An example of potential density on the Hilbert cube of a K3 surface

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To Yuri Prokhorov, for his birthday volume

1 Introduction

An irreducible projective variety X defined over a number field K is said to have potentially dense rational points, or sometimes simply to be "potentially dense", if there is a finite extension L of K such that the set of L-points $X(L)$ is Zariski-dense in X. For example, $\mathbb{P}^n_K$ is potentially dense for any K; it follows that the unirational varieties are potentially dense.

It is also not difficult to prove that abelian varieties are potentially dense (see for example [HT00, Prop. 3.1], and also [ABR11, Prop. 2.11] for an alternative, elementary and self-contained proof). On the contrary, by a celebrated theorem of Faltings, curves of geometric genus at least two are not potentially dense. It is generally believed and fits into Lang–Vojta's conjectural framework that smooth projective varieties of general type are not potentially dense, whereas varieties with negative or trivial canonical bundle are. Campana conjectures that potentially dense varieties are exactly special varieties in the sense of [Cam04] (the class of special varieties includes all rationally connected varieties and all varieties of zero Kodaira dimension). But these conjectures are difficult to verify even in dimension two.

For instance, we still do not know whether a general K3 surface is potentially dense. The potential density has been obtained by Bogomolov and Tschinkel for elliptic K3 surfaces, and also for K3 surfaces with infinite automorphism group ([BT00]). But no example of a potentially dense (or of a not potentially dense) K3 surface with Picard number one is available. Only one example of a family of simply-connected varieties with trivial canonical bundle, such that a sufficiently general member has Picard number one and is potentially dense, is known: such are the varieties of lines contained in cubics in $\mathbb{P}^5$ ([AV08], [ABR11]). These varieties are deformations of the second punctual Hilbert scheme of certain K3 surfaces. This family is complete in the sense that it exhausts all smooth polarized deformations of its general member. So far, this example remains unique.

Potential density of the k-th punctual Hilbert scheme $S^{[k]}$ of a K3 surface S has been studied by Hassett and Tschinkel in [HT00]. Roughly speaking, they prove that $S^{[k]}$ is potentially dense when it admits an abelian fibration (possibly rational rather than regular). In particular, when $S \subset \mathbb{P}^g$ is a K3 surface of genus g, then $S^{[g]}$ is potentially dense (indeed it is birational to the bundle of Pic^g of its hyperplane sections).

Our main result is as follows.

Theorem: Let S be a K3 surface with Néron-Severi lattice of rank 2, with intersection form $6x^2 + 16xy + 6y^2$. Then $X = S^{[3]}$ is potentially dense.

Note that by Corollary 2.9 of [Mor84], every even lattice of signature $(1,1)$ is the Neron-Severi lattice of an algebraic K3 surface. We show that S as above admits two distinct projective embeddings as a complete intersection of a quadric and a cubic in $\mathbb{P}^4$ (Proposition 2.3 below). A general complete intersection of type $(2,3)$ has cyclic Picard group, i.e. the Neron-Severi lattice of rank one, generated by a class z of square 6. Our surfaces are in the so-called Noether-Lefschetz locus, that is the locus of surfaces where some additional cohomology class z' in the second cohomology lattice is of type $(1,1)$. It is a countable union of codimension one[1] subvarieties in the parameter space of complete intersections of type $(2,3)$, which can be indexed by such z', or the intersection forms as above. A surface with a given Néron-Severi lattice of rank 2 corresponds to a "very general" (that is, outside of a countable union of proper subvarieties) point of a component of the Noether-Lefschetz locus, whereas special points may correspond to surfaces of higher Picard number.

This paper can be seen as a sequel to [Ame11], where the first-named author proved, following some remarks by O'Grady and Oguiso in [OGr05], [Ogu09], the potential density for the Hilbert square $F^{[2]}$ of certain K3 surfaces F admitting two embeddings as a quartic in $\mathbb{P}^3$. The surfaces in [Ame11] are chosen so that they have only finitely many automorphisms and no elliptic fibrations (otherwise F itself is known to be potentially dense, and the potential density of $F^{[2]}$ follows immediately). These properties are encoded in the Néron-Severi lattice of F, which she took to be of rank two and with the intersection form $4x^2+14xy+4y^2$.

Note that $F^{[3]}$ is potentially dense by [HT00]. To prove potential density of $F^{[2]}$, the idea was to use an automorphism similar to the one studied by Oguiso in [Ogu09].

Recall that the Hilbert square of a quartic surface F has a natural involution ι, the Beauville involution, sending an unordered pair of points x_1+x_2 to the residual pair of points in the intersection of F and the line through x_1 and x_2. When F contains no line, ι is a regular map. When F admits two different embeddings as a quartic, the product $f=\iota_2\iota_1$ of the corresponding Beauville involutions is an automorphism of $F^{[2]}$ of infinite order. The idea of [Ame11] was to take a hyperplane section $C\subset F$ with one double point and consider the surface $\Sigma = C*C\subset F^{[2]}$ (coming from $\mathrm{Sym}^2 C$, see [HT00] for the precise definition). The surface Σ is birationally abelian and therefore potentially dense, so the potential density of $X=F^{[2]}$ follows once we know that the union of $f^k(\Sigma)$ is Zariski-dense. This in turn means that Σ is not periodic and that the Zariski-closure of $\bigcup_k f^k(\Sigma)$ is not a divisor. Both statements are easily checked by a computation in cohomologies $H^2(X,\mathbb{Q})\supset \mathrm{NS}(X)$ and $H^4(X,\mathbb{Q})=\mathrm{Sym}^2 H^2(X,\mathbb{Q})$.

Our main theorem gives a similar example in dimension six. Namely, we consider a K3 surface S which admits two projective embeddings as an intersection of a quadric and a cubic in $\mathbb{P}^4$ (still with finite automorphism group and no elliptic fibration). This leads to two Beauville involutions on $X=S^{[3]}$, but now these are only rational maps even for S generic: the indeterminacy locus of each is a $\mathbb{P}^3$ formed by triples of colinear points on S. The product f of two Beauville involutions is a birational automorphism of X. We take for $\Sigma\subset S^{[3]}$ the indeterminacy locus of ι_2, and we prove that the union of $f^k(\Sigma)$ is Zariski-dense. This is done in three steps.

The proof that the Zariski closure of $\bigcup_k f^k(\Sigma)$ is not a divisor is just a computation in cohomology, which yields, exactly as in the 4-dimensional case, that no effective divisor can be f-invariant. We also need to prove that Σ is not periodic, and that the Zariski closure of $\bigcup_k f^k(\Sigma)$ is not four-dimensional. But this time it is not so easy to show this by a cohomological computation, because of two reasons: the cohomology itself is more complicated, and

[1] In our particular K3 case, thanks to $h^{2,0}=1$.

especially, f^* is not a ring homomorphism anymore. The proof of non-periodicity in Section 6 is an inductive argument which uses the explicit description of the Mori cone of X (Section 4). The proof that the Zariski closure of the iterates of Σ cannot be four-dimensional (Section 5) uses holomorphic symplectic geometry in an essential way. Our key tool is a generalization of Jouanolou-Ghys theorem by Correa, Maza and Soares [CMS13]. As a byproduct, we obtain a result which might be of independent interest (Proposition 5.1).

2 Preliminaries and construction of the example

An **irreducible holomorphic symplectic**, or **hyperkähler**, manifold is a simply connected compact Kähler manifold with $H^{2,0}$ generated by a single nowhere degenerate form σ.

The n-th punctual Hilbert scheme $S^{[n]}$ of a K3 surface S (including the case $n = 1$, i.e. S itself) is the best-known example ([Bea83, Section 6]).

If X is hyperkähler, then the Néron-Severi lattice of X (and in fact the larger lattice $H^2(X, \mathbb{Z})$) is equipped with an integral quadratic form, the **Beauville-Bogomolov form**. For a K3 surface, this is just the intersection form. For the n-th punctual Hilbert scheme of a K3 surface S, it can be computed as follows.

The class of the exceptional divisor of the Hilbert-Chow morphism $X = S^{[n]} \to \operatorname{Sym}^n S$ in $\mathrm{NS}(X)$ is divisible by two; denote by E its half. The Néron-Severi lattice of the variety X decomposes as the orthogonal direct sum with respect to the Beauville-Bogomolov quadratic form q on X:

$$\mathrm{NS}(X) \cong \mathrm{NS}(S) \oplus \mathbb{Z}E, \qquad q(E) = -2(n-1) \tag{1}$$

and $q|_{\mathrm{NS}(S)}$ coincides with the intersection form on S.

Take a K3 surface S of Picard number 2, with intersection form

$$b(x, y) = 6x^2 + 16xy + 6y^2 \tag{2}$$

in some coordinates (x, y) (such a K3 surface exists by [Mor84, Cor. 2.9]).

From now on and until the end of the paper we denote by X the Hilbert cube of S and by q its Beauville-Bogomolov form.

Throughout this paper, it is more convenient for us to work with column vectors, so we write $(x, y)^{\mathrm{T}}$ meaning the transposition of the corresponding row vector (x, y).

Proposition 2.1. *The surface S has (-2)-curves. This surface is not elliptic. The variety X is not rationally fibered in abelian threefolds.*

Proof: The class $(2, -1)^{\mathrm{T}}$ is an example of a (-2)-class, so the surface S has (-2)-curves. If the surface S is elliptic or the variety X is rationally fibered in abelian threefolds, then (obviously in the first case, by [AC08, Proposition 3.1] in the second one) the form q represents zero, i.e. $3x^2 + 8xy + 3y^2 - 2z^2 = 0$ for some $x, y, z \in \mathbb{Z}$. If $2 \nmid x$, then counting modulo 8 yields the contradiction. If $2 \mid x$, then both y and z are even, so applying the same argument to the triple $(x/2, y/2, z/2)$ implies $x = y = z = 0$. □

Proposition 2.2. *The group $\mathrm{Aut}(S)$ is finite.*

Proof: It is proven in [PS71, section 7] that a K3 surface with Picard number 2 has an infinite automorphism group if and only if its intersection form represents both -2 and 0. □

Proposition 2.3. *Denote by* N *the rank 2 lattice with the bilinear form* b*. There is an isometry* $\varphi : \mathrm{N} \simeq \mathrm{NS}(S)$ *such that the classes* $\varphi\left((1,0)^{\mathrm{T}}\right)$ *and* $\varphi\left((0,1)^{\mathrm{T}}\right)$ *are very ample. In each of the corresponding projective embeddings, the surface* S *is an intersection of a quadric and a cubic in* $\mathbb{P}^4$*, not containing lines.*

Proof: We adapt the argument from [Ame11]. Denote by h_1 and h_2 the classes of line bundles corresponding to the classes $(1,0)^{\mathrm{T}}$ and $(0,1)^{\mathrm{T}}$. Observe that there are no classes $v = (x,y)^{\mathrm{T}}$ with $b(v) = -2$ and $v \cdot h_1 = 0$. Indeed, if such a class v exists, then $6x + 8y = 0$ and $6x^2 + 16xy + 6y^2 = -2$. Using the first equation we get that $3 \mid y$ which contradicts the second equation. Therefore, applying some sequence of reflections in hyperplanes orthogonal to (-2)-classes (called "Picard-Lefschetz reflections" in [Bar+04]) if necessary, we may assume that the class h_1 is ample. Indeed the nef cone of a K3 surface is a fundamental domain for the action of the group generated by the Picard-Lefschetz reflections on the positive cone ([Bar+04, VIII, Prop. 3.10]).

Now, we show that the class h_2 is then automatically ample too. It is enough to verify that intersections of the classes h_1 and h_2 with any (-2)-class have the same sign. Let $v = (x,y)^{\mathrm{T}}$ be a (-2)-class intersecting h_1 positively, i.e. $3x^2+8xy+3y^2 = -1$ and $3x+4y > 0$. Denoting $t = 3x + 4y$ we get that $t^2 - 7y^2 = -3$ and $t > 0$. The equality $t^2 = 7y^2 - 3$ implies $4t > 7y$, so $v \cdot h_2 = \frac{1}{3}(4t - 7y) > 0$. Thus, the class h_2 is ample.

It remains to show that the classes h_1 and h_2 are in fact very ample. Since for any (-2)-class L, we have $L \cdot h_1 \notin \{0,1\}$, then by [Sai74, Subsection 2.7], the divisor h_1 has no fixed components. Any divisor having positive self-intersection without fixed components is either very ample or hyperelliptic (see [Sai74, Th. 4.1]). By [Sai74, Th. 5.2] the latter is not the case, since the form b does not represent 0 and there is no curve C' with $2[C'] = h_1$.

The non-existence of lines follows from $\ell \cdot h_1 \neq 1$ for any class $\ell \in \mathrm{NS}(S)$. □

Using φ we identify the lattices N and $\mathrm{NS}(S)$ and get that both h_1 and h_2 are very ample. The morphism associated to the linear system $|h_1|$ (or to $|h_2|$) embeds S in $\mathbb{P}^4$ as a K3 surface of genus 4, which is well-known to be a complete intersection of a quadric and a cubic (this last fact is easy to check using the exact sequence defining the ideal sheaf of S in $\mathbb{P}^4$).

Lemma 2.4. *The classes* $2h_1 - h_2$ *and* $2h_2 - h_1$ *correspond to* (-2)*-curves.*

Proof: It is enough to prove the statement for the class $2h_1 - h_2$. Since $b(2h_1 - h_2) = -2$, either the class $2h_1 - h_2$ or the class $-(2h_1 - h_2)$ is effective. The class h_1 is ample and $(2h_1 - h_2) \cdot h_1 = 4$, so the class $2h_1 - h_2$ is effective and corresponds to some curve C. Suppose that the curve C is not irreducible or not reduced. Then we can write

$$[C] = [C'] + [C'']$$

for some effective classes $[C'], [C'']$ with $b(C') = -2$. Since

$$4 = h_1 \cdot [C] > h_1 \cdot [C'] > 0$$

and $2 \mid (h_1 \cdot [C'])$, then $h_1 \cdot [C'] = 2$. Now, if we write $[C'] = xh_1 + yh_2$, then this condition reads as $3x + 4y = 1$, and the condition $b(C') = -2$ is equivalent to $(3x + 4y)^2 - 7y^2 = -3$. The equation $7y^2 = 4$ has no integer solutions, therefore, the curve C is reduced and irreducible. □

Proposition 2.5. *There exist K3 surfaces* S *defined over a number field, with Picard number*[2] 2 *and the intersection form* b *as in* (2)*.*

[2]Throughout the paper, we view varieties defined over a number field as complex algebraic varieties, so this is geometric Picard number.

Proof: Such a K3 surface over $\mathbb{C}$ is a general member of a component of the Noether-Lefschetz locus of the family of intersections of a quadric and a cubic in $\mathbb{P}^4$. This component is an algebraic subvariety defined over a number field. Now the only problem is that it could, apriori, happen that every surface from this locus which is defined over a number field has higher Picard number; but this is ruled out by [MP12], which shows that in any family (defined over a number field) of smooth projective varieties, most members defined over a number field have the same Néron-Severi group as the general member. □

3 Birational automorphism and its invariant divisors

The classes h_1 and h_2 define two embeddings $S \subset \mathbb{P}^4$ as an intersection of a quadric and a cubic. Two embeddings induce two Beauville involutions $\iota_1, \iota_2 : S^{[3]} \dashrightarrow S^{[3]}$, see [Bea82, Section 6]. Recall that a Beauville involution of $S^{[3]}$ sends a general triple of points to the residual triple in the intersection of S and the 2-plane through the triple. Denote by $H_1, H_2 \in \mathrm{NS}(X) \simeq \mathrm{NS}(S) \oplus \mathbb{Z}E$ the classes corresponding to $h_1, h_2 \in \mathrm{NS}(S)$ respectively. The induced morphism $\iota_k^* : \mathrm{NS}(X) \to \mathrm{NS}(X)$ ($k \in \{1, 2\}$) is opposite to the reflection in the hyperplane $(H_k - E)^\perp$ ([Deb84, Theorem 4.1]). In other words, the action of the homomorphism ι_k is given by

$$\iota_k^*(L) = -L + 2\frac{q(L, H_k - E)}{q(H_k - E, H_k - E)}(H_k - E) = -L + q(L, H_k - E)(H_k - E).$$

In the basis $\{H_1, E, H_2\}$ the operators ι_1^*, ι_2^* are given by matrices

$$M_1 = \begin{pmatrix} 5 & 4 & 8 \\ -6 & -5 & -8 \\ 0 & 0 & -1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} -1 & 0 & 0 \\ -8 & -5 & -6 \\ 8 & 4 & 5 \end{pmatrix}. \tag{3}$$

The following construction is described in [Bea82, Section 6]. Denote by $\tilde{\Pi}_k$ the indeterminacy set of the involution ι_k. The set $\tilde{\Pi}_k$ consists of length 3 subschemes contained in a line. The image of S under the embedding given by h_k is defined by the intersection of the quadric Q_k and some cubic hypersurface. Any line intersecting S in at least 3 points is contained in Q_k, so the indeterminacy set is parametrized by the lines in Q_k, i.e. $\tilde{\Pi}_k \simeq \mathbb{P}^3$. The involution ι_k is biregular on the set $X \setminus \tilde{\Pi}_k$. Now, denote by $\epsilon_k : \widehat{X}_k \to X$ the blow-up of X in $\tilde{\Pi}_k$. The exceptional divisor E_k of this blow-up is isomorphic to the projectivization of the conormal bundle $\mathbb{P}\left(N^*_{\tilde{\Pi}_k/X}\right)$. Since $\tilde{\Pi}_k \simeq \mathbb{P}^3$ and is lagrangian[3] (there are no non-zero holomorphic forms on $\mathbb{P}^3$ so $\omega|_{\tilde{\Pi}_k} = 0$), we get

$$\mathbb{P}\left(N^*_{\tilde{\Pi}_k/X}\right) \simeq \mathbb{P}\left(T_{\tilde{\Pi}_k}\right) \simeq \left\{(x, H) \in \mathbb{P}^3 \times \left(\mathbb{P}^3\right)^\vee \,:\, x \in H\right\}.$$

The map ι_k extends to a biregular involution of $\widehat{X}_k$ which maps E_k to itself.

Define a birational automorphism

$$f = \iota_2 \circ \iota_1 : X \dashrightarrow X. \tag{4}$$

[3]Recall that a lagrangian subvariety of a holomorphic symplectic $2n$-fold is an isotropic subvariety of maximal dimension n.

Proposition 3.1. *1) The action of f^* on $\mathrm{NS}(X)$ in the basis $\{H_1, E, H_2\}$ is given by the matrix*

$$M_1M_2 = \begin{pmatrix} 27 & 12 & 16 \\ -18 & -7 & -10 \\ -8 & -4 & -5 \end{pmatrix}.$$

2) $(f^n)^* = (f^*)^n$.

Proof: The first part of the proposition is the equality $(\iota_2 \circ \iota_1)^* = \iota_1^* \circ \iota_2^*$, which holds because birational automorphisms of varieties with trivial canonical bundle contract no hypersurfaces. The second part is true by the same reason. □

The eigenvalues of the operator $f^* : \mathrm{NS}(X) \to \mathrm{NS}(X)$ are $1,\ 7-4\sqrt{3},\ 7+4\sqrt{3}$ with eigenvectors

$$2H_1 - 7E + 2H_2,\ (-2+\sqrt{3})H_1 + (1-\sqrt{3})E + H_2,\ (-2-\sqrt{3})H_1 + (1+\sqrt{3})E + H_2$$

respectively. Since $7+4\sqrt{3} > 1$, the map f is of infinite order.

Proposition 3.2. *The map f is not induced by any birational automorphism of S. Moreover, it is not induced by any birational automorphism of a K3 surface S' with $S^{[3]} \simeq (S')^{[3]}$.*

Proof: If f is induced by an automorphism of S', then $f^*(E') = E'$, where $2E'$ is an exceptional divisor of the morphism $(S')^{[3]} \to \mathrm{Sym}^3 S$. Then we get $E' = k(2H_1 - 7E + 2H_2)$ for some $k \in \mathbb{Q}$. However, this is not possible since

$$-4 = q(E') = k^2 \cdot q(2H_1 - 7E + 2H_2) = -84k^2.$$ □

Proposition 3.3. *No effective divisor on X is invariant under f.*

Proof: The only invariant divisors on X are multiples of $2H_1 - 7E + 2H_2$. Such a divisor is not effective since ι_1^* preserves the non-degenerate effective cone and

$$\iota_1^*(2H_1 - 7E + 2H_2) = -(2H_1 - 7E + 2H_2).$$ □

4 The Mori cone

In this section, we describe the closure of the cone of effective curves $\mathrm{NE}(X)$ and the nef cone $\mathrm{Nef}(X)$ of X. Using the form q, we view curve classes as divisor classes with rational coefficients, so that all cones we consider are in the real Néron-Severi group.

The main tool for the computation is the classification of extremal rays established in [HT10] and [HT16]. We define the cone

$$\mathcal{W}(X) = \left\langle E,\ H_1 - \frac{3}{2}E,\ H_2 - \frac{3}{2}E,\ 2H_1 - H_2 - \frac{1}{2}E,\ 2H_2 - H_1 - \frac{1}{2}E \right\rangle_{\mathbb{R}_{\geq 0}}.$$

Lemma 4.1.

$$\overline{\mathrm{NE}}(X) = \mathcal{W}(X).$$

Proof: Let $R = xH_1 + yH_2 + zE$ be a primitive element of $\mathrm{NS}(X)$ (in particular, x, y, z are integers). By [HT10, Section 3] and [HT16, Section 5] (see also [AV22, Corollary 5.7]), the ray generated by R is an extremal ray of $\overline{\mathrm{NE}}(X)$ if and only if it satisfies one of the following properties

- $q(R,R) = -2,\ q(R,H_1) \geq 0$;
- $q(R,R) = -4,\ q(R,H_1) \geq 0$, and x, y are both divisible by 2 but not by for 4;
- $q(R,R) = -4,\ q(R,H_1) \geq 0$, and x, y are both divisible by 4;
- $q(R,R) = -12,\ q(R,H_1) \geq 0$, and x, y are both divisible by 2;
- $q(R,R) = -36,\ q(R,H_1) \geq 0$, and x, y are divisible by 4.

It can be easily checked that the classes generating $\mathcal{W}(X)$ satisfy one of the properties above.

Now, assume that R generates an extremal ray. If $q(R,H_1) = 0$, then $(x,y) = (4t,3t)$ for some $t \in \mathbb{Z}$. Thus $(R,R) = -42t^2 - 4z^2$ which is possible only for $R = E$. Therefore, we may assume that R intersects positively with both H_1 and H_2. Now, write

$$\mathcal{W}(X) = \big\{xH_1 + yH_2 + zE \,:\, 2x + y \geq 0,\ x + 2y \geq 0,\ 3x + 5y + 2z \geq 0,\\ 5x + 3y + 2z \geq 0,\ 3x + 3y + 2z \geq 0\big\}.$$

The transverse section of $\mathcal{W}(X)$ is depicted in Figure 1. Let $R = xH_1 + yH_2 + zE$ $(x, y, z \in \mathbb{Z})$ be an extremal class. We need to check that the numbers x, y, z satisfy all five inequalities above. By symmetry, it is enough to prove that $x+2y \geq 0$, $5x+3y+2z \geq 0$ and $3x+3y+2z \geq 0$. Observe that

$$\frac{3}{2}q(R,R) = t^2 - 7y^2 - 6z^2, \qquad \text{where} \quad t = 3x + 4y = q(R,H_1) > 0.$$

1. Suppose that $x + 2y < 0$. Thus, $t + 2y < 0$, so $0 < t < -2y$ and we get

$$\frac{3}{2}q(R,R) < -3y^2 - 6z^2.$$

   Recall that $q(R,R) \in \{-2,-4,-12,-36\}$, so there are finitely many such R and it can be shown by brute force that in fact there are no such R. Indeed, since $-3y^2 - 6z^2 > -54$, we get $|z| < 3$.

   If $|z| = 2$, then $q(R,R) = -36$ and the equation $t^2 - 7y^2 = -30$ modulo 7 yieds the contradiction.

   If $|z| = 1$, then either $q(R,R) = -36$ or $q(R,R) = -12$, that is, $t^2 - 7y^2 = -48$ or $t^2 - 7y^2 = -12$. Since $0 < t < -2y$, we have $|y| < 3$. It can be easily checked that these equations have no solutions with $|y| < 4$.

   Finally, if $z = 0$, then $-3y^2 > t^2 - 7y^2 \in \{-3,-6,-18,-54\}$. The case $t^2 - 7y^2 = -3$ implies $y = 0$ which gives no solutions. In the case $t^2 - 7y^2 = -6$, one easily gets $y = -1$ and $t = 1$ which is impossible since $3x - 4 = 1$ has no integer solutions. The equation $t^2 - 7y^2 = -18$ has no solutions modulo 7. The only solutions of equation $t^2 - 7y^2 = -54$ with $|y| \leq 5$ are $t = \pm 11, y = \pm 5$ which are impossible by $0 < t < -2y$ and $t = \pm 3, y = \pm 3$ which together with $0 < t < -2y$ imply x is non-integer.

2. Suppose that $3x + 5y + 2z < 0$. Thus, $0 < t < -2z - y$ and we get

$$\frac{3}{2}q(R,R) < 6y^2 - 4yz - 2z^2 \leq -2y^2 - z^2.$$

   Again, $q(R,R) \in \{2,-4,-12,-36\}$, so by the same argument, it can be checked that there are no such R.

3. Suppose that $3x+3y+2z<0$. Thus, $0<t<2z-y$ and we get

$$\frac{3}{2}q(R,R) < 6y^2+4yz-2z^2 \leq -2y^2-z^2.$$

This case is analogous to the previous one.

Remark 4.2. *The classes generating the cone of curves are interpreted geometrically as follows (recall that we use the form q to identify $\mathrm{NE}(X,\mathbb{R})$ and $\mathrm{NS}(X,\mathbb{R})$).*

- *It is well-known that the class of the ruling of the exceptional divisor of the Hilbert-Chow morphism is $\frac{1}{4}E$.*
- *According to [HT16, Example 14], the classes $H_1-\frac{3}{2}E, H_2-\frac{3}{2}E$ are the classes of lines in the projective subspaces $\tilde{\Pi}_1, \tilde{\Pi}_2$ defined in the section 3.*
- *According to [HT10, Example 4.11], the classes $2H_2-H_1-\frac{1}{2}E,\ 2H_1-H_2-\frac{1}{2}E$ are the classes of lines in the projective subspaces $C_1^{[3]}, C_2^{[3]}$, where C_1 and C_2 are (-2)-curves having classes $2h_1-h_2$ and $2h_2-h_1$ respectively (see Lemma 2.4).*

By duality, one obtains the following result.

Proposition 4.3. *The nef cone of X is generated by the following classes*

$$\mathrm{Nef}(X) = \big\langle 5H_1-2H_2,\ 5H_2-2H_1,\ 11H_1-3H_2-7E,\ 11H_2-3H_1-7E,\\ 3H_1+3H_2-7E\big\rangle_{\mathbb{R}_{\geq 0}}.$$

Remark 4.4. *When the nef cone is polyhedral, X has only finitely many automorphisms. Indeed an automorphism induces a permutation of extremal rays of the nef cone, and these are finitely many, so the statement follows from the finiteness of the group of automorphisms acting trivially on $\mathrm{NS}(X)$. The latter is well-known (for example by Fujiki-Lieberman theorem, since $\mathrm{Aut}(X)$ is discrete).*

This explains the necessity to work with a birational, rather than a biregular, automorphism in spite of some technical difficulties. The following proposition shall be useful in dealing with them. Recall that we have the birational automorphism f on X defined in (4) as a composition of two involutions ι_1 and ι_2 with indeterminacy loci $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$.

Proposition 4.5. *The intersection of $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ is empty.*

Proof: Let $C_1\subset S$ be the (-2)-curve in the class $2h_1-h_2$. Observe that $\iota_1\left(C_1^{[3]}\right)=\tilde{\Pi}_2$ since $\iota_1^*(2H_1-H_2-\frac{1}{2}E)=H_2-\frac{3}{2}E$. Since ι_1 restricted to the complement of $\tilde{\Pi}_1$ is a regular automorphism it is enough to check that the intersection $C_1^{[3]}\cap\tilde{\Pi}_1$ is empty. The image of C_1 under the embedding given by the class h_1 has degree $h_1\cdot(2h_1-h_2)=4$. Assume that $C_1^{[3]}\cap\tilde{\Pi}_1\neq\emptyset$. Then there is a trisecant line to a degree 4 rational curve $C_1\subset\mathbb{P}^4$. However, such a curve does not have a trisecant when non-degenerate (indeed, a rational normal curve is an intersection of quadrics). Hence C_1 is contained in a hyperplane, and the class h_1 must be the sum of $2h_1-h_2$ and the effective class of the residual intersection of S and this hyperplane. Since the class h_2-h_1 is not effective, we conclude that the intersection $C_1^{[3]}\cap\tilde{\Pi}_1$ is empty. $\square$

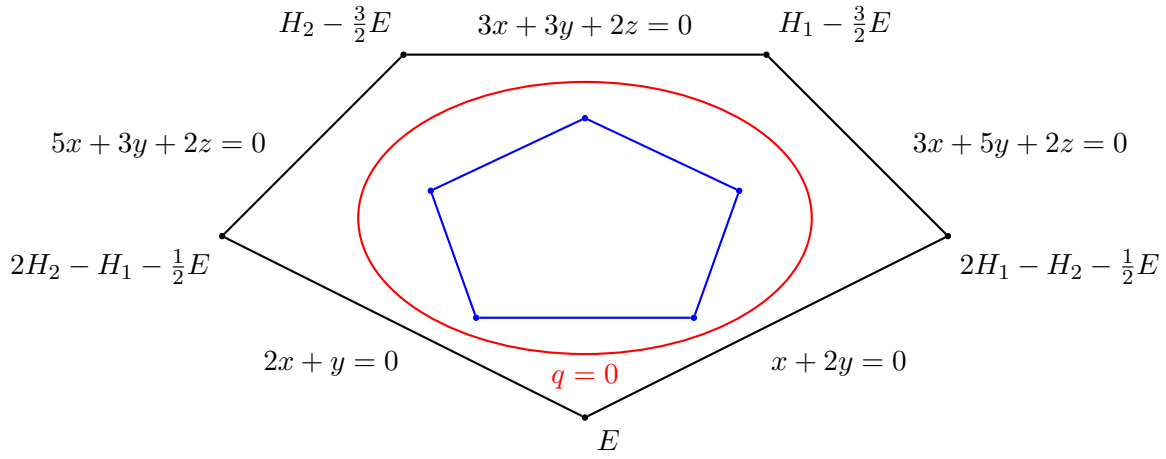


Figure 1: Transverse section of $\overline{\mathrm{NE}}(X)$, the conic given by $q = 0$ and the nef cone (drawn blue) which is dual to $\overline{\mathrm{NE}}(X)$ with respect to q

5 Rationally connected threefolds and Pfaff systems

To prove the potential density, we shall investigate the set of f-iterates of a suitable $\mathbb{P}^3$ in X and prove that it is Zariski-dense. We have already seen that it cannot be 5-dimensional (Proposition 3.3). In this section, we exclude the case of dimension four by proving a more general proposition which can be of independent interest.

Proposition 5.1. *Let X be an irreducible holomorphic symplectic manifold of dimension $2n$. If $Y \subset X$ is a subvariety with $\dim Y = n + 1$, then there is only a finite number of rationally connected n-dimensional subvarieties $\Pi_i \subset X$ contained in Y.*

Before proving the proposition, we recall some definitions.

Definition 5.2. Let M be a smooth compact complex manifold of dimension n. *A Pfaff system* of codimension r on M is a non-trivial section $\omega \in H^0(M, \Omega^r_M \otimes L)$, where Ω^r_M denotes the sheaf of holomorphic r-forms and L is some holomorphic line bundle.

In our setting, we shall have $r = 2$ and L trivial, and our Pfaff system can be visualized as a foliation defined by the kernel of ω, which will be the pullback of the holomorphic symplectic form to a desingularization of Y. We invoke Pfaff systems in order to cite correctly a result by Correa, Maza and Soares below.

Definition 5.3. An irreducible hypersurface $Z \subset M$ is called *ω-invariant* if $\omega|_Z = 0$ (if Z is singular we mean that the restriction of ω to the smooth part of Z is zero or, equivalently, the pullback of ω to the resolution of singularities of Z is zero).

A first integral of the Pfaff system defined by the form ω is a non-constant meromorphic map $\pi : M \to C$ for some projective smooth curve C such that the fibers of π are ω-invariant hypersurfaces.

The well-known Jouanolou-Ghys theorem, usually stated as the fact that a codimension-one holomorphic foliation with infinitely many closed leaves is a fibration, was proven in [Jou78] with some extra assumptions, satisfied for projective manifolds, and then in [Ghy00]. For the case of arbitrary codimension, it is generalized in as follows.

Theorem 5.4. *([CMS13]) Consider a Pfaff system of codimension r on a connected compact complex manifold M defined by the form $\omega \in H^0(M, \Omega^r_M \otimes L)$. If ω does not admit a first integral, then the number of ω-invariant hypersurfaces is not greater than*

$$\dim \left[H^0\left(M, \Omega^{r+1}_M \otimes L\right) / \omega \wedge H^0(M, \Omega^1_{\mathrm{cl}})\right] + \dim H^1(M, \Omega^1_{\mathrm{cl}}) + r + 1,$$

where Ω^1_{cl} denotes the sheaf of closed 1-forms.

For our purpose, we use only the following corollary.

Corollary 5.5. *If the number of ω-invariant hypersurfaces is infinite, then the Pfaff system defined by ω admits a first integral.*

Now we are ready to prove Proposition 5.1.

Proof: Resolving singularities if necessary, we may assume that Y is smooth and maps to X birationally onto its image. Observe that the pull-back of the holomorphic symplectic form σ to Y is non-zero, because σ is non-degenerate on X. Hence it defines a Pfaff system of codimension 2 (with L trivial). Since there are no non-trivial holomorphic forms on a rationally connected variety, the hypersurface $\Pi_i \subset Y$ is ω-invariant for any i. By Corollary 5.5, this Pfaff system admits a first integral $\pi : Y \dashrightarrow C$.

Blowing-up the variety Y again if necessary, we get the following commutative diagram

$$\begin{array}{ccc} \widehat{Y} & & \\ \downarrow{\scriptstyle \epsilon} & \searrow^{\widehat{\pi}} & \\ Y & \overset{\pi}{\dashrightarrow} & C \end{array}$$

The morphism $\widehat{\pi}$ is flat since C is a curve. By [FF12], over the complex numbers the set of points $x \in X$ with rationally connected fiber $\widehat{\pi}^{-1}(x)$ is constructible. Since this set is infinite, it is Zariski-open. We get that there is an open subset $U \subset C$ over which $\widehat{\pi}$ is smooth with rationally connected fibers. Since there are no non-trivial holomorphic forms on a rationally connected variety, any holomorphic two-form on $Y_U = \widehat{\pi}^{-1}(U)$ must be zero, as it can be seen taking the second exterior power of the exact sequence of relative differentials

$$0 \to \Omega^1_{Y_U/U} \otimes \widehat{\pi}^* \Omega^1_U \to \Omega^2_{Y_U} \to \Omega^2_{Y_U/U} \to 0,$$

a contradiction. □

6 Potential density

In this section, we finish the proof of our main result about potential density of rational points on X. Define by induction the family of subvarieties $\{\Pi^i_k\}$ $(i \in \{1,2\}, k \in \mathbb{Z}_{\geq 0})$

$$\Pi^1_0 = \tilde{\Pi}_2, \quad \Pi^2_k = \overline{\iota_1\left(\Pi^1_k \setminus \tilde{\Pi}_1\right)}, \quad \Pi^1_{k+1} = \overline{\iota_2\left(\Pi^2_k \setminus \tilde{\Pi}_2\right)}.$$

Every embedding $\Pi^i_k \hookrightarrow X$ $(i \in \{1,2\}, k \in \mathbb{Z}_{\geq 0})$ induces a map of the Mori cones (recall that we view the Mori cone as a part of $\mathrm{NS}(X, \mathbb{R})$). Our goal is to describe the image of this map. For this purpose, it is convenient to work in two different bases, namely

$$L_i = H_{i+1} - \frac{3}{2}E, \quad K_i = 2H_{i+1} - H_i - \frac{1}{2}E, \quad E$$

(here, we set $i+1=1$ for $i=2$). The action of the morphism

$$\iota_i^* : \langle L_i, K_i, E\rangle \to \langle L_{i+1}, K_{i+1}, E\rangle$$

in these bases is given by the matrix (note that we use two different bases for the source and the target)

$$\begin{pmatrix} 0 & 5 & 4 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Define the subcones

$$J_1 = \big\langle L_1, K_1, E\big\rangle_{\mathbb{R}_{\geq 0}},\ J_2 = \big\langle L_2, K_2, E\big\rangle_{\mathbb{R}_{\geq 0}} \subset \overline{\mathrm{NE}}(X).$$

It is clear that $\iota_i^*(J_i) \subset J_{i+1}$.

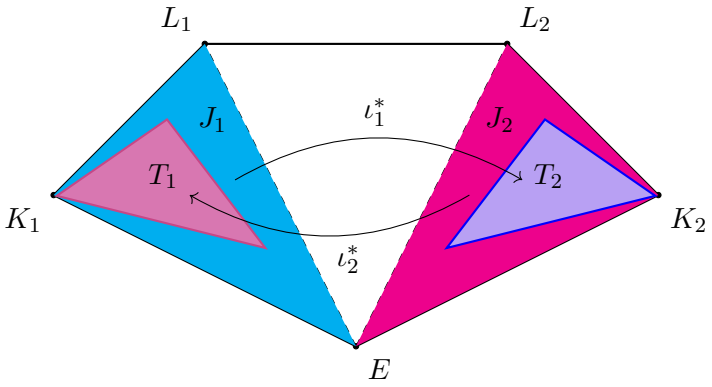


Figure 2: Subcones J_1, J_2 and $T_1 = \iota_1^*(J_1)$, $T_2 = \iota_2^*(J_2)$

Lemma 6.1. *Let $C \subset X$ be a curve not intersecting $\tilde{\Pi}_1$. Then the class of the image curve $\iota_1(C)$ coincides with $\iota_1^*(C)$ (to define $\iota_1^*(C)$ we view $\overline{\mathrm{NE}}(X)$ as a part of $\mathrm{NS}(X,\mathbb{R})$ by means of Beauville-Bogomolov form q).*

Proof: The lemma follows from the following chain of equations for a divisor D:

$$q(\iota_1(C), D) = \iota_1(C)\cdot D = C\cdot \iota_1^*(D) = q(C, \iota_1^*(D)) = q(\iota_1^*(C), D).$$

We are ready to prove the following proposition consisting of two parts A(k) and B(k) for each non-negative integer k.

Proposition 6.2. A(k) : *For any integer $k \geq 1$, the indeterminacy set of the map $\iota_2 : \Pi^2_{k-1} \dashrightarrow \Pi^1_k$ is finite. For any integer $k \geq 0$ the image of the map $\overline{\mathrm{NE}}(\Pi^1_k) \to \overline{\mathrm{NE}}(X)$ is contained in the cone J_1.*

B(k) : *For any integer $k \geq 0$, the indeterminacy set of the map $\iota_1 : \Pi^1_k \dashrightarrow \Pi^2_k$ is finite. For any integer $k \geq 0$ the image of the map $\overline{\mathrm{NE}}(\Pi^2_k) \to \overline{\mathrm{NE}}(X)$ is contained in the cone J_2.*

Proof: We prove both parts of proposition by induction on k. We start with A(0) and B(0) (this is the induction base for all statements but the very first, but the first statement of $A(1)$ shall follow by induction step from $B(0)$). For A(1), it is enough to observe that (the

image of) $\overline{\mathrm{NE}}(\Pi_0^1)$ is generated by L_1, and thus, is contained in J_1. For B(1), observe that ι_2 is regular on Π_0^1 by proposition 4.5. Thus, by lemma 6.1,

$$\overline{\mathrm{NE}}(\Pi_k^2) = \iota_1^* \left(\overline{\mathrm{NE}}(\Pi_k^1)\right) \subset \iota_1^*(J_1) \subset J_2.$$

Let us prove the induction step. Namely, we prove that B(k) implies A($k+1$), and then we prove that A($k+1$) implies B($k+1$). Assume that $B(k)$ is true. We show that $A(k+1)$ is true. If the intersection $\Pi_k^2 \cap \tilde{\Pi}_2$ is of dimension at least 1, then it contains an effective curve whose class is a positive multiple of L_1. However, $\overline{\mathrm{NE}}(\Pi_k^2) \subset J_2$ which contradicts $L_1 \notin J_2$.

It is clear that the set $\Pi_{k+1}^1 \setminus \iota_2(\Pi_k^2 \setminus \tilde{\Pi}_2)$ is contained in $\tilde{\Pi}_2$. Thus, any class of an irreducible curve in Π_{k+1}^1 equals either a positive multiple of L_1 or $\iota_2^* C - \lambda L_1$, where C is a class of some effective curve in Π_k^2 and $\lambda \geq 0$. We know that $C \in J_2$, so $\iota_2^* C \in J_1$. By proposition 4.3, the class of an effective curve has non-negative intersection with $5H_2 - 2H_1$. Suppose that

$$\iota_2^* C - \lambda L_1 = \alpha L_1 + \beta K_1 + \gamma E.$$

It is evident that $\beta, \gamma \geq 0$. Using that $q(5H_2 - 2H_1, K_1) = q(5H_2 - 2H_1, E) = 0$ we get $(5H_2 - 2H_1, \alpha L_2) = 14\alpha \geq 0$. Therefore, the class of any effective curve in Π_k^1 belongs to J_1.

The proof of the induction step from A($k+1$) to B($k+1$) is analogous. □

Proposition 6.3. *The family of subvarieties $\{\Pi_k^1\}$ is not periodic. In other words,*

$$\Pi_n^1 \neq \Pi_m^1$$

for $n \neq m$.

Proof: It is enough to prove the proposition for $n = 0$ (indeed, if $\Pi_n^1 = \Pi_m^1$ for $n, m > 0$ then the intersection of Π_{n-1}^1 and Π_{m-1}^1 must be dense in both of them, so $\Pi_{n-1}^1 = \Pi_{m-1}^1$). Assume the opposite, then $\Pi_m^1 = \Pi_0^1$ for some positive m. Take a curve $C \subset \Pi_0^1$ not passing through the indeterminacy set of the map f^m (this set is finite by proposition 6.2). The class of the image curve is a positive multiple of $[C]$ (since both of them lie in $\Pi_0^1 \simeq \mathbb{P}^3$). Then $[C]$ is an eigenvector of the operator $(f^m)^* = (f^*)^m$ which contradicts the proposition 3.1. □

Finally, we are ready to prove the main theorem.

Theorem 6.4. *Rational points of X are potentially dense.*

Proof: By construction, there exists a finite field extension such that the rational points of all subvarieties Π_k^1 are dense in these subvarieties (it is enough to take a field such that rational point of Π_0^1 are dense which is possible since Π_0^1 is a projective space). So the Zariski-closure of rational points of X over this extension contains the closure of the union $\bigcup \Pi_k^1$. Define

$$Y = \overline{\bigcup_{n \in \mathbb{Z}_{\geq 0}} \Pi_n^1}.$$

By Proposition 6.3, $\dim Y > 3$. By Proposition 5.1, $\dim Y \neq 4$. If $\dim Y = 5$, then Y is an effective divisor such that $f^* Y = Y$ (since $\overline{f(\Pi_n^1)} = \Pi_{n+1}^1$ and f induces a bijection on the set of prime divisors), which contradicts Proposition 3.3. Therefore, $\dim Y = 6$, and $Y = X$. □

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