PARABOLIC AUTOMORPHISMS OF HYPERKÄHLER MANIFOLDS: ORBITS AND BETTI MAPS

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ABSTRACT. We study infinite order automorphisms of irreducible holomorphically symplectic manifolds with an invariant Lagrangian fibration. Such automorphisms act (possibly after taking a positive iterate) by translations on smooth fibers, and their orbits in a general fiber are dense ([1]). We provide a simple proof that the associated Betti map is of maximal rank, in particular, the set of fibers where the induced translation is of finite order is dense as well.

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1. INTRODUCTION

1.1. **The dynamics of parabolic automorphisms.**

1.1.1. Let X be an irreducible hyperkähler (or "holomorphically symplectic") manifold of complex dimension $2g$. This means that

(a) X is a simply-connected, compact, Kähler manifold;
(b) there is a holomorphic 2-form σ on X which is symplectic, i.e. σ^g is a non-vanishing holomorphic form of top degree;
(c) σ is unique up to a nonzero multiplicative factor.

1.1.2. On $H^2(X,\mathbb{Z})$ there is a non-degenerate integral quadratic form q of signature $(3,b_2-3)$, the Beauville-Bogomolov form (see [22], §23.4). The signature of q on $H^{1,1}(X;\mathbb{R})$[1] is $(1,h^{1,1}(X)-1)$, so that the projectivization of the positive cone

$$\{u\in H^{1,1}(X;\mathbb{R})\ ;\ q(u,u)>0\} \tag{1.1}$$

can be viewed as a model of the hyperbolic space. We shall denote by $\mathbb{H}_X$ this hyperbolic space, its dimension is $h^{1,1}(X)-1$. Its boundary $\partial\mathbb{H}_X$ is the projectivization of the isotropic cone $\{u\in H^{1,1}(X;\mathbb{R})\ ;\ q(u,u)=0\}$.

We denote by $\mathrm{NS}(X)$ the Néron-Severi group of X,

$$\mathrm{NS}(X)=H^{1,1}(X;\mathbb{R})\cap H^2(X;\mathbb{Z}). \tag{1.2}$$

If L is a line bundle on X, we denote by $[L]\in\mathrm{NS}(X)$ its Chern class.

1.1.3. The group $\mathsf{Aut}(X)$ acts by isometries on $H^2(X;\mathbb{Z})$ with respect to q and preserves the Hodge decomposition, so that it acts also by isometries on $H^{1,1}(X;\mathbb{R})$ and on $\mathbb{H}_X$. As described in [40] for instance, there are three types of isometries of hyperbolic spaces, hence three types of automorphisms: elliptic, parabolic, and loxodromic. In this article, we study parabolic automorphisms. An automorphism f of X is **parabolic** if the induced automorphism f^* of $H^{1,1}(X;\mathbb{R})$ satisfies the following equivalent properties:

(a) f^* has exactly one fixed point on the boundary $\partial\mathbb{H}_X$ and no fixed point in the interior;
(b) there is a positive iterate $(f^*)^n$ of f^* acting as a unipotent matrix of infinite order on $H^{1,1}(X;\mathbb{R})$ (resp. on $H^2(X;\mathbb{Z})$);
(c) $\|(f^*)^n\|=c(f)n^2+O(n)$ for some positive constant $c(f)$. (Here, $\|\cdot\|$ is any norm on $\mathsf{End}\,(H^{1,1}(X;\mathbb{R}))$ or $\mathsf{End}\,(H^2(X;\mathbb{R}))$.)

[1] We denote by $H^{1,1}(X,\mathbb{R})$ the subspace of $H^2(X,\mathbb{R})$ whose complexification is $H^{1,1}(X)$.

We refer to the Appendix for references and a proof of (c).

1.1.4. Let $f \in \mathsf{Aut}(X)$ be parabolic. Its fixed point on the boundary $\partial\mathbb{H}_X$ corresponds to a line in $H^{1,1}(X;\mathbb{R})$ which is fixed pointwise by f^*; this line is integral: it is generated by some primitive isotropic class $\ell_f \in \mathrm{NS}(X)$. Moreover, the nef cone of X being closed and $\mathsf{Aut}(X)$-invariant, we can choose ℓ_f to be the class of some nef line bundle (i.e. ℓ_f is in the closure of the Kähler cone of X). This uniquely determines ℓ_f. Since $b_1(X) = 0$, $\mathrm{Pic}^0(X) = 0$ and there is a unique nef line bundle L_f such that $[L_f] = \ell_f$; then $f^*L_f = L_f$.

1.1.5. The so-called Lagrangian Conjecture (which has been stated independently by several people, including Hassett and Tschinkel, Huybrechts, and Sawon), also known as the Hyperkähler SYZ Conjecture, says that a nef line bundle L with $q([L],[L]) = 0$ should be semi-ample: this means that $L^{\otimes n}$ should be base-point-free for large positive integers n. This conjecture has been verified in all known examples (see [5]); applied to L_f, it says that the linear system of sections of $L_f^{\otimes n}$ defines a morphism

$$p_f \colon X \to B \tag{1.3}$$

with connected fibers of strictly positive dimension. According to [32], such a morphism is a Lagrangian fibration, which means that the smooth fibers of p_f are Lagrangian tori. The base B of the fibration is a normal projective variety of complex dimension g and of Picard number 1, which a priori can be singular.

Then, there is an automorphism f_B of B such that

$$p_f \circ f = f_B \circ p_f, \tag{1.4}$$

and it can be shown that f_B has finite order (see [29] and Theorem B below). Thus, for some $k \geq 1$,

(1) the action of $(f^k)^*$ on $H^2(X;\mathbb{Z})$ is unipotent, and of infinite order;
(2) $p_f \circ f^k = p_f$;
(3) f^k acts as a translation on each smooth fiber ([1], Proposition 3.8).

1.1.6. Theorem 3.11 of [1] shows that the orbits of f^k must be dense in the euclidean topology on almost all smooth fibers of p_f. A natural question is whether one often encounters smaller orbit closures. For example, is the set of $b \in B$ such that f^k is of finite order on X_b (i.e. acts as a translation by a torsion element) dense in B? Our main theorem answers this question positively (see below for the definitions of translation vector and maximal variation).

Theorem A. *Let X be an irreducible hyperkähler manifold of dimension $2g$. Let f be a parabolic automorphism of X with an invariant fibration $p_f\colon X \to B$ and let k be a positive integer such that $p_f \circ f^k = p_f$. Then,*

(1) *for any $p \in \{1,2,\dots,g\}$, there is a positive constant $c_p(f)$ such that*

$$\|(f^n)^*\|_{H^{p,p}(X;\mathbb{R})} = c_p(f)n^{2p} + O(n^{2p-1});$$

(2) *the translation vector of f^k has maximal variation;*

(3) *for any $s \in \{0,1,2,\dots,2g\}$, the subset of B defined by*

$$D_s(f^k) = \{b \in B\,;\ \textit{the closure of any orbit of } f^k_{|X_b} \textit{ has dimension } s \textit{ in } X_b\}$$

is dense in B for the euclidean topology.

In the definition of $D_s(f^k)$, "closure" means "closure with respect to the euclidean topology". For instance, the following sets are dense in B:

$$D_g = \{b \in B\,;\ \text{every orbit of } f^k \text{ in } X_b \text{ is dense in } X_b\} \tag{1.5}$$

$$D_0 = \{b \in B\,;\ \ f^k_{|X_b} \text{ has finite order}\}. \tag{1.6}$$

Note that we assume in Theorem A that f preserves a Lagrangian fibration; as explained in Section 1.1.5, this is satisfied in all known examples.

1.1.7. When X is projective, Theorem A is not new: it can be derived from results of Bakker, André-Corvaja-Zannier, Gao, and Voisin. This is explained in Section 1.3. Theorem A has also been proven for all surfaces in [7, 9], but it seems difficult to apply the same methods in higher dimension ([2]). Our goal is to describe a new proof of it, and to extend the result to non-projective manifolds; on our way, we also extend a result of Lo Bianco (see Theorems B and C).

1.2. **Betti coordinates, translation vector, maximal variation.**

[2]The surfaces in [9] are Kähler but do not have to be hyperkähler. Indeed, if X is a compact complex surface, the intersection form defines a quadratic form on the second cohomology group of X. If the surface is Kähler, its restriction to $H^{1,1}(X;\mathbb{R})$ is non-degenerate and of signature $(1, h^{1,1}(X)-1)$. Thus, automorphisms of X can also be classified into three types, elliptic, parabolic, or loxodromic. By a theorem of Gizatullin, every parabolic automorphism of a compact Kähler surface preserves a genus 1 fibration (with finite order action on the base except for some automorphisms of some tori).

1.2.1. *A few references.* The Betti coordinates, introduced below, have been used in various situations similar to the one in this paper. A good historical introduction can be found in the first pages of [3]. The references we know about that are closest to our setting are the following: firstly, Manin's study of the Mordell problem over function fields [30]; secondly, Zannier's famous book [46], more precisely its Chapter 3.3 concerning a question of Masser; thirdly, the work of Corvaja, Masser, and Zannier, in particular [11]. Since then, Betti coordinates have become a common tool in Diophantine geometry, with connexions to algebraic dynamics (see [9, 13]). See in particular Section 1.3 below.

1.2.2. *Betti coordinates.* Let $p\colon X \to B$ be a fibration of a compact complex manifold. We shall always denote by B° the subset of regular values of p where, by definition, the singularities of B are put in $B \setminus B^\circ$. Suppose that for every $b \in B^\circ$, the fiber $X_b = p^{-1}(b)$ is a torus, isomorphic to $\mathbb{C}^g/\mathrm{L}(b)$ for some lattice $\mathrm{L}(b) \subset \mathbb{C}^g$ (not uniquely defined, see Remark 1.1 below).

Let U be a simply connected open subset of B°, b_0 a point of U, and $s\colon U \to X$ a section of p. If one fixes a basis of $H_1(X_{b_0};\mathbb{Z})$, it can be propagated continuously to the fibers X_b for $b \in U$ and this gives a trivialization $H_1(X_U;\mathbb{Z}) \simeq \mathbb{Z}^{2g}$. Then, there is a unique diffeomorphism

$$\Phi\colon X_U \to U \times (\mathbb{R}^{2g}/\mathbb{Z}^{2g}) \tag{1.7}$$

such that

(i) $p = \mathrm{pr}_U \circ \Phi$, where pr_U is the projection from $U \times (\mathbb{R}^{2g}/\mathbb{Z}^{2g})$ to U,
(ii) $\Phi \circ s_{|U}(b) = (b,0)$ for all $b \in U$,
(iii) $\Phi\colon X_b \to \{b\} \times (\mathbb{R}^{2g}/\mathbb{Z}^{2g})$ is an isomorphism of Lie groups for all $b \in U$,
(iv) Φ_* maps the basis of $H_1(X_U;\mathbb{Z})$ to the canonical basis of

$$\mathbb{Z}^g \simeq H_1\left(U \times (\mathbb{R}^{2g}/\mathbb{Z}^{2g});\mathbb{Z}\right).$$

This diffeomorphism is real analytic. We shall refer to Φ as the Betti diffeomorphism; points of $U \times (\mathbb{R}^{2g}/\mathbb{Z}^{2g})$ can be written (u,x) with u in U and $x = (x_1,\dots,x_{2g})$ in $\mathbb{R}^{2g}$ modulo $\mathbb{Z}^{2g}$, and we shall refer to these as the **Betti coordinates** (determined by Φ). We refer to the triple given by U, the section $s_{|U}$, and the basis of $H_1(X_U,\mathbb{Z})$ as the **Betti datum** used to define Φ. To simplify notation, we shall simply write $U \times \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ instead of $U \times (\mathbb{R}^{2g}/\mathbb{Z}^{2g})$.

Remark 1.1. The Betti datum can be used to fix a choice of lattice $L(b)$ such that $X_b = \mathbb{C}^g/L(b)$. For this, we fix a trivialization of the tangent bundle to the

fibers of p along $s(U)$. Then, $T_{s(b)}X_b$ can be identified to $\mathbb{C}^g$ for any $b \in U$ and $L(b) \subset \mathbb{C}^g$ can be defined to be the kernel of the exponential map $\mathbb{C}^g \simeq T_{s(b)}X_b \to X_b$. With such a definition, $L(b)$ depends holomorphically on b.

1.2.3. *The translation vector.* Let f be an automorphism of X such that $p \circ f = p$ and f acts by translations on the general fibers of p. Conjugating f by Φ one gets a diffeomorphism $\Phi \circ f \circ \Phi^{-1}$ of $U \times \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ of type

$$(u,x) \mapsto (u, x + t_f(u)) \tag{1.8}$$

for some real analytic function $t_f \colon U \to \mathbb{R}^{2g}$ (or to $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$). By definition, t_f is the **translation vector** of f (in the Betti coordinates defined by Φ). As we shall see in Section 3.2, the generic rank of t_f is an even integer and this integer does not depend on the choice of Betti coordinates. We shall refer to it as the **rank** of the translation vector. The maximal possible rank is $\min(2\dim_{\mathbb{C}}(B), 2g)$. For Lagrangian fibrations $\dim_{\mathbb{C}}(B) = g$ so, in what follows, we assume that $\min(2\dim_{\mathbb{C}}(B), 2g) = 2g$ for simplicity; then, we say that t_f has **maximal rank**, or equivalently that t_f has **maximal variation**, if its generic rank is $2g$. Lemma 3.1 shows that the variations of t_f are maximal if and only if the image of t_f is open in $\mathbb{R}^{2g}$, if and only if t_f is an open mapping.

This explains the meaning of Assertion (2) in Theorem A and shows that this assertion implies Assertion (3) (see Section 3 for more on t_f and a detailed proof of how (3) is derived from (2)).

1.2.4. Now, suppose that $s \colon B \to X$ is a global holomorphic section of p. For every $b \in B^\circ$, we can declare that $s(b)$ is the neutral element of X_b and, doing so, X_b becomes a commutative complex Lie group.

Then, $f \circ s$ is a new section of p, and the action of f on X_b is the translation by $f \circ s(b) - s(b)$ for every $b \in B^\circ$. Let us now set $t = f \circ s$ and forget about f. The dynamical properties of f can be translated into properties of t. More precisely, consider a Betti diffeomorphism Φ (determined by some choice of Betti datum) and set

$$t_f = \mathrm{pr}_{\mathbb{R}^{2g}/\mathbb{Z}^{2g}}(\Phi \circ t) \mod \mathbb{Z}^{2g}. \tag{1.9}$$

This map $u \in U \mapsto t_f(u) = \mathrm{pr}_{\mathbb{R}^{2g}/\mathbb{Z}^{2g}}(\Phi(t(u))) \mod \mathbb{Z}^{2g}$ is usually called the **Betti map** associated to the section t (and the chosen Betti datum). Thus, *the translation vector of f has maximal variation if and only if the Betti map is generically of maximal rank* $2g$. This property of the Betti map has been studied a lot, at least in the case when X is projective, as explained below.

1.3. **General results on Betti maps.** Let us explain how Theorem A can be derived from works of André, Corvaja and Zannier, of Gao, of Voisin, and of Bakker when X is projective.

1.3.1. As mentionned above, Chapter 3.3 of [46] makes use of Betti coordinates to solve a problem of unlikely intersection concerning common torsion points of two sections of an elliptic fibration. It combines the counting estimates of Pila and Wilkie for rational points on a subanalytic set S of $\mathbb{R}^n$ and an argument that relies on the monodromy of the Gauss-Manin connection of the elliptic fibration to control the algebraic part of the subset S involved in the problem (see also [39, 11]). A second crucial reference is [3]: it initiates the systematic study of Betti coordinates in arbitrary dimensions and combines the previous strategy with theorems from functional transcendence theory, notably André's theorem concerning the independence of abelian logarithms [2] and the Ax–Schanuel theorem from [34].

In this section, we rely on the subsequent developments by Gao, because [19] and [18] contain optimal results regarding the variations of the Betti maps (i.e. of the translation vectors). The tools used in [19] being somewhat simpler, we base our explanation on it and describe how it is related to Theorem A.

1.3.2. In [19, Theorem 1.3], Gao considers an abelian scheme $p\colon \mathcal{A} \to S$ of relative dimension g over a smooth complex algebraic variety S, with a section ξ (or more generally a multisection) generating $\mathcal{A}$. By definition, this means that $\mathcal{A}$ is a group scheme (there is a neutral section of p) and $\mathbb{Z}\xi$ is Zariski dense in $\mathcal{A}$. He proves that the associated Betti map is generically of rang $2g$ if the following three properties are satisfied:

(a) the modular map $\mu : S \to \mathcal{A}_g$ is quasi-finite ([3]);
(b) $\dim(S) \geq g$, and
(c) the geometric generic fiber of the family is simple ([4]).

To apply this result to our context, we can take S to be the set of regular values B° of p_f. Then (b) is satisfied by construction. Moreover, a recent theorem of Bakker proving Matsushita's conjecture states that either μ is quasi-finite on

[3]Here, $\mathcal{A}_g$ is the space of polarized abelian varieties in dimension g with respect to some polarization type and some level structure; since they are not relevant, we simply write $\mathcal{A}_g$.

[4]See also [18], where it is established that the non-maximality of the rank of the Betti map associated to a generating ξ implies the existence of a quotient abelian scheme of low variation. For simplicity's sake we prefer to keep [19] as our main reference: it is almost equally quick to get applications to hyperkähler manifolds from [19], see [4].

a dense open subset of S or μ is constant, i.e. the family is isotrivial (see [4]). Thus, in the non-isotrivial case, Gao's theorem is close to establishing Assertion (2) of Theorem A; nevertheless, there is a subtlety here: in general it is not true that the geometric generic fiber is simple, though the scheme-theoretic generic fiber is simple and has Picard number 1 (see [37]).

Example 1.2. Let Y be a K3 surface with a genus one fibration $h : Y \to \mathbb{P}^1$. The Cartesian square of Y is naturally fibered in products of genus one curves over $\mathbb{P}^1 \times \mathbb{P}^1$, and the natural involution exchanges the fiber over (a,b) with that over (b,a). Set $X = Y^{[2]} = \mathrm{Hilb}^2(Y)$. Then X is hyperkähler and is fibered over $\mathbb{P}^2 = \mathrm{Sym}^2(\mathbb{P}^1)$ in such a way that most of the fibers are tori: the fiber over $a+b$, $a \neq b$, is $h^{-1}(a) \times h^{-1}(b)$. The generic fiber of this fibration $h^{[2]} : X \to \mathbb{P}^2$ ceases to be simple after a degree two extension of the function field of $\mathbb{P}^2$, corresponding to the map $(a,b) \mapsto a+b$ from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$.

For most applications, though, Gao's theorem works with some extra argument: see for example [4, Corollary 9] for a density statement similar to what we discuss here.

Finally, coming back to the setting of hyperkähler manifolds, the case of isotrivial Lagrangian fibrations is covered in a paper by Voisin [45] together with the case $\dim(X) \leq 8$ (i.e. $g \leq 4$).

Remark 1.3. In [3], the authors raise the question whether the Betti map associated to a section ξ of $p\colon \mathcal{A} \to S$ is generically of rank $2g$ under the following milder conditions: ξ generates $\mathcal{A}$, $\mathcal{A}$ has no fixed part over any étale finite covering of S, μ is quasifinite, and (as above) $\dim(S) \geq g$. In [18, Example 9.4], a counterexample is given with $g = 4$; clearly, this example is not hyperkähler. However, a positive answer to this question was obtained in [3] when $g \leq 3$, and in any dimension under the additional assumption that the abelian scheme has no non-trivial (that is, other than multiplying by an integer) endomorphism over any finite covering of the base.

1.4. **Strategy of proof.** With the previous results in mind, the reason why we wrote this text is twofold. Firstly, Theorems A and B now hold uniformly, for projective and non-projective hyperkähler manifolds, and for isotrivial and non-isotrivial fibrations. Secondly, the proof follows a new route. Gao obtains his result as a consequence of mixed Ax-Schanuel theorem. On one side, our argument is simpler because it relies on more basic principles; on the other side it applies only to the hyperkähler case, because we rely on Verbitsky's

theorem [42, Theorem 1.5] on the cohomology of hyperkähler manifolds (this is used to get Assertion (1) of Theorem A, which is – in turn – used to derive Assertion (2)). The proof is done for projective hyperkähler manifolds first, and then generalized to the non-projective case. The argument for this last step is of independent interest; it applies recent results of Soldatenkov and Verbitsky.

One of our arguments combines basic volume estimates from riemannian geometry with a theorem of Gauthier and Vigny. This riemannian point of view is quite natural and is reminiscent of both Manin's proof of the geometric Mordell conjecture and Parshin's proof of the geometric Shafarevich conjecture in characteristic zero (see [33, 38]). See also the recent article [10] and the book [17], in which this approach is described in details for elliptically fibered surfaces.

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2. HYPERKÄHLER MANIFOLDS

In this section, X is a irreducible hyperkähler manifold of dimension $2g$, as in Section 1.1.1. We introduce the main basic features of parabolic automorphisms and Lagrangian fibrations.

We denote by σ a holomorphic symplectic form σ, as in Section 1.1.1, and by q the Beauville-Bogomolov form on $H^2(X;\mathbb{Z})$, as in Section 1.1.2. The Beauville-Bogomolov form will be seen either as a quadratic or a bilinear form; we refer to [22] for its main properties. Note that when $g = 1$, X is a K3 surface and q is given by the natural intersection form on $H^2(X;\mathbb{Z})$.

2.1. **The Néron-Severi group.** We denote by q_X the restriction of q to $\mathrm{NS}(X)$. If A is a ring, we set $\mathrm{NS}(X;A) = \mathrm{NS}(X) \otimes_{\mathbb{Z}} A$; hence $\mathrm{NS}(X) = \mathrm{NS}(X;\mathbb{Z})$. When X is projective, there are classes u in $\mathrm{NS}(X)$ with $q_X(u,u) > 0$, for instance Chern classes of ample line bundles. Conversely, a theorem of Huybrechts shows that if such a class $u \in \mathrm{NS}(X)$ exists, then X is projective (see [21, 23]).

There are three possibilities for the signature of q_X on $\mathrm{NS}(X;\mathbb{R})$

(a) q_X is non-degenerate of signature $(1, \rho(X) - 1)$;

(b) q_X is degenerate with one-dimensional kernel, and takes only non-positive values; in this case, following Oguiso (see [36], page 167), we say that q_X is **parabolic**;
(c) q_X is negative definite.

The second and third cases do not appear when X is projective.

2.2. **The transcendental lattice.** The transcendental lattice $\mathrm{T}(X)$ is, by definition, the orthogonal complement of $\mathrm{NS}(X)$ in $H^2(X;\mathbb{Z})$ with respect to q. The Lefschetz theorem on $(1,1)$-classes implies that $\mathrm{T}(X)$ is the smallest subgroup of $H^2(X;\mathbb{Z})$ such that $\mathbb{C}\sigma$ is contained in $\mathrm{T}(X)\otimes_{\mathbb{Z}}\mathbb{C}$ and $H^2(X;\mathbb{Z})/\mathrm{T}(X)$ is torsion free. The intersection $\mathrm{T}(X)\cap\mathrm{NS}(X)$ is reduced to 0, except when q_X is parabolic.

2.3. **Fibrations and polarizations.** Let $p\colon X\to B$ be a holomorphic fibration, that is, a proper surjection with connected fibers and $\dim(X)>\dim(B)\geq 1$. Then by the results of Matsushita, p is a Lagrangian fibration in the sense of holomorphic symplectic geometry (see [22, Sec. 21.4]); moreover,

(1) its fibers are projective, and the generic fiber is an abelian variety of dimension $g=\dim(X)/2$ on which σ vanishes (see [6], Proposition 2.1, which the author attributes to Voisin);
(2) the base B is projective too, indeed it is Kähler and Moishezon with rational singularities (see [25] Theorem 2.8, [35] Corollary 1.7). Moreover B is $\mathbb{Q}$-factorial with Picard number 1 (see [31]). It is generally expected that B is isomorphic to $\mathbb{P}^g$; when B is smooth, this is a theorem by Hwang [24].

Now, set $X^\circ=X_{B^\circ}$, where B° is defined as in Section 1.2.2. The projection $p\colon X^\circ\to B^\circ$ is a proper submersion, the fibers of which are naturally polarized abelian varieties (however, when p does not have a section, the group law on the general fiber is not well defined). Indeed,

(3) the restriction homomorphism $H^2(X;\mathbb{Z})\to H^2(X_b;\mathbb{Z})$ has cyclic image.

This remark has been made by Oguiso in [37] (see [1] for a self-contained proof). Let $R_b\subset H^2(X_b;\mathbb{Z})$ denote this cyclic group. If κ is a Kähler form on X, then there is a unique positive multiple $\alpha\kappa$ such that R_b is generated by the class $[\alpha\kappa_{|X_b}]$; this integral class gives a natural polarization of X_b for each $b\in B^\circ$. When X is projective, we can assume that $[\alpha\kappa]$ is in $\mathrm{NS}(X)$.

2.4. **Automorphisms.** Let $p\colon X \to B$ be a Lagrangian fibration on X. Since $\rho(B) = 1$, there is a unique primitive ample class h_B in $\mathrm{NS}(B)$; we denote by $h \in \mathrm{NS}(X)$ its pull back by p: this class is nef and isotropic (i.e. $q_X(h,h) = 0$).

Let f be an automorphism that preserves the class h. Then f preserves the fibration p, in the following sense: there is an automorphism f_B of B such that $p \circ f = f_B \circ p$. The automorphism f_B preserves h_B. One can also find an embedding $B \subset \mathbb{P}^N(\mathbb{C})$ such that f_B preserves the Fubini-Study form (restricted to B); we denote such a form by κ_B:

$$f_B^* \kappa_B = \kappa_B. \tag{2.1}$$

The existence of such a form κ_B is due to Lo Bianco (see [29], Lemma 3.1). Let us sketch his proof. Since the Picard group of B is cyclic and B is projective, there is an f_B-invariant and very ample line bundle L_B on B. Then, f_B induces a linear transformation F_B of $H^0(B;L_B)$ and the Kodaira-Iitaka embedding $\iota\colon B \to \mathbb{P}(H^0(B;L_B)^\vee)$ is equivariant: $\iota \circ f_B = F_B \circ \iota$. On the other hand, the volume form $\mathrm{vol}_X := (\sigma \wedge \overline{\sigma})^g$ induces a probability measure $\mu_B = p_* \mathrm{vol}_X$ on B with full support which is f_B-invariant. Then, the invariance of $\iota_* \mu_B$ under F_B, the fact that $\iota(B)$ is not contained in a hyperplane of $\mathbb{P}(H^0(B;L_B)^\vee)$, and the equality $\mathrm{Supp}(\iota_* \mu_B) = \iota(B)$ imply that F_B is contained in a compact subgroup of $\mathsf{PGL}\,(H^0(B;L_B)^\vee)$. Thus, up to a linear conjugacy, F_B preserves the Fubini-Study metric.

A priori B can have singularities, but it does make sense to speak of such a differential form as the restriction of a form defined on the ambient space $\mathbb{P}^N(\mathbb{C})$; this is compatible with the definitions of Varouchas as suggested in [41], Remark 2.2 (see also [14]).

Theorem B. *Let $p\colon X \to B$ be a Lagrangian fibration of an irreducible hyperkähler manifold X. Let f be an automorphism of X such that $p \circ f = f_B \circ p$ for some $f_B \in \mathsf{Aut}(B)$. Then there is an integer $k \geq 1$ such that*

(1) *f^k preserves the symplectic form σ, i.e. $(f^k)^* \sigma = \sigma$, and*
(2) *$f_B^k = \mathrm{Id}_B$, i.e. f^k preserves each fiber of p.*

If X is not projective, then $f^* \sigma = \sigma$; in other words, one can take $k = 1$ in the first assertion; it is implied by $h \in \mathrm{T}(X)$, see Theorem 2.4 in [36].

Proof. Let us prove Assertion (1) (see also [8, 36]). Since f is parabolic, all eigenvalues of f^* on $H^2(X;\mathbb{C})$ have modulus 1. Since f^* preserves the lattice $H^2(X;\mathbb{Z})$, its characteristic polynomial is a monic polynomial with integer coefficients. Thus, the eigenvalues of f are roots of unity. On the other hand, σ

is unique up to a scalar factor, so $f^*\sigma = \alpha\sigma$ where α is the eigenvalue of f^* on $H^{2,0}(X;\mathbb{C})$. If k denotes the order of α, then $(f^k)^*\sigma = \sigma$.

When X is projective, Assertion (2) is part of a theorem from [29]([5]). The non-projective case reduces to the projective one as follows: using the form κ_B one defines a family of new complex structures X_t on X (the "degenerate twistor deformations" studied by Verbitsky and Soldatenkov), all fibered over B, such that the map f remains holomorphic on each X_t. Since some of these complex structures are projective, the conclusion will follow from the projective case. Details are provided in Section 7.4. □

3. Maximal variation and Betti coordinates

In this section, we study the variations of the translation vector of a parabolic automorphism of a projective hyperkähler manifold.

3.1. **The setting.** We suppose that X is a projective hyperkähler manifold, with a parabolic automorphism f that preserves a Lagrangian fibration $p_f\colon X \to B$, and use the notations introduced in the previous sections. For simplicity, we replace f by a positive iterate to assume that

$$p_f \circ f = p_f \text{ and } f^*\sigma = \sigma \tag{3.1}$$

as in Theorem B.

Since X is projective, we can find a multisection S of p_f. That is, $S \subset X$ is a smooth, irreducible, g-dimensional subvariety of X which is generically transverse to p_f. Then, $p_{f|S}\colon S \to B$ is generically finite. Moreover, if U is a sufficiently small, non-empty, open subset of B°, we can find such a multisection that is everywhere transverse to p_f above U, i.e. $p_{f|S\cap p_f^{-1}(U)}$ is a non-ramified cover from $S\cap p_f^{-1}(U)$ to U of some degree $d \geq 1$. With such a choice, and if U is simply-connected, there are d holomorphic sections s_i of p_f above U such that $S\cap U$ is the disjoint union of the $s_i(U)$. The degree d is the intersection number $([S]\cdot[X_b])$ for any fiber X_b.

3.2. **Translation vectors.** Let $U \subset B^\circ$ be simply connected and let $s\colon U \to X$ be a holomorphic section of p_f above U. Fix a basis of $H_1(X_U;\mathbb{Z})$ and consider the Betti diffeomorphism Φ and the translation vector t_f associated to these data.

[5]Assertion (2) has been claimed already in [1], with an explanation that Lo Bianco's argument for the projective case was valid in general. That explanation does not seem to be correct, this is why we provide a proof of (2) based on Verbitsky's idea in Section 7.4.

Lemma 3.1. *The following properties are equivalent.*

(1) $t_f\colon U \to \mathbb{R}^{2g}$ *is an open mapping;*
(2) $t_f(U)$ *contains an open subset of* $\mathbb{R}^{2g}$*;*
(3) $t_f\colon U \to \mathbb{R}^{2g}$ *has maximal rank* $2g$ *in the complement of a proper, real analytic subset of* U.

If they are not satisfied the generic rank of t_f *is even and* $\leq 2g-2$.

We just sketch the proof of this lemma because it is already proven in [3, 19, 9]. The first remark is that the fibers of the Betti projection $\pi_2 \circ \Phi\colon X_U \to \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ are complex submanifolds of X_U. The second remark is that, viewed in Betti coordinates, t_f is just the projection of $t = f \circ s - s$ on $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ and t is a holomorphic function (its differential Dt intertwines the complex structure j_U on TU with the complex structure j_X on TX). With this at hand, the first consequence is that t_f is real analytic, and in particular the maximum of the rank of $(Dt)_u$, $u \in U$, is attained on the complement of a proper real analytic subset of U. Then, let $\mathrm{j}(u)$ be the (translation invariant) complex structure on $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ (equivalently, on $\mathbb{R}^{2g}$) obtained from the restriction of j_X to X_u via Φ:

$$\mathrm{j}(u)(v) = \Phi_*(\mathrm{j}_X(\Phi_*^{-1}v)) \tag{3.2}$$

for every vector v tangent to $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$. Then $(\mathrm{j}(u))$ is a real analytic family of complex structures and the second consequence is $(Dt_f)_u \circ \mathrm{j}_U = \mathrm{j}(u) \circ (Dt_f)_u$ for every $u \in U$. Thus, the generic rank of t_f is even. These properties directly imply the lemma.

Lemma 3.2. *The property "*t_f has maximal variation*" does not depend on the Betti datum chosen to define the Betti coordinates.*

Indeed, changing the section s does not change t_f, and changing the basis of $H_1(X_U;\mathbb{Z})$ changes t_f into $A \circ t_f$ for some $A \in \mathsf{GL}_{2g}(\mathbb{Z})$, so in both cases the property "$t_f(U)$ contains an open subset of $\mathbb{R}^{2g}$" is preserved by such a change.

To show that the property does not depend on the choice of U, note that if $U \cap U'$ is non-empty, then Lemma 3.1(3) shows that t_f has maximal variation on U if and only if it has maximal variation on $U \cap U'$, and then this property propagates to U'. Then use that B° is connected.

3.3. **Volumes and variations.** To study the variations of t_f, we shall rely on the following volumic characterization of its maximal variation. If κ is a Kähler form on X, and if W is a complex analytic subset of (some open subset of) X of

dimension m, its volume with respect to κ is equal to

$$\mathrm{vol}_\kappa(W) = \int_W \kappa^m. \tag{3.3}$$

If $W \subset X$ is closed, its volume can be computed in cohomology as the intersection product $\mathrm{vol}_\kappa(W) = ([W] \cdot [\kappa]^m)$, where $[\kappa]$ is the class of κ and $[W]$ is the Poincaré dual of the homology class of W.

Lemma 3.3. *Let U be an open subset of B, the closure of which is contained in B°. Let κ be a Kähler form on X. Let M be a multisection of p_f. Then $\mathrm{vol}_\kappa(f^n(M) \cap X_U) = O(n^{2g})$, and the following properties are equivalent*

(a) *t_f does not have maximal variation;*
(b) *$\mathrm{vol}_\kappa(f^n(M) \cap X_U) = O(n^{2g-1})$ as n goes to $+\infty$;*
(c) *$\|(f^n)^* \kappa^g\|_{X_U} = O(n^{2g-1})$ as n goes to $+\infty$, where $\|\cdot\|_{X_U}$ denotes the uniform norm on X_U (for sections of $\wedge^{g,g}(TX)$).*

Instead of $O(n^{2g-1})$, we could obtain $O(n^r)$ where r is the generic rank of t_f on U, but the weakest estimate will be sufficient for our purpose. This lemma implies again that maximal variation of the translation vector is an intrinsic property that does not depend on the choice of Betti coordinates.

Proof. By compactness of $\overline{U} \subset B^\circ$, we reduce to the case when U is a ball (i.e. viewed in some local chart of B° containing $\overline{U}$, U is a ball in $\mathbb{C}^g$).

As in Section 3.2, $s\colon U \to X$ is a section of p_f above U, Φ is a Betti diffeomorphism and t_f is the translation vector in the Betti coordinates; we set $S = s(U)$. We can assume moreover that s (resp. Φ) extends to a neighborhood of $\overline{U}$ (resp. of $p_f^{-1}(\overline{U})$). We transport the riemannian metric associated to κ by Φ to get a riemannian metric $\|\cdot\|_{\kappa,\Phi}$ on $U \times \mathbb{R}^{2g}/\mathbb{Z}^{2g}$. Let $\|\cdot\|_{euc}$ be the euclidean metric on $\mathbb{C}^g \times \mathbb{R}^{2g}$, restricted to $U \times \mathbb{R}^{2g}$. Since $\overline{U} \subset B^\circ$, there is a constant $A \geq 1$ such that $A^{-1}\|\cdot\|_{euc} \leq \|\cdot\|_{\kappa,\Phi} \leq A\|\cdot\|_{euc}$ uniformly on the tangent space of $U \times \mathbb{R}^{2g}/\mathbb{Z}^{2g}$. Thus, when estimating volumes, we can work with the usual euclidean metric in the Betti coordinates.

Let d be the degree of the multisection M. Let $C \subset U$ be the branch locus of $p_{f|M}\colon M \cap p_f^{-1}(U) \to U$. Let D be a real analytic subset of U containing C such that $U' := U \setminus D$ is simply connected (by this we mean that each connected component of U' is simply connected). Since the Lebesgue measure of D vanishes, the volume of $f^n(M)$ above U' is the same as its volume above U. But over U', M is a union of d sections, so without loss of generality we may replace M by one of them and assume that M is in fact a section.

In the Betti coordinates, f becomes

$$f_\Phi\colon (u,x)\mapsto (u,x+t_f(u)) \tag{3.4}$$

$\Phi(S)$ is parametrized by $u\mapsto(u,0)$, and $\Phi(M)$ by $u\mapsto(u,m^\Phi(u))$ for some real analytic function m^Φ. Thus, $\Phi(f^n(M))$ is parametrized by

$$u\mapsto (u,m^\Phi(u)+nt_f(u)), \tag{3.5}$$

and the question is to estimate the volume of this submanifold of real dimension $2g$ with respect to the euclidean metric.

In the tangent space of $U\subset\mathbb{C}^g=\mathbb{R}^{2g}$, denote by $(v_i)_{i\le 2g}$ the standard orthonormal basis. In the tangent space of $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$, denote by $(e_i)_{i\le 2g}$ the standard orthonormal basis. The image of v_i by the differential of $u\mapsto(u,m^\Phi(u)+nt_f(u))$ is the vector

$$w_i(n;u)=v_i+(Dm^\Phi)_u(v_i)+n(Dt_f)_u(v_i). \tag{3.6}$$

Let us write $t_f(u)=(t_1(u),\dots,t_{2g}(u))$ and $(Dt_f)_u(v_i)=(\partial_i t_j(u))$; similarly, $m^\phi(u)=(m_1^\Phi(u),\dots,m_{2g}^\Phi(u))$ and $(Dm^\phi)_u(v_i)=(\partial_i m_j^\phi(u))$. We see that the exterior product $w_1(n;u)\wedge\cdots\wedge w_{2g}(n;u)$ is a polynomial in n of degree at most $2g$ with coefficients which are uniformly bounded, analytic functions of u. For instance, when $g=1$ we obtain

$$\begin{aligned}
w_1&=v_1+\partial_1 m_1^\Phi e_1+\partial_1 m_2^\Phi e_2+n\partial_1 t_1 e_1+n\partial_1 t_2 e_2\\
w_2&=v_2+\partial_2 m_1^\Phi e_1+\partial_2 m_2^\Phi e_2+n\partial_2 t_1 e_1+n\partial_2 t_2 e_2
\end{aligned}$$

and then setting $v_1'=v_1+\partial_1 m_1^\Phi e_1+\partial_1 m_2^\Phi e_2$ and $v_2'=v_2+\partial_2 m_1^\Phi e_1+\partial_2 m_2^\Phi e_2$ we obtain

$$\begin{aligned}
w_1\wedge w_2=\;&v_1'\wedge v_2'\\
&+n(\partial_2 t_1 v_1'\wedge e_1+\partial_2 t_2 v_1'\wedge e_2-\partial_1 t_1 v_2'\wedge e_1-\partial_1 t_2 v_2'\wedge e_2)\\
&+n^2(\partial_1 t_1\partial_2 t_2-\partial_1 t_2\partial_2 t_1)e_1\wedge e_2
\end{aligned}$$

where the dependence on u is implicit. The monomial n^{2g} appears only in front of $e_1\wedge\cdots\wedge e_{2g}$, and is multiplied by the function $u\mapsto\det((Dt_f)_u)$. Since the euclidean volume is bounded from above by the integral of the function $u\mapsto\left\|w_1(n;u)\wedge\cdots\wedge w_{2g}(n;u)\right\|$ with respect to the Lebesgue measure on $U\subset\mathbb{R}^{2g}$, this proves the first equivalence stated in the lemma.

The computation for $\|(f^n)^*\kappa^g\|_{X_U}$ is similar. □

In the next Section, we extend this type of estimate from compact subset $\overline{U}$ of B° to the whole base B itself. That is, we shall estimate the volumes of $f^n(S) \subset X$, where S is a multisection.

4. Propagation of volume estimates

We show that if the translation vector of f does not have maximal variation, then $\|(f^n)^*\|_{H^{g,g}(X;\mathbb{R})} = O(n^{2(g-1)})$.

Our first goal is the following proposition.

Proposition 4.1. *If the variations of t_f are not maximal then, for any multisection S of $p_f : X \to B$, there is an integer $D \geq 2$ such that*

$$\mathrm{vol}_\kappa(f^{D^n}(S)) = O(D^{2n(g-1)})$$

as n goes to $+\infty$.

This will be achieved in Section 4.3. Then in Section 4.4 we transfer this volume estimate into the upper bound $\|(f^n)^*\|_{H^{g,g}(X;\mathbb{R})} = O(n^{2g-1})$.

The difficulty is to propagate the volume estimate from Lemma 3.3 up to neighborhoods of the singular fibers of p_f because when approaching these fibers, the Betti coordinates may explode. To do this, we rely on pluripotential theory and use a technique that has been developed by Gauthier and Vigny. To refer directly to their work, we translate our problem into a dynamical property of a new (non-invertible, rational) transformation of X.

4.1. Multiplication by D along the fibers.

4.1.1. Let $p : X \to B$ be a fibration of a complex projective variety, the generic fiber of which is isomorphic to an abelian variety of dimension g. Let S_0 be a multisection of p of degree $d = ([S_0] \cdot [X_b])$, as in Section 3.1.

Let $D \geq 2$ be an integer such that d divides $D-1$. Then, there is a well defined dominant, rational transformation $m_D : X \dashrightarrow X$ acting by multiplication by D along the smooth fibers of p. More precisely, pick a point $b \in B^\circ$ and a base point w on the fiber X_b. Using w as neutral element, X_b becomes a commutative group isomorphic to $\mathbb{C}^g/\mathrm{L}(b)$ for some lattice $\mathrm{L}(b)$. Using the group law, the transformation m_D can be defined fiberwise by

$$z \in X_b \mapsto m_D(z) = Dz - \frac{D-1}{d} \sum_{s \in S_0 \cap X_b} s, \tag{4.1}$$

where the points of $S_0 \cap X_b$ are eventually repeated according to their multiplicities. This does not depend on w by the following standard lemma, the proof of which is straightforward.

Lemma 4.2. *Let a_i, $i = 1, \dots, l$ be integers such that $\sum_i a_i = 1$, and P_i, $i = 1, \dots, l$ be points on a complex torus. Then $\sum_i a_i P_i$ does not depend on the choice of the neutral element w.*

Hence it defines a rational transformation of X, regular above B°, preserving p_f, and of topological degree D^{2g}. In Betti coordinates, over some open subset $U \subset B^\circ$, m_D becomes

$$m_{D,\Phi}(u,x) = (u, Dx + t_D(x)) \tag{4.2}$$

for some real analytic map $t_D \colon U \to \mathbb{R}^{2g}$.

4.1.2. If $p \colon X \to B$ is a Lagrangian fibration of a hyperkähler manifold, the natural polarization of the fibers introduced in Section 2.3 is automatically invariant under m_D; that is,

(1) for b in B°, the image of the restriction of $H^2(X;\mathbb{Z})$ to $H^2(X_b;\mathbb{Z})$ is an infinite cyclic subgroup $R_b \subset H^2(X_b;\mathbb{Z})$;
(2) there is an ample line bundle A on X such that (a suitable multiple of) the ample generator of R_b is the Chern class of $A_b := A_{|X_b}$ and $m_D^* A_b = A_b^{\otimes D^2}$.

Thus, m_D is a family of polarized endomorphisms of $X \to B$ in the sense used by Gauthier and Vigny in [20].

To prove (2), let H_b be the ample generator of R_b. The inverse image by m_D multiplies H_b by D^2. If we pick any line bundle L_b in the class H_b, then $m_D^* L_b = L_b^{\otimes D^2} \otimes M_b$ where $M_b \in Pic^\circ(X_b)$. To find a line bundle in the class of H_b, which is taken to the power D^2 by m_D, we have to add to L_b a (D^2-1)-th root of M_b. There are $D^2 - 1$ of them, so equally $D^2 - 1$ line bundles $L_{b,j}$ in the class H_b, and we take the sum of them all to get a monodromy invariant A_b.

Now, suppose we start with a Lagrangian fibration and a multisection S_0, and we do the base change given by $p \colon S_0 \to B$. We get a new variety Y, a map $q \colon Y \to X$ of degree d, and a new fibration $p_Y \colon Y \to S_0$ such that $p \circ q = p_{|S_0} \circ p_Y$. Moreover, p_Y has a natural section $S_0^Y \subset Y$. Then, for each $D \geq 2$ we can construct a rational transformation $m_D^Y \colon Y \dashrightarrow Y$ acting by multiplication by D on the smooth fibers of p_Y and fixing S_0^Y pointwise (we use $S_0^Y \cap Y_b$ as the neutral element of $Y_b := p_Y^{-1}(b)$ for $b \in S_0^\circ$). The natural polarization R_b of the fibers of X_b can be pulled back to Y and it gives an m_D^Y-invariant polarization.

4.2. **Local to global volume estimates.** Let us summarize some of the results of [20]. We fix a fibration $p_Y : Y \to B_Y$ and a rational transformation $g : Y \to Y$. We also fix a Kähler form κ_B on B_Y. We assume that $p_Y \circ g = p_Y$, that g is regular over some dense open subset of B_Y, and that, as above, g is (relatively) polarized. This assumption is equivalent to the existence of an ample line bundle A on Y and an integer $D(g) \geq 2$ such that

$$g^* A_b = A_b^{D(g)} \tag{4.3}$$

for all b in a dense open subset of B_Y. Choose a Kähler form κ representing the Chern class of A and a dense open subset B'_Y of B_Y° over which g is regular and satisfies (4.3) (B'_Y is a *regular part* in the sense of [20]). Set $Y' = p_Y^{-1}(B'_Y)$. Then, on Y', there is a closed positive current $\hat{T}_g$, of type $(1,1)$, such that

$$\frac{1}{D(g)^k}(g^k)^* \kappa \to \hat{T}_g \tag{4.4}$$

in the sense of weak convergence for currents. Moreover, $\hat{T}_g$ has local, continuous potentials (on the open set Y'). For this, we refer to Section 2.3 of [20] ([6]).

Then, Proposition 3.3 in [20] shows that the following properties are equivalent. Let S be a multisection of p_Y and let $\{S\}$ denote the current of integration on S. Let b_Y denote the dimension of B_Y, hence also the dimension of S. The following properties are equivalent

(a) locally above B'_Y the volume of $g^k(S)$ does not grow as fast as $D(g)^{b_Y k}$. This means that for any open subset $U \subset B_Y$ such that $\overline{U} \subset B'_Y$ we have

$$\mathrm{vol}(g^k(S_U)) = o(D(g)^{b_Y k})$$

or equivalently

$$\int_{g^k(S_U)} \kappa^{b_Y} = o(D(g)^{b_Y k})$$

as k goes to $+\infty$;

(b) the intersection of $\hat{T}_g^{b_Y}$ with S over B'_Y vanishes, i.e.

$$\int_{Y'} S \wedge \hat{T}_g^{b_Y} = 0;$$

[6]The construction in [20] differs slightly from what we write. They fix an equivariant embedding $\iota : Y \to B_Y \times \mathbb{P}^N$ such that $\pi = \pi_B \circ \iota$ where π_B is the first projection $B_Y \times Y \to B_Y$. Then, they replace the Kähler form κ by κ_{FS}, the restriction of the Fubini-Study form to $\iota(Y)$. So, their form is not Kähler, but in the limit process (4.4) we obtain the same current.

(c) the global volume of the strict transform $g_*^k(S)$ grows as most as $D(g)^{(b_Y-1)k}$, i.e.

$$\mathrm{vol}(g_*^K(S)) = O(D(g)^{(b_Y-1)k})$$

as k goes to $+\infty$; equivalently

$$\| [g_*^k(S)] \| = O(D(g)^{(b_Y-1)k}).$$

Here, κ is any Kähler form on Y and $[\cdot]$ denotes the class in $H^{2b_Y}(Y;\mathbb{Z})$. The point is that the local Property (a), in which the implicit constant in $o(\cdot)$ might depend on U, gives rise to the global estimates stated in Property (c).

If we apply this result when $b_Y = g$, $\dim(Y) = 2g$, the generic fiber of $Y \to B_Y$ is abelian, and $g = m_D^Y$ acts by multiplication by D along the fibers, then $D(g) = D^2$ and we derive that $\mathrm{vol}(g^k(S_U)) = o(D^{2gk})$ implies

$$\| [(m_D^Y)_*^k(S)] \| = O(D^{(2g-2)k}). \tag{4.5}$$

4.3. **Proof of Proposition 4.1.** Set $X^\circ = X_{B^\circ}$.

4.3.1. First, we assume $p_f \colon X \to B$ has a section $s_0 \colon B \dashrightarrow X$, the image of which is denoted by S_0.

(a). *First Step.*– We fix some integer $D \geq 2$ and denote by $m_D \colon X \dashrightarrow X$ the rational map fixing S_0 and acting by multiplication by D along the fibers of p_f. Let Φ denote local Betti coordinates associated to an open set $U \subset B^\circ$, the section S_0, and some basis of $H_1(X_b;\mathbb{Z})$, $b \in U$. Viewed in the Betti coordinates, the section s_0 corresponds to $s_{0,\Phi}(u) = 0$, while f and m_D correspond to

$$f_\Phi(u,x) = (u, x + t_f(u)) \tag{4.6}$$

$$m_{D,\Phi}(u,x) = (u, Dx). \tag{4.7}$$

Thus, we have

$$f_\Phi^{D^k} \circ s_{0,\Phi}(u) = D^k t_f(u) \tag{4.8}$$

$$= m_{D,\Phi}^k \left(f_\Phi \circ s_{0,\Phi}(u) \right) \tag{4.9}$$

which means that above U we have

$$f^{D^k}(S_{0,U}) = m_D^k \left(f(S_{0,U}) \right). \tag{4.10}$$

From Lemma 3.3 and Section 4.2, we deduce that

$$\| f^{D^k}(S_0) \| = O(D^{(2g-2)k}) \tag{4.11}$$

where $\| \cdot \|$ is any norm on the vector space $H^{2g}(X;\mathbb{R})$.

(b). *Second Step*.– Now, we need to prove an estimate similar to (4.11), but with S_0 replaced by any multisection S of p_f. Consider the fiber product $W = X \times_B X \times_B X$; as a set, this algebraic variety is

$$W = \{(x,y,z) \in X^3 \ ; \ p_f(x) = p_f(y) = p_f(z)\}. \tag{4.12}$$

It comes with a fibration $P_f \colon W \to B$, defined by $P_f(x,y,z) = p_f(x)$, and with a rational map $A \colon W \to X$, defined by

$$A(x,y,z) = x + (y-z), \tag{4.13}$$

using the group law along the fiber X_b, $b = P_f(x,y,z)$. (We do not need the existence of a section or a choice of neutral element in X_b for this definition.)

Let S be a multisection of p_f and consider the sequence of multisections $T_k \subset W$ of P_f defined by

$$T_k = \{(x,y,z) \in W \ ; \ x \in S, \ y \in S_0, \ z = f^{D^k}(y)\}. \tag{4.14}$$

Then,

$$f^{D^k}(S) = A(T_k). \tag{4.15}$$

From the first step, we know that the class $[T_k] \in H^{2g}(W;\mathbb{Z})$ satisfies

$$\| \ [T_k] \ \| \leq C D^{(2g-2)k} \tag{4.16}$$

for some constant $C > 0$ that depends on S. At the level of (co)homology classes, A acts as a linear map between finite dimensional spaces; thus, we obtain $\| \ [A(T_k)] \ \| \leq C' \ \| \ [T_k] \ \|$ for some constant $C' > 0$ and

$$\| \ [f^{D^k}(S)] \ \| \leq C'' D^{(2g-2)k} \tag{4.17}$$

for $C'' = C \cdot C'$. This concludes the proof.

4.3.2. In case $p_f \colon X \to B$ does not have a section, we take for S_0 a multisection and do the base change $p_f \colon S_0 \to B$, as in Section 4.1.2. This provides a new variety $Y \to S_0$ with a natural section S_0^Y, and a map $q \colon Y \to X$ above $S_0 \to B$. Define S_0' to be the locus of points $s \in S_0$ around which $p_f \colon S_0 \to B$ is a local diffeomorphism and $p_f(x) \in B^\circ$. If $V \subset S_0'$ is an open subset and V is small enough, q realizes a diffeomorphism from $Y_V = p_Y^{-1}(V)$ to $p_f^{-1}(p_f(V))$. The automorphism f induces a rational transformation f_Y of Y such that $f \circ q = q \circ f_Y$. For $D \geq 2$, we define $m_{D,Y}$ to be the multiplication by D along the fibers of p_Y fixing the natural section S_0^Y.

Now, if S is a multisection of p_f, we pull back it to Y by q. This gives a multisection S^Y of p_Y for which $\mathrm{vol}(f_Y^{D^k}(S_U^Y)) = O(D^{(2g-1)k})$ as soon as $U \subset S_0'$, because q realizes a local conjugation between f_Y and f. Thus, we can repeat

the argument from Step 1 above in Y. Then, we can repeat the argument from Step 2 by working on $W_Y = Y \times_{S_0} Y \times_{S_0} Y$ and composing the addition map $A\colon W_Y \to Y$ (defined as in Equation (4.13)) with $q\colon Y \to X$. This concludes the proof in the general case.

4.4. **A cohomological estimate.**

Proposition 4.3. *Let f be a parabolic automorphism of a hyperkähler manifold X. If the variations of the translation vector of f are not maximal, then*

$$\|(f^n)^*\|_{H^{g,g}(X;\mathbb{R})} = O(n^{2(g-1)})$$

as n goes to $+\infty$.

Proof. Embed X into a projective space $\mathbb{P}^N$, and intersect it with a suitable linear subspace to get a multisection S of X. The class $[S]$ of S will be considered as an element of $H^{g,g}(X;\mathbb{Z})$ (using Poincaré duality), it is the same as $[\omega_{FS}^g]$ where ω_{FS} denotes the restriction of the Fubini-Study form to X. Proposition 4.1 shows that

$$\|(f^n)^*[S]\| = O(n^{2(g-1)}) \tag{4.18}$$

along the subsequence $n = D^k$, for some $D \geq 2$.

Now, consider a real subspace W of $H^{2g}(X;\mathbb{R})$ together with a closed, convex, and salient cone $C \subset W$, the interior of which is non-empty. Assume that (a) W and C are f^*-invariant and (b) $[S]$ is in the interior of C. Then, $\|(f^*)^n\|_W = O(n^{2(g-1)})$. Indeed, by Birkhoff's version of the Perron-Frobenius theorem, we know that

$$A^{-1}\|(f^*)^n[S]\| \leq \|(f^*)^n\|_W \leq A\|(f^*)^n[S]\| \tag{4.19}$$

for some constant $A \geq 1$. Thus, Equation (4.18) implies that the spectral radius of f_W^* is equal to 1. This, in turn, implies that $\|(f^*)^n\|_W$ grows like a power of n, and then Equation (4.18) shows that this power is $\leq 2(g-1)$.

We apply this scheme to the vector space $W = H^{g,g}(X;\mathbb{R})$ and the cone $C = P^g(X)$, the cone of classes represented by closed positive currents of bidegree (g,g) (see for instance [15]). This is a closed, convex cone, and it is salient because the set of closed positive currents T of bidegree (g,g) with fixed mass $M_\kappa(T) = \langle T|\kappa^g\rangle$ is compact for the weak-$*$ topology. The class $[S] = [\omega_{FS}^g]$ is in the interior of this cone, because any small perturbation of ω^g is a positive (g,g)-form. And $P^g(X)$ is $\mathsf{Aut}(X)$-invariant. This concludes the proof. □

5. Action on the cohomology

In this section, we show that if f is a parabolic automorphism of a hyperkähler manifold, then for every $1 \leq p \leq g$ there is a positive constant $c_p(f)$ such that $\|(f^*)^n\|_{H^{p,p}(X;\mathbb{R})} \simeq c_p(f)n^{2p}$.

Let f be an automorphism of a hyperkähler manifold. Recall from Section 1.1.3 that f is either elliptic, parabolic, or loxodromic. The following result is a concatenation of theorems of Lo Bianco, Oguiso, and Verbitsky.

Theorem C. *Let f be an automorphism of a hyperkähler manifold of dimension $2g$.*

(1) *If f is elliptic, then $f^k = \mathrm{Id}_X$ for some $k \geq 1$;*
(2) *if f is parabolic, then for every $1 \leq p \leq g$ there is a positive constant $c_p(f)$ such that $\|(f^*)^n\|_{H^{p,p}(X;\mathbb{R})} \simeq c_p(f)n^{2p}$;*
(3) *if f is loxodromic, there is a real number $\lambda(f) > 1$ such that for every $1 \leq p \leq g$ there is a positive constant $c_p(f)$ such that $\|(f^*)^n\|_{H^{p,p}(X;\mathbb{R})} \simeq c_p(f)\lambda(f)^{pn}$;*
(4) *by duality, if f is parabolic then for $g \leq p \leq 2g$ there is a positive constant $c_p(f)$ with $\|(f^*)^n\|_{H^{p,p}(X;\mathbb{R})} \simeq c_p(f)n^{2(2g-p)}$, and similarly if f is loxodromic then $\|(f^*)^n\|_{H^{p,p}(X;\mathbb{R})} \simeq c_p(f)\lambda(f)^{n(2g-p)}$.*

We only sketch the proof, because the only part that may be considered to be new is the second assertion.

Let us prove Assertion (1). Since f is elliptic, its eigenvalues on $H^{1,1}(X;\mathbb{R})$, and then on $H^2(X;\mathbb{R})$, all have modulus 1. Thus, being roots of the characteristic polynomial of $f^*_{H^2(X;\mathbb{Z})}$, hence of a monic polynomial with integer coefficients, these eigenvalues must be roots of unity. This implies that a positive iterate of f acts trivially on $H^2(X;\mathbb{Z})$. Then, according to Lieberman's theorem (see [26]), a positive iterate of f must be in the connected component $\mathsf{Aut}(X)^0$ of the identity. Since $\mathsf{Aut}(X)^0$ is trivial for any irreducible hyperkähler manifold (see [22]), we conclude that f has finite order.

Now, suppose that f is parabolic. Then $\|(f^*)^n\|_{H^{1,1}(X;\mathbb{R})} \simeq c_1(f)n^2$ for some $c_1(f) > 0$ (see § 8). Theorem 1.5 of [42] says that $\mathrm{Sym}^p(H^2(X;\mathbb{R}))$ embeds into $H^{2p}(X;\mathbb{R})$ for $p \leq g$ via the cup product, so

$$\|(f^*)^n\|_{H^{p,p}(X;\mathbb{R})} \geq cn^{2p} \tag{5.1}$$

for some positive constant c and for all $n \geq 1$. On the other hand, from the Khovanski-Teyssier inequalities, the numbers

$$s_p(f) = \limsup_{n\to+\infty} \frac{\log(\|(f^*)^n\|_{H^{p,p}(X;\mathbb{R})})}{\log(n)} \tag{5.2}$$

form a concave function of p, for $0 \leq p \leq \dim(X)$ (see [27]); this means that

$$2s_p(f) \leq s_{p-1}(f) + s_{p+1}(f) \tag{5.3}$$

for every p. Since $s_0(f) = 0$ and $s_1(f) = 2$, it follows from the lower estimate (5.1) that $s_p(f) = 2p$ for every $p \leq g$.

When f is loxodromic, we have $\|(f^*)^n\|_{H^{1,1}(X;\mathbb{R})} \simeq c_1(f)\lambda(f)^n$ for some $\lambda(f) > 1$ and the same argument leads to Assertion (3) (for the details, see [28, Appendix] or [36]).

6. Theorem A in the projective case, and application

We prove Theorem A when X is assumed to be projective and give an application to the dynamics of some groups of automorphisms.

6.1. **Proof of Theorem A.** Let f be a parabolic automorphism of a hyperkähler manifold X, as in Theorem A. Assertion (1) of Theorem A is contained in Theorem C. In particular, $\|(f^*)^n\|_{H^{g,g}(X;\mathbb{R})}$ grows like $c_g(f)n^{2g}$. The same estimate holds for f^k, the first positive iterate acting as the identity on the base of the invariant Lagrangian fibration p_f, with constant $c_g(f)$ replaced by $c_g(f^k) = c_g(f)k^{2g}$.

Now, assume that X is projective. By Proposition 4.3, the translation vector of f^k has maximal variations, as stated in Assertion (2) of Theorem A.

Let us derive Assertion (3) from Assertion (2). This final step does not use that X is projective, only the validity of Assertion (2). Let $U \subset B^\circ$ be a small, relatively compact, simply connected open subset. Write f^k on X_U in some Betti coordinates :

$$f^k_\Phi(u,x) = (u, x + t_{f^k}(u)). \tag{6.1}$$

For $b \in U$, the closure $Z(b)$ of $\mathbb{Z}t_{f^k}(b) \in \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ is a Lie subgroup of $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$. Its dimension $r(b)$ and its number of connected components $c(b)$ vary with b. Since t_{f^k} has maximal variations, $t_{f^k}(U)$ contains an open set, so that any pair (r,c) with $0 \leq r \leq 2g$ and $c \in \mathbb{Z}_{\geq 1}$ can be realized as $(r(b), c(b))$ by some b in U. This proves Assertion (3) because orbits of f^k in X_b correspond, in the Betti coordinates $U \times \mathbb{R}^{2g}/\mathbb{Z}^{2g}$, to subsets of type $\{b\} \times (x + Z(b))$.

6.2. **Example.** Consider a compact hyperkähler manifold X, of dimension $2g$, together with two parabolic automorphisms f and g preserving distinct Lagrangian fibrations $p_f\colon X \to B_f$ and $p_g\colon X \to B_g$. By this we mean that p_f and p_g satisfy the following equivalent properties (see [1, Proposition 2.20]): (a) the restriction of p_f to a general fiber of p_g is not constant; (b) the restriction of p_f to a general fiber of p_g is a dominant morphism onto B_f; (c) the restriction of p_g to a general fiber of p_f is a dominant morphism onto B_g. The existence of two distinct Lagrangian fibrations like that implies that X is projective.

For each pair of positive integers (k,ℓ), consider the subgroup $\Gamma_{k,\ell}$ of $\mathsf{Aut}(X)$ generated by f^k and g^l. Fix a distance $\mathsf{dist}(\cdot)$ on X and say that a subset $\Lambda \subset X$ is ε-dense if every point of X is at distance less than ε from a point of Λ.

(1) The set $\{x \in X\ ;$ the orbit $\Gamma_{k,\ell}(x)$ is dense in $X\}$ (for the euclidean topology) is a countable intersection of open dense subsets of X; in particular, it is dense and has full measure for the volume form $(\sigma \wedge \overline{\sigma})^g$.
(2) For every $\varepsilon > 0$, there is an integer $N \geq 1$ such that for every pair (k,ℓ) with k and $\ell \geq N$, the set $\{y \in X\ ;\ \Gamma_{k,\ell}(y)$ is finite$\}$ is an ε-dense set.

To get (1), do as in [1, §6] and [9]: by Theorem A, the locus of points b in B_f such that the orbit closures of f in X_b have codimension ≥ 1 is a countable union of proper real analytic subsets of B_f°. Pick a point $x \in X$ such that $p_f(x)$ is not in this meager set. The closure of $\Gamma_{k,\ell}(x)$ contains the fiber of p_f through x, hence also the orbit of this fiber by $\Gamma_{k,\ell}$; then, using g, it contains all fibers of p_g in the complement of a meager set. This shows that a generic orbit is dense. To conclude, note that being dense is the same as being ε-dense for all $\varepsilon > 0$. (Also, an orbit is dense for some $\Gamma_{k,\ell}$ if and only if it is dense for all $\Gamma_{m,n}$.)

To get Assertion (2), we use the following consequence of Theorem A: the set $F_k := \{b \in B_f\ ;\ f^k_{|X_b} = \mathrm{Id}_{X_b}\}$ becomes ε-dense in B if k is large enough. Similarly, the set $G_\ell \subset B_g$ corresponding to fibers of p_g on which the order of g divides ℓ is ε-dense for large enough ℓ. Then, the set

$$p_f^{-1}(F_k) \cap p_g^{-1}(G_\ell) \tag{6.2}$$

is ε-dense and is made of fixed points of $\Gamma_{k,\ell}$.

Remark 6.1. The second assertion shows the finite exceptional orbits in Theorem 0.2 of [16] can be arbitrarily large (resp. ε-dense) when one reduces the size of the group (see also Corollary 1.2 of [12]).

7. From the projective case to the Kähler case

The purpose of this section is to deduce Theorem A in the non-projective setting from the case of projective hyperkähler manifolds. For this, we apply the method of degenerate twistor deformations developed by Verbitsky and Soldatenkov in [43, 41].

According to Section 6.1, we only need to prove the second assertion of Theorem A, since the first one has already been verified for all hyperkähler manifolds and the third one is a consequence of the second.

7.1. **The non-projective setting.** Let X be a *non-projective*, irreducible, hyperkähler manifold, with a fixed holomorphic structure, equipped with some holomorphic Lagrangian fibration $p\colon X \to B$. Let σ be a holomorphic symplectic form on X. Recall from Section 2.3 that B is projective. We denote by h the pull-back of a fixed very ample class in $\mathrm{NS}(B)$. The class $h \in \mathrm{NS}(X)$ is nef and satisfies $q_X(h) = 0$.

Since X is not projective, Huybrechts' Theorem (see § 2.1) shows that the form q_X is negative semi-definite on $\mathrm{NS}(X;\mathbb{R})$, with one-dimensional kernel $\mathbb{R}h$. In other words, with the vocabulary from Sections 2.1 and 2.2, the lattice $(\mathrm{NS}(X), q_X)$ is parabolic and the class h is an isotropic element of both $\mathrm{NS}(X)$ and the transcendental lattice $\mathrm{T}(X)$.

Consider a parabolic automorphism $f\colon X \to X$ such that $p \circ f = f_B \circ p$ for some automorphism f_B of B. Since $\mathrm{Pic}(B) \simeq \mathbb{Z}$, we have $f^*h = h$. Moreover, according to Section 2.4, there is a Kähler form κ_B on B such that $f_B^*\kappa_B = \kappa_B$; we can assume furthermore that $[\kappa_B] = h$.

In the non-projective case, it follows from Oguiso's results that $f^*\sigma = \sigma$ (see Assertion (1) of Theorem B and the comment after this theorem).

7.2. **Twistor deformations.** *A* C*-symplectic form on a differentiable manifold M of dimension $4n$ is a closed, complex-valued 2-form Ω such that $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega^n}$ is everywhere non-vanishing* (see Definition 2.1 in [41]). The first main properties of such a C-symplectic form are (see [41]):

(1) the kernel of Ω on the complexified tangent space $T_{\mathbb{C}}M$ is everywhere of dimension $2n$ and can be seen as the antiholomorphic tangent bundle of a complex structure J;
(2) with respect to this complex structure, Ω is a holomorphic symplectic form.

Take X as in Section 7.1. On the differentiable manifold X, consider the family of differentiable forms $\Omega_t = \sigma + t p^*\kappa_B$. According to [41, Theorem 2.3],

(3) for any t, Ω_t is a C-symplectic form.

Thus, the family Ω_t defines a family of complex structures J_t, hence a family of complex manifolds (on the same underlying differentiable manifold) such that $X_0 = X$, Ω_t is a holomorphic symplectic form on X_t, and therefore $H^{2,0}(X_t;\mathbb{C})$ is generated by $[\Omega_t] = [\sigma] + th$ for each t. Moreover, by [41, Theorem 2.3],

(4) the map $p : X_t \to B$ remains a holomorphic Lagrangian fibration, and the complex structure on the fibers of p does not change.

7.3. **Conclusion.** Now, the next proposition is almost obvious.

Proposition 7.1. *The diffeomorphism $f : X \to X$ is holomorphic with respect to all the complex structures J_t, $t \in \mathbb{C}$.*

Proof. First, recall that we assume that the parabolic automorphism is symplectic with respect to the initial symplectic structure $\sigma = \Omega_0$. Moreover, f_B preserves κ_B. Thus, f preserves each C-symplectic form $\Omega_t = \sigma + tp^*\kappa_B$. In particular, f preserves its kernel and the conjugate of the kernel, which are the antiholomorphic and holomorphic tangent bundles to X_t, respectively. □

With this construction at hand, the keypoint is that some of the X_t are projective for arbitrarily small parameters $t \in \mathbb{C}$.

Proposition 7.2. *For any $r > 0$, there exists $t \in \mathbb{C}$ with $|t| < 1/r$ such that X_t is projective.*

Proof. For t small, X_t is Kähler, because being Kähler is an open property; then, as in any family of Kähler manifolds, the Hodge numbers stay constant (see [44]). Thus, for small parameters, we obtain a family of irreducible hyperkähler manifolds.

From Huybrechts' theorem, we know that X_t is projective when it carries an integral $(1,1)$-class u with $q(u,u) > 0$. Moreover, an integral class $u \in H^2(X;\mathbb{Z})$ is of type $(1,1)$ with respect to the complex structure J_t when it is q-orthogonal to the class $[\Omega_t]$.

Fix $r \geq 1$ large, and then choose a class $a \in H^2(X,\mathbb{Z})$ such that

$$0 < r|q(a,\sigma)| < q(a,h); \tag{7.1}$$

such a class exists because $q(\cdot,\sigma)$ and $q(\cdot,h)$ are two linearly independent linear forms on $H^2(X;\mathbb{C})$. Then, changing a into $a + mh$ for some integer $m \in \mathbb{Z}$ we can suppose $q(a,a) > 0$. Now, a is of type $(1,1)$ on X_t if and only if $t = -q(a,\sigma)/q(a,h)$. This defines a unique t, of modulus $< 1/r$. □

Since Assertion (2) of Theorem A does not depend on the complex structure J_t but only on the dynamical properties of f, we can now apply Theorem A on X_t to derive the same conclusion on X_0. This concludes the proof of our main theorem in the non-projective setting.

7.4. **Extension of Lo Bianco's theorem.** A similar argument can be applied to extend the second assertion of Theorem B from the projective to the non-projective setting. Indeed, along a twistor deformation (X_t, Ω_t), with $\Omega_t = \sigma + tp^*\kappa_B$, the action of f_t on the base of its invariant fibration does not change. Since for some parameters t we know that X_t is projective, we can apply Lo Bianco's result to conclude that f_B has finite order.

8. Appendix

8.1. **Parabolic isometries.** Let V be a real vector space of (finite) dimension $m+1$ endowed with a non-degenerate quadratic form q_V of signature $(1,m)$. Let $h \in \mathsf{O}(q_V)$ be a linear transformation of V preserving q_V. By definition, h is parabolic if it does not fix any vector $v \in V$ with $q_V(v) = 1$ and if all its eigenvalues have modulus 1. Equivalently, 1 is an eigenvalue of h, but the corresponding eigenspace $\{v\,;\, h(v) = v\}$ does not intersect $\{v\,;\, q_V(v) > 0\}$. When h is parabolic, there is a unique isotropic line $D_h \subset V$ which is h-invariant, and this line is fixed pointwise. For this, we refer to Ratcliffe's book on hyperbolic geometry [40].

Remark 8.1. If h is a parabolic isometry and h preserves a subspace $W \subset V$ on which q_V is non-degenerate and indefinite, then $h_{|W} : W \to W$ is also parabolic.

If $\dim(V) \leq 2$, there is no parabolic isometry; indeed, if $\dim(V) = 2$ the isotropic cone is made of two lines, and a parabolic isometry should preserve each of them, with eigenvalue 1 on one of them, hence with eigenvalue ± 1 on the second (because $\det(\cdot) = \pm 1$ on $\mathsf{O}(q_V)$), but then the isometry would have order 1 or 2.

Proposition 8.2. *Let h be such a parabolic isometry. Then, given any operator norm $\|\cdot\|$ on $\mathsf{End}(V)$, there is a constant $c(h) > 0$ such that $\|h^n\| \simeq c(h)n^2$.*

Proof. The characteristic polynomial of h can be written $P_h(t) = (t-1)^r Q(t)$ where $Q \in \mathbb{R}[t]$ and $Q(1) \neq 0$. From this, we get a decomposition $V = E_1 \oplus E_Q$ where E_1 is the kernel of $(h - \mathrm{Id}_V)^r$ and E_Q is the kernel of $Q(h)$. This is an orthogonal decomposition $E_1^\perp = E_Q$. In particular, the restriction of q_V to E_1 is non-degenerate. The line D_h is contained in E_1. Thus, the signature of q_V on E_1 is $(1, \dim(E_1) - 1)$, and q_V is negative definite on E_Q. In particular, the restriction of h to E_Q is in a compact group.

Thus, we can now assume that V is equal to E_1. In other words, h is unipotent. Then, on $D_h^\perp / D_h$, the endomorphism induced by h is unipotent and preserves a negative definite quadratic form; thus, it is equal to the identity.

Let v_2 be an element of V such that $q_V(v_2) = 1$. Set $v_1 = h(v_2) - v_2$. Then $v_1 \neq 0$ (because h is parabolic) and v_1 is orthogonal to D_h. Thus, $v_1 = h(v_1) - v_0$ for some $v_0 \in D_h$. The vector v_0 is not 0, because otherwise h would induce a parabolic isometry

of the 2-dimensional space $\mathrm{Vect}(v_1, v_2)$. Thus, the vector space $W = \mathrm{Vect}(v_1, v_2, v_3)$ is h-invariant, has dimension 3, and contains D. In the basis (v_1, v_2, v_3), the matrix of h is a Jordan bloc of size 3, and the growth of $\|h^n\|$ is quadratic. On the orthogonal complement $W^\perp$, h is the identity. This concludes the proof. □

8.2. **Parabolic automorphisms.** Combining the two previous sections, we get the notion of parabolic automorphism of hyperkähler manifolds: these are automorphisms $f\colon X \to X$ such that f^* determines a parabolic isometry of $V := H^{1,1}(X;\mathbb{R})$ with respect to the Beauville-Bogomolov quadratic form $q_V := q$. Then, Proposition 8.2 proves the equivalence between Assertions (a) and (c) from Section 1.1.3. The equivalence with (b) comes from the fact that all eigenvalues of f^* are roots of unity because f^* preserves the lattice $H^2(X;\mathbb{Z})$ in $H^2(X;\mathbb{R})$.

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