Reflective lattices and hyperkähler manifolds

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Abstract: Using the results of Nikulin and Vinberg on the groups of isometries generated by reflections, we construct a subvariety called the Nikulin–Vinberg locus in the moduli space of polarized hyperkähler manifolds. It is obtained as a finite union of components of higher Noether–Lefschetz loci which parameterize manifolds with certain special Néron–Severi lattices. The Nikulin–Vinberg locus is the closure of the set of hyperkähler manifolds with Picard number $\geqslant 3$ which have finite groups of birational automorphisms. Using this construction and a refinement of an argument by Oguiso, we show that any non-trivial family of projective deformations of a hyperkähler manifold with $b_2(M) \geqslant 6$ has a dense set of fibers which have an infinite group of birational automorphisms.

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1 Introduction

The goal of this paper is to construct a certain family of special subvarieties in the moduli spaces of polarized hyperkähler manifolds, and to discuss the relation of those subvarieties with the properties of birational automorphism

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groups of the corresponding hyperkähler manifolds. Let us state our main result. We will denote by M a hyperkähler manifold in a fixed deformation class, by $h \in H^2(M, \mathbb{Z})$ the class of a polarization and by $\mathcal{M}_h$ the moduli space of hyperkähler manifolds polarized by h. We denote by $[M] \in \mathcal{M}_h$ the point corresponding to M, by $\mathrm{Aut}(M)$ and $\mathrm{Bir}(M)$ the groups of biregular and birational automorphisms of M, respectively. We will recall all necessary definitions below in Section 1.2.

Theorem 1.1: Assume that $b_2(M) \geqslant 6$. There exists a finite family of special subvarieties $\mathcal{X}_i \subset \mathcal{M}_h$, $i \in \{1, \dots, m\}$ and a countable family of special divisors $\mathcal{Y}_j \subset \mathcal{M}_h$, $j \in \mathbb{N}$ that have the following properties.

1. Assume that M is a hyperkähler manifold with $[M] \in \mathcal{M}_h$, $\mathrm{rk}\,\mathrm{NS}(M) \geqslant 3$ and $|\mathrm{Bir}(M)| < \infty$. Then there exists a unique $i \in \{1, \dots, m\}$ such that:
   (a) $[M] \in \mathcal{X}_i$;
   (b) For any M' such that $[M']$ is a very general point of $\mathcal{X}_i$, there exists an isometry between $\mathrm{NS}(M)$ and $\mathrm{NS}(M')$.
2. The family of divisors $\{\mathcal{Y}_j\}_{j\in\mathbb{N}}$ is strongly dense in $\mathcal{M}_h$ in the sense of Definition 2.8.
3. For every $j \in \mathbb{N}$ there exists a Zariski-open non-empty subset $\mathcal{Y}_j^\circ \subset \mathcal{Y}_j$ with the following property. Assume that M is a hyperkähler manifold such that $[M] \in \mathcal{Y}_j^\circ$ for some $j \in \mathbb{N}$. Then $\mathrm{Aut}(M)$ contains an element of infinite order.

The first part of the theorem above is based on a deep result of Nikulin [Ni] and Vinberg [Vi2] on the finiteness of the number of reflective Lorentzian lattices of rank at least three. That result was one of the major achievements within a vast research program initiated by Vinberg [Vi1], dedicated to the study of discrete groups of isometries generated by reflections in hyperbolic spaces. We will therefore call the union $\cup_i \mathcal{X}_i \subset \mathcal{M}_h$ of the subvarieties appearing in the above theorem the Nikulin–Vinberg locus. We refer to [Bel] for an overview of more recent developments related to Vinberg's program.

The first part of the theorem can be briefly (and imprecisely) summarized by saying that the polarized hyperkähler manifolds with Picard rank at least three having only finitely many birational automorphisms lie in the Nikulin–Vinberg locus, which is a proper closed subvariety of the moduli space.

The second and third parts of the theorem are based on an explicit (and elementary) construction of binary anisotropic[1] Lorentzian lattices of arbitrarily large discriminant, the construction that we will explain below. The above theorem has the following corollary.

[1] A quadratic lattice is called anisotropic if it does not contain a non-zero vector x with $q(x) = 0$.

Corollary 1.2: Let $\pi\colon \mathcal{M} \to B$ be a smooth non-isotrivial family of polarized hyperkähler manifolds with $b_2 \geqslant 6$ over a complex base B. Then there exists a dense subset $B' \subset B$ such that for $t \in B'$ the group $\mathrm{Bir}(M_t)$ is infinite, where $M_t = \pi^{-1}(t)$.

This corollary was first proven for K3 surfaces by Oguiso in [Og], and then extended to all known deformation types of hyperkähler manifolds in [DORV]. The result in fact holds in greater generality and does not rely on any information specific to the known families of hyperkähler manifolds. Both the above theorem and its corollary readily extend, for example, to hyperkähler orbifolds. Our proof is purely lattice-theoretic. The only technical condition that we impose concerns b_2, but we believe that it should be possible to get rid of this condition. We also note that the first part of the theorem does hold without assuming $b_2 \geqslant 6$.

Remark 1.3: So far we don't know whether an analogue of Corollary 1.2 holds with $\mathrm{Bir}(M_t)$ replaced by $\mathrm{Aut}(M_t)$. One way to prove such stronger form of Corollary 1.2 is by showing that the preimage in B of the locus $\cup_j \mathcal{Y}_j^\circ$ under the period map is dense in B. Here $\mathcal{Y}_j^\circ$ are the open subsets of the divisors $\mathcal{Y}_j$ appearing in part 3 of Theorem 1.1. General considerations along the lines of the Zilber–Pink conjecture indicate that this could be a reasonable expectation, at least in the case when the base B is algebraic.

1.1 Lattices

An integral lattice is a free $\mathbb{Z}$-module $\Lambda \simeq \mathbb{Z}^r$ together with a non-degenerate symmetric bilinear form $(\cdot,\cdot)\colon \Lambda \otimes \Lambda \to \mathbb{Z}$. The corresponding quadratic form will be denoted by q, so that $q(v) = (v,v)$ for any $v \in \Lambda$. We will also extend $(\cdot,\cdot)$ to a $\mathbb{Q}$-bilinear form on the vector space $\Lambda \otimes \mathbb{Q}$. The dual lattice $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z})$ is embedded into $\Lambda \otimes \mathbb{Q}$ via the form $(\cdot,\cdot)$, so that

$$\Lambda^* = \{v \in \Lambda \otimes \mathbb{Q} \mid \forall u \in \Lambda \ \ (u,v) \in \mathbb{Z}\},$$

and the quotient Λ^*/Λ is called the discriminant group, denoted $d(\Lambda)$. The exponent of the lattice is the smallest positive integer $e(\Lambda)$ which annihilates the discriminant group.

A vector $v \in \Lambda \otimes \mathbb{Q}$ with $q(v) \neq 0$ (such v is called non-isotropic) defines a reflection s_v, that is the following involution of $\Lambda \otimes \mathbb{Q}$:

$$s_v(u) = u - 2\frac{(u,v)}{(v,v)}v.$$

A primitive non-isotropic element $v \in \Lambda$ is called a root if $s_v(\Lambda) = \Lambda$. Equivalently, v is a root if $q(v)$ divides $2(u,v)$ for all $u \in \Lambda$. It is well-known (and follows directly from the definitions) that in this case $q(v)$ divides $2e(\Lambda)$, see e.g. [SW, section 1]. Any vector $v \in \Lambda$ with $0 < |q(v)| \leqslant 2$ is automatically a

root. Let us denote by $\mathrm{R}(\Lambda)$ the set of all roots $v \in \Lambda$ with $q(v) < 0$. Note our convention for the sign of the roots: the condition $q(v) < 0$ is not standard and not necessary for studying lattices in general, but for our applications only the negative roots will be relevant.

Given a lattice Λ, consider the group $\mathrm{O}(\Lambda)$ of orthogonal transformations of $\Lambda \otimes \mathbb{Q}$ preserving Λ. The reflections s_v, where $v \in \mathrm{R}(\Lambda)$, generate a subgroup $\mathrm{W}(\Lambda) \subset \mathrm{O}(\Lambda)$ called the Weyl group of Λ. If the lattice Λ has no negative roots, the Weyl group is trivial.

A lattice Λ is called Lorentzian if the quadratic form q has signature $(1, r-1)$. Note that in the literature dedicated to the theory of lattices another convention is often used: the Lorentzian signature is $(r-1, 1)$. Our convention is more convenient in the context of algebraic geometry.

A lattice Λ is called reflective if $W(\Lambda)$ is of finite index in $O(\Lambda)$. We refer to [Bel] for an overview of the results related to the classification of reflective Lorentzian lattices.

1.2 Moduli spaces of polarized hyperkähler manifolds

By a hyperkähler (also known as irreducible holomorphically symplectic or IHS) manifold M we will mean a compact simply connected complex $2n$-dimensional manifold that admits a Riemannian metric of holonomy $\mathrm{Sp}(n)$, see [Bea], [Ma] and [Ve] for the definitions and the foundations of the theory of such manifolds.

Let $\Lambda = H^2(M, \mathbb{Z})$ be the second cohomology of a hyperkähler manifold M equipped with the Beauville–Bogomolov–Fujiki (BBF) form q. The latter has signature $(3, b_2(M) - 3)$, and we will assume that it is rescaled to be integral and primitive. All deformations of M that admit Kähler metrics will also be hyperkähler manifolds with the same Λ and q.

From now on we fix a deformation class of hyperkähler manifolds and denote by M one of the manifolds in this deformation class. The moduli theory of the deformations of M is governed by the period domains attached to Λ. Let us recall their definitions.

Given a primitive sublattice $\mathbb{L} \subset \Lambda$ of signature $(1, k)$, its orthogonal complement $\mathbb{L}^\perp$ in Λ has signature $(2, b_2(M) - k)$. Let

$$\mathcal{D}_{\mathbb{L}} = \{x \in \mathbb{P}(\mathbb{L}^\perp \otimes \mathbb{C}) \mid q(x) = 0,\, q(x, \overline{x}) > 0\}.$$

This is a Hermitian symmetric domain whose points parametrize hyperkähler manifolds M with $\mathbb{L} \subset \mathrm{NS}(M)$, where the latter is the Néron–Severi group of M. For a very general (that is, lying outside of a countable union of proper subvarieties) point of $\mathcal{D}_{\mathbb{L}}$ we have equality $\mathbb{L} = \mathrm{NS}(M)$. We will call $\mathcal{D}_{\mathbb{L}}$ the period domain of $\mathbb{L}$-polarized hyperkähler manifolds. Note that the condition on the signature of $\mathbb{L}$ guarantees that all $\mathbb{L}$-polarized hyperkähler manifolds are projective. In particular, if $\mathbb{L} = \mathbb{Z}h$ for some primitive $h \in \Lambda$ with $q(h) > 0$, we will write $\mathcal{D}_h$ instead of $\mathcal{D}_{\mathbb{L}}$ and call it the period domain of h-polarized hyperkähler manifolds. We will denote the orthogonal complement $h^\perp$ in Λ by Λ_h.

Clearly, given two primitive sublattices $\mathbb{L}_1 \subset \mathbb{L}_2$, we have $\mathcal{D}_{\mathbb{L}_2} \subset \mathcal{D}_{\mathbb{L}_1}$. In particular, if $h \in \mathbb{L}$, we have $\mathcal{D}_{\mathbb{L}} \subset \mathcal{D}_h$. We call such $\mathcal{D}_{\mathbb{L}}$ a special subvariety of the h-polarized period domain $\mathcal{D}_h$.

The mapping class group $\Gamma = \mathrm{Diff}(M)/\mathrm{Diff}_0(M)$ acts on the Teichmüller space parametrizing the complex structures on M , see [Ve]. Let Γ^M denote its subgroup preserving the connected component of the Teichmüller space containing M. Recall that the image of Γ^M in $\mathrm{O}(\Lambda)$ under the natural action on the second cohomology is a finite index subgroup of $\mathrm{O}(\Lambda)$ called the monodromy group, denoted $\mathrm{Mon}(M)$. For $\mathbb{L} \subset \Lambda$ we will denote by $\mathrm{Mon}_{\mathbb{L}}(M) \subset \mathrm{Mon}(M)$ the (pointwise) stabilizer of the sublattice $\mathbb{L}$.

The image of $\mathrm{Mon}_{\mathbb{L}}(M)$ is of finite index in $\mathrm{O}(\mathbb{L}^\perp)$. It is well-known (see [BB]) that the quotient $\mathcal{M}_{\mathbb{L}} = \mathcal{D}_{\mathbb{L}}/\mathrm{Mon}_{\mathbb{L}}(M)$ is a quasi-projective variety. We will call this quotient the moduli space of $\mathbb{L}$-polarized hyperkähler manifolds. When $\mathbb{L} = \mathbb{Z}h$, the moduli space of h-polarized hyperkähler manifolds will be denoted by $\mathcal{M}_h = \mathcal{D}_h/\mathrm{Mon}_h(M)$. It comes with the collection of special subvarieties $\mathcal{M}_{\mathbb{L}}$ corresponding to the sublattices $\mathbb{L}$ containing h.

Recall how one can describe the (bimeromorphic) automorphism group of M in terms of $\mathrm{NS}(M)$. There exists a $\mathrm{Mon}(M)$-invariant collection of second cohomology classes in Λ called MBM classes, see [AV2]. We will denote them by $\mathrm{MBM} \subset \Lambda$. Recall from [AV1] that MBM consists of a finite number of $\mathrm{Mon}(M)$-orbits, in particular the BBF squares of the MBM classes are bounded.

Let $\mathbb{L} = \mathrm{NS}(M)$ and $\mathrm{MBM}_{\mathbb{L}} = \mathrm{MBM} \cap \mathbb{L}$. Denote by $\mathcal{C} \subset \mathbb{L} \otimes \mathbb{R}$ the connected component of the positive cone that contains the Kähler classes. Then for $x \in \mathrm{MBM}_{\mathbb{L}}$ the hyperplanes $x^\perp$ cut $\mathcal{C}$ into a locally finite collection of open subcones called Kähler chambers. The automorphism group of M is isomorphic to the subgroup of $\mathrm{Mon}(M)$ that preserves $\mathbb{L}$, the Kähler chamber of M and the period of M.

The divisorial MBM classes on M are elements of $\mathrm{MBM}_{\mathbb{L}}$ which are stably prime exceptional ([Ma], section 6; see also [AV0]). Let $W^+(\mathbb{L})$ be the subgroup of $W(\Lambda)$ that is generated by reflections in divisorial MBM classes. Recall from [Ma] that we have $W^+(\mathbb{L}) \subset \mathrm{Mon}(M)$. Since the roots that we are considering are always q-negative, the action of $W^+(\mathbb{L})$ on $\mathbb{L} \otimes \mathbb{R}$ preserves the cone $\mathcal{C}$. The fundamental domain of $W^+(\mathbb{L})$-action on $\mathcal{C}$ that contains a Kähler class of M is called the bimeromorphic Kähler chamber of M. The group of bimeromorphic automorphisms of M is isomorphic to the subgroup of $\mathrm{Mon}(M)$ that preserves the bimeromorphic Kähler chamber and the Hodge structure of M (in particular, $\mathbb{L} = \mathrm{NS}(M)$ is preserved automatically). It is finite exactly when $W^+(\mathbb{L})$ is of finite index in the orthogonal group ([Ma], theorem 6.18; see also [Vi3] for general lattice-theoretic treatment).

2 Subvarieties in the moduli space

2.1 The Nikulin–Vinberg locus

Here we will construct a family of subvarieties $\mathcal{X}_i \subset \mathcal{D}_h$ appearing in the first part of Theorem 1.1. We fix a positive class $h \in \Lambda$ with $q(h) = d$ that defines a polarization.

We will use the following notation: given a lattice $\mathbb{L}$ we denote by $\mathbb{L}(k)$ the same lattice with the form $(\cdot,\cdot)$ rescaled by an integer $k \neq 0$. Also, let $\mathbb{I}$ denote the rank one lattice with $q(v) = v^2$ for $v \in \mathbb{Z}$.

Proposition 2.1: There exists a finite collection of lattices $\mathbb{M}_i$, $i \in \{1, \dots, m_0\}$ with the following property. Suppose that M is an h-polarized hyperkähler manifold such that $\operatorname{rk} \mathrm{NS}(M) \geqslant 3$ and $\mathrm{Bir}(M)$ is finite. Then $\mathrm{NS}(M) \cap \Lambda_h \simeq \mathbb{M}_i$ for some i.

Proof: We denote by $\mathcal{NV}$ the set of isomorphism classes of integral reflective Lorentzian lattices of rank at least three that are unscaled (i.e. such that the Gram matrix of the lattice is indivisible). It follows form [Ni, Theorem 5.2.1] and [Vi2, Theorems 2 and 4] that the set $\mathcal{NV}$ is finite.

Recall from Section 1.2 that under our assumptions on M the orthogonal group of the lattice $\mathrm{NS}(M)$ contains a finite index subgroup generated by reflections in the roots that are also MBM classes. It follows that the lattice $\mathrm{NS}(M)$ is reflective, though possibly not unscaled. Recall also that the squares of the MBM classes are negative and bounded, the bound being deformation-invariant. Let $-N$ be the lower bound for the squares of the MBM classes. Define $\mathcal{NV}^+ = \cup_{k=1}^{N} \{\mathbb{L}(k) \mid \mathbb{L} \in \mathcal{NV}\}$. The latter is a finite set of isomorphism classes of lattices, and we see that $\mathrm{NS}(M) \in \mathcal{NV}^+$.

To conclude, recall e.g. from [KS, Satz 30.2] that for any lattice $\mathbb{L}$ there exist only finitely many primitive embeddings $\mathbb{I}(d) \hookrightarrow \mathbb{L}$, up to the action of $\mathrm{O}(\mathbb{L})$. For every $\mathbb{L} \in \mathcal{NV}^+$ and for every such primitive embedding $\mathbb{I}(d) \hookrightarrow \mathbb{L}$ consider its orthogonal complement $\mathbb{M} = \mathbb{I}(d)^\perp \subset \mathbb{L}$. Then there are, up to isomorphism, only finitely many lattices that can occur as such orthogonal complements. We denote these lattices $\mathbb{M}_i$, $i \in \{1, \dots, m_0\}$. Since $\mathrm{NS}(M) \in \mathcal{NV}^+$ and M is h-polarized, with $q(h) = d$, we see that $\mathrm{NS}(M) \cap \Lambda_h$ is one of the orthogonal complements constructed above. ■

We now define the family of subvarieties that form the Nikulin–Vinberg locus in the moduli space. Recall from Section 1.2 that $\mathrm{Mon}_h \subset \mathrm{O}(\Lambda_h)$ is a finite index subgroup. Combining this fact with [KS, Satz 30.2], we conclude that for each of the lattices $\mathbb{M}_i$, $i \in \{1, \dots, m_0\}$ from Proposition 2.1 there exist finitely many (up to the action of Mon_h) primitive embeddings $\mathbb{M}_i \hookrightarrow \Lambda_h$. Let us denote by $\mathbb{M}'_j \subset \Lambda_h$, $j \in \{1, \dots, m\}$ the images of such embeddings that represent every of the Mon_h-orbits, for all $\mathbb{M}_i$ from Proposition 2.1.

Definition 2.2: Let $\mathcal{X}_i \subset \mathcal{M}_h$, $i \in \{1, \dots, m\}$ be the image of $\mathcal{D}_{\mathbb{M}'_i}$ under the

quotient map $\mathcal{D}_h \to \mathcal{M}_h$. The union of all $\mathcal{X}_i$ is called the Nikulin-Vinberg locus of M.

2.2 Binary Lorentzian lattices

Given $n \in \mathbb{Z}$, we will say that a lattice $\mathbb{L}$ represents n if there exists a non-zero $v \in \mathbb{L}$ with $q(v) = n$. For a lattice $\mathbb{L}$ let us denote by $\mu(\mathbb{L})$ the following number

$$\mu(\mathbb{L}) = \max\big(\{q(v) \mid v \in \mathbb{L}\} \cap \mathbb{Z}_{<0}\big),$$

i.e. $\mu(\mathbb{L})$ is the largest negative integer represented by $\mathbb{L}$.

Given two lattices, the symbol $\oplus$ will be used to denote their orthogonal direct sum. A binary lattice is a lattice of rank two. We will consider binary Lorentzian lattices $\mathbb{L}_a = (\mathbb{Z}^2, q)$, where $q = \begin{pmatrix} 1 & 0 \\ 0 & -a \end{pmatrix}$ for some integer $a > 0$.

Proposition 2.3: Let $n > 0$ be a fixed integer. Then for any integer $b > 0$ there exist infinitely many integers $a > 0$ such that $\mathbb{L}_{ab}$ does not contain isotropic vectors and $\mu(\mathbb{L}_{ab}) < -n$, i.e. such that the lattice $\mathbb{L}_{ab}$ does not represent $0, -1, -2, \ldots, -n$.

Proof: For any $k \in \mathbb{Z}$ the condition that $\mathbb{L}_{ab}$ represents $-k$ means that there exist $x, y \in \mathbb{Z}$ such that $x^2 - aby^2 = -k$. This implies that $-k \equiv x^2 \,(\mathrm{mod}\, p)$ for any prime p dividing a. For any $k \in \{1, \ldots, n\}$ choose a prime p_k that violates the condition $-k \equiv x^2 \,(\mathrm{mod}\, p_k)$ for any x ($-k$ is not a quadratic residue). By Lemma 2.4 below there exist infinitely many such primes, so we may also assume that p_k are distinct and greater than b. Then define $a = \prod_k p_k$. By construction, the integer ab is not a square, hence $\mathbb{L}_{ab}$ does not represent zero, and since there are infinitely many choices of p_k for every k, there are also infinitely many choices of a as above. ■

Lemma 2.4: For any integer $k > 0$ there exist infinitely many primes p such that $-k$ is not a quadratic residue modulo p.

Proof: We have $k = 2^{a_0} \prod_i q_i^{a_i}$ where q_i are distinct odd primes, $a_i > 0$. Then

$$\left(\frac{-k}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right)^{a_0} \prod_i \left(\frac{q_i}{p}\right)^{a_i}, \tag{2.1}$$

where $\left(\frac{a}{b}\right)$ denotes the Legendre symbol. Imposing the condition $p \equiv -1 \,(\mathrm{mod}\, 8)$ will guarantee that

$$\left(\frac{-1}{p}\right) = -1, \quad \left(\frac{2}{p}\right) = 1,$$

implying that the product of the first two factors in (2.1) is equal to -1 for any a_0. Moreover, assuming that $p \neq q_i$ for all i, the congruence $p \equiv -1 \,(\mathrm{mod}\, 8)$ implies that we have

$$\left(\frac{q_i}{p}\right) = (-1)^{\frac{q_i - 1}{2}} \left(\frac{p}{q_i}\right),$$

by the quadratic reciprocity. Choosing the residues of p modulo q_i we can guarantee that $\left(\frac{q_i}{p}\right) = 1$ for all q_i. Then (2.1) will imply that $\left(\frac{-k}{p}\right) = -1$. By Dirichlet's theorem on arithmetic progressions there exist infinitely many primes p that satisfy the above congruences. ■

Remark 2.5: If one needs to construct a binary Lorentzian anisotropic lattice $\mathbb{M}$ that satisfies the condition $\mu(\mathbb{M}) < -n$ for a given n, then the lattices from Proposition 2.3 may be not the optimal choice from the computational point of view, in the sense that their discriminants are unnecessarily huge. In fact, a more practical solution may be to consider $\mathbb{M} = \mathbb{L}_{a^2-1}$ for $a > 1$. One can check by a direct computation, using the continued fraction expansion of $\sqrt{a^2-1}$, that $\mu(\mathbb{L}_{a^2-1}) = 2 - 2a$, and then one can take any $a > 1 + n/2$.

2.3 Divisors in the period domain

Given a lattice of rank two $\mathbb{L} \subset \Lambda$ that contains the given element $h \in \Lambda$, the manifold $\mathcal{D}_{\mathbb{L}}$ is a divisor in the period domain $\mathcal{D}_h$. We will construct a family of such divisors using the lattices from Section 2.2. In what follows, we will assume that $r = \mathrm{rk}(\Lambda) \geqslant 6$. Recall that h is a primitive vector that defines the polarization and $d = q(h) > 0$.

Let $\Lambda_h \subset \Lambda$ be the orthogonal complement of h. Then Λ_h is a lattice of signature $(2, r-3)$, therefore it represents zero ([Me]). Let $e \in \Lambda_h$ be a primitive isotropic vector, and let $(e, \Lambda_h) = m\mathbb{Z}$. Then Λ_h and $\widetilde{e} = e/m$ span an integral lattice $\widetilde{\Lambda}_h$ in the vector space $\Lambda_h \otimes \mathbb{Q}$. We have an inclusion $\Lambda_h \subset \widetilde{\Lambda}_h$ of index m. Now, $(\widetilde{e}, \widetilde{\Lambda}_h) = \mathbb{Z}$, hence we can find an isotropic element $\widetilde{f} \in \widetilde{\Lambda}_h$ with $(\widetilde{e}, \widetilde{f}) = 1$.

Given an integer a, let $u_a = \widetilde{e} - da\widetilde{f}$. Then $u_a \in \widetilde{\Lambda}_h$ is a primitive vector with $q(u_a) = -2da$, and $\mathbb{Z}u_a \oplus \mathbb{Z}h$ is a primitive sublattice in $\widetilde{\Lambda}_h \oplus \mathbb{Z}h$. By construction, $\mathbb{Z}u_a \oplus \mathbb{Z}h \simeq \mathbb{L}_{2a}(d)$.

Recall from Section 1.2 that we have a collection of MBM classes in Λ and that there exists $N > 0$ such that for any $v \in \mathrm{MBM}$ we have $-N \leqslant q(v) < 0$.

Proposition 2.6: There exists an infinite sequence of primitive binary anisotropic sublattices $\mathbb{M}_j \subset \Lambda$, $j \in \mathbb{N}$, with $h \in \mathbb{M}_j$ and $\mathbb{M}_j \cap \Lambda_h = \mathbb{Z}v_j$, where v_j is primitive with $q(v_j) \to -\infty$ for $j \to +\infty$, and such that $\mathbb{M}_j \cap \mathrm{MBM} = \varnothing$ for all $j \in \mathbb{N}$.

Proof: Consider the lattices $\mathbb{Z}u_a \oplus \mathbb{Z}h \simeq \mathbb{L}_{2a}(d)$ constructed above. Denote by T the index of $\Lambda_h \oplus \mathbb{Z}h$ in Λ. Let $a_j \to +\infty$ be a sequence of integers obtained from Proposition 2.3 with the constant $n = NT^2$, where N is the bound for the squares of the MBM classes, as recalled above.

We define v_j to be a generator of $\mathbb{Z}u_{a_j} \cap \Lambda_h$. Since $[\widetilde{\Lambda}_h : \Lambda_h] = m$, we have $v_j = m_j u_{a_j}$, where m_j is a divisor of m, so $q(v_j) \to -\infty$. By construction $\mathbb{Z}v_j \oplus \mathbb{Z}h$ is a sublattice of $\mathbb{L}_{2a_j}(d)$, so it does not represent the integers $-dNT^2, \dots, -1, 0$. Let $\mathbb{M}_j$ be the primitive sublattice of Λ spanned by $\mathbb{Z}v_j \oplus \mathbb{Z}h$. Then the index of the latter in $\mathbb{M}_j$ is not bigger than T. Therefore

$\mu(\mathbb{M}_j) < -dN$, because if $x \in \mathbb{M}_j$, then $Tx \in \mathbb{Z}v_j \oplus \mathbb{Z}h$, and $q(Tx) = T^2 q(x)$. We conclude that the lattices $\mathbb{M}_j$ do not contain MBM classes. They are anisotropic, because they are commensurable with $\mathbb{L}_{2a_j}(d)$. Also, by construction, $\mathbb{M}_j \cap \Lambda_h = \mathbb{Z}v_j$, with $q(v_j) \to -\infty$. ■

We now use the lattices constructed above to define a family of divisors in the period domain and in the moduli space.

Definition 2.7: Let $\mathbb{M}_j$ be the lattices from Proposition 2.6 and $\mathcal{D}_{\mathbb{M}_j} \subset \mathcal{D}_h$ the corresponding divisors. Note that $\mathcal{D}_{\mathbb{M}_j} = \mathcal{D}_h \cap \mathbb{P}(v_j^\perp \otimes \mathbb{C})$, where v_j is the sequence of vectors appearing in the above proposition. We define $\mathcal{Y}_j$ to be the images of $\mathcal{D}_{\mathbb{M}_j}$ in $\mathcal{M}_h$ under the quotient map $\mathcal{D}_h \to \mathcal{M}_h$.

2.4 Strong density of a family of divisors

We next discuss the density properties of the divisors constructed above.

Definition 2.8: We call a family of divisors $\{\mathcal{Y}_j\}_{j\in\mathbb{N}}$ in a complex analytic space $\mathcal{M}$ strongly dense if the following condition holds: for any non-constant holomorphic map $f\colon \Delta \to \mathcal{M}$, where Δ is the unit disc in $\mathbb{C}$, the subset $f^{-1}(\cup_j \mathcal{Y}_j)$ is dense in Δ.

Remark 2.9: Strong density clearly implies that $\cup_j \mathcal{Y}_j$ is dense in $\mathcal{M}$ in the analytic topology, because through any point $p \in \mathcal{M}$ there pass arbitrarily small holomorphic discs, and any of those discs has to intersect at least one of the divisors $\mathcal{Y}_j$. The converse is not true. For example, consider the linear projection $\pi\colon \mathbb{C}P^2 \dashrightarrow \mathbb{C}P^1$ from a fixed point in $\mathbb{C}P^2$. For a countable and dense set of points in $\mathbb{C}P^1$, their preimages under π form a family of lines that is dense but not strongly dense in $\mathbb{C}P^2$.

Our proof of strong density of the divisors constructed in Definition 2.7 is similar to Oguiso's arguments from [Og]: to find an intersection with $\{\mathcal{Y}_j\}$ in a neighbourhood of a point $f(0)$, we first draw a hyperplane through this point, in such a way that it is transversal to $f(\Delta)$, and then approach this hyperplane by $\{\mathcal{Y}_j\}$. To perform this in our setting we will need to recall the description of $\mathcal{D}_h$ as a homogeneous space of the orthogonal group, as well as an important step in the proof of the main result of [AV1].

The quadratic form q induced on the vector space $V = \Lambda_h \otimes \mathbb{R}$ has signature $(2, r-3)$, where r is the second Betti number of our hyperkähler manifold. Let us fix a subspace $V_0 \subset V$ with $\dim(V_0) = 2$ and $q|_{V_0} > 0$. Choose one of the two isotropic subspaces in $V_0 \otimes \mathbb{C}$ corresponding to one of the two points in $\mathcal{D}_h \cap \mathbb{P}(V_0 \otimes \mathbb{C})$ and denote it by ℓ_0 (equivalently, fix an orientation of V_0). Then $\mathcal{D}_h$ is the orbit of ℓ_0 under the action of $G = \mathrm{O}(V, q) \simeq \mathrm{O}(2, r-3)$[1]. The

[1] This is best seen when one interprets the period domain as the grassmannian of positive oriented 2-planes in the real second cohomology, indeed to give l_0 is the same as to give V_0 by taking V_0 generated by the real and the imaginary part of l_0

stabilizer of ℓ_0 is the compact group K isomorphic to $\mathrm{SO}(2) \times \mathrm{O}(r-3)$.

Denote by $\Gamma \subset G$ the image of $\mathrm{Mon}_h(M)$ under the natural action on the orthogonal complement of h. By the above description we see that $\mathcal{D}_h \simeq G/K$ and $\mathcal{M}_h \simeq \Gamma\backslash G/K$. Choose a point $v_0 \in V_0^\perp$ with $q(v_0) = -1$ and consider its stabilizer $P \subset G$. Then $P \simeq \mathrm{O}(2, r-4)$ and the P-orbit of ℓ_0 in $\mathcal{D}_h$ is a divisor $\mathcal{D}_0$ isomorphic to $P/(P \cap K)$ as a homogeneous space. There exists a sequence of elements $\gamma_j \in G$, $j \in \mathbb{N}$, such that $\gamma_j v_0 = \lambda_j v_j$, where v_j are the vectors appearing in Definition 2.7, and $\lambda_j \in \mathbb{R}$. As for the divisors appearing in that definition, we have $\mathcal{D}_{\mathbb{M}_j} = \gamma_j \mathcal{D}_0$. In other words, $\mathcal{D}_{\mathbb{M}_j}$ and $\mathcal{Y}_j$ are the images of the left coset $\gamma_j P \subset G$ under the projection $G \to G/K$, respectively $G \to \Gamma\backslash G/K$.

The group Γ is countable and we may enlarge the set of vectors v_j appearing in Definition 2.7 adding the Γ-orbit for each of those vectors. Then we may assume that the set $\{v_j\}_{j\in\mathbb{N}}$ is stable under the action of Γ. The condition on the squares of the vectors v_j appearing in Proposition 2.6 guarantees that the set $\{v_j\}_{j\in\mathbb{N}}$ consists of infinitely many Γ-orbits, and therefore we also have infinitely many Γ-orbits in the set $\{\gamma_j\}_{j\in\mathbb{N}}$ of the elements appearing above.

Theorem 2.10: [AV1] In the above setting the union of the left cosets $\bigcup_{j\in\mathbb{N}} \gamma_j P$ is dense in G (and $\gamma_j P$ are dense in the space of left cosets).

Proof: Although not claimed explicitly, this is the main step of the proof of [AV1, Theorem 1.7] applied to our situation. ■

In particular there is a sequence of $\mathcal{D}_{\mathbb{M}_j}$ converging to any translate $g_0 D_0$. It remains to show that we can choose g_0 in such a way that $g_0 D_0$ is transverse to a given image of a disc.

For this we analyze the action of G on $\mathbb{P}(T^*\mathcal{D}_h)$, where $T^*\mathcal{D}_h$ is the holomorphic cotangent bundle. Let us choose a base point $\widetilde{\ell}_0 \in \mathbb{P}(T^*\mathcal{D}_h)$ lying above $\ell_0 \in \mathcal{D}_h$. To do this, observe that $T_{\ell_0}\mathcal{D}_h$ is naturally isomorphic to $V_0^\perp \otimes \mathbb{C}$. The orthogonal complement of v_0 inside $T_{\ell_0}\mathcal{D}_h$ is a hyperplane, and we let $\widetilde{\ell}_0$ to be the corresponding point in the projectivisation of $T^*_{\ell_0}\mathcal{D}_h$. Denote by $\widetilde{\mathcal{D}}_h$ the G-orbit of $\widetilde{\ell}_0$.

The P-orbit of $\widetilde{\ell}_0$, denoted by $\widetilde{\mathcal{D}}_0$, is isomorphic to $P/(P \cap K) \simeq \mathcal{D}_0$, and this orbit is the canonical lifting of the divisor $\mathcal{D}_0$ to the projectivised cotangent bundle of $\mathcal{D}_h$, where each point of the divisor is lifted to the hyperplane (considered as a point in the dual space) tangent to the divisor.

Let us now consider the intersection $\widetilde{\mathcal{D}}_h \cap \mathbb{P}(T^*_{\ell_0}\mathcal{D}_h)$. This intersection is the K-orbit of $\widetilde{\ell}_0$. Note that $K \simeq \mathrm{SO}(2) \times \mathrm{O}(r-3)$, and the first factor acts trivially on $V_0^\perp \otimes \mathbb{C} \simeq T_{\ell_0}\mathcal{D}_h$, while for the second factor the latter space is the complexification of the standard representation. In particular, it follows that $T^*_{\ell_0}\mathcal{D}_h$ is irreducible as a complex representation of K.

Proposition 2.11: In the above setting, the collection of divisors $\{\gamma_j \mathcal{D}_0\}_{j\in\mathbb{N}}$ is strongly dense in $\mathcal{D}_h$, and $\{\mathcal{Y}_j\}_{j\in\mathbb{N}}$ is strongly dense in $\mathcal{M}_h$.

Proof: Let $f\colon \Delta \to \mathcal{D}_h$ be a non-constant holomorphic map. Since the choice of the base point $\ell_0 \in \mathcal{D}_h$ was arbitrary, we may assume that $\ell_0 = f(0)$. Consider the tangent vector $f'(0) \in T_{\ell_0}\mathcal{D}_h$. As we have observed above, the vector space $T^*_{\ell_0}\mathcal{D}_h$ does not contain any K-invariant complex subspaces. It follows that there exists $g_0 \in K$ such that the hyperplane in $T_{\ell_0}\mathcal{D}_h$ defined by $g_0\widetilde{\ell}_0$ does not contain $f'(0)$. In this case the divisor $g_0\mathcal{D}_0$ passes through the point $f(0)$ but does not contain the image of f. By the density result of Theorem 2.10, the divisor $g_0\mathcal{D}_0$ is the limit of a sequence of divisors $\gamma_{j_\alpha}\mathcal{D}_0$, $\alpha \to +\infty$. Note that all of those divisors are intersections of $\mathcal{D}_h$ and some hyperplanes in $\mathbb{P}(V \otimes \mathbb{C})$. Our claim about the divisors $\{\gamma_j\mathcal{D}_0\}_{j\in\mathbb{N}}$ now follows from Lemma 2.12 below. The claim about the divisors $\mathcal{Y}_j$ follows as well, because $\mathcal{Y}_j$ are the images of $\gamma_j\mathcal{D}_0$ under the quotient map $\mathcal{D}_h \to \mathcal{D}_h/\Gamma$. ■

Lemma 2.12: Let $f\colon \Delta \to \mathbb{C}P^n$ be a non-constant holomorphic map and $H \subset \mathbb{C}P^n$ be a hyperplane such that $f(0) \in H$ but $f(\Delta) \not\subset H$. Then for any sequence of hyperplanes H_i, $i \in \mathbb{N}$ that converge to H (as points of the dual projective space) when $i \to +\infty$, there exists $i_0 \in \mathbb{N}$ such that $f(\Delta) \cap H_{i_0} \neq \varnothing$.

Proof: We may pass to a local chart containing $f(0)$ and assume that H_i are defined by degree one polynomials g_i that converge to a degree one polynomial g that defines H. Then $\varphi = g \circ f$ is a non-constant holomorphic function on Δ with $\varphi(0) = 0$, and $\varphi_i = g_i \circ f$ are holomorphic functions converging to φ uniformly on compact neighborhoods of $0 \in \Delta$. The claim now follows from Rouché's theorem. ■

3 Proofs

3.1 Proof of Theorem 1.1

1. The family of subvarieties $\mathcal{X}_i$ is defined in Section 2.1, see Definition 2.2. Using the notations from the proof of Proposition 2.1, we see that $\mathrm{NS}(M) \in \mathcal{NV}^+$. Then the orthogonal complement of h in $\mathrm{NS}(M)$ is one of the lattices $\mathbb{M}_i$ constructed in that proposition. The period of M is contained in the special subvariety $\mathcal{D}_{\mathbb{M}_i}$, and for a very general point $[M'] \in \mathcal{D}_{\mathbb{M}_i}$ we have $\mathrm{NS}(M') \simeq \mathbb{M}_i$. This completes the proof of the first part.

2. The family of divisors $\{\mathcal{Y}_j\}_{j\in\mathbb{N}}$ is given by Definition 2.7. Their strong density is Proposition 2.11.

3. The hyperkähler manifolds parametrized by the divisors $\mathcal{Y}_j$ are projective, and there exist universal families $\pi_j\colon \mathcal{U}_j \to \mathcal{Y}_j^\circ$ over some non-empty Zariski open subsets $\mathcal{Y}_j^\circ \subset \mathcal{Y}_j$ (possibly after passing to a finite covering of those subsets). For a very general point $t \in \mathcal{Y}_j^\circ$, the fibre $M = \pi_j^{-1}(t)$ has $\mathrm{NS}(M)$ isomorphic to one of the rank two lattices constructed in Proposition 2.3. This lattice, by construction, contains no MBM classes, is in-

definite and does not represent zero. Hence its orthogonal group contains an infinite cyclic subgroup, and the same is true about the automorphism group of M, see Section 1.2. So $\mathrm{Aut}(M)$ contains an element of infinite order when $[M] \in \mathcal{Y}_j^\circ$ is a very general point. Using the standard Hilbert scheme argument (using the fact that the family $\mathcal{U}_j$ is projective), we may shrink $\mathcal{Y}_j^\circ$ and guarantee that the same holds for any $[M] \in \mathcal{Y}_j^\circ$.

3.2 Proof of Corollary 1.2

After possibly passing to the universal cover of B we may consider the period map $\psi\colon B \to \mathcal{M}_h$. Let $\mathbb{M}$ be the lattice that represents the isomorphism class of $\mathrm{NS}(M_t)$ for a very general $t \in B$. If the lattice $\mathbb{M}$ is not reflective, then $\mathrm{Bir}(M_t)$ is infinite for a very general $t \in B$, completing the proof. If $\mathbb{M}$ is reflective, then we consider three cases.

First assume that $\mathrm{rk}(\mathbb{M}) \geqslant 3$. Since $\mathbb{M}$ is reflective, $\mathrm{Im}(\psi) \subset \mathcal{X}_{i_0}$ for one of the subvarieties $\mathcal{X}_{i_0}$ from the first part of Theorem 1.1. We may also choose $\mathcal{X}_{i_0}$ to be the of the smallest possible dimension among $\mathcal{X}_i$ containing $\mathrm{Im}(\psi)$. Then, since our family is not isotrivial, we can find a dense subset $B'' \subset B$ such that $\mathrm{rk}\,\mathrm{NS}(M_t) > \mathrm{rk}\,\mathbb{M}$ for $t \in B''$ (see e.g. [Og]). Let

$$J = \{j \mid \dim(\mathcal{X}_j) < \dim(\mathcal{X}_{i_0})\}.$$

The image $\psi(B'') \subset \mathcal{X}_{i_0}$ can not be contained in the union $\cup_{j\in J}\mathcal{X}_j$, indeed otherwise it must be contained in one of $\mathcal{X}_j, j \in J$, and this is not possible by the choice of i_0. Setting $B' = \{t \in B'' \mid \psi(t) \notin \cup_{j\in J}\mathcal{X}_j\}$, we get the desired dense subset in B.

The second case is when $\mathrm{rk}(\mathbb{M}) = 2$. In this case we can find a dense subset $B'' \subset B$ such that $\mathrm{rk}\,\mathrm{NS}(M_t) > 2$ for $t \in B''$. As above, $\psi(B'')$ can not be contained in the Nikulin–Vinberg locus $\cup_i\mathcal{X}_i$, and we conclude by defining $B' = \{t \in B'' \mid \psi(t) \notin \cup_i\mathcal{X}_i\}$.

Finally, if $\mathrm{rk}(\mathbb{M}) = 1$, by the strong density of the divisors $\mathcal{Y}_j$ from Theorem 1.1, the preimage $B'' = \psi^{-1}(\cup_j\mathcal{Y}_j)$ is dense in B. For any point $t \in B''$, either $\mathrm{NS}(M_t)$ is of rank two and non-reflective (by the definition of $\mathcal{Y}_j$), or $\mathrm{rk}\,\mathrm{NS}(M_t) \geqslant 3$. Then, as above, we conclude by defining $B' = \{t \in B'' \mid \psi(t) \notin \cup_i\mathcal{X}_i\}$.

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