an **Ehresmann connection** and integrating it to obtain a flow of diffeo-

morphisms between the fibers. Recall that an Ehresmann connection on a smooth family (i.e. such that π is a submersion with compact fibers) $\pi:\ \mathcal{X} \longrightarrow B$ is a splitting of the exact sequence

$$0 \longrightarrow T_{\mathcal{X}/B}\mathcal{X} \longrightarrow T\mathcal{X} \longrightarrow \pi^*TB \longrightarrow 0, \tag{5.1}$$

where $T_{\mathcal{X}/B}\mathcal{X}$ denotes the sheaf of vector fields tangent to the fibers. It is not hard to see (by integrating local vector fields lifted from the base via the splitting) that the deformation is trivialized over B if and only if it admits an Ehresmann connection. Obstructions to the splitting of (5.1) lie in $\operatorname{Ext}^1(\pi^*TB, T_{\mathcal{X}/B}\mathcal{X}) = H^1(\mathcal{X}, \mathcal{H}om(\pi^*TB, T_{\mathcal{X}/B}\mathcal{X}))$. However, on a real analytic variety higher cohomology of all coherent sheaves vanishes ([Car]), hence this sequence splits, and one can trivialize the deformation.

For a singular family $\pi:\ \mathcal{X} \longrightarrow B$, the splitting does not always exist, even in the real analytic category (Subsection 1.1). However, "locally trivial" (in the sense of Flenner and Kosarew) deformations are trivialized.

Throughout this section, the base B is assumed to be smooth.

Proposition 5.1: Let $\pi:\ \mathcal{X} \longrightarrow B$ be a deformation of compact complex varieties, which is locally trivial in the sense of Definition 4.4. Then the real analytic map $\pi_{\mathbb{R}}:\ \mathcal{X}_{\mathbb{R}} \longrightarrow B_{\mathbb{R}}$ underlying π defines a family which is trivial over any sufficiently small open set $U \subset B$.

Proof: By Artin's analytification theorem ([Ar], Cor. 1.6), it would suffice to trivialize the family $\pi_{\mathbb{R}}$ in a formal neighbourhood $\hat{F}$ of $F := \pi^{-1}(b)$, for all $b \in B$. Denote by $\hat{\pi}:\ \hat{F} \longrightarrow \hat{B}$ the corresponding map in the mixed formal-analytic category (the variety $\hat{F}$ is analytic along F and formal in the transversal direction). [1]

Locally in $\mathcal{X}$, the complex family π is a product. The local-in-$\mathcal{X}$ trivialization of π defines a Čech cocycle $w \in H^1(F, \operatorname{Aut}_F(\hat{F}))$ where $\operatorname{Aut}_F(\hat{F}))$

[1] The mixed formal-analytic setting is natural for the deformation theory of complex analytic varieties, such as in [KV] or in [BK]. The objects of the relevant category are complex varieties formally completed in some directions. To be more rigorous, an object $\mathcal{X}$ of this catefory is a pro-scheme obtained as an inverse limit of complex analytic spaces with the same reduction X. The formal deformation space of X is obtained as such an inverse limit, hence it belongs to this category. If, instead of complex analytic, we start in the category of algebraic (Noetherian) schemes, the same approach gives the usual formal schemes. The analytification of a formal deformation is a complex analytic space containing X as a closed complex analytic subvariety, with the formal completion along X identified with $\mathcal{X}$.

is the group sheaf of automorphisms of $\hat{F}$ trivial on $F \subset \hat{F}$ and commuting with the projection to B. The sheaf $\operatorname{Aut}_F(\hat{F})$) can be obtained as a limit of sheaves of automorphisms of infinitesimal neighbourhood $F_k \subset \hat{F}$ of order k. Therefore, $w \in H^1(F, \operatorname{Aut}(\hat{F}))$ vanishes whenever its finite order representatives $w_k \in H^1(F, \operatorname{Aut}_F(F_k))$ vanish. The Lie groups $\operatorname{Aut}_F(F_k)$ are nilpotent, and fit into exact sequences

$$0 \longrightarrow V_k \longrightarrow \operatorname{Aut}_F(F_k) \longrightarrow \operatorname{Aut}_F(F_{k-1}) \longrightarrow 0$$

where V_k is a sheaf of abelian unipotent groups, that is, a coherent sheaf. In the corresponding exact sequence of first cohomology

$$H^1(V_k) \longrightarrow H^1(\operatorname{Aut}_F(F_k)) \longrightarrow H^1(\operatorname{Aut}_F(F_{k-1}))$$

all terms vanish, which can be shown by induction. Indeed, $\operatorname{Aut}_F(F_1)$ is trivial because the automorphisms commute with the projection to B. On the other hand, higher cohomology of any coherent sheaf on a real analytic variety vanishes ([Car], Théorème 3, completed by [GMT] Theorem 2.7, [Nar] p. 931). We obtain that the group sheaf $\operatorname{Aut}_F(F_k)$ is filtered by normal subgroups with coherent subquotients, hence has vanishing cohomology. ■

In the sequel, a "vector field" on a singular variety S is understood as a section of the sheaf $(\Omega^1_S)^*$ (dual to the Kähler differentials). By the universal property of Kähler differentials, $(\Omega^1_S)^*$ is the sheaf of derivations from $\mathcal{O}_S$ to itself.

Proposition 5.2: Let $\pi : \mathcal{X} \longrightarrow B$ be a deformation of complex varieties, which is locally trivial in the sense of Flenner-Kosarew (Definition 4.4), and $\sigma : \widetilde{\mathcal{X}} \longrightarrow \mathcal{X}$ a simultaneous resolution of singularities. Assume that any vector field on the smooth part of $\mathcal{X}$ can be lifted to a vector field on $\widetilde{\mathcal{X}}$. Then the family $f = \sigma \circ \pi : \widetilde{\mathcal{X}} \longrightarrow B$ admits a real analytic Ehresmann connection such that the corresponding flow of diffeomorphisms preserves the exceptional variety $\mathcal{Z}$ of σ, and moreover does so fibrewise over its image in $\mathcal{X}$.

Proof. Step 1: We start by showing that it suffices to prove existence of an Ehresmann connection preserving $\mathcal{Z}$ locally in $\widetilde{\mathcal{X}}$. An Ehresmann connection in $f : \widetilde{\mathcal{X}} \longrightarrow B$ is the same as a splitting of the exact sequence

$$0 \longrightarrow T_{\widetilde{\mathcal{X}}/B}\widetilde{\mathcal{X}} \longrightarrow T\widetilde{\mathcal{X}} \longrightarrow f^*TB \longrightarrow 0, \tag{5.2}$$

Therefore, a difference between two Ehresmann connections is a section of $\mathcal{H}om(f^*TB, T_{\widetilde{\mathcal{X}}/B}\widetilde{\mathcal{X}})$. Consider the natural pairing

$$\Psi: \ J_{\mathcal{Z}}/J_{\mathcal{Z}}^2 \otimes T\widetilde{\mathcal{X}} \longrightarrow \mathcal{O}_{\mathcal{Z}}$$

obtained if we identify vector fields with derivations and take a derivation of $\alpha \in J_{\mathcal{Z}}$ evaluating it on $\mathcal{Z}$. Clearly, a diffeomorphism associated with a vector field v preserves $\mathcal{Z}$ if and only if $\Psi(v, \cdot) = 0$. This gives a coherent sheaf $\ker \Psi$ denoted by $T_{\mathcal{Z}}\widetilde{\mathcal{X}} \subset T\widetilde{\mathcal{X}}$. This is the sheaf of vector fields preserving $\mathcal{Z}$. Now, if we have found an Ehresmann connection preserving $\mathcal{Z}$ locally in $\mathcal{X}$, the corresponding Čech cocycle w (of differences on intersections) takes values in

$$\operatorname{Ext}^1(f^*TB, T_{\widetilde{\mathcal{X}}/B}\widetilde{\mathcal{X}} \cap T_{\mathcal{Z}}\widetilde{\mathcal{X}}) = H^1((f^*TB)^* \otimes (T_{\widetilde{\mathcal{X}}/B}\widetilde{\mathcal{X}} \cap T_{\mathcal{Z}}\widetilde{\mathcal{X}}));$$

this group vanishes because cohomology of any coherent sheaf on a real analytic variety vanish ([Car]). Therefore, the connections constructed locally on $\widetilde{\mathcal{X}}$ give rise to a global one.

Step 2: In Step 1, we reduced Proposition 5.2 to a statement which is local on $\mathcal{X}$. Since locally in $\mathcal{X}$ we have $\mathcal{X} = B \times F$, we can assume that the family $\pi: \ \mathcal{X} \longrightarrow B$ is trivial, and $\mathcal{X} = B \times F$. This gives a natural embedding $\pi^*TB \hookrightarrow T\mathcal{X}$. Replacing B by an open ball if necessary, we fix the coordinate vector fields $\zeta_1, ..., \zeta_n \in H^0(TB)$. Using the embedding $\pi^*TB \hookrightarrow T\mathcal{X}$, we obtain holomorphic vector fields vector fields $\zeta_1, ..., \zeta_n$ on $\mathcal{X}$ which can be integrated to diffeomorphisms $\mathcal{V}_t^i$. These diffeomorphisms are coordinate translations along B in the decomposition $\mathcal{X} = B \times F$.

The vector fields ζ_i can be lifted to holomorphic vector fields on the simultaneous resolution $\widetilde{\mathcal{X}}$, by assumptions of Proposition 5.2. Denote the corresponding holomorphic diffeomorphism flows on $\widetilde{\mathcal{X}}$ by $\widetilde{\mathcal{V}}_t^i$. These diffeomorphism flows commute with the projection $\sigma: \ \widetilde{\mathcal{X}} \longrightarrow \mathcal{X}$, because $\sigma \circ \mathcal{V}_t^i = \widetilde{\mathcal{V}}_t^i \circ \sigma$ at the general point of $\widetilde{\mathcal{X}}$. Therefore, the diffeomorphism flows $\widetilde{\mathcal{V}}_t^i$ preserve $\mathcal{Z}$, and the corresponding vector fields give a splitting of (5.2). ■

To apply Proposition 5.2 to holomorphic symplectic varieties, we use the following lemma.

Lemma 5.3: Let $\sigma: \ \widetilde{\mathcal{X}} \longrightarrow \mathcal{X}$ be a simultaneous birational contraction in a family of holomorphic symplectic manifolds over a ball B. Then any

vector field on the smooth part of $\mathcal{X}$ can be extended to a vector field on $\widetilde{\mathcal{X}}$.

Proof: Notice that the manifold $\widetilde{\mathcal{X}}$ has trivial canonical bundle, so that both on $\widetilde{\mathcal{X}}$ and on the smooth part of $\mathcal{X}$, vector fields are identified with differentials of degree $\dim \mathcal{X} - 1$. The differentials extend by [KS], Cor. 1.8. ■

Comparing this lemma with Proposition 5.2, we obtain the real analytic Ehresmann connection preserving the exceptional sets of birational contractions:

Theorem 5.4: Let M be a hyperkähler manifold, and $\pi : M \longrightarrow M_1$ a birational contraction associated with a face F of the Kähler cone of M. Assume that $b_2(M_1) \geqslant 5$, and consider the universal family $\mathcal{U} \longrightarrow \mathsf{Teich}_F$ of over the Teichmüller space Teich_F, and the corresponding universal family of birational contractions $\mathcal{U} \stackrel{\sigma}{\longrightarrow} \mathcal{U}_1 \longrightarrow \mathsf{Teich}_F$ constructed by Bakker and Lehn (see 4.2). Then the family $\mathcal{U} \longrightarrow \mathsf{Teich}_F$ admits a real analytic trivialization which preserves the fiberwise exceptional sets of the contraction σ. ■

Our main Theorem 1.8 obviously follows.

6 Applications of Thom-Mather-Verdier theory to the families of MBM loci

We shall now prove a weaker form of Theorem 1.8 for e.g. the family of Barlet spaces. In our previous paper [AV3] which the current one supersedes, this method has been applied to the initial family of MBM loci $Z_I \subset M_I$ for which a stronger result has just been obtained using Bakker-Lehn's theorem. Our old method is based on two ingredients which apply in a great generality, thus permitting to obtain a weaker result for essentially any family related to the geometry of rational curves on an IHSM.

One ingredient is the work by Verdier on the Whitney stratification and Thom-Mather theory in the complex analytic context [Ver]. It implies that the members of any proper complex analytic family (i.e. the fibers X_y of a proper morphism of countable at infinity complex analytic spaces $f : X \to Y$) are homeomorphic and stratified diffeomorphic (with respect

to a strong Witney stratification, [Ver], 2.1) over a complement to a union of closed analytic subvarieties.

The other ingredient is the description of the orbits of the monodromy action on $\mathsf{Teich}_z^{\mathsf{min}}$, or more generally Teich_F, which is the same as the one for the period space but the proofs are somewhat more technical (Theorem 6.2). This description allows to send, by an element of the mapping class group, a point on such a subvariety (along which a topological/stratified differentiable degeneration in a family over $\mathsf{Teich}_z^{\mathsf{min}}$ or Teich_F is supposed to happen) into a small neighbourhood of a general point. As the mapping class group acts by diffeomorphisms, this proves that the degeneration actually does not happen, unless the Picard number at that point is maximal (in this case the mapping class group orbit is closed so the argument does not work).

We now give the details of the argument sketched above, restricting for simplicity of notation to the families over $\mathsf{Teich}_z^{\mathsf{min}}$ (but the argument is the same over Teich_F which is the intersection of several $\mathsf{Teich}_z^{\mathsf{min}}$).

6.1 Mapping class group action on $\mathsf{Teich}_z^{\mathsf{min}}$

The group $\Gamma_z \subset \Gamma$ obviously acts on $\mathsf{Teich}_z^{\mathsf{min}}$. Indeed the action of any $\gamma \in \Gamma$ is just the transport of the complex structure; if $z^\perp$ contains a wall of the Kähler cone in a complex structure I, then so does γz in the complex structure γI. Notice that the same remark applies to rational curves: γC is a rational curve in the structure γI and the minimality is preserved. So the full MBM locus $Z \subset X = (M, I)$ of z is sent by an element of Γ_z to the full MBM locus $Z_{\gamma I} \subset X' = (M, \gamma I)$.

It turns out that the results on the mapping class group action on $\mathbb{P}\mathsf{er}$ "lift" to those on the action on Teich, but if we want to work on a subspace where z remains of type $(1,1)$ this has to be $\mathsf{Teich}_z^{\mathsf{min}}$ rather than Teich_z.

The following theorem from [V2], [V2bis] strengthens Theorem 2.14.

Theorem 6.1: Assume $b_2(M) \geqslant 5$. Let Γ denote the mapping class group. Then there are three types of Γ-orbits on Teich: closed (where the period planes are rational, thus the complex structures have maximal Picard number), dense (where the period planes contain no rational vectors), and such that each irreducible component of the closure is formed by points whose period planes contain a fixed rational vector γv (where $\gamma \in \Gamma$ and v generates the unique rational line in the period plane). In the last case, no neighbourhood of a point c in the orbit closure C_v is contained in a proper complex subvariety of Teich.

The argument in [V2bis] proves that the closure of ΓI is the inverse image of the closure of $\Gamma\,\mathsf{Per}(I)$ when the Picard group of I is not maximal. The key idea is to replace $\mathsf{Per} : \mathsf{Teich} \to \mathbb{P}\mathsf{er}$ by a Γ-equivariant embedding $\mathsf{Teich}_K \to \mathsf{Per}_K$, where Teich_K consists of pairs (I, ω) where $I \in \mathsf{Teich}$ and $\omega \in \mathrm{Kah}(I)$ is of square one, and Per_K consists of pairs $(\mathsf{Per}(I), \omega)$, $\mathsf{Per}(I) \in \mathbb{P}\mathsf{er}$, $\omega \in \mathrm{Pos}(I)$ of square one (note that the positive cone is an invariant of the period point). Since the points of Teich correspond to pairs $(\mathsf{Per}(I), k)$ where $\mathsf{Per}(I) \in \mathbb{P}\mathsf{er}$ and k is a Kähler chamber of $\mathrm{Pos}(I)$, it suffices to prove that the closure of the orbit of $(I, \mathrm{Kah}(I))$ as a set contains the orbit of $(\mathsf{Per}(I), \mathrm{Pos}(I))$. The space Per_K being homogeneous, one uses Ratner theory to do this.

The analogue of Theorem 6.1 in our setting is as follows. The interesting feature is that it is $\mathsf{Teich}_z^{\mathsf{min}}$ rather than Teich_z which replaces Teich.

Theorem 6.2: Assume $b_2(M) > 5$. Let $z \in H^2(M, \mathbb{Z})$ be an MBM class and Γ_z the subgroup of the mapping class group consisting of all elements whose action on the second cohomology fixes z. Then Γ_z acts on $\mathsf{Teich}_z^{\mathsf{min}}$ ergodically, and there are the same three types of orbits of this action as in Theorem 6.1.

Proof: It proceeds along the same lines as in [V2bis]. We introduce the spaces $\mathsf{Per}_{K,z}$ consisting of pairs

$$\{(\mathsf{Per}(I), \omega) | I \in \mathsf{Teich}_z, \omega \in \mathrm{Pos}(I) \cap z^\perp, q(\omega, \omega) = 1\}$$

and $\mathsf{Teich}_{K,z}$ consisting of pairs (I, ω) where $I \in \mathsf{Teich}_z^{\mathsf{min}}$, and ω of square 1 belongs to the wall of $\mathrm{Kah}(I)$ given by $z^\perp$. We denote such a wall by $\mathrm{Kah}(I)_z$, though of course its elements are not Kähler forms on I, but rather nef limits of those. Since the complex structures in $\mathsf{Teich}_z^{\mathsf{min}}$ which have the same period point are in one-to-one correspondence with the walls of the Kähler chambers in which the other MBM classes partition $z^\perp$, $\mathsf{Teich}_{K,z}$ again embeds naturally in $\mathsf{Per}_{K,z}$. We fix a complex structure I with non-maximal Picard number. We need to prove that the closure of the Γ_z-orbit of the subset $(I, \mathrm{Kah}(I)_z)$ contains the orbit of $(\mathsf{Per}(I), \mathrm{Pos}(I) \cap z^\perp)$. This is done exactly in the same way as in [V2bis], proof of theorem 3.1. The key idea in [V2bis] is as follows: the non-maximality of Picard number means that the subspace generated by the integral classes in $H^{1,1}_I(M, \mathbb{R})$ has non-zero orthogonal complement T. Consider a three-dimensional subspace $W \subset H^{1,1}_I(M, \mathbb{R})$ of signature $(1, 2)$ such that $W \cap T \neq 0$. Then

the intersection of the Kähler cone with W has a "round part" ([V2bis], Proposition 3.4) and therefore contains horocycles ([V2bis], subsection 3.3). In our context, the space T is clearly contained in $z^\perp$, meaning that for $W \subset z^\perp$ of signature $(1,2)$ intersecting T, $\mathrm{Kah}(I)_z \cap W$ contains horocycles. As in [V2bis], Proposition 3.5 and the following paragraph, we deduce from Ratner's orbit closure theorem that the closure of the projection of such a horocycle to $\mathsf{Per}_{K,z}/\Gamma_z$ is large, containing an $SO(H^{1,1}(I) \cap z^\perp)$-orbit, which is the projection of $\mathrm{Pos}(I) \cap z^\perp$. ■

6.2 Stratification

Consider the family of hyperkähler mamifolds $\mathcal{X}$ over $\mathsf{Teich}_z^{\mathsf{min}}$ ($X_I = (M, I)$ over each point I, one can introduce it as a pullback of the universal family from ([Ma2]). Throughout this paper we have been interested in the family $\mathcal{Z} \subset \mathcal{X}$ with the fiber over $I \in \mathsf{Teich}_z^{\mathsf{min}}$ obtained as the full MBM locus of z on the complex manifold $X = (M, I)$. This family can be constructed by taking the image of the evaluation map for the union of the components of the relative Barlet space corresponding to cohomology classes proportional to z and dominating Teich_z^{min}.

Another family we shall consider is the dominating part of the relative Barlet space itself: in such a way we obtain a family $\mathcal{B}$ over $\mathsf{Teich}_z^{\mathsf{min}}$, and we call B_I the fiber over I (the Barlet space of minimal rational curves in classes proportional to z, it has compact components because X_I are compact Kähler).

Finally, there is the incidence variety $\mathcal{J} \subset \mathcal{X} \times_{\mathsf{Teich}_z^{\mathsf{min}}} \mathcal{B}$. As $\mathsf{Teich}_z^{\mathsf{min}}$ is not Hausdorff, we shall, whenever necessary, restrict all families to a small neighbourhood U of some point x, or to a small compact K within U, and denote by $\mathcal{X}_U$, $\mathcal{Z}_U$ etc. the restrictions of these families.

Whitney [W] introduced stratifications of analytic varieties by singularity type. Recall that a stratification of Y is a finite filtration by closed subsets Z_i of dimension i such that each difference $Z_i - Z_{i-1}$ is a manifold, in general non-connected; the strata are its connected components. To use stratifications in practice, one needs some "'glueing conditions" of technical nature, the so-called Whitney's A and B conditions, or Verdier W condition (these are equivalent in the complex analytic case). A Whitney stratification is a locally finite stratification satisfying those conditions. Verdier ([Ver], Théorème 2.2) proved that a complex analytic space, countable at infinity, admits a Whitney stratification by complex analytic strata, and moreover such that a given closed analytic subset is a union of strata.

Thom-Mather theory uses stratifications to prove the stratified differen-

tiable local triviality of a family over an open subset of the base, for example via the following **first isotopy lemma** (see [M] for a detailed but accessible account).

Lemma 6.3: ([M], Proposition 11.1) Let $f : Y \to B$ be a smooth mapping of smooth manifolds and W a closed subset of Y admitting Whitney stratification, such that $f : W \to B$ is proper. If the restriction of f to each stratum of W is a submersion, then W is locally trivial over B (topologically and stratified differentiably).

In the situation of this lemma, one says that f is a "controlled submersion", or "transverse to the stratification". Verdier ([Ver], Théorème 3.3) proves a very general result to the effect that a proper morphism of complex analytic spaces $f : X \to Y$ is transverse to a stratification of the source over the complement to a closed analytic subset (more generally, if f is proper in restriction to a closed subset A with a stratification, then f is transverse to the stratification over a dense open subset of $f(A)$).

The first isotopy lemma holds for a morphism of complex analytic spaces $f : X \to Y$, with non-singular Y, proper over a stratified closed subset W. Théorème 4.14 of [Ver] is then an analogue of the first isotopy lemma and implies local topological and stratified differentiable triviality of a proper morphism of complex spaces over a complement to a closed analytic subset ([Ver], Corollaire 5.1, formulated as an example of what one obtains in the algebraic case, but the proof remains valid with the properness hypothesis in the analytic case).

The results of [Ver] applied to our situation yield the following

Lemma 6.4: The family $\mathcal{Z}_U$ is locally topologically and stratified differentiably trivial over a complement to a (lower-dimensional) analytic subset, and so is $\mathcal{B}_U$. Moreover $(\mathcal{X} \times_{\mathsf{Teich}_z^{\mathsf{min}}} \mathcal{B})_U$ admits a trivialization preserving the incidence subset $\mathcal{J}$.

Proof: This follows from the above recollection of Verdier's results, keeping in mind that the family $\mathcal{B}_U$ is proper over U by the compactness of the cycle spaces for compact Kähler manifolds, and so, by the same reason, is $\mathcal{Z}_U$. Finally, the preservation of $\mathcal{J}$ is a consequence of it being a union of strata for a suitable Whitney stratification ([Ver], Théorème 2.2).

Remark 6.5: Concerning $\mathcal{Z}_U$, this result is of course weaker than the one already proved using contractibility. Unfortunately, this other method does

not seem to apply to Barlet and incidence spaces.

6.3 Proof of Theorem 1.9 and closing remarks

Theorem 1.9 is a consequence of the following fact.

Theorem 6.6: If $b_2(M) > 5$, the families $\mathcal{B}$ and $\mathcal{J} \subset \mathcal{X} \times_{\mathsf{Teich}_z^{\mathsf{min}}} \mathcal{B}$ are topologically (and stratified-differentiably) trivial over the whole $\mathsf{Teich}_z^{\mathsf{min}}$, with a possible exception of points corresponding to the complex structures with maximal Picard number.

Proof: We know by Lemma 6.4 that this is the case over the complement to a union (possibly countable, but finite in a neighbourhood of any point in the base) of proper analytic subsets $P = \cup_i P_i \subset \mathsf{Teich}_z^{\mathsf{min}}$. First we pick a point $x \in \mathsf{Teich}_z^{\mathsf{min}}$ which is not in P and whose Γ_z-orbit is dense. Then x has a neighbourhood U_x over which all fibers $\mathcal{B}_b$, $b \in U_x$ are homeomorphic. Moreover the union $\bigcup_{\gamma\in\Gamma_z} \gamma(U_x)$ is a dense open subset of $\mathsf{Teich}_z^{\mathsf{min}}$ and all fibers $\mathcal{B}_b$ over this union are homeomorphic.

Take another point $x' \in \mathsf{Teich}_z^{\mathsf{min}}$ (which now can be in P) with dense Γ_z-orbit (i.e. "ergodic"). Then the orbit of x' hits $\bigcup_{\gamma\in\Gamma} \gamma(U_x)$. But Γ_z is a subgroup of the mapping class group and its action is just the transport of the complex structure. Therefore rational curves in a complex structure I and in $\gamma(I)$ correspond via γ, and so do the MBM loci, Barlet spaces, incidence varieties. So $\mathcal{B}_{x'}$ is homeomorphic to $\mathcal{B}_b$ for $b \in U_x$ and no degeneration happens at x'.

Now take $y \in \mathsf{Teich}_z^{\mathsf{min}}$ such that the corresponding complex structure is not ergodic but does not have maximal Picard number either ("the intermediate orbit" of Theorem 2.14 and Corollary 2.15). If $\mathcal{B}_y$ is not homeomorphic to $\mathcal{B}_b$ for $b \notin P$, the orbit of y should remain in P and so must the orbit closure. Each irreducible component of the orbit closure must be contained in an irreducible component of P, but this is an analytic subvariety. However, the closure of an intermediate orbit is not contained in a proper analytic subset, even locally (Proposition 2.16), so this is impossible.

The proof for the stratified diffeomorphic case is exactly the same, and also the same arguments apply to $\mathcal{J}$. ■

In Theorem 1.9 we prove that the fibers of natural families associated with rational curves are homeomorphic and stratified diffeomorphic. However, there is a version of the Thom-Mather theory which gives bi-Lipschitz equivalence of the fibers over open strata of Thom-Mather stratification

([Pa3, Theorem 1.6]; see also [Pa1], [Pa2]). Then the same arguments as above prove that the homeomorphisms constructed in Theorem 1.9 are bi-Lipschitz.

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References

[AV1] Amerik, E., Verbitsky, M. *Rational curves on hyperkähler manifolds*, Int. Math. Res. Notices 2015, no.23, 13009 – 13045 (Cited on pages 7, 8, 9, 10, 11, 16, 17, and 21.)

[AV2] Amerik, E., Verbitsky, M. *Morrison-Kawamata cone conjecture for hyperkähler manifolds*, Ann. Sci. ENS 50 (2017), no. 4, 973–993. (Cited on page 16.)

[AV3] Amerik, E., Verbitsky, M. *MBM loci in families of hyperkähler manifolds and centers of birational contractions,* preprint arxiv:1804.00463. (Cited on page 25.)

[AV4] Amerik, E., Verbitsky, M., MBM classes and contraction loci on low-dimensional hyperkähler manifolds of K3 type, preprint arXiv:1907.13256 (Cited on page 9.)

[Ar] Artin, M. *On the solutions of analytic equations*, Invent. Math. 5 1968 277-291. (Cited on page 22.)

[BL1] Bakker, B., Lehn, C. *A global Torelli theorem for singular symplectic varieties*, arXiv:1612.07894. (Cited on pages 3, 4, 6, 9, 11, 19, and 20.)

[BL2] Bakker, B., Lehn, C. *The global moduli theory of symplectic varieties,* arXiv preprint 1812.09748 (Cited on page 19.)

[BK] Barannikov, S., Kontsevich, M. *Frobenius Manifolds and Formality of Lie Algebras of Polyvector Fields*, Internat. Math. Res. Notices **1998**, no. 4, 201–215. (Cited on page 22.)

[Bea] Beauville, A. *Varietes Kähleriennes dont la première classe de Chern est nulle,* J. Diff. Geom. **18**, pp. 755–782 (1983). (Cited on page 12.)

[Bes] Besse, A. *Einstein Manifolds*, Springer-Verlag, New York (1987) (Cited on page 12.)

[Bo2] Bogomolov, F. A. *Hamiltonian Kähler manifolds*, Sov. Math. Dokl. **19** (1978), 1462–1465. (Cited on pages 13 and 14.)

[Bou] Boucksom, S. *Le cône kählérien d'une variété hyperkählérienne*, C. R. Acad. Sci. Paris Ser. I Math. 333 (2001), no. 10, 935–938. (Cited on pages 5, 16, and 19.)

[Cam] Campana, F. *Connexité rationnelle des variétés de Fano*, Ann. Sc. E. N.S. 25 (1992), 539-545. (Cited on page 11.)

[Car] Cartan, H. *Variétés analytiques réelles et variétés analytiques complexes,* Bull. SMF t. 85 (1957), 77- 99 (Cited on pages 4, 22, 23, and 24.)

[Cat] Catanese, F. *A Superficial Working Guide to Deformations and Moduli*, in *Handbook of moduli, Vol. I, 161-215*, Int. Press, Somerville, MA, 2013. (Cited on page 13.)

[FK] Flenner, H., Kosarew, S. *On locally trivial deformations*, Publ. Res. Inst. Math. Sci. 23 (1987), no. 4, 627-665. (Cited on pages 4, 6, and 19.)

[F] Fujiki, A. *On the de Rham Cohomology Group of a Compact Kähler Symplectic Manifold*, Adv. Stud. Pure Math. 10 (1987), 105-165. (Cited on page 12.)

[GMT] Guaraldo, F., Macri, P., Tancredi, A. *Topics on real analytic spaces*, Advanced lectures in mathematics, Braunschweig: F. Vieweg, 1986. (Cited on pages 4 and 23.)

[HT] Hassett, B., Tschinkel, Y. Moving and ample cones of holomorphic symplectic fourfolds, Geom. Funct. Anal. 19 (2009), no. 4, 1065–1080. (Cited on pages 5 and 16.)

[H1] Huybrechts, D. *Compact hyperkähler manifolds: Basic results*, Invent. Math. 135 (1999), 63-113, alg-geom/9705025 (Cited on pages 5, 14, and 16.)

[H2] Huybrechts, D. *Erratum to the paper: Compact hyperkähler manifolds: basic results*, Invent. math. 152 (2003), 209-212, math.AG/0106014. (Cited on pages 5 and 16.)

[H3] Huybrechts, D. *Finiteness results for hyperkähler manifolds*, J. Reine Angew. Math. 558 (2003), 15–22, arXiv:math/0109024. (Cited on pages 2 and 13.)

[KV] D. Kaledin, M. Verbitsky, *Period Map for Non-Compact Holomorphically Symplectic Manifolds*, GAFA vol. 12, 1265-1295 (2002). (Cited on page 22.)

[K1] Kawamata, Y. *Pluricanonical systems on minimal algebraic varieties,* Invent. Math., 79, (1985), no. 3, 567-588. (Cited on page 18.)

[K2] Kawamata, Y. *On the length of an extremal rational curve,* Invent. Math. 105 (1991), no. 3, 609–611. (Cited on pages 9 and 19.)

[KS] Kebekus, S., Schnell, C. *Extending holomorphic forms from the regular locus of a complex space to a resolution of singularities,* arXiv preprint 1811.03644, to appear in JAMS. (Cited on page 25.)

[KMM] Kollar, J., Miyaoka, Y., Mori, S. *Rationally connected varieties*, J. Algebraic Geom. 1 (1992), no. 3, 429-448 (Cited on page 11.)

[Mag] Magnússon, J. *Lectures on Cycle Spaces,* Schriftenreihe des Graduiertenkollegs Geometrie und Mathematische Physik, 2005. (Cited on page 11.)

[Ma1] Markman, E. *A survey of Torelli and monodromy results for holomorphic-symplectic varieties*, Proceedings of the conference "Complex and Differential Geometry", Springer Proceedings in Mathematics, 2011, Volume 8, 257–322, arXiv:math/0601304. (Cited on pages 5, 8, 12, and 17.)

[Ma2] Markman, E. *On the existence of universal families of marked hyperkahler varieties*, arXiv:1701.08690, 11 pages. (Cited on pages 3 and 28.)

[Ma3] Markman, E. *Prime exceptional divisors on holomorphic symplectic varieties and monodromy reflections,* Kyoto J. Math. 53 (2013), no. 2, 345–403. (Cited on page 8.)

[M] Mather, J. Notes on topological stability, Harvard University, 1970 (Cited on pages 4 and 29.)

[Mo] Moore, C. C. *Ergodicity of Flows on Homogeneous Spaces*, American Journal of Mathematics Vol. 88, No. 1 (Jan., 1966), pp. 154-178 (Cited on page 14.)

[Mon] Mongardi, G. *A note on the Kähler and Mori cones of hyperkähler manifolds,* Asian J. Math. 19 (2015), no. 4, 583–591. (Cited on page 16.)

[N] Namikawa, Y. *On deformations of Q-factorial symplectic varieties,* J. Reine Angew. Math. 599 (2006), 97–110. (Cited on page 19.)

[Nar] Narasimhan, R. *Imbedding of Holomorphically Complete Complex Spaces,* American Journal of Mathematics, 82 (1960), p. 917–934 (Cited on pages 4 and 23.)

[Pap] Papayanov, G. *Cohomology of real analytic coherent sheaves*, 07.12.2017, https://mathoverflow.net/questions/317121/cohomology-of-real-analytic-coherent-sheaves. (Cited on page 31.)

[Pa1] Parusiński, A. *Lipschitz properties of semi-analytic sets*, Ann. Inst. Fourier (Grenoble) 38 (1988), no. 4, 189-213 (Cited on pages 6 and 31.)

[Pa2] Parusiński, A. *Lipschitz stratification*, Global analysis in modern mathematics (Orono, ME, 1991; Waltham, MA, 1992), 73-89, Publish or Perish, Houston, TX, 1993. (Cited on pages 6 and 31.)

[Pa3] Parusiński, A., *Lipschitz stratification of subanalytic sets*, Annales scientifiques de l'É.N.S. 4e série, tome 27, no 6 (1994), p. 661-696 (Cited on pages 4, 6, and 31.)

[R] Ran, Z., *Hodge theory and deformations of maps*, Compositio Math. 97 (1995), no. 3, 3 09–328. (Cited on page 7.)

[V1] Verbitsky, M. *A global Torelli theorem for hyperkähler manifolds,* Duke Math. J. Volume 162, Number 15 (2013), 2929-2986. (Cited on pages 13 and 14.)

[V2] Verbitsky, M. *Ergodic complex structures on hyperkähler manifolds*, Acta Mathematica, Sept. 2015, Volume 215, Issue 1, pp 161-182. (Cited on pages 15 and 26.)

[V2bis] Verbitsky, M. *Ergodic complex structures on hyperkähler manifolds: an erratum*, arXiv preprint 1708.05802. (Cited on pages 15, 26, 27, and 28.)

[Ver] Verdier, J.-L. *Stratifications de Whitney et theoreme de Bertini-Sard,* Invent. Math. 36 (1976), 295–312. (Cited on pages 6, 25, 26, 28, and 29.)

[W] Whitney, H. *Local properties of analytic varieties,* Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), p. 205–244, Princeton Univ. Press, Princeton, N. J. (Cited on page 28.)

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