ON ALGEBRAICALLY COISOTROPIC SUBMANIFOLDS OF HOLOMORPHIC SYMPLECTIC MANIFOLDS

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To Claire Voisin, for her birthday volume

Abstract. We investigate algebraically coisotropic submanifolds X in a holomorphic symplectic projective manifold M. Motivated by our results in the hypersurface case, we raise the following question: when X is not uniruled, is it true that up to a finite étale cover, the pair (X, M) is a product $(Z \times Y, N \times Y)$ where N, Y are holomorphic symplectic and $Z \subset N$ is Lagrangian? We prove that this is indeed the case when M is an abelian variety and give some partial answer when the canonical bundle K_X is semi-ample. In particular, when K_X is nef and big, X is Lagrangian in M (using a recent text of Taji, we could also obtain this for X of general type). We also remark that Lagrangian submanifolds do not exist on a sufficiently general Abelian variety, in contrast to the case when M is irreducible hyperkähler.

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1. Introduction and main results

1.1. **Algebraically coisotropic submanifolds.** Let M be a complex projective manifold equipped with a holomorphic symplectic form σ. In particular, M is of even dimension $2n$. Let X be an irreducible complex submanifold of M. By linear algebra, at every $x \in X$, the corank of the restriction $\sigma|_X$ at x is at most $c = codim(X)$. The equality holds exactly when $T_{X,x}$ contains its σ-orthogonal.

Definition 1. *X is* **coisotropic** *if the corank of $\sigma|_X$ is equal to c at each*[1] *point of X. In this case, the kernel of $\sigma|_X$ defines a regular foliation*[2] *$\mathcal{F}$ of rank c on X, and the rank of $\sigma|_X$ is $2(n-c)$. The foliation $\mathcal{F}$ is called the* **characteristic foliation** *of X.*

It follows from the definition that the codimension c of a coisotropic subvariety X is at most n, and if it is equal to n, then the restriction of σ to X is zero. In this case, X is called **lagrangian**. Recall also that a subvariety Y is **isotropic** if $\sigma|_Y = 0$; the dimension of an isotropic subvariety is at most n, and to be lagrangian is the same as to be isotropic and coisotropic at the same time.

In general, the set of the coisotropic submanifolds of M depends on the choice of the form σ; when σ is not specified, we say that a submanifold of M is coisotropic when it is coisotropic for some σ on M. One important and most studied case is when M is **irreducible holomorpic symplectic (IHS)**, or **hyperkähler**: that is, M is simply-connected and $h^{2,0}$ is one-dimensional. In this case, obviously the notion of the coisotropy is independent of the form. When M is a product of IHS manifolds, by Künneth formula, any symplectic form is a product of forms lifted from the factors, and so the notion of coisotropy does not depend on the choices in the case when X is itself a compatible product. Recall that by Bogomolov decomposition theorem, a finite étale covering of a holomorphic symplectic M is a product of several IHS manifolds and a complex torus.

A regular foliation on an algebraic manifold is said to be **algebraic** (or **algebraically integrable**) if its leaves are algebraic submanifolds. If the characteristic foliation on X is algebraic, X is said to be **algebraically coisotropic**. In this case, there is a fibration $f : X \to B$ which has the leaves of $\mathcal{F}$ as fibers. This fibration is 'quasi-smooth' in the sense that all of its fibres have smooth reduction. We call it the **characteristic fibration** of X.

Example 2. *Any smooth hypersurface D of M is coisotropic, since $\sigma|_D$ is of even rank and therefore degenerate.*

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[1] This definition coincides with those of [23] and [29] for X smooth. More generally, when X is a possibly singular subvariety, one calls X coisotropic when $T_{X,x}$ contains its σ-orthogonal for every smooth point $x \in X$.

[2] Apriori only a distribution, it is a foliation because σ is d-closed. See [23], §2.1.

Lagrangian subvarieties of M are algebraically coisotropic: the characteristic foliation has then a single leaf, X itself. In particular, curves on a holomorphic symplectic surface (either $K3$ or Abelian) are algebraically coisotropic.

1.2. **Divisors.** It is easy to see that a smooth uniruled hypersurface is algebraically coisotropic with rational curves as leaves. On the opposite side, Hwang and Viehweg [17] have shown that a smooth algebraically coisotropic hypersurface of general type is a curve in a holomorphically symplectic surface. Several important observations have been made in [23]. In [2], we have described the algebraically coisotropic divisors as follows.

Theorem 3. *A smooth algebraically coisotropic hypersurface D in a holomorphic symplectic manifold M is either uniruled, or, up to a finite étale cover of M, of the following form: $D = C \times Y \subset M = S \times Y$, where Y is a holomorphic symplectic manifold, S is a holomorphic symplectic surface and C is a curve on S.*

1.3. **Higher codimension.** It is natural to ask whether a similar description is still valid for algebraically coisotropic subvarieties of higher codimension.

The analogues of curves in holomorphically symplectic surfaces are Lagrangian subvarieties. The most optimistic higher-codimensional analogue of Theorem 3 would be the positive answer to the following.

Question 4. *Let $X \subset M$ be a non-uniruled algebraically coisotropic submanifold in a projective holomorphic symplectic manifold M. Up to a finite étale cover, is $X = L \times Y$ and $M = N \times Y$, where Y and N are holomorphically symplectic, and $L \subset N$ is Lagrangian in N?*

In particular, is X Lagrangian if K_X is big? Is X Lagrangian if M is either a simple Abelian variety, or an irreducible hyperkähler manifold?

Remark 5. *While every hypersurface is coisotropic, most of higher-codimensional submanifolds are not, in fact coisotropy is a cohomological condition (see* [29]*, lemma 1.4). Notice however that not all coisotropic submanifolds are of the type described in the question 4: e.g. when M is irreducible hyperkähler, a complete intersection of two hypersurfaces with zero Beauville-Bogomolov intersection is coisotropic, essentially by definition of the Beauville-Bogomolov form. In particular when M has a Lagrangian fibration $f : M \to \mathbb{P}^n$, the inverse image of a codimension-two complete intersection (and more generally, of any submanifold) $Y \subset \mathbb{P}^n$ is coisotropic. See Proposition 17 for a result on its characteristic foliation.*

Remark 6. *A positive answer to question 4 would reduce the description of algebraically coisotropic $X \subset M$ to the Lagrangian submanifolds of symplectic manifolds. Up to a finite covering, these are products*

$M = \Pi_i M_i$, where the M_i are either simple Abelian varieties or IHS. However, the symplectic form does not need to be a product of forms on the factors if there is a non-trivial torus part (see [2]*).*

The aim of this note is to prove several very partial results on these questions. Concerning Lagrangian submanifolds, we provide a brief survey on IHS in §4 and make some remarks on abelian varieties in §5.

Theorem 7. *Let X be an algebraically coisotropic submanifold of a holomorphic symplectic manifold M. Let $f : X \to B$ be its characteristic fibration. Suppose that K_X is semiample. Then f is isotrivial, and $\kappa(X) = \kappa(F)$, where F is a smooth fibre of f.*

The proof of Theorem 7 is a direct adaptation of the one given in [2] as the main step of the proof of Theorem 3.

Using [27], one can weaken the semi-ampleness condition as follows:

Theorem 8. *Let X be an algebraically coisotropic submanifold of a holomorphic symplectic manifold M. Let $f : X \to B$ be its characteristic fibration. Suppose that the smooth fibres of f have good minimal models. Then f is isotrivial, and $\kappa(X) = \kappa(F)$, where F is a smooth fibre of f.*

One may expect that the pseudo-effectivity of K_X is sufficient for the conclusions of Theorem 8, but new techniques are certainly required. See remark 4.3 for a related question.

Corollary 9. *Let M be a holomorphic symplectic manifold and $X \subset M$ an algebraically coisotropic submanifold. Assume that K_X is nef and big, or more generally that X is of general type. Then X is lagrangian.*

Indeed if K_X is nef and big, then so is $K_F = K_X|_F$, and Kawamata base point free theorem implies that K_F is semiample. Hence $\kappa(X) = \kappa(F)$, and by bigness of K_X we must have $X = F$, that is, X is lagrangian. If X is of general type so is F, in particular it has a good minimal model, and we obtain the same conclusion from Theorem 8.

Note that this is exactly the analog of Hwang-Viehweg theorem in higher codimension.

Corollary 10. *If M is a simple abelian variety, and $X \subset M$ is algebraically coisotropic, then X is Lagrangian.*

Indeed all positive-dimensional submanifolds in a simple abelian variety have ample canonical bundle ([16], Proposition 4.1).

Examples of Lagrangian submanifolds in simple abelian varieties of dimension $2n \geq 4$ seem difficult to construct. O. Debarre and C. Voisin have informed us of one such construction for $n = 2$, due to Schoen, see [24], [13]. We do not know any higher-dimensional examples. They do not exist on sufficiently general ("Hodge-general") abelian varieties, by Corollary 21.

The answer to the question 4 is positive when M is an Abelian variety.

Theorem 11. *Let M be an abelian variety, and $X \subset M$ an algebraically coisotropic submanifold. Then after a finite étale cover , there are subtori D, N, C, P of M such that $M = D \times C \times N \times P$, and a submanifold $Z \subset N$ such that:*

1. Z generates N;

2. $X = D \times C \times Z$;

3. The restriction σ_N of σ to N is symplectic, and Z is Lagrangian in N;

4. The characteristic fibration of X is the projection $f : X \to D$ with fibres $C \times Z$, and the restriction σ_D is symplectic;

5. $dim(C) = dim(P)$; in particular if X generates M, then $C = 0$ and moreover $\sigma = \sigma_D \oplus \sigma_N$.

Conversely, given the tori D, C, N, P, the symplectic forms σ_N and σ_D, and a σ_N-lagrangian submanifold of general type Z generating N, then $X = D \times C \times Z$ is algebraically coisotropic in $D \times C \times N \times P$ for a suitable σ which restricts as σ_N resp. σ_D to N resp. D.

Remark 12. *We shall describe all such σ in the proof of Theorem 11.*

2. Proof of Theorems 7 and 8.

The proof of these theorems uses several auxiliary results. First of all, let us recall some general background. We follow [2] where we considered only algebraically coisotropic hypersurfaces, but many starting remarks are valid in general.

Let X be algebraically coisotropic, then the family of leaves of the characteristic foliation $\mathcal{F}$ defines a morphism from X to its Chow variety. Normalizing the image if necessary, we obtain a fibration $f : X \to B$ with fibers which are leaves of $\mathcal{F}$ (the "characteristic fibration"). We recall that the codimension c of X is equal to the relative dimension of f, and we usually denote by F the general fiber of f. Each (set-theoretic) fiber is smooth but some of them can be multiple as f is not necessarily a smooth morphism. Equivalently, some leaves of f have non-trivial holonomy. The base B can be singular at the corresponding points, but has only quotient singularities. In fact all holonomy groups of the foliation are finite, and locally in the neighbourhood of the fiber F_0, X is diffeomorphic to the quotient of $T \times F$ by the holonomy group G, where F is the Galois covering of F_0 with Galois group G, T is a local transverse and G acts diagonally. This result is known in foliation theory as **Reeb stability**. The map f in the neighbourhood of F_0 is the projection to T/G.

Complex-analytically, it is not always true that a neighbourhood of F_0 in X is a quotient of the product, because the complex structure

on the general leaves F can vary; but it remains true that the quotient T/G gives a local model for B.

The following lemma is due to J. Sawon (see [23], lemma 6, for the case of hypersurfaces). The argument is purely local on X, depending only on the d-closedness of σ.

Lemma 13. *Let X be smooth and algebraically coisotropic, and let $f : X \to B$ be the characteristic fibration. Denote by $f : X^0 \to B^0$ the restriction to the smooth locus (that is, $B^0 = B - E$, where $b \in E$ iff the fiber X_b is a multiple fiber). Then there is a holomorphic symplectic form η on B^0 with $f^*\eta = \sigma|_X$.*

Proof: As in [23], we remark that the restriction of σ gives a holomorphic symplectic form on any local transverse, and that if U_1 and U_2 are two local transverses over the same small $U \subset B^0$, then the natural isomorphism $\phi : U_1 \to U_2$ (sending the intersection point of U_1 with some leaf L to that of U_2 with L) can be viewed as a $t = 1$ map of a flow ϕ_t preserving $\sigma|_X$. Using the d-closedness of σ, one obtains as in [23] that the forms agree on local models: $\sigma|_{U_1} = \phi^*\sigma|_{U_2}$, yielding a global holomorphic symplectic form η on B^0.

The next result has been obtained for hypersurfaces in [2], Lemma 2.3, and its proof in the general case is analogous.

Proposition 14. *In the situation of the preceding lemma, f has no multiple fibers in codimension one, so that σ descends to a symplectic form η outside of a subset of codimension at least two in B. Moreover, the highest exterior power $\alpha = \eta^{n-c}$ trivializes K_B, and B has only canonical singularities. Therefore:*

1. Any smooth model of B has $\kappa = 0$, and in particular B is not uniruled.

2. The smooth locus B^0 of B is special in the sense of [3].

Proof: The argument for the absence of multiple fibers in codimension one is that of [2], Lemma 2.3, we only adapt the notations. We set $\omega = \sigma^{n-c}$, so that $f^*\alpha = \omega$ over B^0. Suppose there is a divisor of multiple fibers, then, replacing B by a small open neighbourhood of a general point of the image E of this divisor, we may assume that B is a polydisk with coordinates $(u_1, \dots, u_{2(n-c)}) = (u, u')$, and E is given by $u = 0$ (u' denotes the $2(n-c)-1$-tuple of remaining coordinates) . Moreover, we can choose local coordinates $(z_1, \dots, z_{2(n-c)}, w_{2(n-c)+1}, \dots w_{2n-c}) = (z, z', w)$ on X such that f is locally given by $u = z^m, u' = z'$. Then the same calculation as in [2], Lemma 2.3, shows that $m = 1$.

Hence α is a nonvanishing holomorphic $2(n-c)$-form defined over a complement of a subset of codimension at least two, and by definition it trivializes K_B. Moreover, since B has only quotient singularities, a lemma due to Freitag [14] shows that α extends to any resolution

$r : B' \to B$ of B, and so the singularities of B are canonical, hence $\kappa(B') = 0$. If E is the reduced exceptional set of r, then $h^0(B', m(K_{B'} + E)) = h^0(B', mE') = 1$, for each $m > 0$, since $K_{B'} + E := E' \geq E$ is an effective divisor supported on E. Thus $\kappa(B', K_{B'} + E) = 0$, and (B', E) is special.

Notice also that in the exact sequence $0 \to \mathcal{F} \to T_X \to N_F \to 0$, the normal bundle N_F is equipped with a non-degenerate two-form, hence it has trivial determinant. It follows that $K_{\mathcal{F}} = K_X$.

Proof of Theorem 7:

When K_F is semiample, the results of [3] (Theorem 5.1, Corollary 5.2) apply, yielding that f is an isotrivial fibration. Moreover by proposition 14, f has no multiple fiber in codimension one. Therefore $K_X = K_{X/B} + f^*K_B$. By the same proposition, K_B is trivial and so $K_X = K_{X/B}$.

By isotriviality, the family $f : X \to B$ trivializes after a finite covering $B' \to B$: the base change $f' : X' \to B'$ is the projection from $X' = F \times B'$ to the second factor. Let g be the natural base-change morphism from X' to X and h the projection from X' to F. Then $h^*(K_F) = K_{X'/B'} = g^*K_{X/B}$. Hence $\kappa(F)$ equals to the Iitaka dimension of $K_{X/B}$ which is equal to $\kappa(X)$, q. e. d.

Proof of Theorem 8:

We repeat the preceding arguments, replacing [3] (Theorem 5.1, Corollary 5.2) by [27], Theorem 1.1,applied to the special base B^0.

Proof of Corollary 10:

The structure of a submanifold of an abelian variety is described in [28]: if X is such a submanifold, then there exists an abelian subvariety $A \subset M$ such that $X = p^{-1}(Z)$, where p is the projection from M to the quotient abelian variety $L = M/A$ and $Z \subset L$ is a subvariety of general type. Now if M is simple then A is trivial and $Z = X$, so that Corollary 9 applies (note that K_Z is nef and big, and even ample when M is simple).

3. Proof of Theorem 11.

Let M be an abelian variety and let X be an algebraically coisotropic submanifold of M. We derive from Ueno's classification result [28], Theorem 10.9, that K_X is semiample: indeed in the notations right above, K_X is the inverse image of K_Z which is nef and big hence semiample by Kawamata base point freeness. Clearly, in the same notations, the restriction $g : X \to Z$ of the projection $p : M \to L = M/A$ is the Iitaka fibration of X, the fibers of g are isomorphic to A and $\kappa(X) = \kappa(Z) = dim(Z)$. Denote by N the subtorus of L generated by Z.

Since K_X is semiample, Theorem 7 applies. Let $f : X \to B$ be the characteristic fibration on X, it is isotrivial with general fiber F. Since $\kappa(X) = \kappa(F)$, the restriction of g to F is the Iitaka fibration of F. Applying Ueno's theorem again, we see that the fiber of $g|_F$ is a subtorus C of A.

Now Poincaré complete reducibility yields that passing to a finite étale covering, we may assume $A = D \times C$, $L = N \times P$ and $M = D \times C \times N \times P$, so that $Z \subset N$ induces the embedding of $X = D \times C \times Z$ into M, and $g : X = D \times C \times Z \to Z$ and $f : D \times C \times Z \to D$ are natural projections.

We shall now analyze the form σ according to the decomposition $M = D \times C \times N \times P$. We view any torus as the quotient of its tangent space by a lattice, and by abuse of notation denote by the same letter a subtorus of M and its tangent space at any point. By the coisotropy condition $C + T_{Z,z}$ is orthogonal (with respect to σ) to $C + D$ at each point $z \in Z$. Thus C and D are orthogonal, and $C + D$ is orthogonal to $T_{Z,z}, \forall z \in Z$. Since Z generates the subtorus N, the spaces $T_{Z,z} \subset N$ generate the vector space N (that is: T_N), by the Gauss map. Thus $C + N$ and $C + D$ are orthogonal. Equivalently, C is isotropic, and C, N and D are pairwise orthogonal.

Künneth formula then implies that $\sigma = \sigma_D \oplus \sigma_N \oplus \sigma_P \oplus s$, where σ_N is the restriction of σ to N, similarly for D, P, and s comes from an element in $H^{1,0}(P) \otimes H^{1,0}(D \times C \times N)$. This means the following: we have a pairing $s(p, q) = \sum_i \alpha_i(p)\beta_i(q)$ for $p \in P,\ q \in Q := D \times C \times N$ and linear forms $\alpha_i,\ \beta_i$, and we associate to it the alternating bilinear form s' on $P \times Q$ defined by: $s'(p + q, p' + q') := s(p, q') - s(p', q)$. To simplify the notation we still write s for s', keeping the definition $s(p + q, p' + q') := s(p, q') - s(p', q)$. Any alternating 2-form on $P \times Q$ is uniquely written as: $\sigma_P \oplus \sigma_Q \oplus s$, for alternating 2-forms σ_P, σ_Q on P, Q, and s as above on $P \times Q$, characterised by $s(P, P) = s(Q, Q) = 0$.

Remark that the restriction σ_D of σ to D is a symplectic form, for example by lemma 13. The following claim is crucial:

Claim 15. *The restriction σ_N of σ to N is a symplectic form, and Z is coisotropic in N.*

Indeed let K be the kernel of σ_N. It gives a foliation $\mathcal{K}$ on N, which is a trivial subbundle in the tangent bundle T_N. The orthogonality relations established above imply that K is contained in the σ-orthogonal to $D \times C \times N$, hence also in the σ-orthogonal to T_X at any point $x \in X$. So K is also contained in the σ_N-orthogonal to T_Z at any point $z \in Z$. In other words, $\mathcal{K}|_Z \subset T_Z$ (that is, Z is invariant by the foliation $\mathcal{K}$). But since Z is of general type, it has only finitely many automorphisms. So $H^0(Z, T_Z) = 0$ and T_Z cannot have a trivial subbundle of positive rank, hence $K = 0$ and σ_N is symplectic.

To deduce the second part of the claim, consider the restriction σ' of σ to $D \times C \times N$. Since the σ'-orthogonal to T_X is contained in its σ-orthogonal, X is also coisotropic with respect to σ' (by definition, we say that X is coisotropic with respect to an arbitrary, not necessarily nondegenerate form σ', if T_X contains its σ'-orthogonal). By the orthogonality relations above, this implies that Z is σ_N-coisotropic in N, finishing the proof of the claim.

We now give two different proofs of the fact that $Z \subset N$ is Lagrangian and $dim(C) = dim(P)$. The first one is shorter, the second one additionally gives information on σ.

First proof: We start by recalling that X is σ-coisotropic in M and remark that $Q = D \times C \times N$ is σ-coisotropic in M as well: indeed since $X \subset Q$, the σ-orthogonal to Q is contained in the σ-orthogonal to T_X at every point of X, so it is also contained in Q at every point of X. By triviality of the tangent bundle of a torus, the orthogonal to a subtorus at a point does not depend on this point. Hence the σ-orthogonal to Q is contained in Q at every point, meaning that Q is coisotropic.

By definition of coisotropy, the codimension of $Q = D \times C \times N$ in M is equal to the corank of the restriction of σ to $D \times C \times N$, that is, $dim(P) = dim(C)$. In the same way, the coisotropy of X in M gives $dim(C \times Z) = codim(X, M)$ (indeed the fiber of the coisotropic fibration is $C \times Z$), hence $dim(C) + dim(Z) = codim(Z, N) + dim(P)$ and so again $dim(Z) = codim(Z, N)$, meaning that Z is σ_N-Lagrangian in N.

Second proof:

Lemma 16. *The restriction $s_{P\times C}$ is symplectic on $P\times C$, hence $dim(C) = dim(P)$ and $Z \subset N$ is σ_N-Lagrangian.*

Proof. We fix $x \in X$, and compute the σ-orthogonal $TX_x^\perp$ of TX_x, which is, by assumption, $C \times TZ_z$, if z is the Z-component of x.This orthogonal consists of the quadruples $(d, c, n, p) \in D \times C \times N \times P$ such that $\sigma((d, c, n, p), (d', c', n')) = 0, \forall (d', c', n') \in D \times C \times TZ_z$.This value of σ is equal to $\sigma_D(d, d') + \sigma_N(n, n') + s(p, (d', c', n'))$. By the linearity of this expression, for (d, c, n, p) fixed, in (d', c', n'), this vanishing is equivalent to the following system of equations:

1. $s(p, c') = 0, \forall c' \in C$.
2. $\sigma_D(d, d') + s(p, d') = 0, \forall d' \in D$.
3. $\sigma_N(n, n') + s(p, n') = 0, \forall n' \in TZ_z$.

Assume the first equation had a nonzero solution p. Since σ_N and σ_D are symplectic, this solution p could be uniquely lifted to a solution d (resp. n) of the second (resp. third) equation, and we would have elements $(d, c, n, p \neq 0)$, with $c \in C$ arbitrary, in $TX_x^\perp$. Contradiction, since $TX^\perp \subset TX \subset Q$. The only solution p to equation 1 is thus $p = 0$. The map from P to the dual of TC induced by $s_{P\times C}$ is therefore

injective, and so $dimP \leq dimC$. Since moreover $s(c, c') = 0$ for $c, c' \in C$, the kernel of $s_{P\times C}$ is zero, that is, $s_{P\times C}$ is symplectic. As C is isotropic for $s_{P\times C}$, also $dimC \leq dimP$, that is $dimC = dimP$.

Since $p = 0$ if $(d, c, n, p) \in TX_x^{\perp}$, from the second and third equations, we get: $d = 0$, and n is any element of the σ_N-orthogonal of TZ_z, $c \in C$ being arbitrary. We thus get also: $C \times TZ_z = TX_x^{\perp} = C \times TZ_z^{\perp}$, the first equality following from the characteristic fibration, $TZ_z^{\perp}$ being the σ_N-orthogonal. We deduce that $TZ_z = TZ_z^{\perp}$, so that Z is Lagrangian. □

Conversely, if we are given any $Z, N, , D, C, P, \sigma_N, \sigma_D, \sigma_P, s$ such that:
1.σ_N, σ_D symplectic.
2. $Z \subset N$, Z of general type generating N, σ_N-Lagrangian (so that $dimN$ is even).
3. $s(p, c)$ symplectic on $P \times C$, C, P maximal isotropic (so that $dimC = dimP$).

Then $X := D \times C \times Z$ is σ-coisotropic, with characteristic fibration $f : X \to D$. This is easy to check, using the preceding arguments.

4. The Irreducible Hyperkähler case.

4.1. **Some known examples of Lagrangian submanifolds.** Lagrangian submanifolds of IHS manifolds have been studied to some extent, especially when the dimension of the ambient IHS manifold is 4. Many examples are exhibited in [30]. C. Voisin also shows that the codimension, in the component of the deformation space of an IHS manifold M, of the locus where the lagrangian submanifold $X \subset M$ deforms together with M is equal to the rank of the restriction map $H^2(M, \mathbb{Q}) \to H^2(X, \mathbb{Q})$. In particular, a lagrangian $\mathbb{P}^n \subset M$ deforms together with M in codimension 1.

For instance the Hilbert square of a K3 surface S of genus two contains a lagrangian $\mathbb{P}^2$, indeed S is a double covering of a plane and the fibers of the covering define a plane in the Hilbert square. In the same spirit, an Enriques surface S' has a K3 covering S and so one obtains an embedding of S' into $S^{[2]}$ with Lagrangian image (because S' does not carry holomorphic 2-forms).

Within M, a lagrangian $\mathbb{P}^n$ is isolated. But there are also examples of dominating families of lagrangian submanifolds. It is easy to see that any lagrangian torus deforms in a covering family (and in fact is a fiber of a lagrangian fibration, see [1] for dimension 4, and [18] in general). One can also have a dominating family of lagrangian submanifolds of general type[3]. Indeed the variety of lines $M = F(V)$ of a cubic fourfold V is an IHS fourfold of K3 type and the surface of lines S_Y contained in a hyperplane section Y of V is Lagrangian. This results from the fact

[3]E. Macri informed us of a conjecture stating that actually every projective IHS manifold is covered by lagrangian submanifolds

that $[\sigma]$ comes from a class $\eta \in H^{3,1}(V)$ via the "Abel-Jacobi map", that is the composition of the pullback to the universal family of lines on V with subsequent projection to $F(V)$ (see e.g. [4]). One also computes that these surfaces are of general type, using the fact that these are zero-loci of sections of the restriction of the universal bundle from the grassmannian to M. Notice that the surfaces S_Y cover $F(V)$.

Another source of examples is described in [10]: the fixed locus of an antisymplectic involution of a holomorphic symplectic manifold is Lagrangian. This applies for instance to the double EPW sextic (a double covering of a sextic of a certain type in $\mathbb{P}^5$ which turns out to be an IHS fourfold of K3 type, see e.g. [21]). The fixed locus of the covering involution is a smooth lagrangian surface Σ of general type which does not move in a family. However, E. Macri communicated to us that there is a covering family of lagrangian surfaces in the cohomology class $2[\Sigma]$.

In conclusion, on IHS manifolds we know a considerable variety of lagrangian submanifolds, with negative, zero, torsion or positive canonical class. These submanifolds can move or be fixed. The situation is very different for abelian varieties. See §5.

4.2. **Lagrangian fibrations.** Let $f : M \to B$ be a Lagrangian fibration on an IHS manifold M, with discriminant hypersurface $\Delta \subset B$. Let X be a "vertical" smooth subvariety, that is, $X = f^{-1}(S)$ for some $S \subsetneq B$. Assume that X is algebraically coisotropic. If Question 4 has positive answer, then X must be a smooth fibre of f. Indeed a stronger statement holds under the following hypothesis (this is essentially due to R. Abugaliev, [5]).

Proposition 17. *In the notations above, assume that $\pi_1(S-(S\cap\Delta))$ surjects onto $\pi_1(B-\Delta)$. If $X = f^{-1}(S)$ is a vertical submanifold, then the leaves of the characteristic foliation of X are Zariski-dense in the fibers of f. In particular, X cannot be algebraically coisotropic unless S is a point.*

Proof: We briefly sketch the proof and refer to [5] for details (he treats the case $codim(X) = 1$ but the general one is analogous). It is easy to see that the leaves are tangent to the fibers of f and the closures of the leaves in the smooth fibers are subtori. When these subtori are proper, this gives a fibration in subtori on a smooth fiber $F \subset X$, and this family of subtori is invariant under the monodromy $\pi_1(S - (S \cap \Delta))$. By a remark of Oguiso (who combined results by Voisin and Matsushita) from [22], the image of the restriction map $H^2(M, \mathbb{Q}) \to H^2(F, \mathbb{Q})$ is one-dimensional, hence, up to proportionality, only one class in $H^2(F, \mathbb{Q})$ can be invariant under $\pi_1(B - \Delta)$. This is the restriction of an ample class on M, itself ample. On the other hand, a fibration of F into proper subtori as above provides a nef non-ample class in $H^2(F, \mathbb{Q})$, invariant under $\pi_1(S - (S \cap \Delta))$ (the

pull-back of an ample class on the base). Therefore the closure of a general leaf must be the whole fiber of f.

4.3. **Nefness and non-uniruledness.** A recent observation by Abugaliev and Pereira shows that a smooth non-nef hypersurface D in a holomorphic symplectic M is uniruled ([6], Theorem 8.4); if M is an IHS, the Beauville-Bogomolov square of such a D is negative ([6], Corollary 8.5). In general, for a submanifold $X \subset M$ of higher codimension, the non-nefness of K_X implies the existence of rational curves by bend-and-break. It would be interesting to see whether for algebraically coisotropic X, the non-nefness of K_X implies uniruledness as well.

5. Lagrangian submanifolds on abelian varieties.

There are obvious "linear" examples of Lagrangian submanifolds (subtori), as well as other trivial ones (curves in surfaces and products of such). In addition to Schoen's lagrangian surfaces in certain simple abelian fourfolds ([24]), which we have mentioned in the introduction, some nontrivial examples have been constructed using correspondences between K3 surfaces (see [9]) or Galois closures of certain coverings (see [7]). In these examples the abelian variety is not simple. However in [7] the authors show that the square of a sufficiently general $(1,2)$-polarized abelian surface admits a lagrangian surface of general type, not fibered in curves (Theorem 0.2 of [7]). The following problem is interesting but probably difficult and out of scope of this note.

Problem: Find examples (if any) of lagrangian submanifolds in simple abelian varieties of dimension ≥ 6.

5.1. **Hodge-general Abelian Varieties.** The purpose of this section is to remark that a sufficiently general abelian variety M does not have lagrangian submanifolds. The genericity condition is stronger than simpleness and is of Hodge-theoretic nature (Definition 19).

Notice the sharp constrast with the case of projective IHS manifolds. Indeed we know that in some maximal families of projective IHS, every manifold is covered by lagrangian submanifolds. The reason is as follows: on a Lagrangian submanifold of a sufficiently general manifold (abelian or IHS), all holomorphic 2-forms must vanish, and on an abelian variety there are too many 2-forms for this (see corollary 21 and its proof).

We now give the details (see e.g. [8],§17 for more).

Let M be an abelian variety of dimension g equipped with a polarization, and V its first cohomology $H^1(X, \mathbb{Q})$ equipped with its polarized Hodge structure. We view the polarization as a skew form ϕ on V. There is a natural representation $h : S^1 \to SL(V_{\mathbb{R}})$ associated to the

Hodge structure on V: here S^1 is the unit circle acting with weight 1 on $H^{1,0}$ and -1 on $H^{0,1}$.

Definition 18. *The Hodge group $Hg(M)$ is the smallest algebraic subgroup of $SL(V)$ defined over $\mathbb{Q}$, such that its group of $\mathbb{R}$-points contains $h(S^1)$.*

The Hodge group is always an algebraic subgroup of $Sp(V, \phi)$) ([8], Proposition 17.3.2). We say that M is Hodge general when $Hg(M)$ is as large as possible.

Definition 19. *A polarized abelian variety M is Hodge-general if $Hg(M) = Sp(V, \phi)$.*

By [8], Proposition 17.4.2, M is Hodge-general outside of a countable union of strict analytic subsets of the Siegel upper half plane parametrising n-dimensional Abelian varieties with a polarisation H of a given type (see [8], §8.1, this coarse version is sufficient for our purpose). Moreover the Jacobian of a sufficiently general curve is Hodge general. Many explicit examples of Hodge general Jacobians, even for particular types of curves (e.g. hyperelliptic curves) have been constructed by Zarhin (see for instance [31]).

A Hodge-general abelian variety is simple and its Picard number is 1. This immediately follows from the following more precise statement.

Proposition 20. *Let (M, H) be a polarised abelian variety. Denote by $H^2(M, \mathbb{Q})_{prim}$ the primitive part of the cohomology, that is the orthogonal complement to $H \in H^2(X, \mathbb{Q})$ for the pairing induced by the polarization. If (M, H) is Hodge-general, then $\mathbb{Q}H$ and $H^2(M, \mathbb{Q})_{prim}$ are the only Hodge substructures in $H^2(M, \mathbb{Q})$.*

Proof. Since $H^2(M, \mathbb{Q}) = \wedge^2 H^1(M, \mathbb{Q})$ as polarised Hodge structures, one only needs to check that $H^2(M, \mathbb{Q})_{prim}$ is an irreducible $Sp(M, H)$-module. This follows for example from [15], §17, p.260, which asserts that for the standard representation W of the symplectic group, the kernel of the natural contraction map $\wedge^k W \to \wedge^{k-2} W$ is irreducible (take $k = 2$). □

Corollary 21. *Let M be a Hodge-general abelian variety (for some polarization H) and σ a non-zero holomorphic 2-form on M. Let X be an irreducible subvariety of M of dimension at least 2, and $j : X' \to M$ the natural map of a desingularization of X to M. Then the holomorphic 2-form $j^*\sigma$ on X' is non-zero. In particular, M does not contain any Lagrangian subvariety when $dim(M) > 2$.*

Proof. The natural morphism $j^* : H^2(M, \mathbb{Q}) \to H^2(X', \mathbb{Q})$ is a morphism of Hodge structures, so its kernel is a Hodge substructure of $H^2(M, \mathbb{Q})$. If M is Hodge-general, there are only two Hodge substructures in $H^2(M, \mathbb{Q})$. So in this case, if $j^*\sigma = 0$, this substructure must

be $H^2(M,\mathbb{Q})_{prim}$, and so all holomorphic 2-forms on M vanish on X'. This is not possible if $d = dim(X) \geq 2$. Indeed: if $x \in X$ is a smooth point, and $(z_1, \ldots, z_n)$ are global coordinates on M (i.e. linear on its universal cover) such that the tangent space of X at x is defined by $dz_{d+1} = \cdots = dz_n = 0$, the global two-form $dz_1 \wedge dz_2$ on M does not vanish on X. □

Remark 22. *The same argument proves that if $f : Y \to M$ is a morphism to a Hodge-general M, then $f^*\sigma \neq 0$ unless $dim(f(Y)) \leq 1$.*

5.2. **Products of simple Abelian varieties.** Consider now a product $M = \Pi_i M_i$ of Abelian varieties, where the factors M_i are pairwise non-isogeneous, and each $M_i = S_i^{d_i}$ is a power of a simple Abelian variety S_i. We would like to ask the following questions:

1. For which d_i and $dim(S_i)$ such that $d_i dim(S_i)$ is even, can M_i contain a Lagrangian subvariety (for some symplectic σ)?

2. If none of the M_i contains a Lagrangian subvariety, can M contain some?

As a variant, one may assume S_i to be Hodge-general (for a certain polarisation), instead of just simple.

Examples for 1 indeed exist (see [7]) if $dim(S_i) = 2$ and $d_i = 2$, also with Hodge-general S_i equipped a $(1,2)$-polarisation. It is shown in [7] that such Abelian surfaces admit a map $\gamma : S \dashrightarrow F$ of degree 3 to a Hirzebruch surface F. The component T of $S \times_F S \subset S \times S$ residual to the diagonal is smooth and Lagrangian for a suitable symplectic 2-form on $S \times S$.

Let us notice that T can also be described as arising from a rational (hence Lagrangian) surface F' in the generalized Kummer IHS $K^{[2]}(S)$. Indeed: let us consider the Zariski-closure T' in S^3 of the set of pairwise distinct triples (p,q,r) which form a fibre of $\gamma : S \dashrightarrow F$. By definition, T' is invariant by the symmetric group S_3 acting on S^3 by permutation of the factors. Thus $T'/S_3 \cong F' \subset S^3/S_3$ is Lagrangian in S^3/S_3, and so is T' in S^3. The image of T'/S_3 is contained in a fibre of $S^3/S_3 \to S$ deduced from the addition on S: indeed, since F is rational, the sum in S of such a triple is constant. Since $K^{[2]}(S)$ is birational to a fibre of the Albanese map of S^3/S_3, we get the claim.

5.3. **Relation to fundamental groups.** This is also related to the questions about the fundamental groups of d-dimensional submanifolds $X \subset M$, when M is a $2d$-dimensional simple Abelian variety, $d \geq 2$.

Recall that if $a_X : X \to A_X$ is the Albanese map of a complex projective manifold, it induces a natural map: $H^2(A_X,\mathbb{C}) = \wedge^2 H^1(X,\mathbb{C}) \to H^2(X,\mathbb{C})$. The kernel of this map describes the nilpotent completion[4] of $\pi_1(X)$. Examples of $X \subset M$ with torsionfree nilpotent but non-abelian $\pi_1(X)$ have been so produced ([26], [12]). These examples

[4]Said differently: the tower of torsion-free nilpotent quotients of $\pi_1(X)$.

all have A_X not simple. Moreover the kernels are then contained in $H^{1,1}(M)$. Abelian fourfolds M containing an X such that there is a non-zero holomorphic 2-form in this kernel, and such that X is not fibered over a curve of general type[5], are constructed in [9], [7], [24]. The following is suggested by [12], Remarque 1.5.

Corollary 23. *Let M be a general abelian variety, and let $h : X \to M$ be a holomorphic map. If $dim(h(X)) \geq 2$, $h^* : H^2(M, \mathbb{C}) \to H^2(X, \mathbb{C})$ is injective, and either:*

1. $q(X) = dim(M)$: any torsion-free nilpotent quotient of $\pi_1(X)$ is equal to $H_1(X, \mathbb{Z})/Torsion$, or:

2. $q(X) > dim(M)$, and A_X is not simple.

Proof. Notice that $h(X)$ generates M, since M is simple. The injectivity claim is remark 22. Assume that $q(X) = dim(M)$. We then have an isogeny: $u : A_X \to M$ such that $h = u \circ a_X$, we thus get the injectivity of $a_X^* : H^2(A_X, \mathbb{C}) \to H^2(X, \mathbb{C})$, and so the conclusion in case 1, by [12], Théorème 3.10. In case 2, we still have a factorisation $h = u \circ a_X$ as before, but then the kernel of $u : A_X \to M$ is a positive-dimensional abelian subvariety of A_X, which is therefore not simple. □

More generally, we ask:

Question 24. *Let M be a simple abelian variety of even dimension $2n > 2$. Are there restrictions on the fundamental groups of its smooth submanifolds Y of dimension n?*

When $dim(Y) > n$, the fundamental group of Y is abelian (isomorphic to $\pi_1(A_Y)$), by [25]. When $dim(Y) < n$, there does not seem to be any restriction on $\pi_1(Y)$ (take the inverse image of a suitable Z by a finite projection $M \to \mathbb{P}^{2n}$). In middle dimension, no example of unusual $\pi_1(Y)$ seems to be known if $M = A_Y$ is simple, this being possibly due to the fact that most of them are not computed. If M is simple, the Sommese-Van de Ven construction, based on the sporadic Horrocks-Mumford abelian surfaces, gives examples in dimension 4, but with $M \neq A_Y$.

6. A non-projective example

Proposition 25. *There exist two-dimensional (non-projective) complex tori T admitting an automorphism g such that $g^*(\sigma) = \lambda.\sigma$, for any symplectic 2-form σ on T, with λ a complex number not a root of unity.*

Remark 26. *$|\lambda| = 1$, and T cannot be projective, see* [28]*, §14.*

[5]This property is remarkable because X admits a fibration over a curve of genus ≥ 2 iff a decomposable 2-form on A_X vanishes on the image of X, by Castelnuovo – de Franchis theorem.

Corollary 27. *Let T, λ be as in Proposition 25 above, and let S be the smooth Kummer surface associated to T. Let s be a symplectic form on either T or S. On $T \times T$ and $S \times S$ there exist smooth Lagrangian surfaces for the symplectic form $(s, \lambda s)$.*

Proof. (of corollary 27)Take the graph of g in $T \times T$, and its image in $S \times S$. □

Remark 28. *We do not know if there exists an abelian surface T and a Lagrangian surface $S \subset T \times T$ for a symplectic form $(s, \lambda s)$ with λ of infinite order in $\mathbb{C}^*$.*

Proof. (of Proposition 25). The construction follows closely the one due to S. Iitaka in dimension 3, as exposed in [28], Remark 14.6, p. 179.

Let $P(X) := X^4+X+1$: it is irreducible over $\mathbb{Q}$ by reduction modulo 2. The roots $a, \bar{a}, b, \bar{b}$ of $P(X)$ in $\mathbb{C}$ are all distinct and non-real. We define $T = \mathbb{C}^2/R$ as the quotient of $\mathbb{C}^2$ with $\mathbb{C}$-basis (e, e') by the lattice R generated over $\mathbb{Z}$ by the four elements $R_i := a^i.e + b^i.e', i = 0, 1, 2, 3$. That R is indeed a lattice follows from the fact that, otherwise there would exist $Q(X) \in \mathbb{R}[X]$ a nonzero polynomial of degree less that 4 vanishing on a and b. Thus Q had to vanish on their conjugates, and so be divisible by $P(X)$, so that $deg(Q) \geq 4$, a contradiction.

The torus T has an automorphism g acting by mutiplication by a on e, and by b on e', and so g acts on any symplectic form σ on T by multiplication by $\lambda := ab$. We have $|ab| = 1$ since the constant coefficient of $P(X)$ is $1 = |ab|^2$. It remains to see that λ is not a root of unity, or equivalently that λ has a Galois conjugate not of modulus 1. It is a standard exercise to check that the Galois group G of $P(X)$ is S_4. There is thus an element of G which fixes a and sends b to $\bar{a}$. Thus ab is G-conjugate to $a\bar{a}$. If $|a|^2 = 1$, $ab = 1$, since it is a G-conjugate of $a\bar{a}$, and so $\bar{a} = b$. Contradiction. □

Remark 29. *The torus T just constructed is simple of algebraic dimension 0. Indeed, since T is not projective, if its algebraic dimension were 1 (or equivalently, if it were not simple), it would have a quotient fibration $f : T \to B = T/E$ for elliptic curves E, B. This fibration would be preserved by $Aut(T)$, which is a finite extension of $Aut^0(T)$ since $Aut(E)$ and $Aut(B)$ have finitely many components.*

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