MBM classes and contraction loci on low-dimensional hyperkähler manifolds of K3$^{[n]}$ type

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Abstract
We describe the extremal rays and the exceptional loci of extremal contractions on a hyperkähler manifold of K3$^{[n]}$-type for small n by deforming to the Hilbert scheme of a non-algebraic K3 surface.

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1 Introduction

In the last twenty years or so, a considerable effort has been made to understand the Mori cone, or, dually, the ample (or Kähler in the non-projective case) cone of irreducible holomorphic symplectic (IHS) manifolds. For K3 surfaces it has been known for a long time that the extremal rays correspond to (-2)-curves (that is, irreducible curves of square -2, automatically

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smooth and rational), so that the ample (or Kähler) classes are characterized, among positive classes, as those which are positive on all (-2)-curves. In higher dimension, though, until recently the only general result, due to Huybrechts and Boucksom, was that one could still test the Kähler property on rational curves. However, no boundedness statement for such curves was available. A boundedness statement (equivalent to the Morrison-Kawamata cone conjecture) was obtained in [AV2], Corollary 1.6, and a more precise version in [AV4], Corollary 1.4.

In [HT1] Hassett and Tschinkel conjectured a description of the Mori cone for projective IHS manifolds which are deformation equivalent to the Hilbert square of a K3 surface. Such manifolds are also called holomorphic symplectic fourfolds of $\mathrm{K3}^{[n]}$ type. The conjecture by Hassett and Tschinkel (Conjecture 3.1 of [HT1]) stated that the extremal rays are generated by the classes with some particular numerical properties. For the Hilbert square of a K3 surface, these numerics are either those of the rational curves contracted by the Hilbert-Chow morphism, or of the rational curves arising from a smooth rational curve C on the K3 surface, or of the line in the projective plane which is the symmetric square of such a C. Hassett and Tschinkel have partially proved this conjecture in [HT2] (Theorem 1, a sufficient condition for ampleness).

For high-dimensional holomorphic symplectic manifolds of $\mathrm{K3}^{[n]}$ type the analogous initial conjecture by Hassett and Tschinkel (Conjecture 1.2 of [HT3] turned out to be false. The description of the Mori cone was provided by Bayer and Macri ([BM], Section 12) using advanced technology around Bridgeland's stability conditions.

Another conjecture by Hassett and Tschinkel (Conjecture 3.6 of [HT1], and still a conjecture 24 in [HT2]) was concerned with the geometry of the loci covered by the minimal rational curves according to the numerical properties of the corresponding extremal rays (by Kawamata base-point-free theorem, one can interpret those loci as exceptional sets of birational contractions). As it is natural, one expects that such a locus is either a lagrangian plane or a $\mathbb{P}^1$-bundle over a K3 surface, the former covered by curves with Beauville-Bogomolov square $-5/2$, the latter either $-1/2$ (the type of the exceptional divisor of the Hilbert-Chow map) or -2 (the type of the divisor of subschemes with some support on a -2-curve C on the K3-surface). One can also expect a similar description in higher dimension. Hassett and Tschinkel have given lists of possible types of extremal rays and corresponding contraction loci in [HT3] and then completed a couple of missing types following Bayer and Macri in [HT4].

Bayer and Macri's technology allows, in principle, to describe contrac-

tion loci as fibrations in projective spaces over K3-related manifolds (up to birational equivalence). Those manifolds arise as moduli spaces of sheaves on a K3 surface in question, and the projective spaces are projectivisations of Ext-groups between the sheaves. But this approach is not very explicit, whereas at least in low dimension it seems possible and useful to describe the extremal rays and the subvarieties covered by the corresponding curves "by hand".

The purpose of this note is to provide a particularly simple description of possible extremal rays and contraction loci for holomorphic symplectic manifolds of $\mathrm{K3}^{[n]}$ type in low dimension, particularly four and six (i.e. for manifolds which are deformation equivalent to the Hilbert square or the Hilbert cube of a K3 surface). Rather than studying extremal rays, we work with a slightly different notion of MBM classes. This notion was introduced in [AV1], Section 5, in order to put things in the deformation-invariant context, and indeed the crucial point in our argument is the deformation to the Hilbert scheme of a non-projective K3 surface.

The MBM classes are the classes which correspond to extremal rays of the Mori cone on some deformation of X, or, equivalently, the classes of extremal rays up to monodromy and birational equivalence (the name derives from "monodromy birationally minimal"). From a uniruled subvariety Z of an IHS X we obtain an MBM class as follows: let C be a rational curve which is minimal among the rational curves in Z (in the sense that C does not bend-and-break). If its class is Beauville-Bogomolov negative, then it is MBM.

The parameter space $\mathsf{Teich}_\alpha^{min}$ for complex structures where an MBM class α is represented by a minimal rational curve coincides with the whole space where α stays of type $(1,1)$ up to inseparability issues. Such curves can always be contracted ([AV3, Theorem 4.6]).

In [AV3] we proved some deformation-invariance statements about the corresponding contraction loci (i.e. exceptional sets of morphisms contracting the class α). We have, in particular, shown that the diffeomorphism type of such a contraction locus is constant over $\mathsf{Teich}_\alpha^{min}$. Moreover, so is the biholomorphism type of the fibers of the rational quotient fibration of the exceptional locus, with a possible exception of the points of $\mathsf{Teich}_\alpha^{min}$ corresponding to the complex structures with maximal Picard number (Theorems 1.8 and 1.10 of [AV3]).

The idea now is as follows. Let α be a negative integral $(1,1)$-class on a hyperkähler fourfold X of $\mathrm{K3}^{[n]}$ type. Using Markman's description of the monodromy group, we prove that X can be deformed to a non-projective manifold $X' \cong \mathrm{Hilb}^2(S')$ where S' is a K3 surface with cyclic Picard group

generated by a negative class, and in such a way that α stays of type $(1,1)$. Since rational curves on X' are easy to describe, we obtain the classification of subvarieties which can be covered by rational curves in the class α, by deformation-invariance.

If X is an IHS of $\mathrm{K3}^{[n]}$ type of dimension six, it turns out that X still deforms to the Hilbert cube of a non-projective K3 surface S, but its Picard group can now be generated by an element with negative or zero Beauville-Bogomolov square. The geometry in this last case is also well-understood. Namely, S is an elliptic $K3$ surface with singular fibers which are nodal rational curves, and there is no other curve at all. All rational curves (up to deformation) on the symmetric cube of S are obtained from linear systems $g^1_k, k \leqslant 3$ on the fibers,[1] so again easy to describe. We obtain a table of five types of MBM classes, their squares and the types of the corresponding contraction loci, which agrees with the data by Hassett and Tschinkel (one may notice here that the MBM class initially missing in the tables of [HT3] and exhibited in [HT4] while working out Bayer and Macri's formulae on this example, comes in our picture from the Hilbert cube of a surface with zero intersection form on the Picard group).

The same calculation works in fact for eightfolds (bringing into the picture another case missing in the tables of [HT3]) and tenfolds, but not for higher-dimensional IHS of $\mathrm{K3}^{[n]}$ type. It would be interesting to see whether the present elementary method could be adapted to derive Bayer and Macri's description and the classification of exceptional loci in all dimensions.

2 Preliminaries

Recall that an **irreducible holomorphically symplectic (IHS)** manifold, or a **simple hyperkähler** manifold M is a compact Kähler manifold M with $\pi_1(M) = 0$ and such that $H^{2,0}(M)$ is generated by a nowhere-degenerate form.

The basic series of examples is that of punctual Hilbert schemes of K3 surfaces. As observed by Beauville in [Bea], for any K3 surface S, the Hilbert scheme $S^{[n]}$ of ideal sheaves of colength n on S is an IHS. An IHS which is a deformation of $S^{[n]}$ for a K3 surface S is said to be **of $\mathrm{K3}^{[n]}$ type**.

The second cohomology of an IHS carry an integral but possibly not unimodular quadratic form, the **Bogomolov-Beauville-Fujiki form** (BBF form) which we denote by q. Throughout this paper we use it to embed

[1] As defined in [Har, §IV.5], g^d_k denotes a linear system of dimension d and degree k on a curve. In particular, g^1_k correspond to k-sheeted maps to $\mathbb{P}^1$.

$H_2(M,\mathbb{Z})$ into $H^2(M,\mathbb{Q})$ as an overlattice of $H^2(M,\mathbb{Z})$. This allows us to view the classes of curves on M as rational $(1,1)$-classes and compute the BBF square of a curve which is a rational number (cf. [HT1]).

Example 2.1: If M is a K3 surface, the BBF form is just the intersection form. If $M = S^{[n]}$ for a K3 surface S, we have an orthogonal direct sum decomposition of lattices $H^2(M,\mathbb{Z}) = H^2(S,\mathbb{Z}) \oplus \mathbb{Z}e$, where $H^2(S,\mathbb{Z})$ is equipped with the intersection form and $q(e) = -2(n-1)$. In fact e is one half of the class of the diagonal, that is, the divisor of non-reduced subschemes of length n in S (this divisor is also the exceptional set of the Hilbert-Chow morphism for $S^{[n]}$). Moreover, since the BBF form is of topological origin (Fujiki's formula, [F]), such a decomposition also holds for all IHS of $\mathrm{K3}^{[n]}$ type, but the class e may cease to be algebraic.

2.1 Extremal curves and MBM classes

Recall that if M a K3 surface, $\alpha \in H^{1,1}(M)$ is Kähler iff it has positive square and is positive on all (-2)-curves (and analogously for ample classes). Therefore the orthogonal hyperplanes to the classes of the (-2)-curves cut out $\mathrm{Kah}(M)$ inside the positive cone $\mathrm{Pos}(M)$ (one of the two connected components of the set of classes with positive square). In higher dimension the first analogue is the following

Theorem 2.2: (Huybrechts, Boucksom, see [Bou]) The Kähler cone of M is the set of all $\eta \in \mathrm{Pos}(M)$ such that $q(\eta, C) > 0$ for all rational curves C.

So the Kähler cone is cut out inside the positive cone by the orthogonal hyperplanes to the classes of rational curves. These rational curves have negative BBF square (by Hodge index theorem, since they are orthogonal to a positive square class) and obviously we can consider only the curves in a class which cannot be represented as a sum of two other, non-proportional such classes.

Definition 2.3: A rational curve C is extremal if the orthogonal hyperplane to its class defines a wall[1] of $\mathrm{Kah}(M)$, and if moreover C is minimal in the following sense: C is of minimal degree (with respect to some fixed Kähler form) among the rational curves passing through a general point of the subvariety $Z \subset M$ covered by the deformations of C.

[1] That is, a codimension-one face

By Hodge index theorem, extremal rational curves must have negative self-intersection. Also, minimal rational curves have the following remarkable properties (see for instance [AV1], Section 4).

Proposition 2.4: Let X be IHS of dimension $2n$, then a minimal rational curve C deforms together with X whenever its cohomology class remains of type $(1,1)$. The space of deformations of C within X is of dimension $2n-2$. The subvariety Z covered by deformations of C is coisotropic and can be of any dimension between n and $2n-1$. Its codimension in X is equal to the relative dimension of its rational quotient fibration.

Definition 2.5: A rational curve is called **MBM** if it is minimal and its cohomology class has negative self-intersection.

Note that an MBM curve C is extremal on a generic deformation of M as in Proposition 2.4, as only its class, up to proporionality, survives as a $(1,1)$-class there (so that the Kähler classes are exactly the positive classes which have positive intersection with C).

The origin of this term is as follows. Extremal rational curve may cease to be extremal under a deformation. To put the things into a deformation-invariant context, we have introduced the **MBM classes** in [AV1]. The simplest way to define them is as follows.

Definition 2.6: A class $\eta \in H^2(M,\mathbb{Z})$ is MBM if its rational multiple can be represented by an extremal curve on a deformation on M.

In the simplest example of K3 surfaces, extremal rational curves are (-2)-curves and MBM classes are classes in $H^2(X,\mathbb{Z})$ of square (-2). A class of square -2 is not, in general, represented by a -2-curve, but it is, up to $\pm$ sign, on a deformation where it generates the Picard group.

The original definition from [AV1] is as follows: an MBM (short for "monodromy birationally minimal") class is, up to a rational multiple, a monodromy transform (see next subsection for the definition of the monodromy) a class of an extremal rational curve on an IHS birational model of M. The following reformulation of [AV1, Theorem 1.17] explains why this is the same.

Theorem 2.7: Let η be the cohomology class of an extremal curve on a hyperkähler manifold, and (M_1,η) be obtained as a deformation of (M,η) in

a family such that η remains of type $(1,1)$ for all fibers of this family. Then M_1 is birational to a hyperkähler manifold M_1' such that $\gamma(\eta)$ is a class of an extremal curve on M_1' for some γ in the monodromy group.

Remark 2.8: Birational hyperkähler manifolds are deformation equivalent by a result of Huybrechts. In fact an extremal rational curve remains extremal on an M_1' unseparable from M_1 in the Teichmüller space (as observed for instance in [AV3], Section 3; see next section for more details).

The set of MBM classes is thus the same for a given deformation type. If a complex structure I is fixed, some of them are of type $(1,1)$, and among those, some correspond to extremal curves (that is, determine the Kähler cone), others correspond to extremal curves on birational models and their monodromy transforms. The following result further explains the connection between MBM classes and the Kähler cone.

Theorem 2.9: ([AV1], Theorem 1.19) Let (M,I) be a hyperkähler manifold and S the set of all MBM classes of type (1,1) on (M,I). Let $S^\perp$ the union of all orthogonal complement to all $s \in S$. Then $\mathrm{Kah}(M,I)$ is a connected component of $\mathrm{Pos}(M)\backslash S^\perp$.

Following E. Markman, we call these connected components **the Kähler-Weil chambers**.

2.2 Deformation spaces and monodromy

The deformation space we consider in this paper is the connected component of the **Teichmüller space** (see [V1]) containing the parameter point for our complex structure on M, or, equivalently, the connected component of the **marked moduli space** (see e.g. [Ma1]). We denote it by Teich. This is a finite-dimensional complex manifold, not necessarily Hausdorff. The Torelli theorem for IHS manifolds [V1] states that the **period map** is the Hausdorff reduction of Teich.[2] Recall that the period map $\mathsf{Per}: \mathsf{Teich} \longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ maps a complex structure J to its $H^{2,0}$ which is a point in $\mathbb{P}H^2(M,\mathbb{C})$, and the image $\mathbb{P}\mathsf{er}$ of Per is an open subset of a quadric (the period space)

$$\mathbb{P}\mathsf{er} := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \mid \ q(l,l)=0, q(l,\bar{l})>0\}.$$

The **monodromy group** also can be defined in two equivalent ways: following [V1], as the image in $O(H^2(M,\mathbb{Z}))$ of the part of the mapping class

[2] i.e. it is exactly unseparable points which have the same image.

group preserving our connected component Teich, or, following [Ma1], as the subgroup of $O(H^2(M,\mathbb{Z}))$ generated by parallel transports in families.

Huybrechts in [H1] observed that unseparable points of Teich correspond to birational IHS, up to the action of the mapping class group. The following precise description of unseparable points is due to Markman.

Theorem 2.10: ([Ma1], Theorem 1.4) The points in $\mathsf{Per}^{-1}(\mathsf{Per}\, I)$ correspond to the Kähler-Weyl chambers (that is, to the connected components of $\mathrm{Pos}(M)\backslash\cup z^\perp$ where z runs through the MBM classes of type $(1,1)$). Each chamber is the Kähler cone of the corresponding point in $\mathsf{Per}^{-1}(\mathsf{Per}\, I)$.

In particular, as a general hyperkähler manifold has no curves (the ones which have curves belong to a countable union of divisors in Teich), Teich is generically Hausdorff.

Let now α be an MBM class, we are interested in the complex structures I from Teich such that α is of type $(1,1)$ in I. Denote the subspace of such structures in Teich by Teich_α. It is well-known that $\mathsf{Teich}_\alpha = \mathsf{Per}^{-1}(\alpha^\perp)$. From Theorem 2.10 it is clear that Teich_α is not Hausdorff even generically (indeed each complex structure I in Teich_α has at least two Kähler-Weil chambers in which $\alpha^\perp \subset H^{1,1}(I)$ divides the positive cone). On the other hand, Teich_α is naturally divided in two halves, according to whether α is negative or positive on the Kähler classes. In the "positive" half Teich^+_α take the part formed by the complex structures whose Kähler cones are adjacent to $\alpha^\perp$.

Definition 2.11: The space thus defined is called $\mathsf{Teich}^{\mathsf{min}}_\alpha$. In other words $\mathsf{Teich}^{\mathsf{min}}_\alpha$ parameterizes the complex structures such that α is the class of a minimal rational curve, up to a positive multiple.

Remark 2.12: It is clear from this description that $\mathsf{Teich}^{\mathsf{min}}_\alpha$ is generically Hausdorff and that Per gives the Hausdorff reduction of $\mathsf{Teich}^{\mathsf{min}}_\alpha$ with image $\alpha^\perp \subset \mathbb{P}\mathsf{er}$.

2.3 Monodromy for manifolds of K3$^{[n]}$ type

The monodromy group is very important for understanding the geometry of IHS manifolds, but in the general ("classification-free") case its structure remains somewhat mysterious. One knows, thanks to Markman [Ma2], that the reflections in the classes of uniruled divisors are in Mon, and this already has nontrivial consequences. But it is not clear how to decide whether a

given MBM class is "divisorial" or not (that is, whether the corresponding rational curves cover a divisor or a subvariety of higher codimension). One also knows that it is of finite index in $O(H^2(M,\mathbb{Z}))$ ([V1]; see [V1-err], Theorem 2.6, for the corrected version).

A lot of work has been done on the known examples of IHS. For manifolds of $\mathrm{K3}^{[n]}$ type, Markman proved the following.

Theorem 2.13: ([Ma3, Theorem 1.2]) The monodromy group Mon is the subgroup of $O(H^2(M,\mathbb{Z}))$ generated by the reflections in the classes of square -2 and and the antireflections in those of square 2, that is, by the transformations $\rho_u(x) = \frac{-2x}{q(u)} + q(x,u)u$, where the Beauville-Bogomolov square $q(u) = \pm 2$, and the same notation q is used for the bilinear form associated to q. Up to $\pm$ sign, it also coincides with the **stable orthogonal group** $\widetilde{O}(H^2(M,\mathbb{Z}))$ of isometries acting trivially on the discriminant group.

Here the **discriminant group** of a lattice L is the quotient L^*/L where L^* is the dual lattice and L is naturally embedded into L^*. If L is $H^2(M,\mathbb{Z})$ for M a $2n$-dimensional IHS of $\mathrm{K3}^{[n]}$ type, the discriminant group is cyclic of order $2n-2$ (see Example 2.1). We shall often view it as a subgroup of $\mathbb{Q}/\mathbb{Z}$, like in Theorem 2.15 just below.

Remark 2.14: When $n-1$ is a power of a prime number, one can show that Mon coincides with the orientation-preserving part of the orthogonal group itself ([Ma3], lemma 4.2). We shall not need this as the group $\widetilde{O}$ has also been studied. Notably we shall use the following.

Theorem 2.15: (see e.g. [GHS], Lemma 3.5) Let (L,q) be a lattice containing two orthogonal isotropic planes. The orbit of $l \in L$ under $\widetilde{O}(L)$ is determined by $q(l)$ and the image of $l/d(l)$ in the discriminant group. Here we view the dual lattice as an overlattice of L in $L \otimes \mathbb{Q}$, and $d(l)$ is the *divisibility* of l, that is, the pairing of l with L gives the ideal $d(l)\mathbb{Z}$.

3 Birational contractions and invariance by deformation

On a projective IHS manifold, it follows immediately from Kawamata base-point-free theorem ([K1, Theorem 4.3]) that extremal rational curves are contractible. Indeed Kawamata's theorem implies that the sections of a suitable power of a big and nef bundle L on such manifolds define a bi-

rational contraction (that is, a birational morphism with connected fibers) $\varphi : M \longrightarrow Y$. Suppose that M is a projective IHS manifold, α is an MBM class and the given complex structure I on M is such that $I \in \mathsf{Teich}^{\mathsf{min}}_{\alpha}$, so that α is in fact the class of an extremal curve: the hyperplane $\alpha^{\perp}$ in $H^{1,1}(M, \mathbb{R})$ contains a wall of the Kähler cone. Since M is projective, the interior of this wall contains an integral $(1,1)$-class which by Lefschetz theorem is $c_1(L)$ for a line bundle L. This line bundle is nef, and also big. Indeed its BBF square is positive, and hence so is $c_1(L)^{2n}$. Therefore a power of L defines a birational morphism contracting exactly the curves of class proportional to α.

It turns out that extremal curves can be contracted also in the non-projective case. This is essentially due to Bakker and Lehn in [BL]. More precisely, putting together some results in [AV1], [BL] and [AV3] gives the following.

Theorem 3.1: ([BL, Proposition 4.10], [AV3, Corollary 5.3])
Let M be an IHS manifold and α the class of an extremal rational curve on M. Then there exists a birational contraction $\varphi : M \longrightarrow Y$ contracting to points the curves whose class is proportional to α. Moreover, for every component Z of the exceptional set, the fibers of $\varphi|_Z$ are rationally connected, and $\varphi|_Z$ is the rational quotient fibration of Z.

In particular the uniruled subvarieties of M covered by deformations of an extremal rational curve are irreducible components of the exceptional sets of birational contractions. We sometimes refer to these as **MBM loci**.

The following is the main result of [AV3] (summarizing theorems 1.8-1.10).

Theorem 3.2: Let α be an MBM class of type $(1,1)$ on a hyperkähler manifold M with $b_2(M) > 5$. Assume that M and its deformation M' are both in $\mathsf{Teich}^{\mathsf{min}}_{\alpha}$, so that α is a class of an extremal rational curve on both. Then the corresponding contraction centers on M and M' are sent one to another by a diffeomorphism $\Psi : M \to M'$ preserving the fibers of the contraction. If moreover neither M nor M_1 has maximal Picard rank, Ψ can be chosen identifying the contracted extremal curves and thus induces the biholomorphism of the fibers provided that they are normal.

Remark 3.3: Clearly, $H^{1,1}(M)$ is obtained as orthogonal complement to the 2-dimensional space $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega \rangle$, where Ω is the cohomology class of the holomorphic symplectic form. Then $\operatorname{Pic}(M) = H^{1,1}(M) \cap H^2(M, \mathbb{Z})$ has

maximal rank only if the plane $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega\rangle$ is rational. There is at most countable number of such M.

4 MBM loci for low-dimensional manifolds of $K3^{[n]}$ type

The diffeomorphism theorem allows one to give an explicit description of birational contraction centers and MBM loci in terms of the period spaces and lattices. The easiest illustration is the (well-known, cf. Hassett and Tschinkel's works, but seemingly with no direct proof ever written) description of the MBM loci on an IHS fourfold of $\mathrm{K3}^{[2]}$ type.

Theorem 4.1: On an IHS fourfold of $\mathrm{K3}^{[2]}$ type, there are only three types of extremal rational curves l (notation: $\alpha = [l]$, Z is the MBM locus of l):

(a) $q(\alpha) = -5/2$, 2α is integral, $Z \cong \mathbb{P}^2$

(b) $q(\alpha) = -2$, α is integral, Z is a $\mathbb{P}^1$-bundle over a K3 surface;

(c) $q(\alpha) = -1/2$, 2α is integral, Z is a $\mathbb{P}^1$-bundle over a K3 surface.

Taking into account Theorem 3.2, this is a consequence of the following precision:

Theorem 4.2: Let α be the class of an extremal rational curve on an IHS fourfold M of $\mathrm{K3}^{[2]}$ type and Z the corresponding MBM locus. Then there exists a deformation M' of M within Teich_α, isomorphic to the Hilbert square of a non-projective K3 surface S with cyclic Picard group generated by a negative class z, such that one of the following holds:

(a) S contains a smooth rational curve C and Z deforms to its Hilbert square, isomorphic to $\mathbb{P}^2$; α is a line in this $\mathbb{P}^2$;

(b) S contains a smooth rational curve C and Z deforms to $Z_C \subset M'$, the divisor of all length-two subschemes of S with support intersecting C; Z_C is a ruled divisor singular along $C^{[2]} \cong \mathbb{P}^2$ and α is the class of the ruling.

(c) Z deforms to the exceptional divisor of the Hilbert-Chow map $S^{[2]} \to \operatorname{Sym}^2(S)$, and α is the class of the ruling.

Remark 4.3: In the second case α is clearly not a class of an extremal rational curve, that is, M' is not in $\mathsf{Teich}^{min}_{\alpha}$. As we know, this means that some complex structure unseparable from M is in $\mathsf{Teich}^{min}_{\alpha}$. In fact, to render α extremal, one performs a flop in $C^{[2]}$, which also resolves the singularities of Z_C and makes it a $\mathbb{P}^1$-bundle.

Once the existence of a deformation $M' \cong S^{[2]}$, $\operatorname{Pic}(S) = \langle z \rangle$, $q(z) < 0$, is shown, the rest is easy. Indeed one has the following description of rational curves on M'.

Proposition 4.4: Let S be a K3 surface with the Picard group $\operatorname{Pic}(S)$ of rank 1 generated by a vector $z \in H^{1,1}(S, \mathbb{Z})$ with negative square, and Z an MBM locus on $M^{[2]}$. Then Z is described by (a), (b) or (c) of Theorem 4.2.

Proof: If S has no curves, the symmetric square $\operatorname{Sym}^2 S$ also has no curves, and the bimeromorphic contraction $S^{[2]} \longrightarrow \operatorname{Sym}^2 S$ contracts the exceptional divisor E to the singular set of $\operatorname{Sym}^2 S$. This means that $E = Z$, case (b) of Theorem 4.2.

If S has a curve, it is a smooth (-2)-curve C which is necessarily unique. Indeed, any curve with negative self-intersection is a smooth rational curve with square -2 by adjunction, and there cannot be two different curves in the proportional classes, again because $(z, z) < 0$. Then the (-2)-curve C can be blown down to an ordinary double point. Let S_1 be the result of this blow-down. Then $\operatorname{Sym}^2(S_1)$ has no curves, and the natural map $S^{[2]} \longrightarrow \operatorname{Sym}^2(S_1)$ blows down all curves in $S^{[2]}$. The corresponding exceptional set V is a union of two divisors, the exceptional divisor E and the set $W \subset S^{[2]}$ of all ideal sheaves of colength two $I \subset \mathcal{O}_S$ with support of I intersecting C. Clearly, these two divisors intersect in $P^2 = C^{[2]}$, and outside of $C^{[2]}$ each of these divisors is covered by a unique family of rational curves, hence the classification. ■

As for Beauville-Bogomolov squares, we recall the computations by Hassett and Tschinkel for convenience. We have the decomposition into the orthogonal direct sum $H^2(M, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z}e$ where e is half of the class of the diagonal E, and $e^2 = -2$.

- If α is the class of the ruling of the exceptional divisor, $q(e, \alpha) = -1$ since the exceptional divisor restricts as $\mathcal{O}(-2)$ to the ruling. Hence $\alpha = e/2$.

- If α is the class of the ruling of Z_C, then α is the class of C.
- If α is the class of the line in the Lagrangian plane $C^{[2]}$, then $\alpha = [C] - e/2$ ([HT3], Example 4.11; for $S^{[n]}$ this would be the same, indeed $q(e, \alpha) = n - 1$)

We postpone the existence of M' until the next section and deal with the next example of an IHS sixfold of $\mathrm{K3}^{[3]}$ type. There are five types of extremal rational curves/MBM classes/contraction loci. Four of them appear in the tables of [HT3] and are described geometrically exactly in the same way as those on a fourfold: on such an M which is the Hilbert cube of a K3 surface S containing a (-2)-curve C, we have four natural subvarieties covered by deformations of rational curves in BBF-negative classes, that is, the exceptional divisor of the Hilbert-Chow map (ruled over the big diagonal), the ruled divisor of subschemes with some support on C (ruled over the symmetric square of S), the fourfold of subschemes with at least length-two part supported on C (birational to a $\mathbb{P}^2$-bundle over S), and $\mathbb{P}^3 = C^{[3]}$.

After Bayer and Macri have given their description of the Mori cone in [BM], Hassett and Tschinkel observed in [HT4] that one case was actually missing from their table in dimension 6. The cohomology class of the extremal curve in question is half-integer and has square -1. We would like to present the following simple way to find this MBM class.

Theorem 4.5: Let α be a class of an extremal rational curve on an IHS M of $\mathrm{K3}^{[n]}$ type of dimension at most ten and let Z be the corresponding MBM locus. Then there exists a deformation M' of M within Teich_α, isomorphic to the Hilbert scheme of a non-projective K3 surface S with cyclic Picard group (thus generated by a non-positive class z).

Again, let us describe the geometry we get in dimension six before going to the proof.

First case: If $\operatorname{Pic} S$ is generated by a negative class, the rational curves on M' are described as in Proposition 4.4: one has the four types listed above.

Second case: If $\operatorname{Pic} S$ is generated by a class of square zero, then one deduces from Riemann-Roch theorem that S carries an elliptic pencil with some degenerate fibers which are rational curves with a singularity, and no other curves. For Hilbert cube this means that the extremal rational curves arise from linear systems of degree $k \leqslant 3$ on the (eventually normalized) curves from the elliptic pencil: one fixes $3 - k$ points on S and lets the

remaining k-uplet run through a pencil in the system. Of course, not all curves arising in this way are extremal, and some of them are already on the list obtained in the negative case.

Note that the computation of the BBF square is particularly easy since only the intersection with e counts. Indeed the intersection form is zero on its orthogonal complement (which is naturally identified with $\operatorname{Pic} S$).

For instance, the curve obtained by fixing two points and letting the third one run through the singular fiber is of BBF square zero, so not extremal. Considering a rational curve R coming from a g^1_3 on a smooth elliptic curve, we see that $q(R) = -9/4$ and R is quarter-integer (e comes with coefficient $-3/4$), by the following well-known lemma (see [CK], Lemma 2.1, for a version which also applies to singular curves).

Lemma 4.6: The class of a rational curve coming from a g^1_k on a curve C of genus g is $C - \frac{g+k-1}{2(n-1)}e$.

The idea is that the intersection with the diagonal is computed by the Hurwitz formula, as this is roughly speaking the ramification of the morphism to the projective line defined by the g^1_3.

Such an extremal class (quarter-integer with square $-9/4$) is already obtained by Hassett-Tschinkel's computation when S contains a (-2)-curve C. It appears as the class of a line in the projective plane $C^{[2]}$. By the description of monodromy orbits, the two classes are conjugate by a monodromy transform, so we get no new geometry here.

On the contrary, fixing one point on S and considering a g^1_2 on an elliptic curve, one gets a half-integer class of square -1, exactly the one omitted in the table of [HT3].

The following corollary results from the above computations.

Corollary 4.7: There are 5 types of MBM classes on an IHS sixfold of $\mathrm{K3}^{[3]}$ type:

(a) $q(\alpha) = -3$, 2α is integral, $Z \cong \mathbb{P}^3$

(b) $q(\alpha) = -9/4$, 4α is integral, Z is birationally a $\mathbb{P}^2$-bundle over a K3 surface;

(c) $q(\alpha) = -2$, α is integral, Z is birationally a $\mathbb{P}^1$-bundle over the Hilbert square of a K3 surface;

(d) $q(\alpha) = -1/4$, 4α is integral, Z is birationally a $\mathbb{P}^1$-bundle over a product of two K3 surfaces;

(e) $q(\alpha) = -1$, 2α is integral, Z is birationally a $\mathbb{P}^1$-bundle over a product of two K3 surfaces.

It could be interesting to see whether one can make this more precise with the help of flops, as in the fourfold case, though there is no hope that one could suppress the word "birationally" altogether. Indeed the diagonal divisor (case (d)) is a contraction locus but not a $\mathbb{P}^1$-bundle, the fibers over the small diagonal are more complicated (cones over twisted cubics, [T]).

In the next section we perform a monodromy computation and use it to prove theorems Theorem 4.5 and Theorem 4.2. In dimension 8 and 10, one derives a similar description of MBM classes and their loci. Theorem 4.5 does not hold beyond dimension ten. Nevertheless one can pursue the explicit description of extremal curves along these lines.

5 Monodromy computations

Theorem 5.1: Let S be a K3 surface and let Λ be the lattice $H^2(S^{[k]}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z}e$ with its BBF form. If $k = 2$, then for any primitive $z \in \Lambda$ with $q(z, z) < 0$, there exists $\gamma \in \widetilde{O}(\Lambda)$ such that the rank 2 lattice $\langle \gamma(z), e\rangle$ is negative definite, unless z and e belong to the same $\widetilde{O}(\Lambda)$-orbit. If $3 \leqslant k \leqslant 5$, there exists $\gamma \in \mathsf{Mon}$ such that the rank 2 lattice $\langle \gamma(z), e\rangle$ is negative semidefinite.

Remark 5.2: In this statement $\widetilde{O}(\Lambda)$ can be replaced by "monodromy", we just want to stress that the statement is about lattices.

This implies Theorem 4.5 and Theorem 4.2 because of the following proposition.

Proposition 5.3: Let $Z \subset X$ be a birational contraction center on a deformation X of the n-th Hilbert scheme of a K3 surface. Suppose that Theorem 5.1 holds for its second cohomology lattice. Then the pair (X, Z) can be deformed to $(S^{[n]}, Z')$, where S is a non-algebraic K3 surface with (necessarily BBF nonpositive) cyclic Picard group.

Proof: Let η be the MBM class associated with Z and Teich_η the Teichmüller space of deformations of X such that η remains of type (1,1). From the global Torelli theorem it follows that any lattice $\Lambda_1 \subset \Lambda$ with $\Lambda_1^\perp$ containing a positive 2-plane and $\Lambda_1 \ni \eta$ can be realized as a Picard lattice of $I \in \mathsf{Teich}_\eta$. Moreover all structures unseparables from I (the points of $\mathsf{Per}^{-1}(\mathsf{Per}(I))$ have this Picard lattice, and the corresponding manifolds are all birational. Applying this to the lattice $\langle \eta, \gamma^{-1}e\rangle$ from Theorem 5.1 , we obtain deformations of (X, Z) with Picard lattice $\langle \eta, \gamma^{-1}e\rangle$. On such deformations, η remains of type $(1, 1)$. Finally, some of them are Hilbert schemes of K3 surfaces: indeed the period points of Hilbert schemes of K3 surfaces in $\mathbb{P}\mathsf{er}$ form the union of the orthogonal hyperplanes to the monodromy transforms of e (each point of this union is a period point by Torelli theorem for K3 surfaces). ■

We denote by $\delta(z)$ the image of $\frac{z}{d(z)}$ in the discriminant group, where $d(z)$ denotes the divisibility of z (see Theorem 2.15). Recall from Theorem 2.15 that $z, z' \in \Lambda = (H^2(S^{[n]}, \mathbb{Z}), q)$ belong to the same orbit of $\widetilde{O}(\Lambda)$ if and only if $q(z) = q(z')$ and $\delta(z) = \delta(z')$.

Example 5.4: Let S be a K3 surface, and $\Lambda = (H^2(S^{[2]}, \mathbb{Z}), q)$, so that $\Lambda = H^2(S, \mathbb{Z}) \oplus \mathbb{Z}e$, where $q(e) = -2$, and the discriminant group of Λ is $\mathbb{Z}/2\mathbb{Z}$. The primitive vectors $z \in \Lambda$ are $ax + be$, where $x \in H^2(S, \mathbb{Z})$ is primitive and $a, b \in \mathbb{Z}$ relatively prime. Since the lattice $H^2(S, \mathbb{Z})$ is unimodular, $d(x) = 1$ for any primitive $x \in H^2(S, \mathbb{Z})$. Therefore, $d(z) = 2$ if and only if a is even, and $\delta(z) \neq 0$ if and only if a is even and b odd. For such z, one has $q(z) = a^2q(x) - 2b^2$. More generally, if $\Lambda = (H^2(S^{[n]}, \mathbb{Z}), q)$ and e denotes half the diagonal class, then the divisibility of $ax + be$, $x \in H^2(S, \mathbb{Z})$ primitive, is equal to g.c.d. $(a, 2(n - 1)b)$.

Proof of Theorem 5.1 in the case $k = 2$: We consider two subcases, $\delta(z) = 0$ and $\delta(z) \neq 0$. If $\delta(z) = 0$, we can find $z' \in H^2(S, \mathbb{Z}) \subset H^2(S^{[2]}, \mathbb{Z})$ with $q(z) = q(z')$, and Theorem 2.15 implies that $z' = \gamma(z)$ for some $\gamma \in \widetilde{O}(\Lambda)$. Clearly, the lattice $\langle z', e\rangle$ is negative definite.

Now, let $\delta(z) \neq 0$. Then $q(z) = a^2q(x) - 2b^2$, where $a = 2a_1$ is even, and $q(z, z) = 4a_1^2q(x) - 2b^2$ is equal to -2 modulo 8. Moreover, any number $r = -2 \mod 8$ can be realized as $q(z') = 4q(x) - 2$ for $z' = 2x' - e$, because any even number can be realized as a square of $x' \in H^2(S, \mathbb{Z})$. Since $q(z) < 0$, we have $q(z) \leqslant -10$ unless $q(z) = -2$. In the latter case z belongs to the same $\widetilde{O}(\Lambda)$-orbit as e. In the first case z belongs to the same $\widetilde{O}(\Lambda)$-orbit as

$z' = 2x' - e$, with $q(x') \leqslant -2$. In this case, the determinant of the lattice $\langle z', e\rangle$ is equal to $q(z')q(e) - q(z', e)^2 = -2q(z') - 4 \geqslant 16$, hence the lattice $\langle z', e\rangle$ is also negative definite. ■

Proof of Theorem 5.1 in the case $k = 3$: Now $e^2 = -4$ and the discriminant group of the lattice is the cyclic group of order 4 which we view as a subgroup of $\mathbb{Q}/\mathbb{Z}$, and we have three possible subcases.

The first one $\delta(z) = 0$ is the same as above.

If $\delta(z) = \pm 1/4$, z is conjugate to e when $q(z) = -4$. Otherwise we write z as $4x + be$ where $x \in H^2(S, \mathbb{Z})$ and b is odd. Since $q(z) = 16q(x) - 4b^2$ is equal to -4 modulo 16 and is negative, we have $q(z) \leqslant -20$. Moreover, z is conjugate to some element of the form $z' = 4x' \pm e$, as $b^2 - 1$ is divisible by 8, and hence $16q(x) - 4b^2 + 4$ is divisible by 32. Now the lattice generated by e and z' is negative definite.

Some novelty appears when $\delta(z) = 1/2$. Then one still finds a negative definite lattice whenever $q(z) \neq -4$, but when $q(z) = -4$ then z is conjugate to z' of the form $2x' + e$ and the determinant of the lattice $\langle z', e\rangle$ is zero. ■

Proof of Theorem 5.1 in the case of greater k: We set $t = 2(k-1)$ so that $q(e) = -t$, and we eventually replace z by a multiple in order to write uniformly $z = tx + be$, $b \in \mathbb{Z}$, x primitive. We first remark is that one may suppose that $|b| \leqslant t/2$. Indeed $tx+be$ has the same square as $tx'+(b\pm t)e$ for some $x' \in H^2(S, \mathbb{Z})$ (the equation $t^2q(x) - tb^2 = t^2q(x') - t(b^2 \pm 2tb + t^2)$ on $q(x')$ has a solution since the terms tb^2 cancel and the other ones are divisible by $2t^2$), and when x' is primitive the two have the same divisibility: indeed by Example 5.4 this divisibility is g.c.d.(t, bt) resp. g.c.d. $(t, (b \pm t)t)$, and the two are clearly equal. Therefore $tx + be$ and $tx' + (b \pm t)e$ are in the same monodromy orbit.

We know that $q(z)$ is negative. For small t it means that $q(z) \leqslant -tb^2$: indeed this is the case when $q(x)$ is non-positive, and when $t \leqslant 8$ (that is, $k \leqslant 5$), already for $q(x) = 2$ one has $2t^2 - b^2t \geqslant 0$ provided $|b| \leqslant t/2$. Therefore the matrix of q on $\langle z, e\rangle$ is negative semidefinite, and one concludes in the same way as before.

Starting from $k = 6$, the statement fails. Indeed, let $k = 6$, so $t = 10$, and $z = 2x - e$ where x is primitive of square 2, then $z^2 = -2$ but the determinant of q on $\langle z, e\rangle$ is negative and will remail negative when we replace z by a conjugate. ■

Remark 5.5: In the last case mentioned, that is, $k = 6$, the computation above shows that the only case when the pair (M, z) cannot be deformed

to the Hilbert scheme of a nonprojective surface with cyclic Picard group is exactly as described: $z^2 = -2$, $z = 2x - e$ with x orthogonal to e, x of square 2. Under a generic deformation where e is a Hodge class, M becomes the Hilbert scheme of a $K3$ surface S with Picard group generated by x of square 2. Such an S is a double covering of $\mathbb{P}^2$. The interesting feature of this example is that we have two types of MBM curves with proportional MBM classes.

Namely consider the class $z/2$. By Lemma 4.6, there are rational curves in this class coming from g^1_4 on the inverse image of a line under the covering map (this inverse image is of genus 2). Let us describe the corresponding MBM locus. The complete linear system of degree 4 on a curve of genus 2 is a projective plane, so that the MBM locus is of codimension two and is generically fibered in projective planes over the product of the parameter space of possible linear systems and the Hilbert square of S (this factor is responsible for the two remaining points, as we are on $\mathrm{Hilb}^6(S)$). The parameter space in its turn is fibered in the jacobians of genus-two curves over the dual $\mathbb{P}^2$. This is well-khown to be birational to $\mathrm{Hilb}^2(S)$ as well. So up to birational equivalence, the MBM locus is a $\mathbb{P}^2$-bundle over the square of $\mathrm{Hilb}^2(S)$. Denote it by Z.

Now consider the class z. Again by Lemma 4.6, we have rational curves in this class coming from g^1_6 on the inverse image of a conic in $\mathbb{P}^2$, which is a curve of genus 5. The locus of such curves is now a divisor D, generically fibered in projective lines over a tenfold. The tenfold is rationally fibered over $\mathbb{P}^5$ (the space of plane conics) in the jacobians of genus 5 curves. As communicated to the authors by Emanuele Macri, one can check that it is actually birational to O'Grady's 10-dimensional example of an IHS.

Note that D contains Z, indeed "4 points on a line" is a degenerate case of "6 points on a conic" when the conic degenerates to a pair of lines.

By the unicity statement (Theorem 2.15), all MBM loci on an IHS twelvefold of K3 type which cannot be obtained from non-projective K3 surfaces are of the above form.

The computations evoke the paper [KLM] where many rational curves on Hilbert schemes of a K3 surface and on generalized Kummer varieties are constructed from special linear systems on (possibly nodal) curves via Brill-Noether theory. Notice that in this particular case no Brill-Noether theory is needed, since obviously any curve of genus 2 carries a g^1_4 and any curve of genus 5 carries a g^1_6. It seems likely and could be interesting to verify that such a description can be given in general.

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