Birational geometry of hyperkähler manifolds and the Hu-Yau conjecture

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Abstract: Wierzba and Wiśniewski proved that for $\dim_{\mathbb{C}} M = 4$, every bimeromorphic map $M \dashrightarrow M'$ of hyperkähler manifolds is represented as a composition of Mukai flops. Hu and Yau conjectured that this result can be generalized to arbitrary dimension. They defined "Mukai's elementary transformation" as the blow-up of a subvariety ruled by complex projective spaces, composed with the contraction of the ruling. Hu and Yau conjectured that any bimeromorphic map $M \dashrightarrow M'$ can be decomposed into a sequence of Mukai's elementary transformations, after possibly removing subvarieties of codimension greater than 2. We prove this conjecture for compact hyperkähler manifolds of maximal holonomy by decomposing any bimeromorphic map into a composition of wall-crossing flops associated with MBM contractions.

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1 Introduction

Let M and M' be two bimeromorphic compact hyperkähler manifolds. In [HY] Hu and Yau have studied the most basic type of bimeromorphic maps between M and M' known as Mukai flops or Mukai's elementary transformations. Hu and Yau conjectured that every bimeromorphic map between M and M' may to a certain extent be described as a sequence of Mukai flops. More precisely, given a bimeromorphic map $\varphi\colon M \dashrightarrow M'$, it was conjectured that after removing some subvarieties of codimension greater than two from M and M' the map φ decomposes into a finite composition of Mukai's elementary transformations.

In [WW] (and also partially in [BHL]) the conjecture was proven in the case when M and M' are of dimension four. The results of [WW] are stronger than what was stated above: in the case of fourfolds one does not need to remove any subvarieties from M and M' to decompose φ into a sequence of flops. The central result [WW, Theorem 1.1] is the description of the contraction loci in holomorphic symplectic fourfolds, claiming that the contraction locus is always a disjoint union of complex projective planes.

In the present paper we address the Hu–Yau conjecture in arbitrary dimension. Our aim is to give a precise statement of the conjecture, drawing attention to the possible ambiguity in the formulation. The ambiguity originates from the need to remove subvarieties of codimension three and higher from the original hyperkähler manifolds. To make things precise, we introduce in Section 2.2 the notion of a Hu–Yau transformation. Our main result, Theorem 2.5, claims that any bimeromorphic map φ as above is the composition of a finite number of Hu–Yau transformations.

Let us make a few comments on the comparison of our proof with that of [WW]. Our proof relies on [WW, Theorem 1.1] in an essential way. On the other hand, our methods give a new proof of [WW, Theorem 1.2] in the case when the varieties are compact hyperkähler. The novelty here is a different method of decomposing bimeromorphic maps into compositions

of flops. Whereas loc. cit. relies on the general arguments of the minimal model program providing the termination of log-flips, our proof uses completely different methods based on [AV1, Theorem 3.17]. The latter theorem (see also [HT0, Proposition 17] provides a locally finite wall-and-chamber structure for the positive cone of an arbitrary compact hyperkähler manifold (under mild conditions on its second Betti number), and therefore the possibility to decompose an arbitrary bimeromorphic map into a finite sequence of wall-crossing transformations, see Section 4 for details. Unlike [WW, Theorem 1.2], our proof works in the non-projective setting, therefore generalizing the main results of loc. cit. to the case of non-projective hyperkähler fourfolds.

In Section 7 we provide an example to justify our reformulation of the Hu–Yau conjecture. The example deals with the geometry of intersections of the centers of Mukai's elementary transformations in codimension two, and to the dynamical behaviour of the composition of such transformations.

2 Main results

In this section, we give the key definitions, in order to state the main result of this paper, Theorem 2.5.

2.1 Mukai flops

Given a complex $(k+1)$-dimensional vector space V, let $M_{2k} = \mathrm{Tot}(\Omega^1_{\mathbb{P}(V)})$ and $M'_{2k} = \mathrm{Tot}(\Omega^1_{\mathbb{P}(V^*)})$ be the total spaces of the cotangent bundles of $\mathbb{P}(V)$ and of its dual. It is well-known that the zero sections of the bundles M_{2k} and M'_{2k} can be contracted, and there is an isomorphism between the resulting varieties that gives rise to a birational map between M_{2k} and M'_{2k} called the Mukai flop.

Let us briefly recall some details of this construction. Starting from the Euler sequence on $\mathbb{P}(V)$:

$$0 \to \mathcal{O}_{\mathbb{P}(V)} \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to T_{\mathbb{P}(V)} \to 0,$$

we pass to its dual and use the canonical map $\mathcal{O}_{\mathbb{P}(V)}(-1) \hookrightarrow V \otimes \mathcal{O}_{\mathbb{P}(V)}$ to get the embedding of vector bundles

$$\Omega^1_{\mathbb{P}(V)} \hookrightarrow \mathrm{End}(V) \otimes \mathcal{O}_{\mathbb{P}(V)}.$$

Given a point $[u] \in \mathbb{P}(V)$ recall that $\Omega^1_{\mathbb{P}(V),[u]} \simeq \mathrm{Hom}(V/\mathbb{C}u, \mathbb{C}u)$. We see that M_{2k} can be described as the following closed subset of $\mathbb{P}(V) \times \mathrm{End}(V)$:

$$M_{2k} = \{([u], A) \in \mathbb{P}(V) \times \mathrm{End}(V) \mid \mathrm{im}(A) \subset \mathbb{C}u,\ A^2 = 0\}.$$

The projection $\pi_{2k}\colon M_{2k} \to \mathrm{End}(V)$ contracts the zero section

$$Z_{2k} = \{([u], A) \in M_{2k} \mid A = 0\}$$

to the origin. The image $N_{2k} = \pi_{2k}(M_{2k})$ is the closure of the minimal nilpotent $\mathrm{GL}(V)$-orbit in $\mathrm{End}(V)$ consisting of square-zero endomorphisms of rank at most one. The symplectic form on M_{2k} is the pullback of the Kirillov–Kostant form on N_{2k}. Analogously,

$$M'_{2k} = \{([\alpha], B) \in \mathbb{P}(V^*) \times \mathrm{End}(V^*) \mid \mathrm{im}(B) \subset \mathbb{C}\alpha,\ B^2 = 0\}.$$

There is the analogous projection $\pi'_{2k}\colon M'_{2k} \to \mathrm{End}(V^*)$ onto the closed affine cone $N'_{2k} \subset \mathrm{End}(V^*)$. Identifying $\mathrm{End}(V)$ with $\mathrm{End}(V^*)$ by taking adjoints of the endomorphisms gives an isomorphism between N_{2k} and N'_{2k}.

Definition 2.1: The morphism $\mu\colon M_{2k} \setminus Z_{2k} \to M'_{2k}$ defined by $\mu([u], A) = ([\ker(A)], A^*)$ gives a birational map

$$\mu\colon M_{2k} \dashrightarrow M'_{2k} \tag{2.1}$$

called the **Mukai flop** in codimension k. It is clear from the above description that $\mu = (\pi'_{2k})^{-1} \circ \pi_{2k}$, where $\pi_{2k}\colon M_{2k} \to N_{2k}$ and $\pi'_{2k}\colon M'_{2k} \to N'_{2k} \simeq N_{2k}$ are the contractions described above.

Given a hyperkähler fourfold M and a projective plane $S \simeq \mathbb{C}P^2 \subset M$, an open analytic neighborhood of S in M is isomorphic to a neighbourhood of the zero section Z_4 in M_4, see [WW] and references therein, hence the Mukai flop construction can be performed on any hyperkähler fourfold. In [WW], J. Wierzba and J. Wiśniewski prove that a small contraction of a quasiprojective holomorphic symplectic fourfold contracts a disjoint union of $\mathbb{P}^2$'s, and it follows that any bimeromorphic map between hyperkähler manifolds of complex dimension four can be decomposed into a composition of Mukai flops ([BHL]).

More generally, for hyperkähler manifolds of arbitrary dimension, we would like to describe the building blocks of arbitrary bimeromorphic maps. The local model of our bimeromorphic transformations will be given by taking the product of a Mukai flop with a polydisc (possibly up to codimension three or higher, see below).

More precisely, let $U_{2k} \subset N_{2k}$ be an open analytic neighborhood of the vertex $0 \in N_{2k}$, where N_{2k} is the affine cone from Definition 2.1. Using the morphisms π_{2k} and π'_{2k} from Definition 2.1, we define the local model of a bimeromorphic transformation of a d-dimensional complex manifold by the following diagram:

$$\begin{array}{ccc} \pi_{2k}^{-1}(U_{2k}) \times \Delta^{d-2k} & \xrightarrow{\quad\mu\times\mathrm{id}\quad} & (\pi'_{2k})^{-1}(U_{2k}) \times \Delta^{d-2k} \\ & \searrow \qquad \swarrow & \\ & U_{2k} \times \Delta^{d-2k} & \end{array} \tag{2.2}$$

In the above diagram, the diagonal arrows define contractions with the exceptional sets isomorphic to $\mathbb{P}^k$-bundles over the polydisc Δ^{d-2k}. We introduce the following notion, using the terminology of [HY].

Definition 2.2: Let $\varphi\colon X \dashrightarrow X'$ be a bimeromorphic map between arbitrary complex manifolds of complex dimension d. We will say that φ is **Mukai's elementary transformation** in codimension k, if there exists a normal complex-analytic variety Y, two closed non-empty complex submanifolds $E \subset X$, $E' \subset X'$ and two proper bimeromorphic morphisms $\pi\colon X \to Y$, $\pi'\colon X' \to Y$ that satisfy the following conditions.

1. $\varphi = (\pi')^{-1} \circ \pi$;
2. π and π' are contractions of E and E' onto the singular locus $Y^s \subset Y$;
3. for any point $y \in Y^s$ there exists an open neighborhood $U \subset Y$ of y isomorphic to $U_{2k} \times \Delta^{d-2k}$, and over U the map φ is of the form (2.2).

The manifolds E and E', which are $\mathbb{P}^k$-bundles over Y^s, are called the **centers** of the transformation φ and its inverse, respectively.

To summarize, Mukai's elementary transformation is the composition of a contraction of a $\mathbb{P}^k$-bundle and of the inverse of another such contraction, and transversally to the singular locus of the variety obtained by contraction the transformation is modeled on the usual Mukai flop of the total space of the cotangent bundle of $\mathbb{P}^k$ in its zero section, as in Definition 2.1.

2.2 Hu–Yau transformations and the Hu–Yau conjecture

In order to study bimeromorphic transformations of hyperkähler manifolds of arbitrary dimension, we will need to use bimeromorphic maps more

general than the Mukai's elementary transformations introduced in Definition 2.2. Following the idea of Hu and Yau [HY], we will allow arbitrary bimeromorphic transformations that generically look like Mukai flops in codimension two, at the same time ignoring what happens in codimension greater than two. We formalize this by introducing the following notion.

Definition 2.3: Let $\varphi\colon X \dashrightarrow X'$ be a bimeromorphic map between two complex manifolds. We will say that φ is a **Hu–Yau transformation** if there exist closed subvarieties $Z \subset X$ and $Z' \subset X'$ of codimension strictly greater than two, such that over the open subsets $X_0 = X \setminus Z$ and $X'_0 = X' \setminus Z'$

$$\varphi|_{X_0}\colon X_0 \dashrightarrow X'_0$$

is a Mukai elementary transformation in codimension two in the sense of Definition 2.2.

We reproduce the following conjecture made by Hu and Yau in [HY, Conjecture 7.3], stating it only for compact hyperkähler manifolds, since this will be the only case considered in the present paper. The definition of a compact hyperkähler manifold is recalled below, in Section 3.

Conjecture 2.4: Let $\varphi : M \dashrightarrow M'$ be a bimeromorphic map of compact hyperkähler manifolds. Then, after removing subvarieties of codimension greater than two, φ is a sequence of Mukai's elementary transformations.

The above conjecture predicts a description of codimension two structure of bimeromorphic maps between hyperkähler manifolds. Since we are allowed to remove subvarieties of codimension greater than two, the conjecture does not predict anything about bimeromorphic maps that are regular in codimension two. Our main result is the following statement. We use the notion of a Hu–Yau transformation introduced above in Definition 2.3.

Theorem 2.5: Let M and M' be compact hyperkähler manifolds, and $\varphi : M \dashrightarrow M'$ a bimeromorphic map that is not regular in codimension two. Then φ is a composition of a finite number of Hu–Yau transformations between compact hyperkähler manifolds.

The proof of the above theorem will be given in Section 6.

3 Hyperkähler manifolds and MBM classes

In this section we fix the notation related to hyperkähler manifolds that will be used throughout the paper.

3.1 Hyperkähler manifolds and their Teichmüller spaces

By a compact hyperkähler manifold we will mean a compact simply connected complex manifold that admits a hyperkähler metric of maximal holonomy, see [Hu]. To study bimeromorphic maps between hyperkähler manifolds it is sufficient to work with a fixed deformation class, because by [Hu, Proposition 27.8] bimeromorphic hyperkähler manifolds are deformation equivalent. Therefore we fix the underlying C^∞-manifold M. Henceforth, a hyperkähler manifold is the same thing as a complex structure of hyperkähler type (i.e. admitting a hyperkähler metric) on M. Given such a complex structure I, we will write (M, I) for the corresponding hyperkähler manifold. If the complex structure is clear from the context or irrelevant, we will simply write M instead of (M, I). The manifold M is holomorphically symplectic and irreducible, i.e. the holomorphic symplectic form on M is unique up to multiplication by a constant. We will denote by $\sigma_I \in H^0(M, \Omega^2_M)$ a symplectic form of unit volume, where the volume is defined as the integral of the form $\sigma_I^n \wedge \overline{\sigma}_I^n$. Here $2n$ is the complex dimension of M. We will also assume that $b_2(M) \geqslant 6$.

We will denote by q the Beauville-Bogomolov-Fujiki (BBF for short) quadratic form on $H^2(M, \mathbb{Q})$. Recall from [Hu] that q is of signature $(3, b_2(M)-3)$ and satisfies the Fujiki relations:

$$q(a)^n = c_M \int_M a^{2n}$$

for any $a \in H^2(M, \mathbb{Q})$, where $c_M \in \mathbb{Q}$ is a positive constant that can be chosen so that q is integral and primitive on $H^2(M, \mathbb{Z})$, see [So] for details.

Since M is simply connected, its second homology is torsion-free, hence $H_2(M, \mathbb{Z}) \simeq \mathrm{Hom}(H^2(M, \mathbb{Z}), \mathbb{Z})$. The form q defines an embedding of lattices $q\colon H^2(M, \mathbb{Z}) \hookrightarrow H_2(M, \mathbb{Z})$. This induces an embedding of $H_2(M, \mathbb{Z})$ into the vector space $H^2(X, \mathbb{Q})$, in particular q restricts to a rational quadratic form on the homology classes of degree two. In what follows we will implicitly use this extension of q to $H_2(M, \mathbb{Z})$.

The Teichmüller space of all complex structures of hyperkähler type on M will be denoted by $\mathsf{Teich}(M)$. The point corresponding to the complex

structure I will be denoted by $[I] \in \mathsf{Teich}(M)$. Recall the definition of the period domain:

$$\mathcal{D} = \{x \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(x) = 0,\, q(x, \overline{x}) > 0\}$$

and the period map

$$\rho\colon \mathsf{Teich}(M) \to \mathcal{D},$$

sending a point $[I] \in \mathsf{Teich}(X)$ to $[\sigma_I]$. According to [Hu], if (M, I_1) and (M, I_2) are bimeromorphic, then there exists a diffeomorphism $\alpha\colon M \to M$ such that the points $[I_1]$ and $[\alpha^* I_2]$ are non-separated in $\mathsf{Teich}(M)$.

3.2 MBM classes and the Kähler cone

Bimeromorphic hyperkähler manifolds correspond to non-separated points in the Teichmüller space, so for our purposes we may fix a connected component of the latter. This connected component will be denoted by $\mathsf{Teich}^\circ(M)$. The non-separated points of the Teichmüller space are distinguished by the Kähler cones of the corresponding manifolds. For $[I] \in \mathsf{Teich}(M)$ the Kähler cone of (M, I), denoted by $\mathcal{K}_I$, is the set of cohomology classes in $H^{1,1}_I(M, \mathbb{R})$ that can be represented by Kähler forms. To describe the structure of the Kähler cone we will use the notion of an MBM class which we will now recall.

Given a cohomology class $\eta \in H^2(M, \mathbb{Q})$, we define

$$\mathsf{Teich}^\circ_\eta(M) = \{[I] \in \mathsf{Teich}^\circ(M) \mid q(\rho(I), \eta) = 0\},$$

in other words, the subspace of the complex structures where η is of type $(1,1)$.

Definition 3.1: A cohomology class $\eta \in H^2(M, \mathbb{Z})$ is called **an MBM class** if it satisfies the following conditions:

1. η is primitive;
2. $q(\eta) < 0$;
3. for any $[I] \in \mathsf{Teich}^\circ_\eta(M)$ and any $\kappa \in \mathcal{K}_I$ we have $q(\kappa, \eta) \neq 0$.

The set of MBM classes on M will be denoted by $\mathrm{MBM}(M)$ (see [AV0] for various equivalent characterizations of MBM classes).

The notion of an MBM class has a deep connection with the geometry of rational curves on M. We will partially recall this below, but see [AV3, section 3] for a more detailed discussion. Recall that by [AV1, Theorem 3.17] the mapping class group of M acts on $\mathrm{MBM}(M)$ with finitely many orbits, in particular the BBF squares of the MBM classes are bounded.

Given $[I] \in \mathsf{Teich}^\circ(M)$, recall that the restriction of q to $H^{1,1}_I(M,\mathbb{R})$ has signature $(1, b_2(M)-3)$, therefore the set $\{x \in H^{1,1}_I(X,\mathbb{R}) \mid q(x) > 0\}$ has two connected components. We define the positive cone $\mathcal{C}_I$ to be the connected component that contains the Kähler cone $\mathcal{K}_I$. Given $[I'] \in \mathsf{Teich}^\circ(M)$ and a bimeromorphic map $\varphi\colon (M,I) \dashrightarrow (M,I')$, recall from [Hu] that φ^* is a well defined map on $H^2(M,\mathbb{C})$. The set $\varphi^*\mathcal{K}_{I'}$ is an open convex cone contained $\mathcal{C}_I$.

Definition 3.2: The union of the cones $\varphi^*\mathcal{K}_{I'}$ for all bimeromorphic maps φ as above is called **the birational Kähler cone** of (M,I), denoted by $\mathcal{BK}_I$.

Note that the birational Kähler cone is not convex in general. To describe its structure we introduce the wall and chamber decomposition of the positive cone. Given $\eta \in \mathrm{MBM}^{1,1}_I(X)$, denote by W_η the corresponding wall in the positive cone, i.e. the intersection of $\mathcal{C}_I$ with the q-orthogonal complement of η. Define

$$\mathcal{C}^\circ_I = \mathcal{C}_I \setminus \left(\cup_{\eta\in\mathrm{MBM}^{1,1}_I(X)} W_\eta\right).$$

Definition 3.3: In the notation introduced above, **a Kähler chamber** is a connected component of $\mathcal{C}^\circ_I$.

The following statement summarizes a description of the cones introduced above. It is a combination of results proven in [AV0], [AV1], [Ma] and [Bo].

Theorem 3.4: Assume that $b_2(M) \geqslant 5$ and let $[I] \in \mathsf{Teich}^\circ(M)$. Then we have the following:

1. The collection of walls W_η, where $\eta \in \mathrm{MBM}^{1,1}_I(X)$, is locally finite in $\mathcal{C}_I$[1] ;
2. The Kähler cone $\mathcal{K}_I$ is a connected component of $\mathcal{C}^\circ_I$;

[1] Remark that walls may accumulate towards the the border of CC_I, the isotropic cone.

3. For every Kähler chamber there exists a unique $[I'] \in \mathsf{Teich}^\circ(M)$ such that $\rho(I') = \rho(I)$ and $\mathcal{K}_{I'}$ is equal to that chamber. The manifolds (M, I) and (M, I') are bimeromorphic. This gives a bijection between the Kähler chambers and the points $[I'] \in \mathsf{Teich}(M)$ with $\rho(I) = \rho(I')$ and (M, I') bimeromorphic to (M, I);

4. The birational Kähler cone $\mathcal{BK}_I$ is the union of some of the Kähler chambers. The closure $\overline{\mathcal{BK}_I}$ is the cone dual to the pseudo-effective cone of M, in particular $\overline{\mathcal{BK}_I}$ is convex.

We close this section by outlining the connection between the MBM classes and the structure of extremal contractions of hyperkähler manifolds. For more details we refer to [AV3].

Definition 3.5: Given a class $\eta \in \mathrm{MBM}_I^{1,1}(M)$ we will say that η **is extremal**, or that η defines **a wall of the Kähler cone**, if $W_\eta \cap \overline{\mathcal{K}_I}$ contains a non-empty open subset of W_η.

Any MBM class is extremal on some deformation of the ambient complex structure; in a given one, MBM classes are extremal up to monodromy and birational transformation, see [AV0].

By a rational curve C in a hyperkähler manifold M we will mean the image of a non-constant holomorphic map from $\mathbb{P}^1$ into M. Its homology class will be denoted by $[C]$. In the following definition we implicitly use the embedding $H^2(M, \mathbb{Z}) \hookrightarrow H_2(M, \mathbb{Z})$ given by the BBF form, as was explained above.

Definition 3.6: Let $\eta \in \mathrm{MBM}_I^{1,1}(M)$ be an extremal MBM class. The corresponding **full MBM locus** E_η is the closure of the union of all rational curves C such that $[C]$ is proportional to η.

Note that the full MBM locus need not be irreducible (or even connected). We will need to talk about the codimension of the MBM loci. By this we will mean the following.

Definition 3.7: Let $\eta \in \mathrm{MBM}_I^{1,1}(M)$ be an extremal MBM class. The **codimension** of η (and of E_η) is the minimum of the codimensions of the irreducible components of E_η. We will say that η is **divisorial** if it is of codimension one, otherwise η will be called **non-divisorial**.

Lemma 3.8: Let η be a divisorial MBM class. Then E_η is an irreducible divisor.

Proof: By the definition of the codimension of η, E_η contains at least one irreducible component of codimension one, call this component D. The divisor D is uniruled, covered by deformations of a rational curve C with $[C]$ proportional to η, by the definition of E_η. It is known (see e.g. [AV2, Lemma 2.11]) that $[C]$ is proportional to $[D]$, therefore $[D]$ is proportional to η. Assume that E' is another irreducible component of E_η. Then E' contains a rational curve C' that does not lie inside of D and with $[C']$ proportional to η. Since C' is not contained in D, their intersection index is non-negative, contradicting the fact that $q(\eta) < 0$. ■

In the terminology of [Bo], the divisorial MBM loci are the prime exceptional divisors, and their cohomology classes span the extremal rays of the pseudo-effective cone of M. Therefore the divisorial MBM classes define the walls of $\overline{\mathcal{BK}_I}$, i.e. for a divisorial MBM class η the intersection $W_\eta \cap \overline{\mathcal{BK}_I}$ contains a non-empty open subset of W_η that lies on the boundary of $\overline{\mathcal{BK}_I}$. The non-divisorial extremal MBM classes define the internal walls that cut the birational Kähler cone into Kähler chambers.

We close the section by recalling the following result (see also [BL, Corollary 5.9]).

Theorem 3.9: [AV3, Theorem 4.6] Let $\eta \in \mathrm{MBM}_I^{1,1}(M)$ be an extremal MBM class on a hyperkähler manifold with $b_2(M) \geqslant 5$. Then there exists a normal symplectic variety N and a proper bimeromorphic morphism $\pi\colon M \to N$ such that the exceptional set of π is the full MBM locus E_η.

4 Wall-crossing flops

In this section we study the bimeromorphic transformations corresponding to passing through a wall W_η defined by an extremal non-divisorial MBM class η.

Consider a point $[I] \in \mathsf{Teich}^\circ(M)$ and the corresponding birational Kähler cone $\mathcal{BK}_I$.

Definition 4.1: We will say that two chambers $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{BK}_I$ are **adjacent** if the intersection $\overline{\mathcal{K}_1} \cap \overline{\mathcal{K}_2}$ contains a non-empty open subset of one of the walls W_η for some $\eta \in \mathrm{MBM}_I^{1,1}(M)$.

Definition 4.2: Let $\varphi\colon M_1 \dashrightarrow M_2$ be a bimeromorphic map between hyperkähler manifolds with Kähler cones $\mathcal{K}_1$ and $\mathcal{K}_2$. If $\mathcal{K}_1$ and $\varphi^*\mathcal{K}_2$ are adjacent, then φ is called a **wall-crossing flop**.

It will be convenient for us to work with wall-crossing flops that satisfy an additional condition, namely that φ^* acts as the identity on the second cohomology of M. It is always possible to arrange this due to the following statement.

Proposition 4.3: Assume that $\mathcal{K}' \subset \mathcal{BK}_I$ is a chamber. Then there exists a complex structure I' of hyperkähler type on M, unique up to isotopy, and a bimeromorphic map $\varphi\colon (M,I) \dashrightarrow (M,I')$ such that φ^* is the identity map on $H^2(M,\mathbb{Z})$ and $\mathcal{K}' = \mathcal{K}_{I'}$.

Proof: By the definition of $\mathcal{BK}_I$, there exists $[\widetilde{I}] \in \mathsf{Teich}^\circ(M)$ and a bimeromorphic map $\psi\colon (M,I) \dashrightarrow (M,\widetilde{I})$ such that $\mathcal{K}' = \psi^*\mathcal{K}_{\widetilde{I}}$. By [Hu, Proposition 27.8] there exist two families $\pi_1\colon \mathcal{M} \to \Delta$ and $\pi_2\colon \widetilde{\mathcal{M}} \to \Delta$ of hyperkähler manifolds over the unit disc and a bimeromorphic map $\Psi\colon \mathcal{M} \dashrightarrow \widetilde{\mathcal{M}}$ with the following properties. First, we have $\mathcal{M}_0 \simeq (M,I)$ and $\widetilde{\mathcal{M}}_0 \simeq (M,\widetilde{I})$, where $\mathcal{M}_t$ and $\widetilde{\mathcal{M}}_t$ denote the fibers over $t \in \Delta$. Second, Ψ induces a fiberwise isomorphism over $\Delta\setminus\{0\}$, and $\Psi_0\colon \mathcal{M}_0 \dashrightarrow \widetilde{\mathcal{M}}_0$ equals ψ.

Trivializing the families $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ by Ehresmann's theorem, we may identify $\mathcal{M}_{t_0}$ and $\widetilde{\mathcal{M}}_{t_0}$ with M as C^∞-manifolds, where $0 \neq t_0 \in \Delta$ is some base point. Then Ψ_{t_0} defines a diffeomorphism of M. By the second property mentioned above, the action of Ψ_{t_0} on $H^2(M,\mathbb{Z})$ coincides with the action of ψ.

Define $I' = \Psi_{t_0}^*\widetilde{I}$. Then $\Psi_{t_0}^{-1}\colon (M,\widetilde{I}) \to (M,I')$ is a biholomorphic isomorphism, hence $\mathcal{K}_{\widetilde{I}} = (\Psi_{t_0}^{-1})^*\mathcal{K}_{I'}$. Define $\varphi = \Psi_{t_0}^{-1} \circ \psi$. Then by construction φ^* acts on H^2 as the identity. The claim about uniqueness of the complex structure I' up to isotopy follows from the description of the fibers of the period map, as was recalled in Theorem 3.4. ■

Corollary 4.4: Let $\varphi\colon M_1 \dashrightarrow M_2$ be a bimeromorphic map, where $M_1 = (M,I_1)$ and $M_2 = (M,I_2)$. Then there exists a unique $[I_2'] \in \mathsf{Teich}^\circ(M)$ and an isomorphism $\xi\colon M_2 \to M_2'$, where $M_2' = (M,I_2')$, such that $\varphi' = \xi \circ \varphi\colon M_1 \dashrightarrow M_2'$ is a bimeromorphic map that acts as the identity on the second cohomology of M. If φ is a wall-crossing flop, then so is φ'.

Proof: Let $\mathcal{K}' = \varphi^*\mathcal{K}_{I_2}$. By Proposition 4.3 there exists a complex structure I_2', unique up to isotopy, and a bimeromorphic map $\varphi'\colon M_1 \dashrightarrow M_2' = (M, I_2')$ such that φ' acts as the identity on H^2 and and $\mathcal{K}' = \mathcal{K}_{I_2'}$. The composition $\varphi \circ (\varphi')^{-1}$ is a bimeromorphic map from M_2' to M_2, and it maps the Kähler cone of M_2 to the Kähler cone of M_2', hence by [Hu, Proposition 27.6] it is biholomorphic. Defining ξ to be the inverse of this map proves the first claim. If φ is a wall-crossing flop, then by Definition 4.1 $\mathcal{K}'$ is adjacent to the Kähler cone of M_1. By construction $(\varphi')^*\mathcal{K}_{I_2'} = \mathcal{K}'$, hence φ' is also a wall-crossing flop. ■

Lemma 4.5: Let $\varphi\colon M_1 \dashrightarrow M_2$ and $\varphi'\colon M_1 \dashrightarrow M_2$ be two bimeromorphic maps acting as the identity on the second cohomology. Then $\varphi' = \xi \circ \varphi$, where ξ is a biholomorphic automorphism of M_2 that acts as the identity on the second cohomology.

Proof: Define $\xi = \varphi' \circ \varphi^{-1}$ and conclude that it is a biholomorphic map by [Hu, Proposition 27.6]. ■

Now can describe the nature of wall-crossing flops, showing that they are compositions of a blow-down followed by a blow-up, similarly to Mukai's elementary transformations.

Proposition 4.6: Let $\varphi\colon M_1 \dashrightarrow M_2$ be a wall-crossing flop. If M_i are non-projective, assume additionally that $b_2(M_i) \geqslant 6$. Then there exists a compact symplectic variety N and two small contractions $\pi_1\colon M_1 \to N$ and $\pi_2\colon M_2 \to N$ such that $\varphi = \pi_2^{-1} \circ \pi_1$.

The proof of the above proposition will be divided in two parts, depending on whether M_i are projective.

4.1 The projective case

We prove Proposition 4.6 assuming that M_i are projective. Let $M_1 = (M, I_1)$ and $M_2 = (M, I_2)$. By Corollary 4.4 we may assume that the map φ acts as the identity on $H^2(M, \mathbb{Z})$. By the definition of a wall-crossing flop, there exists a wall W_z of the positive cone $\mathcal{C}_{I_1}$ such that the intersection $\overline{\mathcal{K}_{I_1}} \cap \overline{\mathcal{K}_{I_2}}$ contains a non-empty open subset of W_z. Since the manifolds M_i are projective, this open subset contains an integral element $c \in H^2(M, \mathbb{Z})$. The class c is of Hodge type $(1,1)$ both on M_1 and M_2, so there exist

$L_1 \in \mathrm{Pic}(M_1)$ and $L_2 \in \mathrm{Pic}(M_2)$ with $c_1(L_i) = c$. By construction, the class c is big and nef both on M_1 and M_2, so by a theorem of Kawamata [Kw, Theorem 6.1] the bundles L_i are semiample and define small contractions $f_1\colon M_1 \to N_1$ and $f_2\colon M_2 \to N_2$ onto projective symplectic varieties N_1 and N_2. Let $\psi\colon N_1 \dashrightarrow N_2$ be the composition $f_2 \circ \varphi \circ f_1^{-1}$. Note that $L_i \simeq f_i^*\mathcal{O}_{N_i}(1)$, where $\mathcal{O}_{N_i}(1)$ are very ample line bundles on N_i, and $\varphi^* L_2 \simeq L_1$ by construction. We see that $\psi^*\mathcal{O}_{N_2}(1) \simeq \mathcal{O}_{N_1}(1)$, therefore by the Matsusaka–Mumford criterion [Ko, Theorem 11.39] we conclude that ψ is an isomorphism. Identifying N_1 with N_2 via ψ and letting $N = N_1 = N_2$, we finish the proof.

4.2 The non-projective case

To prove Proposition 4.6 in the case when M_i are non-projective we will use ergodic theory, as in [AV3]. We first make the following observation.

Lemma 4.7: Let $\pi\colon M \to N$ be an extremal MBM contraction with the exceptional set E_η, as in Theorem 3.9, and $\xi\colon M \to M$ an automorphism that acts trivially on H^2. Then ξ descends to an automorphism of N.

Proof: Consider the contraction $\pi' = \pi \circ \xi$. Since ξ acts trivially on the second cohomology, it preserves homology classes of the rational curves that cover the full MBM locus E_η. By the definition of a full MBM locus, π' preserves E_η and contracts all the fibers of π. Consider the map $\pi \times \pi'\colon M \to N \times N$ and let Z be its image. The variety Z is finite of degree one over both factors of $N \times N$, and since N is normal, Z is isomorphic to N, therefore being the graph of an automorphism of N. ■

Fix $\eta \in \mathrm{MBM}^{1,1}_I(M)$ and recall that given a Kähler chamber $\mathcal{K} \subset \mathcal{C}^\circ_I$ we say that η defines a wall of $\mathcal{K}$ if $\overline{\mathcal{K}} \cap W_\eta$ contains a non-empty open subset of the wall W_η. Define the following open subspaces of $\mathsf{Teich}^\circ_\eta(M)$:

$$\mathsf{Teich}^+_\eta(M) = \{[I] \in \mathsf{Teich}^\circ_\eta(M) \mid \eta \text{ defines a wall of } \mathcal{K}_I \text{ and } q(\eta, \mathcal{K}_I) > 0\}$$
$$\mathsf{Teich}^-_\eta(M) = \{[I] \in \mathsf{Teich}^\circ_\eta(M) \mid \eta \text{ defines a wall of } \mathcal{K}_I \text{ and } q(\eta, \mathcal{K}_I) < 0\}$$

Note that there is a one-to-one correspondence between the points of the two spaces above: given $[I^+] \in \mathsf{Teich}^+_\eta(M)$, the intersection $\overline{\mathcal{K}}_{I^+} \cap W_\eta$ contains a non-empty open subset of W_η, so there exists a unique chamber $\mathcal{K}_{I^-}$ on the opposite side of the wall W_η, so that $\overline{\mathcal{K}}_{I^-} \cap W_\eta = \overline{\mathcal{K}}_{I^+} \cap W_\eta$ and $[I^-] \in \mathsf{Teich}^-_\eta(M)$. The two Kähler cones $\mathcal{K}_{I^\pm}$ are adjacent and by

Proposition 4.3 there exists a wall-crossing flop $\varphi\colon (M, I^+) \dashrightarrow (M, I^-)$. It is clear that all wall-crossing flops across the walls defined by η arise this way.

We have two period maps $\rho^\pm\colon \mathsf{Teich}^\pm_\eta(M) \to \mathcal{D}_\eta$, where $\mathcal{D}_\eta = \mathcal{D} \cap \mathbb{P}(\eta^\perp)$. We see that the fibers of $\rho^\pm$ over a point $[\sigma_I] \in \mathcal{D}_\eta$ are in one-to-one correspondence with the chambers of the wall $W_\eta = \mathcal{C}_I \cap \eta^\perp$. Those chambers are cut out by the orthogonal complements to the MBM classes in $\mathrm{MBM}^{1,1}_I(M)$ different from $\pm\eta$.

Denote by G the mapping class group of M and by $G_\eta \subset G$ the stabilizer of η. Recall from [AV3, Theorem 2.14] the classification of G_η-orbits in $\mathcal{D}_\eta$. It follows from the classification that given a period $[\sigma_I] \in \mathcal{D}_I$ of a non-projective hyperkähler manifold (M, I), its orbit $G_\eta \cdot [\sigma_I]$ contains a period of some projective hyperkähler manifold in its closure.

Let us finish the proof of Proposition 4.6 in the non-projective case. Let $\varphi\colon M^+ \dashrightarrow M^-$ be a wall-crossing flop that acts trivially on H^2, where $M^\pm = (M, I^\pm)$ for some $[I^\pm] \in \mathsf{Teich}^\pm_\eta(M)$. Let $\pi^\pm\colon M^\pm \to N^\pm$ be the extremal contractions that exist by Theorem 3.9. We need to show that the bimeromorphic map $N^+ \dashrightarrow N^-$ induced by φ is an isomorphism. Assuming that $I^\pm$ are non-projective, we use the G_η-orbit classification [AV3, Theorem 2.14] and [V2, Theorem 3.1] to find a sequence of diffeomorphisms γ_i such that $[\gamma_i^* I^+] \to [I_0^-]$, $i \to \infty$, for some projective complex structures $[I_0^\pm] \in \mathsf{Teich}^\pm_\eta(M)$. Clearly, also $[\gamma_i^* I^-] \to [I_0^-]$. Denote $M_0^\pm = (M, I_0^\pm)$ and let $\varphi_0\colon M_0^+ \dashrightarrow M_0^-$ be a wall-crossing flop acting trivially on H^2. By the projective case of the proposition proven above, we can decompose φ_0 into the composition of a contraction $\pi_0^+\colon M_0^+ \to N_0$ and the inverse of a contraction $\pi_0^-\colon M_0^- \to N_0$.

We refer to [BL] and references therein for the notion of a locally trivial deformation of a complex variety. Let $\nu\colon \mathcal{N} \to B$ be the universal locally trivial deformation of N_0 over a polydisc, and $\Pi^\pm\colon \mathcal{M}^\pm \to \mathcal{N}$ the corresponding deformations of the contractions $\pi_0^\pm$ that exist by [BL, Theorem 1.1]. The morphisms $\Pi^\pm$ induce a family $\Phi\colon \mathcal{M}^+ \dashrightarrow \mathcal{M}^-$, $\Phi_0 = \varphi_0$, of wall-crossing flops (each Φ_t over $\mathcal{N}_t$), where $\mu^\pm\colon \mathcal{M}^\pm \to B$ are the deformations of $M_0^\pm$ preserving the contractions. The latter induce embeddings $B \hookrightarrow \mathsf{Teich}^\pm_\eta(M)$, making B an open neighborhood of $[I_0^\pm]$ in $\mathsf{Teich}^\pm_\eta(M)$. The convergence $[\gamma_i^* I^\pm] \to [I_0^\pm]$, $i \to \infty$ implies that we can find $t \in B$ and $\gamma \in G_\eta$ such that $\mathcal{M}_t^\pm \simeq (M, \gamma^* I^\pm)$.

Now we have two wall-crossing flops Φ_t and $\gamma^{-1} \circ \varphi \circ \gamma\colon \mathcal{M}_t^+ \dashrightarrow \mathcal{M}_t^-$. By Lemma 4.5 we have $\gamma^{-1} \circ \varphi \circ \gamma = \xi \circ \Phi_t$ for some automorphism ξ of $\mathcal{M}_t^-$ acting trivially on H^2. By construction, the flop Φ_t induces an isomorphism

between the bases $\mathcal{N}_t^\pm$ of the extremal contractions $\Pi_t^\pm\colon \mathcal{M}_t^\pm \to \mathcal{N}_t^\pm$. By Lemma 4.7 the map ξ induces an automorphism of $\mathcal{N}_t^-$. Therefore the flop $\gamma^{-1}\circ\varphi\circ\gamma$ induces an isomorphism between $\mathcal{N}_t^\pm$, and the same is true about φ, since $\gamma\colon \mathcal{M}_t^\pm \to (M, I^\pm)$ is an isomorphism. This completes the proof of Proposition 4.6.

4.3 Decomposition

We now show how to decompose an arbitrary bimeromorphic map into a composition of wall-crossing flops.

Proposition 4.8: Assume that $\varphi\colon M \dashrightarrow M'$ is a bimeromorphic map between compact hyperkähler manifolds with $b_2 \geqslant 5$. Then there exists a finite sequence of wall-crossing flops $\varphi_i\colon M_i \dashrightarrow M_{i+1}$, $i = 0, \dots, m-1$ such that $M_0 = M$, $M_m = M'$ and $\varphi = \varphi_m \circ \dots \circ \varphi_0$,

Proof: By Proposition 4.3 we may assume that $\varphi\colon (M, I) \dashrightarrow (M, I')$ acts as the identity on H^2 and $\mathcal{K}_I$, $\mathcal{K}_{I'}$ are two chambers of the birational Kähler cone $\mathcal{BK}_I$. Choose two points $a \in \mathcal{K}_I$, $b \in \mathcal{K}_{I'}$ and connect them by a straight segment $\ell \subset H^{1,1}_I(M, \mathbb{R})$. By [AV1, Theorem 3.17] and [SSV, Proposition 5.14] the collection of walls W_z, $z \in \mathrm{MBM}^{1,1}_I(M)$ is locally finite in $\mathcal{C}_I$, and since the segment ℓ is compact, it intersects only a finite number of walls $W_{z_1}, \dots, W_{z_m}$. Moreover, since the chambers $\mathcal{K}_I$ and $\mathcal{K}_{I'}$ are open, we may, by perturbing the endpoints a and b, make sure that all points of intersection $W_i \cap \ell$ are distinct. In this case, whenever the segment ℓ crosses a wall W_{z_i}, it passes to an adjacent chamber. We therefore get a sequence of adjacent chambers $\mathcal{K}_0 = \mathcal{K}_I, \mathcal{K}_1, \dots, \mathcal{K}_m = \mathcal{K}_{I'}$. For every chamber $\mathcal{K}_j$ there exists a complex structure I_j such that $\mathcal{K}_{I_j} = \mathcal{K}_j$ and a wall-crossing flop $\varphi_j\colon (M, I_j) \dashrightarrow (M, I_{j+1})$ acting as the identity on H^2. Consider the composition $\alpha = \varphi^{-1} \circ \varphi_m \circ \dots \circ \varphi_0$, so that $\alpha\colon (M, I) \dashrightarrow (M, I)$. By construction, the map α acts as the identity on H^2, hence it preserves the Kähler cone $\mathcal{K}_I$, therefore by [Hu, Proposition 27.6] the map α is a regular automorphism. Replacing φ_0 with $\varphi_0 \circ \alpha^{-1}$, we finish the proof. ■

4.4 Wall-crossing flops in codimension two

We apply the main theorem of [WW] to establish the structure of a wall-crossing flop in codimension two. The main step is to reduce the question to the case of four-dimensional symplectic manifolds. We do this by taking a slice transversal to the indeterminacy locus of the flop.

Assume that M is a holomorphically symplectic manifold and $\pi\colon M \to N$ a proper bimeromorphic map onto a normal variety N. We recall that according to [Kl] the variety N is symplectic and admits a unique stratification $N = \coprod N_i$, where N_i are locally closed non-singular symplectic subvarieties of N. We will call it **the Kaledin stratification**. Recall also that the symplectic forms on M and N induce Poisson brackets on their structure sheaves, so that π is a morphism of Poisson varieties.

The following proposition essentially follows from [Kl, Theorem 2.3]. In loc. cit., both the slice of N transversal to a stratum of the Kaledin stratification and a local product decomposition are constructed on the level of formal germs. Our proposition contains two minor modifications: first, our slices and the decomposition are constructed locally in the analytic topology; second, we construct them in the presence of a crepant resolution, and show that the resolution also admits a local product decomposition. The proof is the same as in loc. cit., we give it here for the sake of completeness.

Proposition 4.9: Let M be a holomorphically symplectic manifold and $\pi\colon M \to N$ a proper small contraction onto a normal symplectic variety. Let $Z \subset N$ be a stratum of the Kaledin stratification. Then for any point $z \in Z$ there exists an open subset $U \subset N$ containing z and a closed symplectic subvariety $N' \subset U$ called **the Kaledin slice** such that:

1. $z \in N'$ and $\dim(N') = \operatorname{codim}_N(Z)$;
2. $M' = \pi^{-1}(N')$ is a non-singular symplectic variety and π induces a small contraction of M' onto N';
3. The subset U and its preimage in M admit product decompositions as symplectic varieties:

$$U \simeq N' \times \Delta^{2d}, \quad \pi^{-1}(U) \simeq M' \times \Delta^{2d},$$

where $2d$ is the dimension of Z and the polydisc Δ^{2d} is endowed with the standard symplectic structure.

Proof: Let $E \subset M$ be the exceptional locus of π. Since π is a small contraction, all irreducible components of E are of codimension at least two in M. We will prove the proposition by induction on d. The case $d = 0$ being trivial, we assume that $d \geqslant 1$.

We let U be a small analytic neighborhood of z in N that we may shrink when necessary. Given a holomorphic function $f \in \mathcal{O}(U)$, we will denote

by H_f its Hamiltonian vector field, i.e. the vector field corresponding to the derivation $g \mapsto \{f, g\}$, where $\{\cdot, \cdot\}$ is the Poisson structure on U coming from the symplectic form. We will also let $\widetilde{f}$ be the pullback of f to $\pi^{-1}(U)$ and $H_{\widetilde{f}}$ its Hamiltonian vector field. Observe that since π is a morphism of Poisson varieties, we have $d\pi(H_{\widetilde{f}}) = H_f$.

Given a tangent vector $v \in T_z Z$, we claim that there exists a holomorphic function $f \in \mathcal{O}(U)$ such that $H_f(z) = v$. Indeed, Z is a smooth manifold with a symplectic form σ induced by the Poisson structure on N. We can therefore find a function $g \in \mathcal{O}(Z \cap U)$ such that $dg_z = v \lrcorner \sigma$. The function g is a restriction of some $f \in \mathcal{O}(U)$, and since Z is a Poisson submanifold of N, we have by construction $H_f(z) = v$.

Given a function f as above, denote by N_f its zero locus, and by M_f the preimage of N_f in M, i.e. the zero locus of $\widetilde{f} = f \circ \pi$. Observe that after shrinking U we may assume that H_f is nowhere vanishing, and hence the same is true of $H_{\widetilde{f}}$. It follows that M_f is a non-singular subvariety of $\pi^{-1}(U)$, indeed, df and $d\widetilde{f}$ are non-zero by duality. Given a point $x \in E \cap \pi^{-1}(U)$, we have $d\pi(H_{\widetilde{f}}(x)) = H_f(\pi(x))$ and since H_f is tangent to the strata of the Kaledin stratification, the vector $H_{\widetilde{f}}(x)$ is tangent to E. Moreover, $H_{\widetilde{f}}(x)$ is not contained in the kernel of $d\pi$, therefore $E_f = E \cap M_f$ is of codimension one in E.

We now choose two linearly independent vectors $v_1, v_2 \in T_z Z$ with $\sigma(v_1, v_2) = 1$, and the corresponding functions f_1 and f_2 as above. Shrinking U we may assume that the function $\{f_1, f_2\}$ is nowhere vanishing in U, so that H_{f_1} and H_{f_2} are linearly independent in U. Then the hypersurfaces M_{f_1} and M_{f_2} intersect transversally and their intersection M'' is a submanifold of $\pi^{-1}(U)$. The tangent bundle of M'' is the symplectic orthogonal complement to the span of $H_{f_1}|_{M''}$ and $H_{f_2}|_{M''}$, and since $\{f_1, f_2\}$ is nowhere vanishing, M'' is a symplectic submanifold of $\pi^{-1}(U)$. Observe that by the analogous argument $E'' = E_{f_1} \cap E_{f_2}$ is of codimension two in $E \cap \pi^{-1}(U)$, and $\pi|_{M''}$ is a small contraction onto $N'' = N_{f_1} \cap N_{f_2}$.

We may apply the induction hypothesis and obtain the product decompositions $N'' = N' \times \Delta^{2d-2}$ and $M'' = M' \times \Delta^{2d-2}$. To complete the proof, observe that $\widetilde{F} = (\widetilde{f_1}, \widetilde{f_2}) \colon \pi^{-1}(U) \to \mathbb{C}^2$ defines a submersion onto a neighborhood of the origin in $\mathbb{C}^2$. The fiber over the origin is M'', and the submersion factors through $F = (f_1, f_2) \colon U \to \mathbb{C}^2$. We may replace U by the preimage of a small bidisc in $\mathbb{C}^2$ centered at the origin. The vector fields H_{f_1} and H_{f_2} give a splitting of the exact sequence defining the relative tangent bundle of the submersion $\widetilde{F}$. So, we have $\pi^{-1}(U) \simeq M'' \times \Delta^2$ as soon as the vector fields $H_{\widetilde{f}_1}$ and $H_{\widetilde{f}_2}$ commute, that is, generate a foliation.

To achieve this we need to modify the function f_2 without changing its zero locus.

We will replace f_2 with another function f_2' such that $\{f_1, f_2'\} = 1$ in U and $N_{f_2} = N_{f_2'}$. We do it as follows. On $\pi^{-1}(U)$, the vector field $H_{\widetilde{f}_1}$ is transversal to the hypersurface M_{f_2}, since by construction $\{f_1, f_2\}$ is nowhere vanishing. Therefore there exists a unique function $\widetilde{f}_2'$ in a neighborhood of M_{f_2} such that $\widetilde{f}_2'$ vanishes on M_{f_2} and $\{\widetilde{f}_1, \widetilde{f}_2'\} = \mathcal{L}_{H_{\widetilde{f}_1}}(\widetilde{f}_2') = 1$ (one can see this by using the local canonical form of a vector field in a neighborhood of its regular point). After possibly shrinking U, the function $\widetilde{f}_2'$ is the pullback of some $f_2' \in \mathcal{O}(U)$, because π is an isomorphism in codimension one and N is a normal variety. Now we have $[H_{f_1}, H_{f_2'}] = H_{\{f_1, f_2'\}} = 0$, so the vector fields $H_{\widetilde{f}_1}$ and $H_{\widetilde{f}_2'}$ commute and define a regular foliation transversal to M''. The leaves of this foliation give the necessary splitting. ■

We next describe the structure of wall-crossing flops in codimension two, showing that they are Mukai's elementary transformations in the sense of Definition 2.2.

Proposition 4.10: Let $\varphi\colon M_1 \dashrightarrow M_2$ be a wall-crossing flop between hyperkähler manifolds. If M_i are non-projective, assume additionally that $b_2(M_i) \geqslant 6$. Then there exist two closed subvarieties $Z_1 \subset M_1$ and $Z_2 \subset M_2$ of codimension strictly greater than two, such that $\varphi\colon M_1 \setminus Z_1 \dashrightarrow M_2 \setminus Z_2$ is Mukai's elementary transformation in codimension two (or an isomorphism).

Proof: We deduce from Proposition 4.6 that $\varphi = (\pi_2)^{-1} \circ \pi_1$ where $\pi_i\colon M_i \to N$ are small contractions onto a symplectic variety N. Let $E_i \subset M_i$ be the exceptional loci of the contractions. Since the MBM class defining the flop φ is non-divisorial, all irreducible components of E_i are of codimension strictly greater than one, and the singular locus of N is of codimension strictly greater than three. We are interested in the structure of the flop over the codimension four strata of the Kaledin stratification.

Let Z be the union of the strata of codimension six and higher (recall that the strata of the Kaledin stratification are even-dimensional, being symplectic manifolds). Let $Z_i = \pi_i^{-1}(Z)$. The contractions π_i being semismall, the subvarieties Z_i are of codimension at least three in M_i. Let $N^\circ = N \setminus Z$ and $M_i^\circ = M_i \setminus Z_i$.

Note that the singular locus of N° is either empty or the union of strata of codimension four in the Kaledin stratification. If it is empty, by semis-

mallness our flop is an isomorphism in codimension two. Otherwise, given a singular point $z \in N^\circ$ we apply Proposition 4.9 to construct the Kaledin slice N' passing through the point z. The slice N' is four-dimensional and z is its isolated singular point. It has two crepant resolutions $M_1' = \pi_1^{-1}(N')$ and $M_2' = \pi_2^{-1}(N')$. We now have to show that the preimages of z in M_i' are isomorphic to $\mathbb{P}^2$, since in that case the product decomposition from Proposition 4.9 implies that $\varphi|_{M_1^\circ}$ is Mukai's elementary transformation in codimension two.

We apply [WW, Theorem 1.1] to the small contractions $\pi_i\colon M_i' \to N'$ to conclude the proof. Note that in loc. cit. the result is stated for quasi-projective varieties (however, see the remark following the statement of Theorem 1.1. in loc. cit.). To reduce to the quasi-projective case, we use deformation invariance of the contraction centers [AV3, Theorem 1.8] and deform M_i preserving the wall that defines φ to make M_i projective. Next we note that in the proof of Proposition 4.9 the stratum of the Kaledin stratification is a quasi-projective manifold. In the induction step of the proof, to construct the slice N', we may choose the functions f_1 and f_2 to be algebraic, defined on a Zariski-open subset of N. The slice N' and its preimages M_i' constructed this way will also be quasi-projective, and we may apply [WW, Theorem 1.1] to them, concluding the proof. ■

The following statement follows directly from the above proposition and Definition 2.3.

Corollary 4.11: Any wall-crossing flop between hyperkähler manifolds (assuming $b_2 \geqslant 6$ in the non-projective case) that is not regular in codimension two is a Hu–Yau transformation.

5 The symplectic rank and bimeromorphic maps

5.1 Definition of the symplectic rank

Let M be a holomorphic symplectic manifold with symplectic form σ and $Z \subset M$ an irreducible subvariety. If $z \in Z$ is a smooth point, then the rank of the two-form $\sigma|_{T_{Z,z}}$ is an even integer. This integer is constant on a dense Zariski-open subset of Z, i.e. for a general point of Z.

Definition 5.1. *For an irreducible subvariety $Z \subset M$ its symplectic rank* $\mathrm{rk}_\sigma(Z)$ *is the rank of the form $\sigma|_{T_{Z,z}}$ for a general smooth point $z \in Z$.*

For a subvariety $Z = \cup_i Z_i$ with irreducible components $Z_i \subset M$ define $\mathrm{rk}_\sigma(Z) = \max_i \mathrm{rk}_\sigma(Z_i)$.

We record a few useful properties of the symplectic rank.

Proposition 5.1: Let M be a holomorphic symplectic manifold with symplectic form σ and $Z \subset M$ an irreducible subvariety.

1. Z is isotropic if and only if $\mathrm{rk}_\sigma(Z) = 0$;
2. We always have $\mathrm{rk}_\sigma(Z) \geqslant \dim(Z) - \mathrm{codim}(Z)$ and the equality holds if and only if Z is coisotropic;
3. If $Z' \subset Z$ is another irreducible subvariety, then $\mathrm{rk}_\sigma(Z') \leqslant \mathrm{rk}_\sigma(Z)$;
4. If W is an irreducible variety and $f\colon W \to Z$ a surjective morphism, then $\mathrm{rk}_\sigma(Z)$ is the rank of $f^*\sigma$ at a general smooth point of W.

Proof: *(1), (2) and (3):* Obvious from the analogous statements about a subspace in a symplectic vector space.

(4): Resolving the singularities of W we may assume that W is smooth. Then the morphism f is generically smooth over the smooth locus of Z. We may find a point $w \in W$ such that $df_w\colon T_{W,w} \to T_{Z,f(w)}$ is surjective and $\sigma|_{T_{Z,f(w)}}$ is of rank $\mathrm{rk}_\sigma(Z)$. Then $f^*\sigma|_{T_{W,w}}$ is of the same rank as $\sigma|_{T_{Z,f(w)}}$. The points w with this property form a non-empty open subset of W, hence the claim. ■

We next observe that the symplectic rank is invariant under bimeromorphic transformations.

Proposition 5.2: Let $\varphi\colon M \dashrightarrow M'$ be a bimeromorphic map between hyperkähler manifolds with symplectic forms σ and σ', $Z \subset M$ a closed subvariety and $Z' = \varphi(Z)$ its full transform under φ. Then $\mathrm{rk}_\sigma(Z) = \mathrm{rk}_{\sigma'}(Z')$.

Proof: Resolve the indeterminacies of φ by a bimeromorphic morphism $\pi_1\colon \widetilde{M} \to M$, so that $\varphi = \pi_2 \circ \pi_1^{-1}$ for some bimeromorphic morphism $\pi_2\colon \widetilde{M} \to M'$. Since the space of holomorphic two-forms on a hyperkähler manifold is one-dimensional, the forms $\pi_1^*\sigma$ and $\pi_2^*\sigma'$ are proportional, we may assume that they are equal, and denote this two-form by $\widetilde{\sigma}$.

Let $\widetilde{Z} = \pi_1^{-1}(Z)$, then $Z' = \pi_2(\widetilde{Z})$. Every irreducible component of Z is dominated by an irreducible component of $\widetilde{Z}$, so by part (4) of Proposition

5.1, we have $\mathrm{rk}_\sigma(Z) \leqslant \mathrm{rk}_{\widetilde{\sigma}}(\widetilde{Z})$. The inverse inequality follows from parts (3) and (4) of Proposition 5.1, because every irreducible component of $\widetilde{Z}$ is mapped into some irreducible component of Z. By an analogous argument applied to Z' and $\widetilde{Z}$, we have $\mathrm{rk}_{\sigma'}(Z') = \mathrm{rk}_{\widetilde{\sigma}}(\widetilde{Z})$ and this completes the proof. ■

5.2 The symplectic rank of MBM loci and flops

We use the notion of symplectic rank to study compositions of wall-crossing flops. This is the key technical step in the proof of the Hu–Yau conjecture. We start from the following observation.

Proposition 5.3: Let η be an extremal MBM class on a hyperkähler manifold M and E_η its full MBM locus. Then $\mathrm{rk}_\sigma(E_\eta) = \dim(M) - 2\,\mathrm{codim}(E_\eta)$.

Proof: By [AV2, Proposition 2.2], for every irreducible component E' of E_η the codimension of E' equals the dimension of the kernel of $\sigma|_{E'}$ at a general point of E'. It follows that $\mathrm{rk}_\sigma(E') = \dim(E') - \mathrm{codim}(E') = \dim(M) - 2\,\mathrm{codim}(E')$, which implies the claim, since the symplectic rank of E_η is defined by taking the maximum over all irreducible components, and the codimension is defined by taking the minimum over all irreducible components. ■

The following proposition is a strengthening of Proposition 4.10.

Proposition 5.4: Let $\varphi\colon M_1 \dashrightarrow M_2$ be a wall-crossing flop between hyperkähler manifolds of dimension $2n$. If M_i are non-projective, assume additionally that $b_2(M_i) \geqslant 6$. Let $W_1 \subset M_1$ and $W_2 \subset M_2$ be two closed subvarieties with $\mathrm{rk}_{\sigma_i}(W_i) \leqslant 2n-6$. Then there exist two closed subvarieties $Z_1 \subset M_1$ and $Z_2 \subset M_2$ of codimension strictly greater than two with $W_i \subset Z_i$, $\mathrm{rk}_{\sigma_i}(Z_i) \leqslant 2n-6$ and such that $\varphi\colon M_1 \setminus Z_1 \dashrightarrow M_2 \setminus Z_2$ is Mukai's elementary transformation in codimension two. In particular, this applies when W_i are MBM loci of codimension strictly greater than two, or the full transforms of such loci under some bimeromorphic maps.

Proof: First note that by part (2) of Proposition 5.1 we have $2\,\mathrm{codim}(W_i) \geqslant 2n - \mathrm{rk}_{\sigma_i}(W_i)$, and by our assumptions this implies $\mathrm{codim}(W_i) \geqslant 3$.

Repeating the proof of Proposition 4.10 we decompose φ into the composition of a contraction $\pi_1\colon M_1 \to N$ and the inverse of another contraction $\pi_2\colon M_2 \to N$. We let $W = \pi_1(W_1) \cup \pi_2(W_2)$ and $Z \subset N$ be the union of W

and all strata of the Kaledin stratification of codimension six and higher. Define $Z_i = \pi_i^{-1}(Z)$. Note that since the symplectic rank of W is at most $2n-6$, the variety W can not contain any stratum of codimension four in the Kaledin stratification. Therefore the varieties Z_1 and Z_2 do not dominate any stratum of codimension four, so we have $\mathrm{rk}_\sigma(Z_i) \leqslant 2n-6$. The latter implies that $\mathrm{codim}(Z_i) \geqslant 3$, as was explained above, and we conclude the proof as in Proposition 4.10. ■

We are now ready to prove that under certain conditions the compositions of wall-crossing flops are Hu–Yau transformations. This will give us a basic step in the proof of the Hu–Yau conjecture.

Proposition 5.5: Consider a sequence of wall-crossing flops between $2n$-dimensional hyperkähler manifolds

$$M_1 \overset{\varphi_1}{\dashrightarrow} M_2 \overset{\varphi_2}{\dashrightarrow} \dots \overset{\varphi_k}{\dashrightarrow} M_{k+1},$$

such that exactly one of the MBM loci defining the flops has codimension two. Then the composition $\varphi = \varphi_k \circ \dots \circ \varphi_1$ is a Hu–Yau transformation.

Proof: Assume that φ_{i_0} is a flop in codimension two. Let $W_{i_0+1} \subset M_{i_0+1}$ be the union of the full transforms of E_j for $j > i_0$ and let $W_{i_0} \subset M_{i_0}$ be the union of the full transforms of E_j for $j < i_0$. Then by Proposition 5.2 the symplectic rank of W_{i_0} and W_{i_0+1} is a most $2n-6$. Applying Proposition 5.4 to the flop $\varphi_{i_0}\colon M_{i_0} \dashrightarrow M_{i_0+1}$ and the two subvarieties W_{i_0} and W_{i_0+1}, we get two subvarieties Z_{i_0} and Z_{i_0+1} of symplectic rank at most $2n-6$ such that $\varphi_{i_0}\colon M_{i_0} \setminus Z_{i_0} \dashrightarrow M_{i_0+1} \setminus Z_{i_0+1}$ is Mukai's elementary transformation in codimension two, and we have the inclusions $W_{i_0} \subset Z_{i_0}$ and $W_{i_0+1} \subset Z_{i_0+1}$. We let $Z_1 \subset M_1$ be the full transform of Z_{i_0} and $Z_{k+1} \subset M_{k+1}$ be the full transform of Z_{i_0+1}. By Proposition 5.2 the symplectic ranks of Z_1 and Z_{k+1} are still a most $2n-6$, hence they are of codimension at least three. The composition $\varphi_{i_0-1} \circ \dots \circ \varphi_1\colon M_1 \setminus Z_1 \to M_{i_0} \setminus Z_{i_0}$ is a biholomorphic morphism, because by construction Z_1 contains the indeterminacy locus of that map. Analogously, the composition $\varphi_k \circ \dots \circ \varphi_{i_0+1}\colon M_{i_0+1} \setminus Z_{i_0+1} \to M_{k+1} \setminus Z_{k+1}$ is a biholomorphic morphism. We therefore conclude that $\varphi\colon M_1 \setminus Z_1 \dashrightarrow M_{k+1} \setminus Z_{k+1}$ is a composition of two biholomorphic morphisms and Mukai's elementary transformation in codimension two. Therefore φ is a Hu–Yau transformation. ■

6 Proof of the main theorem

We use the results of the previous section to prove Theorem 2.5. Given a bimeromorphic map $\varphi\colon M \dashrightarrow M'$ between compact hyperkähler manifolds, we apply Proposition 4.8 and decompose φ into a finite sequence of wall-crossing flops $\varphi_i\colon M_i \dashrightarrow M_{i+1}$, $i = 1, \dots, k$, such that $M_1 = M$, $M_{k+1} = M'$, so that $\varphi = \varphi_k \circ \dots \circ \varphi_1$. We split the sequence of flops $\varphi_1, \varphi_2, \dots, \varphi_k$ into groups of consecutive flops, each group containing exactly one flop in codimension two (all the flops in the sequence can not be in codimension greater than two by the assumptions of the theorem). Applying Proposition 5.5 to each of the groups of the flops, we see that the composition of the flops forming the group is a Hu–Yau transformation. Hence the claim of the theorem.

7 Compositions of flops in codimension two

The key step of the proof of Theorem 2.5 is Proposition 5.4 that allows us to show that a wall-crossing flop in codimension two is a Mukai elementary transformation on an open subset with complement of codimension three or higher, even if we require that the complement should contain a given subvariety (under some suitable conditions on the latter). The purpose of this section is to show that it is in general not possible to achieve this for a composition of several flops, i.e. there need not exist an open subset with complement of codimension higher than two where such map is a composition of successive Mukai's elementary transformations in codimension two. We will show this on an example.

To construct the example, we start from the K3 surface S that is a quartic hypersurface in $\mathbb{P}^3$ containing a line C, and generic with this condition. The pencil of planes passing through C defines an elliptic fibration $f\colon S \to \mathbb{P}^1$ whose general fibre will be denoted by F. We take $M = S^{[3]}$.

Let us recall the description of the MBM classes on M following [AV4], and the explicit description of the MBM loci that can be found in [HT1], [HT3] and [AV4].

Let us denote by Λ the K3 lattice and let $\mathrm{Pic}(S) \subset \Lambda$ be the Picard lattice of S. The latter is spanned by the classes of the curves C and F and has the intersection matrix

$$\mathrm{Pic}(S) = \begin{pmatrix} -2 & 3 \\ 3 & 0 \end{pmatrix}$$

Let Λ' be the lattice $H^2(M, \mathbb{Z})$ with the BBF form. Then $\Lambda' \simeq \Lambda \oplus \mathbb{Z}\delta$,

where the direct sum is orthogonal and $\delta = [\Delta]/2$ with Δ the exceptional divisor of the contraction $S^{[3]} \to S^{(3)}$. We have $\delta^2 = -4$. Consequently, we also have $\mathrm{Pic}(M) \simeq \mathrm{Pic}(S) \oplus \mathbb{Z}\delta$. The lattice Λ is unimodular and the discriminant group of the lattice Λ' is clearly isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

According to [AV4] the MBM classes on M form five orbits under the action of the diffeomorphism group of M: three orbits consisting of divisorial classes, one orbit of codimension two classes and one orbit of codimension three classes. The orbits are distinguished by the square of the class $x \in \Lambda'$ and by the image of $x/d(x)$ in the discriminant group, where $d(x)$ is the divisibility of x as an element of the dual lattice $(\Lambda')^\vee$. More precisely, x_1 and x_2 are in the same orbit if and only if $x_1^2 = x_2^2$ and $x_1/d(x_1) \equiv \pm x_2/d(x_2) \,(\mathrm{mod}\ \Lambda')$.

The representatives of the orbits in our situation are the following:

1. The divisorial class δ; we have $\delta^2 = -4$ and $\delta/d(\delta) = \delta/4$ is the class of the ruling on the divisor $\Delta = E_\delta$;

2. The divisorial class $\alpha = [C]$; we have $\alpha^2 = -2$ and $\delta(\alpha) = 1$. The divisor E_α consists of subschemes that intersect C;

3. The divisorial class $\varepsilon = 2[F] - \delta$; we have $\varepsilon/d(\varepsilon) = [F] - \delta/2$ is the class of the ruling on the divisor E_ε. The latter divisor is the closure of the image of the natural map $\mathrm{Hilb}^2(S/\mathbb{P}^1) \times S \dashrightarrow S^{[3]}$, where $\mathrm{Hilb}^2(S/\mathbb{P}^1)$ is the relative Hilbert scheme of the elliptic fibration $f\colon S \to \mathbb{P}^1$;

4. Let us describe three classes of codimension two.

   (a) The class $\beta = 4[C] - \delta$ with $\beta^2 = -36$ and $d(\beta) = 4$. The MBM locus E_β is the closure of the image of the natural map $C^{(2)} \times S \dashrightarrow S^{[3]}$;

   (b) The class $\eta = 4[F] - 3\delta$ with $\eta^2 = -36$ and $d(\eta) = 4$. The MBM locus E_η is the image of the natural embedding $\mathrm{Hilb}^3(S/\mathbb{P}^1) \hookrightarrow S^{[3]}$;

   (c) The class $\zeta = 4[C] + 4[F] - 5\delta$ with $\zeta^2 = -36$ and $d(\zeta) = 4$. The MBM locus E_ζ consists of subschemes of S that lie on a line in $\mathbb{P}^3$; see Remark 7.1 below;

5. The codimension three class $\gamma = 2[C] - \delta$; we have $\gamma^2 = -12$ and $\delta(\gamma) = 2$. The MBM locus E_γ is the set of subschemes supported on C, isomorphic to $C^{(3)} \simeq \mathbb{P}^3$.

Remark 7.1. Descriptions of all MBM classes and the corresponding rational curves from the above list have already appeared in the literature, see [AV4, section 4] and references therein. One exception is, possibly, the class ζ. Let us indicate how one can find the expression of this class given above. A rational curve in the MBM locus E_ζ can be described as follows. Let us fix a line $L_0 \subset \mathbb{P}^3$ such that $L_0 \cap S = \{p, q, r, s\}$, where $p \in C$ and $\{q, r, s\}$ are three other distinct point on S. Let $P_0 \subset \mathbb{P}^3$ be a plane passing through the line L_0 and not containing C. Consider all lines $L \subset P_0$ passing through the point q. For a generic such line the intersection $S \cap L$ consists of the point q and three other points on S, the latter defining a point of $S^{[3]}$. This gives a rational curve C_ζ in $S^{[3]}$. To find its cohomology class, we can compute its intersection indices with the divisor classes $[C]$, $[F]$ and ε. We have chosen the line L_0 in a way to make sure that the points of intersection are smooth points of the divisors and the intersections are transversal. The divisor with the class $[C]$ is E_α, it consists of all subschemes of S intersecting the curve C. By construction, there is only one such subscheme $\{p, r, s\}$ on the curve C_ζ, hence $C_\zeta \cdot [C] = 1$. Analogously, to compute $C_\zeta \cdot [F]$, consider a generic plane $P_1 \subset \mathbb{P}^3$ containing C. Let F_1 be the irreducible component of $P_1 \cap S$ of degree three. The class $[F]$ on $S^{[3]}$ is represented by the divisor consisting of subschemes intersecting F_1. Note that $F_1 \cap P_0$ are three points on a line in P_0. For a generic choice of P_1 the latter line is different from L_0, and therefore C_ζ has three distinct points of intersection with the above divisor, therefore $C_\zeta \cdot [F] = 3$. To compute $C_\zeta \cdot \varepsilon$, note that the unique point of intersection of C_ζ and E_ε is the subscheme $\{p, r, s\}$ corresponding to the line L_0, hence $C_\zeta \cdot \varepsilon = 1$. Using these intersection indices, we find that $[C_\zeta] = [C] + [F] - \frac{5}{4}\delta$, hence the expression for the MBM class ζ given in the list above.

In the above listing we have slightly abused the notation for the MBM loci: not all of the MBM classes on the list are extremal, so not all of the loci described above are contractible on $S^{[3]}$. In order to make them contractible one may need to pass to a different birational model. The birational ample cone of $S^{[3]}$ (i.e. the intersection of the birational Kähler cone with the subspace spanned by the Picard lattice) and its wall-and-chamber structure is depicted in Figure 1.

In Figure 1 we have the projectivisation of the positive cone, which is the hyperbolic plane in our case, and the walls described above. We will consider the Kähler chambers numbered 1, 2, 3 and 4, and the corresponding birational models M_i for $i = 1, 2, 3, 4$. The model M_3 is the Hilbert scheme $S^{[3]}$. The walls are colored depending on the image of $x/d(x)$ in

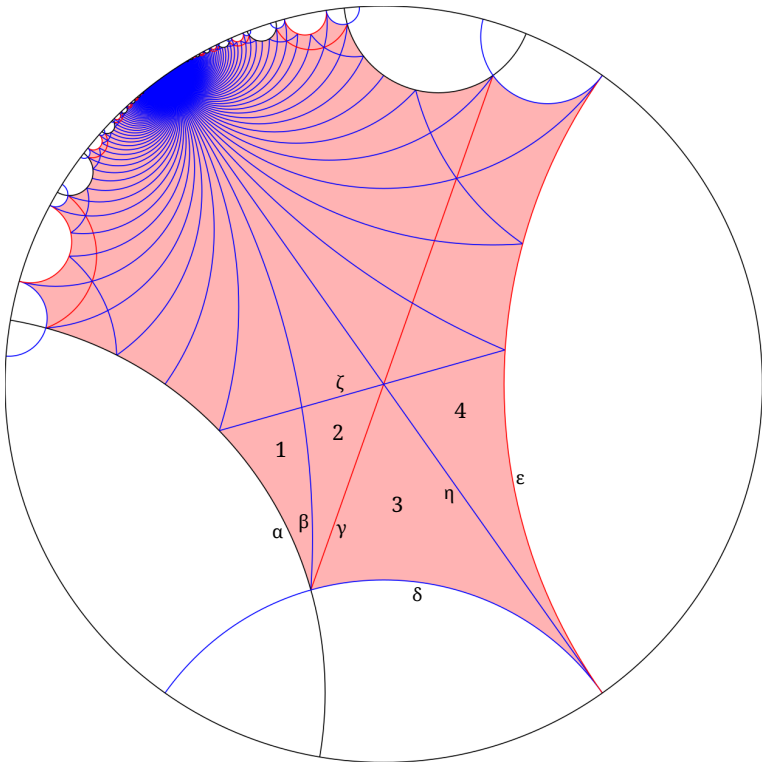


Figure 1: The birational ample cone of $S^{[3]}$

the discriminant group for an MBM class x: the black walls correspond to $0 \pmod 4$, the blue walls to $\pm 1 \pmod 4$ and the red walls to $2 \pmod 4$.

The manifold $S^{[3]}$ admits a Lagrangian fibration induced by the elliptic fibration $f\colon S \to \mathbb{P}^1$, the nef isotropic class defining the fibration being equal to $[F]$ (the lower right cusp in Figure 1). It also admits a rational Lagrangian fibration defined by the isotropic class $[C]+[F]-\delta$ (the point of concentration of the walls in the upper left part of Figure 1). This fibration is due to the fact that $S^{[3]}$ is birational to the Beauville–Mukai system with the Mukai vector $(0,[C]+[F],1)$; it can be described explicitly as the map $S^{[3]} \dashrightarrow \mathbb{P}^3$ sending a triple of points on S to the plane in $\mathbb{P}^3$ spanned by these points. One can see that the indeterminacy locus of this map is the MBM locus E_ζ, and one passes to a model where the fibration becomes regular by making two flops: in $E_\gamma \simeq \mathbb{P}^3$ (which is the singular locus of E_ζ) and then in E_ζ. The Beauville–Mukai system in our example admits an infinite group of birational automorphisms, hence the infinite family of

walls passing through the isotropic class $[C] + [F] - \delta$.

We will consider the following sequence of flops:

$$M_1 \overset{\varphi_\beta}{\dashrightarrow} M_2 \overset{\varphi_\gamma}{\dashrightarrow} M_3 \overset{\varphi_\eta}{\dashrightarrow} M_4,$$

denoting by φ their composition. Both φ_β and φ_η are wall-crossing flops in codimension two, while φ_γ is a wall-crossing flop in codimension three.

To understand the geometry of the map φ, we need to study the intersections of the MBM loci that define the flops. First, consider the subvariety E_β. It is non-normal and birational to $\mathbb{P}^2 \times S \simeq C^{(2)} \times S$. The singular locus of E_β is $E_\gamma \simeq C^{(3)} \simeq \mathbb{P}^3$. Denote by $\widetilde{E}_\beta$ the normalization of E_β, then $\widetilde{E}_\beta$ is isomorphic to the blow-up of $C^{(2)} \times S$ in the incidence subvariety $Z \subset C^{(2)} \times C \subset C^{(2)} \times S$, where $((x,y),z) \in Z$ iff $x = z$ or $y = z$. The proper transform of $C^{(2)} \times C$ under the blow-up is a divisor $D \subset \widetilde{E}_\beta$. The proper transform of E_β on the model M_2, that we will denote by E'_β, is obtained by blowing-down the divisor D. Then variety E'_β is a $\mathbb{P}^2$-bundle over S and it is the center of the flop φ_β^{-1}.

The center of the flop φ_η is $E_\eta \simeq \mathrm{Hilb}^3(S/\mathbb{P}^1)$, which is a $\mathbb{P}^2$-bundle over another K3 surface $S' = \mathrm{Pic}^3(S/\mathbb{P}^1)$, the latter being the relative Picard group of the elliptic fibration $f\colon S \to \mathbb{P}^1$. For $t \in \mathbb{P}^1$ we will denote by F_t the fiber of f over t. We will implicitly fix an origin on F_t making it an elliptic curve and also identifying the fiber of S' over t with F_t.

Let $E_{\beta\eta} = E_\beta \cap E_\eta \subset M_3$. Then we have a morphism $E_{\beta\eta} \to S'$ that is the composition of the embedding into E_η and the projection onto S'. Analogously, we get a rational map $E_{\beta\eta} \dashrightarrow S$ given by the embedding into E_β and the rational projection onto S. We will denote by $\Sigma \subset S \times S'$ the closure of the image of the product of the two maps above. Note that Σ is fibered over $\mathbb{P}^1$ and denote by $\Sigma_t = \Sigma \cap (F_t \times F_t)$ a fiber.

Lemma 7.1: The subvariety $\Sigma \subset S \times S'$ is dominant and finite of degree three both over S and S'. If the fiber $F_t \subset S$ intersects the curve C in three distinct points e_0, e_1 and e_2, then $\Sigma_t \subset F_t \times F_t$ consists of three connected components that are the translations of the diagonal by $(0, e_0 + e_1)$, $(0, e_1 + e_2)$ and $(0, e_2 + e_0)$.

Proof: Let us fix a point $t \in \mathbb{P}^1$ so that F_t satisfies the conditions of the lemma. Let $x \in F_t$ be a point different from e_0, e_1, e_2. Then the fiber of E_β over x is $\mathbb{P}^2 \simeq C^{(2)} \times \{x\}$. It intersects E_η in three points: $((e_0, e_1), x)$, $((e_1, e_2), x)$ and $((e_2, e_0), x)$ that are mapped by the projection $E_\eta \to S'$ to the points $e_0 + e_1 + x$, $e_1 + e_2 + x$ and $e_2 + e_0 + x$ respectively. This proves

the claim about Σ_t, because the latter is closed in $F_t \times F_t$. Hence Σ_t is dominant and finite of degree three over both factors, and since this holds for a generic $t \in \mathbb{P}^1$, we also get the first claim of the lemma. ■

Denote by p and q the projections from Σ to the two factors of $S \times S'$. Given $x, y \in S$ we will write $x \sim y$ if $q(p^{-1}(x)) \cap q(p^{-1}(y)) \neq \emptyset$. Generating an equivalence relation on the points of S by $\sim$, we will call the corresponding equivalence classes Σ-orbits.

Lemma 7.2: Let $x \in F_t$. Then $x \sim x + e_1 - e_0$, $x \sim x + e_2 - e_1$ and $x \sim x + e_0 - e_2$. Whenever at least one of $e_1 - e_0$, $e_2 - e_1$ or $e_0 - e_2$ is non-torsion, the orbit of x is infinite, hence Zariski dense in F_t.

Proof: From Lemma 7.1 we see that

$$q(p^{-1}(x)) = \{x + e_0 + e_1, x + e_1 + e_2, x + e_2 + e_0\},$$

and

$$q(p^{-1}(x + e_1 - e_0)) = \{x + 2e_1, x + 2e_1 - e_0 + e_2, x + e_1 + e_2\},$$

which shows that $x \sim x + e_1 - e_0$. Analogously for the other two points. The last claim of the lemma is obvious. ■

Lemma 7.3: For a very general $t \in \mathbb{P}^1$ the points $e_1 - e_0$, $e_2 - e_1$ and $e_0 - e_2$ are non-torsion in F_t.

Proof: Let $U \subset \mathbb{P}^1$ be the complement of the discriminant of the elliptic fibration f and let $g\colon J \to U$ be the Jacobian fibration corresponding to $f|_U$. Let $J^t \subset J$ be the subset of points that are torsion in their respective fibers. Then J^t is a countable union of closed subvarieties of J, each of them being an unramified finite cover of U.

For a generic choice of the surface S the curve C is ramified over $\mathbb{P}^1$ so that at least one of the ramification points is contained in a smooth fiber of the elliptic fibration f. The pairwise differences of the points e_0, e_1 and e_2 (that are the intersection points of F_t with C) define a curve $C' \subset J$ that is a ramified covering of U. Hence C' can not be contained in J^t, therefore $J^t \cap C'$ and its image in U are countable sets. This proves the claim. ■

As the final step of our discussion let us show that the birational map φ can not be represented as a composition of Mukai flops in codimension two,

even after removing some subvarieties of higher codimension from M_1 and M_4, see Proposition 7.4 below. Before proving this claim, let us introduce some notation. Let Z_β be the center of the flop φ_β, Z_η be the proper transforms of E_η on the model M_1, and Z_γ be the proper transform of the center of the flop φ_γ on the model M_1. By construction, Z_β, Z_γ and Z_η are the irreducible components of the indeterminacy locus of the map φ. Note that Z_γ is of codimension three, being birational to $\mathbb{P}^3$, and the other two components are of codimension two. The subvarieties Z_β, Z_γ and Z_η are swept out by rational curves representing the corresponding MBM classes.

Assume that there exist two open subsets $M_1^\circ \subset M_1$ and $M_4^\circ \subset M_4$ such that $\varphi|_{M_1^\circ} = \varphi_m \circ \ldots \circ \varphi_0$, where $\varphi_i \colon U_i \dashrightarrow U_{i+1}$ are Mukai's elementary transformations in codimension two, $U_0 = M_1^\circ$ and $U_{m+1} = M_4^\circ$. Every φ_i is the composition of a proper contraction $U_i \to V_i$ with the exceptional set E_i and the inverse of a proper contraction $U_{i+1} \to V_i$ with the exceptional set E'_{i+1}. Both E_i and E'_{i+1} are smooth $\mathbb{P}^2$-bundles over the singular locus of V_i. We may also assume that E_i and E'_{i+1} are irreducible.

Let us denote by Z_i the proper transforms of the varieties E_i on the model M_1°. By this we mean the following. Given an irreducible subvariety of codimension two $Z \subset U_{i+1}$, we have two possibilities. Either Z is not contained in E'_{i+1}, and then we take its proper preimage under φ_i in the usual sense: the closure in U_i of the preimage of $Z \setminus E'_{i+1}$ under φ_i. Or Z coincides with E'_{i+1}, and in this case by the proper preimage of Z under φ_i we will mean the subvariety E_i. Iterating this procedure over the maps φ_i, we get a proper preimage of Z in M_1°.

The proper preimages Z_i are codimension two irreducible subvarieties of M_1° that are swept out by rational curves C_i, the latter being proper preimages of the lines in the fibers of the $\mathbb{P}^2$-bundles E_i. Let us denote by $\overline{Z}_i$ the closure of Z_i in M_1. Now we can prove the following statement.

Proposition 7.4: There do not exist open subsets $M_1^\circ \subset M_1$ and $M_4^\circ \subset M_4$ with complements of codimension strictly greater than two and such that $\varphi \colon M_1^\circ \dashrightarrow M_4^\circ$ is a composition of Mukai's elementary transformations in codimension two.

Proof: Assume the contrary and use the notation introduced above. To arrive at a contradiction, we will make two steps. First, we will show that the subvariety Z_γ is contained in the complement of M_1°. Second, we will use Lemma 7.1, Lemma 7.2 and Lemma 7.3 to show that Z_β and Z_η also have to be in the complement of M_1° contradicting the assumption on the codimension.

Step 1. Note that the indeterminacy locus of the map $\varphi|_{M_1^\circ}$ has to be contained in the union of the subvarieties Z_i constructed before the statement of the proposition. Assume that $Z_\gamma^\circ = Z_\gamma \cap M_1^\circ \neq \emptyset$. Then Z_γ° has to be contained in one of Z_i. Let i_0 be the smallest index such that $Z_\gamma^\circ \subset Z_{i_0}$, so that $Z_\gamma \subset \overline{Z}_{i_0}$. Let $Z'_\gamma \subset U_{i_0}$ be the proper transform of Z_γ° under the sequence of flops $\varphi_0, \dots, \varphi_{i_0-1}$. By construction, $Z'_\gamma \subset E_{i_0}$.

We claim that Z'_γ does not dominate the base of the $\mathbb{P}^2$-bundle E_{i_0}. To prove the claim, let $\widetilde{Z}_{i_0}$ be the resolution of singularities of $\overline{Z}_{i_0}$. Observe that the singular locus of Z_{i_0} is contained in the union of Z_i with $i < i_0$. Therefore, our choice of i_0 guarantees that Z_γ° is not contained in the singular locus of Z_{i_0}. We can then consider the proper preimage $\widetilde{Z}_\gamma$ of Z_γ in $\widetilde{Z}_{i_0}$, and the variety $\widetilde{Z}_\gamma$ is then birational to Z_γ. Since Z_γ is rational, the variety $\widetilde{Z}_\gamma$ can not dominate the base of the MRC-fibration of $\widetilde{Z}_{i_0}$. The latter is birational to the $\mathbb{P}^2$-bundle E_{i_0}, hence the claim that Z'_γ does not dominate the base of that $\mathbb{P}^2$-bundle.

The previous paragraph implies that the variety Z'_γ contains one of the fibers of the $\mathbb{P}^2$-bundle E_{i_0}. It follows that one of the rational curves C_{i_0} defined above may be chosen to lie inside the variety Z_γ. Then the image of C_{i_0} on the model M_3 is contained in the MBM locus $E_\gamma \simeq \mathbb{P}^3$, therefore all deformations of C_{i_0} stay inside E_γ. On the other hand, by construction, the deformations of C_{i_0} sweep out a subvariety of codimension two. This contradiction completes the first step of the proof.

Step 2. Denote by $\pi_1 \colon M_1 \to N_1$ and $\pi_4 \colon M_4 \to N_4$ the extremal contractions of the MBM loci that define the wall-crossing flops φ_β and φ_η. Note that the exceptional set of π_1 is the variety Z_β, and since it has codimension two, it has to coincide with $\overline{Z}_i$ for some i. The variety $Z_i = \overline{Z}_i \cap M_1^\circ = Z_\beta \cap M_1^\circ$ is a proper $\mathbb{P}^2$-bundle, hence M_1° has to be proper over its image in N_1. Analogously, M_4° has to be proper over its image in N_4.

The singular locus of N_1 is the K3 surface S. Assume that $x \in S$ is a point in the complement of $\pi_1(M_1^\circ)$. Our definition of the correspondence $\Sigma \subset S \times S'$ from Lemma 7.1 implies that the entire Σ-orbit of x has to lie in the complement of $\pi_1(M_1^\circ)$. By Lemma 7.2 and Lemma 7.3 such orbit will be dense in the fiber F_t of the elliptic fibration f if $t \in \mathbb{P}^1$ is very general.

Now recall that by the first step of the proof Z_γ has to be contained in the complement of M_1°. Therefore the curve $C \subset S \subset N_1$ is in the complement of $\pi_1(M_1^\circ)$. The curve C intersects all fibers of the elliptic fibration f, so by the previous paragraph the complement of $\pi_1(M_1^\circ)$ contains a very general fiber F_t. Therefore it has to contain the whose surface S, because very

general fibers form a dense subset of S. We conclude that the entire MBM locus Z_β has to be in the complement of M_1°. But Z_β has codimension two in M_1, hence the claim of the proposition. ■

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References

[AV0] E. Amerik, M. Verbitsky, *Rational curves on hyperkähler manifolds*, Int. Mat. Res. Not., Volume 2015, Issue 23 (2015), 13009 – 13045 (Cited on pages 8, 9, and 10.)

[AV1] E. Amerik, M. Verbitsky, *Collections of orbits of hyperplane type in homogeneous spaces, homogeneous dynamics, and hyperkähler geometry*, Int. Math. Res. Not. 2020, no. 1, 25–38 (Cited on pages 3, 9, and 16.)

[AV2] E. Amerik, M. Verbitsky, *Rational curves and MBM classes on hyperkähler manifolds: a survey*, Rationality of varieties, Progr. Math., 342 (2021), 75–96 (Cited on pages 11 and 22.)

[AV3] E. Amerik, M. Verbitsky, *Contraction centers in families of hyperkähler manifolds*, Selecta Math. (N.S.) 27 (2021), no. 4, Paper No. 60, 26 pp (Cited on pages 9, 10, 11, 14, 15, and 20.)

[AV4] E. Amerik, M. Verbitsky, *MBM classes and contraction loci on low-dimensional hyperkähler manifolds of $K3^{[n]}$ type*, Algebr. Geom. 9 (2022) 252–265 (Cited on pages 24, 25, and 26.)

[Be1] A. Beauville, *Varietes Kähleriennes dont la première classe de Chern est nulle*, J. Diff. Geom. 18 (1983), p. 755–782 (Not cited.)

[Be2] A. Beauville, *Symplectic singularities*, Invent. Math. 139 (2000), no. 3, 541–549 (Not cited.)

[BL] B. Bakker, C. Lehn, *A global Torelli theorem for singular symplectic varieties*, J. Eur. Math. Soc. 23 (2021), no. 3, 949–994 (Cited on pages 11 and 15.)

[Bo] S. Boucksom, *Divisorial Zariski decompositions on compact complex manifolds*, Ann. Sci. École Norm. Sup. (4) 37 (2004), no. 1, 45–76 (Cited on pages 9 and 11.)

[BHL] D. Burns, Y. Hu, T. Luo, *HyperKähler manifolds and birational transformations in dimension 4*, Vector bundles and representation theory (Columbia, MO, 2002), 141–149. Contemp. Math., 322 (Cited on pages 2 and 4.)

[HY] Y. Hu, S.-T. Yau, *HyperKähler manifolds and birational transformations*, Adv. Theor. Math. Phys. 6 (2002), no. 3, 557–574 (Cited on pages 2, 5, and 6.)

[HT0] B. Hassett, Y. Tschinkel, *Moving and ample cones of holomorphic symplectic fourfolds*, GAFA 19 (2009), 1065-1080. (Cited on page 3.)

[HT1] B. Hassett, Y. Tschinkel, *Intersection numbers of extremal rays on holomorphic symplectic varieties*, Asian J. Math. 14 (2010), no. 3, 303–322 (Cited on page 24.)

[HT2] B. Hassett, Y. Tschinkel, *Flops on holomorphic symplectic fourfolds and determinantal cubic hypersurfaces*, J. Inst. Math. Jussieu 9 (2010), no. 1, 125–153 (Not cited.)

[HT3] B. Hassett, Y. Tschinkel, *Extremal rays and automorphisms of holomorphic symplectic varieties*, in K3 Surfaces and their Moduli, Progr. Math., vol. 315 (Birkhäuser/Springer, Cham, 2016), 73–95 (Cited on page 24.)

[Hu] D. Huybrechts, *Compact hyperkähler manifolds*, Calabi-Yau manifolds and related geometries (Nordfjordeid, 2001), 161–225, Universitext, Springer, Berlin, 2003 (Cited on pages 7, 8, 9, 12, 13, and 16.)

[Kl] D. Kaledin, *Symplectic singularities from the Poisson point of view*, J. Reine Angew. Math. 600 (2006), 135–156 (Cited on page 17.)

[Ko] J. Kollár, *Families of varieties of general type*, Cambridge Tracts in Math., 231, Cambridge University Press, 2023 (Cited on page 14.)

[Kw] Y. Kawamata, *Pluricanonical systems on minimal algebraic varieties*, Invent. Math. 79, (1985), no. 3, 567–588 (Cited on page 14.)

[Ma] E. Markman, *A survey of Torelli and monodromy results for holomorphic-symplectic varieties*, Complex and differential geometry, 257–322, Springer Proc. Math., 8, Springer, Heidelberg, 2011 (Cited on page 9.)

[SSV] N. Sibony, A. Soldatenkov, M. Verbitsky, *Rigid currents on compact hyperkähler manifolds*, arXiv:2303.11362 (Cited on page 16.)

[So] A. Soldatenkov, *On the Hodge structures of compact hyperkähler manifolds*, Math. Res. Lett. 28 (2021), no. 2, 623–635 (Cited on page 7.)

[V1] M. Verbitsky, *Mapping class group and a global Torelli theorem for hyperkähler manifolds*, Duke Math. J. 162 (2013), no. 15, 2929–2986 (Not cited.)

[V2] M. Verbitsky, *Ergodic complex structures on hyperkähler manifolds: an erratum*, arXiv:1708.05802 (Cited on page 15.)

[WW] J. Wierzba, J. Wiśniewski, *Small contractions of symplectic 4-folds*, Duke Math. J. 120 (2003), no. 1, 65–95 (Cited on pages 2, 3, 4, 16, and 20.)

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