São Paulo Journal of Mathematical Sciences
https://doi.org/10.1007/s40863-024-00426-7

MEMORIAL VOLUME FOR SASHA ANAN'IN

Normal form of bimeromorphically contractible holomorphic Lagrangian submanifolds

Ekaterina Amerik[1,2] · **Misha Verbitsky**[1,3]

Accepted: 4 April 2024
© Instituto de Matemática e Estatística da Universidade de São Paulo 2024

Abstract
Let M be a holomorphically symplectic complex manifold, not necessarily compact or quasiprojective, and $X \subset M$ a compact Lagrangian submanifold. We construct a deformation to the normal cone, showing that a neighbourhood of X can be deformed to its neighbourhood in T^*X. This is used to study Lagrangian submanifolds which can be bimeromorphically contracted to a point. We prove that such submanifolds are biholomorphic to $\mathbb{C}P^n$, and show that a certain neighbourhood of X is symplectically biholomorphic to a neighbourhood of the zero section of its cotangent bundle. This gives a holomorphic version of the Weinstein's normal neighbourhood theorem.

Keywords Hyperkähler manifold · Lagrangian submanifold · Normal form · Holomorphically symplectic manifold

Mathematics Subject Classification 53C26

In memory of Sasha Ananin, our dear friend and a colleague. We miss you so much.

Communicated by Ivan Cheltsov.

Ekaterina Amerik and Misha Verbitsky acknowledge support of HSE University basic research program; also partially supported by ANR (France) project FANOHK. Partially supported by FAPERJ E-26/202.912/2018 and CNPq - Process 310952/2021-2.

✉ Ekaterina Amerik
Ekaterina.Amerik@gmail.com

Misha Verbitsky
verbit@impa.br

[1] Department of Mathematics, Laboratory of Algebraic Geometry, National Research University HSE, 6 Usacheva Str., Moscow, Russia

[2] Université Paris-11, Laboratoire de Mathématiques, Campus d'Orsay, Bâtiment 425, 91405 Orsay, France

[3] Instituto Nacional de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina, 110, Jardim Botânico, Rio de Janeiro, RJ 22460-320, Brazil

Published online: 17 May 2024

1 Introduction

1.1 Holomorphically symplectic Moser's lemma and the deformation to the normal cone

A smooth complex manifold M is **holomorphically symplectic** if it carries a closed holomorphic 2-form Ω, non-degenerate at each point. Clearly, in this case M is of even dimension $2n$ and the canonical bundle of M is trivial. A subvariety $X \subset M$ is called **Lagrangian** if the restriction of Ω to X is zero and $\dim(X) = n$, in other words, X is isotropic of maximal possible dimension.

It is not hard to see that the holomorphically symplectic form defines the complex structure in a unique way (Theorem 3.4). This gives a way to describe the holomorphically symplectic structures symplectically, without referring to the complex structure. We define **a C-symplectic structure** (Definition 3.3) on a real manifold of dimension $4n$ as a closed complex-valued form Ω such that $\Omega^{n+1} = 0$ and $\Omega^n \wedge \bar{\Omega}^n$ is non-degenerate. Treated in this fashion, the holomorphic symplectic structure gains the features we could expect from the symplectic form. In [23], a C-symplectic version of Moser's lemma was proven. Just like in case of Moser's lemma, it was shown that a smooth family Ω_t of cohomologous C-symplectic structures is trivialized by an appropriate flow of diffeomorphisms. However, a significant caveat applies: Moser's lemma fails unless one requires the vanishing of the cohomology group $H^1(\mathcal{O}_{M,I_t})$, where I_t the family of complex structures on M induced by the C-symplectic forms Ω_t, and $\mathcal{O}_{M,I_t}$ is the sheaf of holomorphic functions on M with the complex structure I_t.

Weinstein normal neighbourhood theorem is one of the most immediate and important applications of Moser's lemma. This theorem, stated in the traditional symplectic context, identifies a tubular neighbourhood of a Lagrangian submanifold with its neighbourhood in its cotangent space, equipped with the standard symplectic structure.

This paper is an attempt to apply this reasoning to C-symplectic manifolds. However, the condition $H^1(\mathcal{O}_{M,I_t}) = 0$ is quite restrictive. It is not always true, and can otherwise be hard to establish, that $H^1(\mathcal{O}_{M,I_t}) = 0$ for a neighbourhood of a holomorphic Lagrangian submanifold.[1] One exception is when the holomorphic Lagrangian submanifold $S \subset M$ is bimeromorphically contractible, that is, there exists a holomorphic, bimeromorphic map $M \xrightarrow{\pi} M_0$, bijective on $M \backslash S$, which contracts S to a point s. As follows from Grauert-Riemenschneider theorem, in this situation the preimage $V := \pi^{-1}(U) \subset M$ of a Stein neighbourhood U of $s \in M_0$ satisfies $H^1(\mathcal{O}_V) = 0$ (Corollary 2.16).

To establish the existence of a normal form, we prove that the neighbourhood $V \supset S$ can be deformed to a neighbourhood of S in its cotangent bundle. For complex manifolds this construction, called "deformation to the normal cone", is quite standard. In Theorem 4.3, we show that the deformation to the normal cone

[1] To give an example, let M be holomorphically Lagrangian fibered over a disc. The general fiber of this fibration is a complex torus, and one sees from the Leray spectral sequence that $H^1(\mathcal{O}_M)$ does not vanish.

can be performed in the holomorphically symplectic category. We construct a smooth family of holomorphic symplectic manifolds connecting a neighbourhood of a holomorphic Lagrangian submanifold $S \subset M$ to its neighbourhood in T^*S.

The content of this paper is split between Sects. 3 and 4, where the applications of C-symplectic Moser's lemma are discussed and the deformation to the normal cone is constructed, and Sect. 2, where we show that any bimeromorphically contractible Lagrangian submanifold is biholomorphic to the complex projective space.

1.2 Holomorphic Weinstein theorem

Recently, Lagrangian subvarieties have attracted considerable attention in algebraic geometry of holomorphic symplectic manifolds. One of the reasons is their relation to the birational geometry: Lagrangian subvarieties appear as exceptional loci of birational contractions of certain type. These are well-known to be isomorphic to the projective space whenever smooth (a brief sketch of proof is given in the beginning of Sect. 2). However, the general, non-algebraic and non-compact case, apparently has not been studied up to now. In this situation, the exceptional subvariety of a contraction is defined as follows.

Definition 1.1 Let $X \subset Y$ be a complex subvariety in a complex manifold Y. We say that X **can be bimeromorphically contracted** if there exists a proper morphism of complex varieties $Y \longrightarrow Y_0$ mapping X to a point, which is an isomorphism outside of X.

Our main result is an analogue of Weinstein tubular neighbourhood theorem for bimeromorphically contractible Lagrangian subvarieties, as follows.

Theorem 1.2 *Let (M, I, Ω) be a holomorphically symplectic manifold (not necessarily compact or quasi-projective), and $X \subset (M, I)$ a compact holomorphic Lagrangian submanifold. Assume that X can be bimeromorphically contracted to a point. Then X is isomorphic to $\mathbb{C}P^n$. Moreover, X has a neighbourhood which is biholomorphically symplectomorphic to a neighbourhood of the zero section in $T^*\mathbb{C}P^n$.*

The first statement shall be proved in the next section, and the proof of the second part shall be given in Sect. 4.2, after recalling Moser's isotopy lemma and its holomorphic version in Sect. 3.

Remark 1.3 Weinstein tubular neighbourhood theorem fails when X is a fiber of a non-isotrivial holomorphic Lagrangian fibration on a hyperkähler manifold, say, on an elliptic K3 surface. Indeed, the normal bundle NX is trivial, but the elliptic curve in the elliptic family varies, hence its neighbourhood cannot be isomorphic to a neighbourhood of the zero section in the total space of $T^*X = X \times \mathbb{C}$. If the fibration is isotrivial, then, by definition, the tubular neighbourhood theorem holds for a neighbourhood of a smooth fiber. Hence, the above-mentioned vanishing condition for $H^1(\mathcal{O}_M)$ is not necessary (but sufficient) for this theorem.

Remark 1.4 The Kähler condition plays no role in our considerations. Examples and counter-examples to Weinstein tubular neighbourhood might occur in the non-Kähler case as well. For instance, the standard Lagrangian fibration on a non-Kähler holomorphically symplectic manifold, called Bogomolov-Guan manifold ([6]), is isotrivial, hence the Weinstein tubular neighbourhood holds in this case, too.

2 Contractible Lagrangian submanifolds

The main result of this section is the following theorem.

Theorem 2.1 *Let X be a bimeromorphically contractible compact Lagrangian submanifold of a holomorphic symplectic manifold M of dimension $2n$, not necessarily compact or quasiprojective. Then X is isomorphic to $\mathbb{C}P^n$.*

This section is devoted to the proof, in fact we shall provide two versions of the argument. These versions have a common beginning which proves that X is Moishezon. Then one may proceed either by adaptation of log-MMP[2] following [15, 18], or by using the positivity properties of the cotangent bundle restricted to a movable curve (see [8, 9]).

To begin with, we recall that in the projective setting, this result is well-known, see e.g. [7], chapter 8. The main steps of the proof in the projective case (that is, assuming the contraction morphism $f : M \to M_0$ projective, as well as M and M_0) are briefly sketched as follows.

- *Step 1:* The contraction morphism is a log-extremal contraction with respect to a certain boundary divisor Δ on M. More precisely, one takes H ample om M_0, then f^*H is big, and writes $f^*H = A + E$ where A is ample on M and E effective on M. Then $(M, \epsilon E)$ is a klt-pair for a small rational number ϵ, and the contraction is log-extremal for this pair, since E is f-negative. By the log Minimal Model theory, the contraction locus X is uniruled, i. e. covered by rational curves (see e.g. [15]).
- *Step 2:* According to Ran [20], Corollary 5.1, any rational curve in a holomorphic symplectic manifold M (not necessarily algebraic or compact) deforms in a family of dimension at least $2n - 2$.
- *Step 3:* If $C \subset X$ is a rational curve, it is contracted by f, so that any deformation of C inside M must be contracted as well and therefore lies in X. Hence any rational curve in X deforms in a family of dimension at least $2n - 2$. In particular

[2] This part has been added after the first version of this paper has been written, and is suggested by the subsequent discussions with C. Shramov and Yu. Prokhorov.

this is true for a covering family of minimal rational curves.[3] Now the results by S. Kebekus [16] imply that this is possible only when $X \cong \mathbb{P}^n$.[4]

In this section we give a proof of Theorem 2.1 which does not require any projectivity assumption. We start with a contraction criterion by Ancona and Vo Van Tan [2], and obtain in particular that X is Moishezon (i.e. bimeromorphic to projective). Then the results by Campana–Peternell–Toma and Campana–Paun [8] (formulated in the projective context in [8], but remarked by the authors to be of bimeromorphic nature) imply uniruledness. Applying Step 2, we obtain a $2n-2$ -dimensional family of minimal rational curves on X. It remains to remark that Kebekus' proof is also valid when X is Moishezon.

Alternatively (we thank Yuri Prokhorov and Costya Shramov for many indications on this argument), once we know that X is Moishezon we can remark that either X has rational curves, or X is projective (or both) ([22]). Then in the first case we repeat Step 2 and Step 3, and in the second case we make a small adjustment to reduce to the projective case.

Remark 2.2 As soon as we know that $X \cong \mathbb{P}^n$, we can apply Grauert's cohomological criterion ([10], §4, Satz 7, Corollar) to show that a neighbourhood of X in M is biholomorphic to a neighbourhood of the zero section in the cotangent bundle of $\mathbb{P}^n$: see [7], Example 8.1 and Lemma 8.2.

Later on in this paper we also prove the Weinstein normal form theorem for the contractible Lagrangian submanifold (Sect. 4.2), i.e., that there exists a biholomorphism taking the symplectic form on M into the standard symplectic form on the cotangent bundle.

2.1 Vo Van Tan's characterization of contractible subvarieties

Here we recall the main results about the bimeromorphically contractible subvarieties, following [2, 3, 10, 26]. All varieties and analytic spaces are assumed to be connected.

Definition 2.3 A complex analytic space X is called **1-convex** if it admits a proper holomorphic, bimeromorphic map $p : X \to Y$ to a Stein space Y, and $p_* \mathcal{O}_X = \mathcal{O}_Y$.

Remark 2.4 By Remmert Reduction theorem ([21]), a complex variety is holomorphically convex if and only if it admits a proper, holomorphic map with

[3] I.e. such that all of its members passing through a general point are irreducible. Informally, X should not covered by rational curves splitting off some curves in this family.

[4] See [7], Theorem 4.2. Since [7] is known to be not always accurate, we prefer to rely on [16], who proves in Proposition 3.1 that the base H_x of a family of minimal rational curves passing through the general point x and covering the variety X is $\mathbb{P}^{n-1}$, and in the following Sect. 3.2 computes that the total space U_x of this family is $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1))$, so that the evaluation map contracts the exceptional section onto $\mathbb{P}^n$.

connected fibers to a Stein variety. The 1-convexity is a stronger property which requires this map to be bimeromorphic.

Definition 2.5 Given a coherent sheaf A on X, denote by $L(A)$ the complex analytic space obtained as the relative spectrum of the sheaf of rings $\bigoplus \mathrm{Sym}^k(A)$ over X.

Remark 2.6 When A is a vector bundle, $L(A)$ is the total space of the dual bundle A^*.

Definition 2.7 A coherent sheaf A on a compact complex analytic space X is called **ample** if for any coherent sheaf $\mathcal{F}$ on X there exists $k > 0$ such that $\mathrm{Sym}^k(A) \otimes \mathcal{F}$ is globally generated. It is called **cohomologically positive** if there exists $k > 0$ such that $H^i(X, \mathrm{Sym}^k(A) \otimes \mathcal{F}) = 0$ for all $i > 0$, and **weakly positive** if the zero section of $L(A)$ admits a 1-convex neighbourhood.

Claim 2.8 *Suppose that A is a torsion-free coherent sheaf on X. Then A is weakly positive if and only if the zero section $X \subset L(A)$ is bimeromorphically contractible.*

Proof *Step 1:* The group $\mathbb{C}^*$ acts on $L(A)$ by taking any compact subvariety which is not contained in the zero section to a non-compact family of compact subvarieties. Since 1-convexity implies that all positive-dimensional compact subvarieties of $L(A)$ are contained in a compact subset, this is impossible, and any bimeromorphically contractible subset of $L(A)$ is contained in the zero section.

Step 2: The 1-convexity means that $L(A)$ is equipped with a bimeromorphic contraction f to a Stein variety. Since the zero section $X \subset L(A)$ is compact, and a Stein variety has no compact complex subvarieties, the image $f(X)$ is zero-dimensional, hence it is contracted. As follows from Step 2, nothing else is contracted. □

Weak positivity has been introduced by Grauert in [10], who proved that for line bundles on compact complex manifolds weak positivity is equivalent to positivity, i.e. ampleness, and that a compact complex space carrying a weakly positive vector bundle is projective algebraic ([10], §3, Satz 1 and 2). These results were subsequently extended by Vo Van Tan, [26], as follows.

Theorem 2.9 ([26, Theorem 1]) *Let A be a coherent sheaf on a compact irreducible complex space X. Then the three conditions defined above* (*ampleness*, *weak positivity*, *cohomological positivity*) *are equivalent.*

Theorem 2.10 ([26, *Corollary* 6]) *A compact irreducible complex space carries a torsion-free weakly positive coherent sheaf if and only if it is Moishezon.*

Remark 2.11 Grauert proved in [10, §3, Satz 8], that a compact submanifold X in an analytic space M with weakly positive conormal bundle is bimeromorphically

contractible, but observed that the converse is not true, that is, the conormal bundle of a contractible submanifold does not have to be weakly positive. He asked whether the contractibility of X implies that the conormal sheaf of *some* analytic space structure on X is weakly positive. This question has been answered affirmatively by Vo Van Tan and Ancona.

More precisely, there is the following contractibility criterion:

Theorem 2.12 ([26, *Corollary* 7], [2, *Corollary* 3]) *Let M be a complex analytic space and $X \subset M$ a compact subvariety. Then X is bimeromorphically contractible if and only if there exists an ideal sheaf $\mathcal{J} \subset \mathcal{O}_M$ such that* $\mathrm{Supp}(\mathcal{O}_M/\mathcal{J}) = X$ *and the conormal sheaf, which is defined as $\mathcal{J}/\mathcal{J}^2$, is ample on X.*

Remark 2.13 To say that X carries an ample coherent torsion-free sheaf is equivalent to saying that there is an ample coherent sheaf with sheaf-theoretic support equal to X.

Remark 2.14 In particular, it follows that a bimeromorphically contractible subvariety is Moishezon.

2.2 Contractible subvarieties are isotropic

Recall the following classical theorem of complex geometry by Grauert and Riemenschneider [11], Satz 2.4 (see also [24], Corollary I).

Theorem 2.15 *Let $f : U \longrightarrow V$ be a bimeromorphic contraction of complex spaces with U smooth. Then $R^i f_*(K_U) = 0$ for $i > 0$, where K_U is the canonical bundle of U.*

Corollary 2.16 *Let $X \subset M$ be a bimeromorphically contractible subvariety of a manifold with trivial canonical bundle. Then any open neighbourhood of X in M contains a tubular neighbourhood $U \supset X$ such that $H^i(\mathcal{O}_U) = 0$ for any $i > 0$.*

Proof Let $f : M \longrightarrow M_0$ be the bimeromorphic contraction, mapping X to a point $x \in M_0$. Consider a Stein neighbourhood $V \ni x$, and let $U := f^{-1}(V)$. By assumption, $K_M = \mathcal{O}_M$. Grauert-Riemenschneider theorem (Theorem 2.15) implies that $R^i f_*(\mathcal{O}_U) = 0$. The Grothendieck spectral sequence with E_2-table $H^j(R^i f_*(\mathcal{O}_U))$ converges to $H^{i+j}(\mathcal{O}_U)$, giving $H^k(\mathcal{O}_U) = H^k(f_*\mathcal{O}_U) = H^k(\mathcal{O}_V) = 0$ because V is Stein. □

The fact that a contractible subvariety is Moishezon, together with Grauert-Riemenschneider theorem, yields the following result which is well-known in the projective case (see e. g. [14]):

Proposition 2.17 *Let M be a holomorphic symplectic manifold and X a contractible subvariety. Then X is isotropic, that is, the restriction of the holomorphic symplectic form Ω to X is zero.*

Proof We may assume that M is a neighbourhood of X as above, so that $H^2(\mathcal{O}_M) = 0$. Let $h : X' \to X \subset M$ be a resolution of singularities of X with X' projective (it exists because X is Moishezon, Remark 2.14). The Dolbeault cohomology class of the $(2, 0)$-form $h^*\Omega$ is zero, because the Dolbeault cohomology class of Ω itself is zero on M. Since X' is smooth projective, by Hodge theory $h^*\Omega = 0$, so that Ω must be zero on X. □

2.3 An MMP-style proof of Theorem 2.1

Let $f : M \to M_0$ be our bimeromorphic contraction. Without loss of generality, we replace M_0 by a small Stein neighbourhood of $x = f(X)$. In particular, we may assume that M retracts on X. We have established that X is Moishezon. It is known, see [19] in dimension 3, [22, 27] in arbitrary dimension, that all Moishezon varieties either contain rational curves, or are projective, or both.

Case 1: X contains a rational curve. Since X is isotropic, we know that $\dim(X) \leq n$. By [20],[5] any rational curve in M deforms in a family of dimension at least $2n - 2$. Since M_0 is Stein, all rational curves in M are contained in the contraction locus X. We claim they cover X and the dimension of X is n. Indeed without loss of generality we may consider a family of "minimal", or "generically unsplit" curves: this means that all curves through a general point z of the subvariety $Z \subset X$ which they cover, are irreducible. Then by bend-and-break there is only a finite number of minimal curves through z and any other point of Z. Dimension count gives $\dim(Z) \geq n$ and $Z = X$. One proceeds to the proof of Theorem 2.25 to finish the argument.

Case 2: X is projective.

Claim 2.18 *Let M be a complex manifold with trivial canonical bundle admitting a holomorphic birational contraction $f : M \longrightarrow M_0$, contracting a submanifold $X \subset M$ to a point x. If X is projective, then some ample line bundle L on X extends, together with its sections, to a line bundle L' on $f^{-1}(U)$, where U is a suitable neighbourhood of x.*

[5] See also [1] for a simple argument using twistor spaces, which should be generalizable to some extent.

Proof We choose U in such a way that it is Stein and X is a deformation retract of $f^{-1}(U)$. To simplify the notation, we may assume that $U = M_0$, and $f^{-1}(U) = M$. Consider the exponential exact sequence $0 \longrightarrow \mathbb{Z}_M \xrightarrow{2\pi\sqrt{-1}} \mathcal{O}_M \longrightarrow \mathcal{O}_M^* \longrightarrow 0$. From this exact sequence (and a similar sequence on X) we obtain the following diagram, with all rows exact.

$$\begin{array}{ccccccc} H^1(M,\mathcal{O}_M) & \longrightarrow & H^1(M,\mathcal{O}_M^*) & \longrightarrow & H^2(M,\mathbb{Z}) & \longrightarrow & H^2(M,\mathcal{O}_M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(X,\mathcal{O}_X) & \longrightarrow & H^1(X,\mathcal{O}_X^*) & \longrightarrow & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathcal{O}_X) \end{array}$$

The vertical arrow $H^2(M,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{Z})$ in this diagram is an isomorphism, and $H^2(M,\mathcal{O}_M) = 0$ by Grauert-Riemenschneider (Theorem 2.15). This implies that any $\eta \in H^2(X,\mathbb{Z})$ which belongs to the image of $H^1(X,\mathcal{O}_X^*)$ can be lifted to an element of $H^2(M,\mathbb{Z})$ which belongs to the image of $H^1(M,\mathcal{O}_M^*)$. Therefore, the restriction map is surjective on the set of connected components of $H^1(X,\mathcal{O}_X^*)$. Since ampleness is a numerical condition, a connected component of $\mathrm{Pic}(X)$ containing an ample line bundle consists of ample line bundles.

Therefore some ample L on X extends to a line bundle L' on M. The proof of [18], Proposition 1.4, shows that for a sufficiently large k, the restriction map $(L')^{\otimes k} \to L^{\otimes k}$ is surjective on the global sections. □

Hence the contraction f is projective. Now we obtain the uniruledness by the proof of [15], Theorem 2. Indeed, this theorem is a consequence of [15], Lemma 2, where the crucial step is the vanishing of higher direct images from [18], Theorem 3.6, valid in the complex-analytic context. Once the uniruledness is established, we again proceed to Theorem 2.25.

Remark 2.19 As noted in the proof of [18], Theorem 3.6, we are in fact dealing with a version of log-MMP here. Namely, to apply log-MMP one needs a boundary divisor negative on X. In the projective setting, it is given by decomposing the big divisor f^*H, H ample, in the sum of ample and effective. In the complex-analytic setting one remarks that $(L')^{-1}$ is effective, indeed $f_*((L')^{-1})$ has sections because M_0 is Stein. This gives the boundary divisor negative on X.

In the rest of this section, we give another proof of Theorem 2.1, where the uniruledness of X is obtained directly from Ancona-Vo Van Tan criterion with the help of [8].

2.4 Lagrangian contractible submanifolds are uniruled

Here we apply Vo Van Tan's results to a bimeromorphically contractible Lagrangian submanifold X and prove that it is uniruled (that is, covered by rational curves).

Lemma 2.20 *i) A quotient of an ample sheaf is ample*;

ii) A tensor power of an ample sheaf is again ample;

iii) If $\mathrm{Supp}(\mathcal{A}) = X$, *the restriction of an ample sheaf* $\mathcal{A}$ *to a curve on* X, *which is not contained in the singular set of* $\mathcal{A}$, *is of strictly positive degree*.

Proof The first assertion is true because the k-th symmetric power of a sheaf surjects onto the k-th symmetric power of its quotient, the second is proved using the representation theory, exactly as in the vector bundle case treated in [12], and the third results from the fact that a globally generated sheaf of positive rank on a curve has strictly positive degree, unless it is trivial. □

Lemma 2.21 *Let* $X \subset M$ *be a smooth submanifold, and* $\mathcal{J} \subset \mathcal{O}_M$ *an ideal sheaf defining an analytic subspace of* M *with support* X (*that is*, $\mathrm{Supp}(\mathcal{O}_M/\mathcal{J}) = X$). *Denote by* J_X *the ideal of* X. *Then there exists a non-zero map from* $\mathcal{J}/\mathcal{J}^2$ *to* $\mathrm{Sym}^r(J_X/J_X^2) = J_X^r/J_X^{r+1}$ *for some* r.

Proof We have $\mathcal{J} \subset J_X$. Choose r such that $\mathcal{J} \subset J_X^r$ and $\mathcal{J} \subsetneq J_X^{r+1}$. Then the natural map $\mathcal{J} \longrightarrow J_X^r/J_X^{r+1}$ is non-zero. However, since $\mathcal{J} \subset J_X$, this map vanishes on $\mathcal{J}^2$, defining a non-zero sheaf morphism $\mathcal{J}/\mathcal{J}^2 \longrightarrow J_X^r/J_X^{r+1}$.

Corollary 2.22 *If* X *is a contractible Lagrangian submanifold in a holomorphic symplectic* M, *then for some* r, $\mathrm{Sym}^r TM$ *contains an ample subsheaf.*

Proof For such an X, the holomorphic symplectic form provides an isomorphism between the conormal bundle and the tangent bundle. Take $\mathcal{J}$ as in Theorem 2.12 and apply the above lemma.

In the sequel, we will need the following theorem, implicit in the work of Campana and Paun [8] and the preceding work of Campana, Peternell and Toma [9]. Define a **movable curve** C on X as a member of a dominating family of irreducible curves.

Theorem 2.23 *Let* X *be a Moishezon manifold. Assume that for some* $r > 0$, $\mathrm{Sym}^r TX$ *has a subsheaf* $\mathcal{E}$ *such that* $deg(\mathcal{E}|_C) > 0$ *for a movable curve* C. *Then* X *is uniruled.*

Proof Let α be the cohomology class of C, then, in the terminology of [8], $\mu_{\alpha,max}(\mathrm{Sym}^r TX) > 0$. Then also $\mu_{\alpha,max}(TX) > 0$ ([8], Theorem 2.9), and the maximal

destabilizing sheaf $\mathcal{F}$ of TX has positive α-slope[6] (together with all its quotients). If X was projective, we would conclude immediately that X is uniruled, by [8], Theorem 1.1. Otherwise, we let $\pi : X' \to X$ be a projective modification. The class $\pi^*\alpha$ is a movable class on X'. Indeed, by [5] a movable class β on a projective manifold is characterized by $D\beta \geq 0$ for all effective divisor classes D. Setting $\beta = \pi^*\alpha$, we get that this condition is satisfied by the projection formula: indeed $\alpha\pi_*D = 0$ when D is contracted by π, and is non-negative otherwise. Now in the same way as in Lemma 2.12 of [8], the subsheaf $\mathcal{F}' \subset TX'$ defined by $\mathcal{F}' = \pi^*\mathcal{F} \cap TX'$ is of positive slope with respect to the movable class $\pi^*\alpha$, and we conclude by the main results of [8] that X', and hence also X, is uniruled. □

Corollary 2.24 *A contractible Lagrangian submanifold of a holomorphic symplectic manifold is uniruled.* □

2.5 Uniruled Lagrangian submanifolds

Following [7, 13, 16], we use the uniruledness (Theorem 2.23) to prove the following theorem.

Theorem 2.25 *Let $X \subset M$ be a bimeromorphically contractible Lagrangian submanifold of a holomorphically symplectic $2n$-dimensional manifold M. Then X is biholomorphic to $\mathbb{C}P^n$.*

Proof By [20], Corollary 5.1, any rational curve C on M deforms in a family of dimension at least $2n - 2$. When $C \subset X$, any deformation of C also lies in X, because X is contractible. We conclude that any family of rational curves in X, $\dim(X) = n$, has dimension at least $2n - 2$. Such families exist because X is uniruled.

We apply this to a complete family of minimal rational curves covering X. "Minimal" in this context means that all curves from the family passing through a sufficiently general point x of X are irreducible. It is easy to see by noetherian induction that such minimal rational curves exist. We obtain a family of irreducible curves through x of dimension at least $n - 1$. These curves must cover X, by bend-and-break (yielding that there are only finitely many curves connecting x to another point y) and by compacity of families of cycles on Moishezon manifolds. Now we are in the situation studied by Kebekus for projective manifolds, see [16]. The arguments carry over verbatim to the Moishezon case and prove that X is isomorphic to the projective space. □

[6] This is what $\mu_{\alpha,max}(TX) > 0$ means by definition.

3 Moser's lemma and normal form of Lagrangian submanifolds

3.1 Moser's lemma for holomorphic symplectic and C-symplectic structures

The complex structure on a holomorphically symplectic manifold is uniquely determined by the holomorphic symplectic form. This was the starting point of [23]. In [23] the authors obtain a version of holomorphically symplectic Moser's lemma which accounts for the deformations of complex structure as well as for the deformations of the holomorphic symplectic form.

Moser's lemma is the following fundamental result of symplectic geometry.

Theorem 3.1 *Let ω_t, $t \in [0,1]$ be a smooth family of symplectic structures on a compact manifold M. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. Then there exists a smooth family $\Psi_t \in \mathsf{Diff}_0(M)$ of diffeomorphisms such that $\Psi_t^* \omega_0 = \omega_t$.*

Proof [17].

Remark 3.2 The proof proceeds by constructing a vector field and integrating it to a flow of diffeomorphisms. It is for integration that one needs compacity. In particular, one can obtain a similar statement replacing a compact manifold M by a neighbourhood of a compact submanifold $X \subset M$.

For a holomorphically symplectic version of this lemma, we recall the notion of C-symplectic structures. A C-symplectic structure is a holomorphically symplectic form understood abstractly, that is, without fixing a complex structure (which is uniquely determined by the holomorphically symplectic form nevertheless).

Definition 3.3 ([4]) Let M be a smooth $4n$-dimensional manifold. A closed complex-valued form Ω on M is called **C-symplectic** if $\Omega^{n+1} = 0$ and $\Omega^n \wedge \bar{\Omega}^n$ is a non-degenerate volume form.

Theorem 3.4 *Let $\Omega \in \Lambda^2(M, \mathbb{C})$ be a C-symplectic form, and $T^{0,1}_\Omega(M) := \ker \Omega$, where*

$$\ker \Omega := \{v \in TM \otimes \mathbb{C} \mid \Omega \lrcorner v = 0\}.$$

Then $T^{0,1}_\Omega(M) \oplus \overline{T^{0,1}_\Omega(M)} = TM \otimes_{\mathbb{R}} \mathbb{C}$, hence the sub-bundle $T^{0,1}_\Omega(M)$ defines an almost complex structure I_Ω on M. Moreover, I_Ω is integrable, and Ω is holomorphically symplectic on (M, I_Ω).

Proof [4, Proposition 2.12], [25, Theorem 3.5].

Now we can state the C-symplectic version of Moser's lemma.

Theorem 3.5 *Let (M, I_t, Ω_t), $t \in [0,1]$ be a family of C-symplectic forms on a compact manifold, with corresponding complex structures I_t. Assume that the cohomology class $[\Omega_t] \in H^2(M, \mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M,I_t)})$ is the first cohomology of the sheaf of holomorphic functions. Then there exists a smooth family of diffeomorphisms $V_t \in \mathsf{Diff}_0(M)$, such that $V_t^* \Omega_0 = \Omega_t$.*

Proof [23, Theorem 2.5 (2)].

For non-compact manifolds, a slightly more cumbersome version of Moser's lemma can be stated.

Theorem 3.6 *Let $\pi : \mathcal{X} \to \Delta$ be a smooth family of holomorphic symplectic manifolds (not necessarily compact) over the unit disc, locally trivial as a family of C^∞ manifolds. Denote by $\mathcal{X}_t = \pi^{-1}(t)$ its fiber, and let $\Omega_t \in H^0(\mathcal{X}_t, \Omega^2_{\mathcal{X}_t})$ be its holomorphic symplectic form, smoothly depending on t. Using the C^∞ trivialization to identify cohomology groups of the fibres, assume that the cohomology class of Ω_t does not depend on $t \in \Delta$, and $H^1(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$. Let $K \subset \mathcal{X}_{t_0}$ be a compact subset. Then there exists an open neighbourhood $U \subset \Delta$ of $t_0 \in \Delta$, and an open subset $\tilde{U} \subset \pi^{-1}(U)$, with $K \subset \tilde{U}$, with the following property. The set $\tilde{U}$ is locally trivially fibered over U, with all fibres $\tilde{U} \cap \pi^{-1}(t)$, $t \in U$ isomorphic as holomorphically symplectic manifolds.*

Proof [23, Theorem 2.5 (1)].

We will use a slightly different version of this result.

Theorem 3.7 *Let $\pi : \mathcal{X} \to \Delta$ be a smooth family of holomorphic symplectic manifolds (not necessarily compact) over the unit disc, trivial as a family of C^∞ manifolds. Denote by $\mathcal{X}_t = \pi^{-1}(t)$ its fiber, and let $\Omega_t \in H^0(\mathcal{X}_t, \Omega^2_{\mathcal{X}_t})$ be its holomorphic symplectic form, smoothly depending on t. Using the C^∞ trivialization to identify cohomology groups of the fibres, assume that the cohomology class of Ω_t does not depend on $t \in \Delta$, and $H^1(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$. Let $K_t \subset \mathcal{X}_t$ be a locally trivial family of compact subsets, and $\mathcal{K} \subset \mathcal{X}$ their union. Then there exists an open neighbourhood $U \subset \Delta$ of $t_0 \in \Delta$, and an open subset $\tilde{U} \subset \pi^{-1}(U)$, with $K_t \subset \tilde{U}$, with the following property. The set $\tilde{U}$ is locally trivially fibered over U, with all fibres $\tilde{U} \cap \pi^{-1}(t)$, $t \in U$ isomorphic as holomorphically symplectic manifolds.*

Proof Chose $U \subset \Delta$ and $\tilde{U} \subset \mathcal{X}$ as in Theorem 3.6. Denote by Δ_λ the disc of radius λ centered in 0. We need to show that $\mathcal{K} \cap \pi^{-1}(\Delta_\lambda) \subset \tilde{U}$ for λ sifficiently small. Introduce a Riemannian metric on $\mathcal{X}$. Since $\tilde{U} \subset \mathcal{X}$ is open, it contains an ϵ-neighbourhood of K_0, for a sufficiently small $\epsilon > 0$. Choose λ in such a way that K_t belongs to an ϵ-neighbourhood $K_0(\epsilon)$ of K_0 for all $t \in \Delta_\lambda$; this is possible to do, because the Hausdorff distance $d_H(K_t, K_0)$ converges to 0 as t converges to 0. For such λ, clearly, one has

$$\mathcal{K} \cap \pi^{-1}(\Delta_\lambda) \subset K_0(\epsilon) \subset \tilde{U}.$$

3.2 Moser's lemma for a neighbourhood of a holomorphic Lagrangian submanifold

We will apply Theorem 3.7 when K is bimeromorphically contractible Lagrangian submanifolds. We need the following version of Moser's lemma.

Lemma 3.8 *Let (M, I_t, Ω_t), $t \in [0,1]$ be a smooth family of C-symplectic manifolds (not necessarily compact), with all Ω_t exact, and $E_t \subset (M, I_t)$ be a family of compact holomorphic Lagrangian subvarieties. Assume that $H^{0,1}(M, I_t) = 0$. Then E_t have open neighbourhoods U_t in M such that $(U_t, I_t, \Omega_t, E_t)$ is trivialized by a flow of diffeomorphisms.*

Proof Follows immediately from Theorem 3.7.

In the situation described by Corollary 2.16, any deformation of a neighbourhood of a holomorphic Lagrangian subvariety is locally trivial by Lemma 3.8, because the relevant cohomology groups vanish. This brings

Corollary 3.9 *Let (M, I, Ω) be a holomorphically symplectic manifold (not necessarily compact), and $E \subset (M, I)$ a compact holomorphic Lagrangian subvariety biholomorphic to $\mathbb{C}P^n$. Assume that a neighbourhood of E can be smoothly deformed to a neighbourhood of the zero section in T^*E as a C-symplectic manifold. Then E has a neighbourhood which is isomorphic to a neighbourhood of E in T^*E as a holomorphically symplectic manifold.*

Proof We apply Lemma 3.8 to the deformation $(U_t, I_t, \Omega_t, E_t)$ connecting a neighbourhood U_0 of E in M to a neighbourhood U_1 of E in T^*E. Note that since $E \cong \mathbb{P}^n$, all E_t are isomorphic to $\mathbb{P}^n$ as well. Therefore they are all bimeromorphically contractible inside U_t: indeed $N^*E_t \cong TE_t$ is isomorphic to the cotangent bundle of $\mathbb{P}^n$ and hence ample, and we conclude by Remark 2.11. Hence $H^i(\mathcal{O}_{U_t})$, $i > 0$, of a tubular neighbourhood of E_t in (M, I_t) vanish by Corollary 2.16, and Lemma 3.8 applies. □

Remark 3.10 The exactness condition on the symplectic forms is satisfied automatically in a sufficiently small neighbourhood U_t of E_t, because U_t retracts on E_t. Indeed the retraction induces an isomorphism between the second cohomologies of U_t and E_t, and since the form vanishes on E_t, its cohomology class on U_t must be zero.

4 Deformation to the normal cone of a Lagrangian submanifold

To apply Corollary 3.9, we need to deform a neighbourhood of a contractible Lagrangian submanifold to its neighbourhood in its cotangent space. This construction is well known in algebraic geometry under the name "deformation to the normal cone"; however, we have to make sure it works in the holomorphic symplectic category.

4.1 Deformation to the normal cone: an introduction

Let $X \subset M$ be a complex submanifold in a complex manifold. **Deformation to the normal cone** is a holomorphic deformation of a neighbourhood of $X \subset M$ over the disk such that its central fiber is the total space of the normal bundle NX, and the rest of the fibers are M. It is obtained as follows.

Let $X \subset M$ be a complex subvariety. Consider a product $M_1 := M \times \Delta$ of M with the disk Δ, and let $\tilde{M}_1$ be the blow-up of M_1 in $X \times \{0\}$. Denote by $\tilde{\pi}_1 : \tilde{\mathcal{X}} \longrightarrow \Delta$ the blow-down composed with the projection. The preimage $\tilde{\pi}_1^{-1}(0)$ is a union of two irreducible components, the proper preimage of $M \times \{0\}$, denoted D_1, and the blow-up divisor, denoted D_2.

Definition 4.1 Let $X \subset M$ be a complex subvariety and $M_1 := M \times \Delta$, where Δ is a disk. Let $\tilde{M}_1$ be the blow-up of M_1 in $X \times \{0\}$. Denote by $\tilde{\pi}_1 : \tilde{M}_1 \longrightarrow \Delta$ the blow-down composed with the projection. The preimage $\tilde{\pi}_1^{-1}(0)$ is a union of two irreducible components, the proper preimage of $M \times \{0\}$, denoted D_1, and the blow-up divisor, denoted D_2. The **deformation to the normal cone** is the complement $\tilde{M} := \tilde{M}_1 \backslash D_1$ considered as a fibration over the disk Δ.

Clearly, the central fiber of the natural projection $\tilde{M} \longrightarrow \Delta$ is $D_2 \backslash (D_1 \cap D_2)$.

Claim 4.2 *When X is smooth, and M is a neighbourhood of X in some ambient manifold (and a tubular neighbourhood in the differentiable sense), the complement $D_2 \backslash (D_1 \cap D_2)$ is naturally isomorphic to NX, and the "deformation to the normal cone" family $\tilde{\pi} : \tilde{M} \longrightarrow \Delta$ is locally trivial in the smooth category.*

Proof *Step 1:* The blow-up divisor $D_2 = \mathbb{P}N_{M_1}X = \mathbb{P}(NX \oplus \mathcal{O}_X)$, and it contains $\mathbb{P}(NX)$ as the subspace of normal directions without the horizontal component, i.e. tangent to $M \times \{0\}$. Clearly, the intersection of D_2 with the proper preimage D_1 of $M \times \{0\}$ is also the set of all $l \in \mathbb{P}(NX \oplus \mathcal{O}_X)$ tangent to $M \times \{0\}$. So we identify this intersection with $\mathbb{P}(NX)$. This gives an isomorphism $D_2 \backslash (D_1 \cap D_2) = \mathbb{P}(NX \oplus \mathcal{O}_X) \backslash \mathbb{P}(NX) = \mathrm{Tot}(NX)$.

Step 2: Now, the tubular neighbourhood M of X is diffeomorphic to $\mathrm{Tot}(NX)$, hence all fibers of $\tilde{\pi} : \tilde{M} \longrightarrow \Delta$ are diffeomorphic. Indeed the fibers over the points other than 0 are by construction equal to M, and we have $\mathrm{Tot}(NX)$ over 0. □

4.2 Deformation to the normal cone in holomorphic symplectic category

The following theorem, together with Corollary 3.9, implies the holomorphic version of the Weinstein normal neighbourhood theorem (Theorem 1.2).

Theorem 4.3 *Let (M, Ω) be a holomorphically symplectic manifold, and $X \subset M$ a holomorphically Lagrangian submanifold. Then there exists a smooth, holomorphic deformation of a neighbourhood of X in M over the disk Δ, such that its central fiber is biholomorphic to a neighbourhood of X in T^*X, the rest of the fibers are biholomorphic to a neighbourhood of X in M, and the holomorphic symplectic form on T^*X can be smoothly extended to the holomorphic symplectic form on the rest of the fibers.*

Remark 4.4 Note that this deformation in complex analytic category is already constructed: it is the "deformation to the normal cone" family. However, to apply the C-symplectic Moser lemma, we need to have a smooth family of holomorphically symplectic forms on its fibers.

Proof of Theorem 4.3 *Step 1:* Let $\tilde{M} \xrightarrow{\tilde{\pi}} \Delta$ be the deformation to the normal cone family, and t the coordinate on Δ. Locally on M we can describe X by a system of holomorphic equations $q_1 = q_2 = ... = q_n = 0$, and the holomorphically symplectic form as $\Omega = \sum_{i=1}^{n} dp_i \wedge dq_i$. We are going to prove that $t^{-1}\Omega$ is extended to a non-degenerate form on the central fiber of $\tilde{\pi}$.

Since this extension is unique, if would suffice to prove that $t^{-1}\Omega$ can be extended to the central fiber locally in X, and to check that it is non-degenerate.

Step 2: The coordinates on the central fiber of the deformation to the normal cone family $\tilde{M} \xrightarrow{\tilde{\pi}} \Delta$ are given by $p_1, ..., p_n, \tilde{q}_1, ..., \tilde{q}_n$. Trivializing the neighbourhood of $x \times \Delta \in X \times \Delta$ along Δ in the usual way, we write $\tilde{q}_i = t^{-1} q_i$: this is the standard way to write coordinates on the blow-up.

Locally, we can always find a $\theta \in \Omega^1 M$ such that $d\theta = \Omega$ and $\theta\Big|_X = 0$. For example, we can take $\theta = \sum_i q_i dp_i$ in the above coordinates. Writing θ in the coordinates $\tilde{q}_i, p_i$, we get $\theta = t \sum_i \tilde{q}_i dp_i$. Then $d(t^{-1}\theta) = \sum_i d\tilde{q}_i \wedge dp_i$ is a holomorphic form on $\tilde{M}$. Restricted to the fibers $\tilde{\pi}^{-1}(u)$ of the projection $\tilde{M} \xrightarrow{\tilde{\pi}} \Delta$, this form is equal to $t^{-1}\Omega + d(t^{-1}) \wedge \theta$; the second term restricted to $\tilde{\pi}^{-1}(u)$ vanishes, hence the restrictions of $d(t^{-1}\theta)$ extend $t^{-1}\Omega$ holomorphically to $\tilde{M}$.

Step 3: It remains to show that $d(t^{-1}\theta)$ is non-degenerate on the central fiber of $\tilde{M} \xrightarrow{\tilde{\pi}} \Delta$. Writing $\Omega = \sum_{i=1}^{n} dp_i \wedge dq_i$ as above and passing to the coordinates $q_i = t\tilde{q}_i$, we obtain $\Omega = \sum_{i=1}^{n} t dp_i \wedge d\tilde{q}_i + \sum_{i=1}^{n} dp_i \wedge \tilde{q}_i dt$. Since the last term vanishes on the fibers, the form $t^{-1}\Omega = \sum_{i=1}^{n} dp_i \wedge d\tilde{q}_i$ is smooth, non-degenerate on the central fiber, and equal to $t^{-1}\Omega$ on the general fibers. This proves Theorem 4.3. □

Acknowledgements We are grateful to Frédéric Campana, Andreas Höring, Dmitry Kaledin, Yuri Prokhorov, Costya Shramov and Andrey Soldatenkov for their insightful discussions, and to Arnaud Beauville for his valuable email communication.

Declarations

Conflict of interest No conflict of interest.

References

1. Amerik, E., Verbitsky, M.: Rational curves on hyperkähler manifolds. Int. Math. Res. Not. IMRN **23**, 13009–13045 (2015)
2. Ancona, V., Vo Van Tan.: On the blowing-down problem in $\mathbb{C}$-analytic geometry. J. Reine Angew. Math. **350**, 178–182 (1984)
3. Ancona, V., Vo Van Tan.: Embedding Moishezon spaces into 1-convex spaces. Math. Ann. **247**, 143–147 (1980)
4. Bogomolov, F., Deev, R., Verbitsky, M.: Sections of Lagrangian fibrations on holomorphically symplectic manifolds and degenerate twistorial deformations. Adv. Math. **405**(27), 108479 (2022)
5. Boucksom, S., Demailly, J.-P., Paun, M., Peternell, Th.: The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. J. Algebraic Geom. **22**(2), 201–248 (2013)
6. Bogomolov, F., Kurnosov, N., Kuznetsova, A., Yasinsky, E.: Geometry and automorphisms of non-Kähler holomorphic symplectic manifolds. Int. Math. Res. Not. IMRN **16**, 12302–12341 (2022)
7. Cho, K., Miyaoka, Y., Shepherd-Barron, N.I.: Characterizations of projective spaces and applications and applications to complex symplectic manifolds. Adv. Stud. Pure Math. **2002**, 1–88 (2002)
8. Campana, F., Paun, M.: Foliations with positive slopes and birational stability of orbifold cotangent bundles. Publ. Math. Inst. HautesÉtudes Sci. **129**, 1–49 (2019)
9. Campana, F., Peternell, T.: Geometric stability of the cotangent bundle and the universal cover of a projective manifold, with an appendix by M. Toma. Bull. Soc. Math. France, Tome **139**(1), 41–74 (2011)
10. Grauert, H.: Über Modifikationen und exzeptionelle analytische Mengen. Math. Ann. **146**, 331–368 (1962)
11. Grauert, H., Riemenschneider, O.: Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen. Invent. Math. **11**, 263–292 (1970)
12. Hartshorne, R.: Ample Vector Bundles, Publications mathématiques de l'I.H.É.S., tome 29, p. 63–94, (1966)
13. Hu, Y., Yau, S.-T.: HyperKahler manifolds and birational transformations. Adv. Theor. Math. Phys. **6**, 557–574 (2002)
14. Kaledin, D.: Symplectic singularities from the Poisson point of view. J. Reine Angew. Math. **600**, 135–156 (2006)
15. Kawamata, Y.: On the length of an extremal rational curve. Invent. Math. **105**(3), 609–611 (1991)
16. Kebekus, S.: Characterizing the projective space after Cho, pp. 147–155. Complex geometry (Göttingen, Miyaoka and Shepherd-Barron (2000)
17. McDuff, D., Salamon, D.: Introduction to symplectic topology, 2nd edn. Oxford Univ. Press, Oxford (1998)
18. Nakayama, N.: The lower semicontinuity of the plurigenera of complex varieties. Algebraic geometry, Sendai, 1985, 551–590. Adv. Stud. Pure Math., 10, Amsterdam, (1987)
19. Peternell, Th.: Rational Curves in Moishezon 3-folds. Lecture Notes in Math. **1194**, 133–144 (1986)
20. Ran, Z.: Hodge theory and deformations of maps. Compositio Math. **97**(3), 309–328 (1995)
21. Remmert, R.: Sur les espaces analytiques holomorphiquement séparables et holomorphiquement convexes. C. R. Acad. Sci. Paris **243**, 118–121 (1956)
22. Shokurov, V.: Letters of a bi-rationalist. I. A projectivity criterion, Birational algebraic geometry (Baltimore, MD,: Contemp. Math., vol. 207, Amer. Math. Soc. Providence, RI **1997**, 143–152 (1996)
23. Soldatenkov, A., Verbitsky, M.: The Moser isotopy for holomorphic symplectic and C-symplectic structures. arXiv:2109.00935
24. Takegoshi, K.: Relative vanishing theorems in analytic spaces. Duke Math. J. **52**, 273–279 (1985)
25. Verbitsky, M.: Degenerate twistor spaces for hyperkähler manifolds. J. Geom. Phys. **91**, 2–11 (2015)
26. Vo Van Tan: On Grauert's conjecture and the characterization of Moishezon spaces. Commentarii Math. Helvetici **58**, 678–686 (1983)
27. Villalobos-Paz, D.: Moishezon spaces and projectivity criteria. Preprint. arxiv:2105.14630

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.