Parabolic automorphisms of hyperkähler manifolds

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Abstract

A parabolic automorphism of a hyperkähler manifold M is a holomorphic automorphism acting on $H^2(M)$ by a non-semisimple quasi-unipotent linear map. We prove that a parabolic automorphism which preserves a Lagrangian fibration acts on almost all fibers ergodically. The existence of an invariant Lagrangian fibration is automatic for manifolds satisfying the hyperkähler SYZ conjecture; this includes all known examples of hyperkähler manifolds. When there are two parabolic automorphisms preserving two distinct Lagrangian fibrations, it follows that the group they generate acts on M ergodically. Our results generalize those obtained by S. Cantat for K3 surfaces.

Résumé

Un automorphisme parabolique d'une variété hyperkählerienne M est un automorphisme holomorphe dont l'action sur $H^2(M)$ est quasi-unipotente et non semi-simple. Nous montrons que pour un automorphisme parabolique préservant une fibration lagrangienne, l'action sur presque toute fibre est ergodique. L'existence d'une fibration lagrangienne invariante est automatique lorsque la conjecture SYZ hyperkählerienne est satisfaite, en particulier pour tous les exemples connus. Quand on a deux automorphismes paraboliques préservant deux fibrations différentes, on obtient que l'action du groupe qu'ils engendrent est ergodique sur M. Nos résultats généralisent ceux obtenus par S. Cantat pour les surfaces K3.

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1 Introduction

1.1 K3 surfaces

Let S be a projective K3 surface and let $f : S \to S$ be an automorphism. The Neron-Severi lattice $\mathsf{NS}(S)$ is of signature $(1, \rho_S - 1)$ where ρ_S is the Picard number of S, and f induces an automorphism f^* of $NS(S)$. When $\rho_S \geqslant 3$, the projectivization of the cone of vectors of positive square in $V = \mathsf{NS}(S) \otimes \mathbb{R}$ can be viewed as a model of the hyperbolic space which we denote by $\mathbb{H}_S$. The projectivization of the isotropic cone is the boundary of the hyperbolic space, called the absolute, and $f^* \in O(V)$ induces an isometry of $\mathbb{H}_S$. Replacing f by f^2, we may assume $f^* \in SO^+(V)$[1]. The following theorem is well-known (see [Ka2] or [R] for details).

Theorem-definition: Let $n > 1$, and $\alpha \in SO^+(1, n)$ a non-trivial isometry acting on (V, q), q quadratic form of signature $(1, n)$. Then one and only one of these three cases occurs:

(i) α has an eigenvector x with $q(x) > 0$ (α is an **elliptic isometry)**

(ii) α has two eigenvectors x, y with $q(x) = q(y) = 0$ and real eigenvalues λ_x, $\lambda_y = \lambda_x^{-1}$ satisfying $|\lambda_x| > 1$ (α is a **loxodromic, or hyperbolic, isometry)**

(iii) α has a single (up to a constant) eigenvector x with $q(x) = 0$. In this case the eigenvalue $\lambda_x = 1$, and α is called a **parabolic isometry**).

In terms of the action on the hyperbolic space $\mathbb{H}$, an elliptic isometry has a fixed point in $\mathbb{H}$, a parabolic one has one fixed point on the absolute, and a loxodromic one has two fixed points on the absolute (and no fixed points inside $\mathbb{H}$).

Remark 1.1: All eigenvalues of elliptic and parabolic isometries have absolute value 1. Hyperbolic and elliptic isometries are semisimple (that is, the corresponding linear operators are diagonalizable over $\mathbb{C}$), parabolic are not. When (V, q) has some underlying integral structure, for example if $V = \mathsf{NS}(S) \otimes \mathbb{R}$, and q is the intersection form, the elliptic isometries preserving the integral structure are of finite order. Parabolic ones are of infinite order; indeed, any linear homomorphism of finite order is semisimple.

[1] Here $SO^+(V)$ denotes the connected component of the unity of $SO(V)$.

We call the automorphism f **parabolic** if f^* is parabolic. In this case, f^* preserves a class of self-intersection zero, which one may clearly assume nef (by invariance of the nef cone) and integral (as f^* is integral).

Remark that the classification of automorphisms into elliptic, parabolic and loxodromic types also makes sense for compact Kähler surfaces, when one replaces $\mathsf{NS}(S)\otimes\mathbb{R}$ by $H^{1,1}_{\mathbb{R}}(X)$. Though the latter space has no natural integral structure, it is easy to see that elliptic isometries are of finite order[2] and that the parabolic isometries fix a nef integral class, using the integral structure on $H^2(X,\mathbb{R})$ (see for example [M], p. 31).

By Gizatullin's theorem ([G, Grv]), any parabolic automorphism p of a projective surface admits an invariant elliptic fibration, or "elliptic pencil", that is, a fibration $\pi : S \to \mathbb{P}^1$ with general fiber a smooth curve of genus 1 (we do not assume that this fibration has a section), and such that p sends fibers to fibers. For a K3 surface this is easy to see directly, indeed, one deduces from the Riemann-Roch theorem that the linear system of the sections of a nef line bundle of square zero is an elliptic pencil. Moreover, by topological reasons π has at least three singular fibers, which must be permuted by the action of p. So p is of finite order on the base, hence a power of p preserves the fibers[3].

These observations have been made by Serge Cantat in [C]. In Proposition 2.2, Proposition 2.1 of [C], he has shown that a parabolic automorphism of a projective K3 surface has dense orbits on almost all genus 1 fibers. Moreover ([C, Corollaire 3.2]) if two automorphisms f_1, f_2 preserve distinct elliptic fibrations π_1, π_2, then the group they generate acts on S ergodically, that is, every measurable invariant subset has zero or full measure[4]. The ergodicity statement follows from the density of orbits by Fubini theorem, so only the density statement requires work to be done.

Our aim is to generalize these results to hyperkähler manifolds of arbitrary dimension (not necessarily projective).

Let us recall some more remarks about elliptic pencils on K3 surfaces and parabolic automorphisms from [C], some of them well-known.

Let S be a K3 surface. If the intersection form on $\mathsf{NS}(S)$ represents zero and does not represent -2, then S carries an elliptic pencil π. If, moreover, the rank of $\mathsf{NS}(S)$ is greater than two, then there is always a

[2] And so, by Fujiki-Liebermann theorem, are the elliptic automorphisms when S has no vector fields, in particular in the K3 case.

[3] In what follows, we say that p **preserves a fibration** π when $\pi\circ p = \pi$. When p just sends fibers to fibers, we call π **an invariant fibration.**

[4] Here the measure is the one coming from the volume form $\Omega\wedge\overline{\Omega}$, where Ω is the holomorphic 2-form on S, unique up to a constant

parabolic automorphism preserving π. Indeed,[5] $\mathrm{Aut}(S)$ is a lattice, hence the stabilizer of an isotropic class has infinite order: otherwise the quotient of $\mathbb{H}_S$ by $\mathrm{Aut}(S)$ would have an end of infinite volume.

As an example of a surface admitting two parabolic automorphisms with distinct nef eigenvectors we may take a sufficiently general complete intersection of type $(2,2,2)$ in $(\mathbb{P}^1)^3$. Indeed each of the three projections to $\mathbb{P}^1$ is an elliptic pencil and each of the three projections to $(\mathbb{P}^1)^2$ is a double covering. The product of any two of the three covering involutions is parabolic (see e.g. [C], Exemple 5, and [W] for details).

1.2 Holomorphic symplectic manifolds

The purpose of the present article is to generalize this picture to higher dimension, that is, for arbitrary **irreducible holomorphic symplectic manifolds**, also called **maximal holonomy hyperkähler manifolds**.[6] The role of the intersection form is played by the Beauville-Bogomolov-Fujiki form, which we denote by q (see Subsection 2.1). With this form, the projectivization of the positive cone in $H^{1,1}_{\mathbb{R}}(X)$ is again viewed as a model of the hyperbolic space.

The classification of automorphisms of a hyperbolic space plays here the same role as for complex surfaces: elliptic automorphisms have finite order, loxodromic automorphisms have positive entropy ([Y]), and parabolic automorphisms, up to taking a power, preserve a fibration in complex algebraic tori (this is Lo Bianco theorem, Theorem 3.6, [LoB2]) and act ergodically in almost all of its fibers (Theorem A). The existence of the fibration is in fact conditional on the following statement, known as the **hyperkähler SYZ conjecture** ([V1]) or **Lagrangian conjecture**, and verified for all known families of hyperkähler manifolds (see Remark 2.19 for references).

Conjecture 1.2: Every nef Beauville-Bogomolov isotropic line bundle on an irreducible holomorphic symplectic manifold M is semiample, that is, if L is such a line bundle, then $L^{\otimes N}$ is base-point-free for some $N > 0$.

Under this conjecture, the sections of $L^{\otimes N}$, after eventually replacing N by a multiple, define a fibration $\pi : M \to X$ (Iitaka fibration associated to L, see [U], chapter 5, and also [L], theorem 2.1.27, where the result is

[5] For any K3 surface S, the group generated by the reflexions in the (-2)-curves and the image of $\mathrm{Aut}(S)$ is a lattice in $O(\mathsf{NS}\otimes_{\mathbb{Z}}\mathbb{R})$ ([PSh]).

[6] We use both names interchangeably, so that irreducible holomorphic symplectic manifolds are always assumed compact and Kähler.

formulated for projective varieties but the proof does not use projectivity). By a famous result of Matsushita, π is a Lagrangian fibration, all smooth fibers of π are tori and the base is "very similar" to the projective space, in particular it is a Fano variety of Picard number 1, possibly with mild singularities. If it is smooth, it is biholomorphic to $\mathbb{P}^n$ ([Hw]).

Our main results can be stated as follows.

Theorem A (see Theorem 3.11): Let M be an irreducible holomorphic symplectic manifold and $f : M \to M$ a parabolic automorphism preserving a Lagrangian fibration $\pi : M \to X$. Then the orbits of f on a sufficiently general fiber of π are dense in the euclidean topology.

Theorem B: Let $G \subset \mathrm{Aut}(M)$ be a subgroup containing such an f. Then G acts ergodically unless π is G-invariant, in other words, unless G fixes the fixed point of f on the absolute. This exception occurs if and only if G is virtually abelian, or equivalently G does not contain any loxodromic automorphisms.

A slightly weaker formulation proved in the text is Theorem 3.3: see Remark 3.4 which explains how to deduce Theorem B.

We refer to [VO], [M] for an introduction to ergodic theory and dynamics. Recall nevertheless the definition.

A hyperkähler manifold M of maximal holonomy and of dimension $2n$ has a natural invariant probability measure μ coming from the volume form $\Omega^n \wedge \overline{\Omega^n}$, where Ω generates $H^{2,0}(M)$.

Definition 1.3: Let Γ be a group acting on a manifold M and preserving a probability measure μ. We say that the action of Γ is **ergodic** if any Γ-invariant measurable subset of M is full measure or measure 0, with respect to μ.

Remark 1.4: Equivalently, (M, Γ) is ergodic iff any Γ-invariant integrable function is constant almost everywhere.

Remark 1.5: From ergodicity it follows that almost all orbits of Γ are dense, but the converse is not true ([Fur]). However if $g : K \to K$ is a translation of a compact group K, the density of orbits implies the ergodicity (and in fact even a stronger property of unique ergodicity, [VO], section 6.3.3).

Theorem A quickly implies Theorem B: when π is not G-invariant, G contains two parabolic automorphisms p_1, p_2 preserving distinct Lagrangian fibrations π_1, π_2, and we derive in the same way as in Cantat's paper that the group generated by p_1 and p_2 acts ergodically.

1.3 Existence of parabolic automorphisms

Concerning the existence of parabolic automorphisms in the presence of Beauville-Bogomolov isotropic nef line bundles, the role of (-2)-curves on a K3 surface is played in higher dimension by MBM classes (see [AV1] for their definition and properties). If the Neron-Severi lattice of M is of rank at least three and contains no MBM classes, then the automorphism group of M is a lattice ("a lattice" means in this context "a discrete subgroup of finite covolume in $SO(\mathsf{NS}(M))$"), as follows from the global Torelli theorem. When $\mathsf{NS}(M)$ represents zero, any lattice in $SO(\mathsf{NS}(M))$ contains a parabolic element; this implies the existence of a parabolic automorphism.

To construct maximal holonomy hyperkähler manifolds with no MBM classes, we recall from [AV4, Corollary 1.4] that the MBM classes have bounded square: $0 > q(z) > -N$, where N depends only on the deformation class of M. Therefore to exhibit a lattice with a parabolic automorphism we construct one representing zero and not representing numbers between zero and $-N$. A primitive embedding of such a lattice into $H^2(M, \mathbb{Z})$ gives a deformation of M with exactly this Neron-Severi lattice.

Such a construction has been first carried out in [AV2], for loxodromic as well as parabolic automorphisms. We give a particularly simple parabolic version of these results in section 2.2: we observe that the absence of MBM classes orthogonal to L, rather than the total absence, suffices to produce parabolic automorphisms. This substantially simplifies the construction (see also [Mats3] for another version of the argument).

As indicated to us by the referee, a further study of dynamics of groups containing parabolic automorphisms of surfaces preserving distinct fibrations is carried out in [CD]. It would be interesting to see whether it generalizes to hyperkähler case.

2 Basic notions

2.1 Hyperkähler manifolds and Beauville-Bogomolov-Fujiki form

For preliminaries on hyperkähler manifolds, we refer to [Bes] and [Bea]. Throughout the paper, we denote by M a complex manifold of the following kind.

Definition 2.1: A **maximal holonomy hyperkähler manifold**, or an **irreducible holomorphic symplectic manifold (IHS)** M is a simply connected compact Kähler manifold such that $H^{2,0}(M)$ is generated by a nowhere degenerate form[1] Ω.

Theorem 2.2: (Fujiki, [Fuj2]). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ a rational number.

Definition 2.3: This form is called **Bogomolov-Beauville-Fujiki form**. It is defined by the Fujiki's relation uniquely if we also ask that the square of a Kähler class should be positive.

Remark 2.4: The form q has signature $(3, b_2-3)$, and signature $(1, b_2-3)$ on the space of $(1,1)$-classes (see [Bea]) . It is negative definite on primitive (1,1)-classes, and positive definite on $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega\rangle$, where ω is any Kähler form. If M is projective, q has signature $(1,\rho-1)$ on the space $\mathsf{NS}(M)\otimes\mathbb{R} = H^{1,1}_{\mathbb{R}}(M)$, where $\mathsf{NS}(M) = H^{1,1}(M)\cap H^2(M,\mathbb{Z})$ is the Neron-Severi group and $\rho = \rho(M)$ the Picard number.

Remark 2.5: Recall that **the polarization** of a homogeneous polynomial map $P:\ W \longrightarrow \mathbb{Q}$ of degree d is a symmetric polynomial $Q \in \mathbb{Q}[z_1,...,z_d]$, $z_i \in W$ of degree 1 in each variable such that $P(z) = Q(\underbrace{z,...,z}_{d \text{ times}})$. Such a polynomial function Q exists and is unique. The polynomial $Q(z_1,...,z_d)$ is proportional with a positive rational coefficient to the homogeneous component of $P\left(\sum_{i=1}^d z_i\right)$ which has degree 1 in each variable. Applying this to the Fujiki relation $\int_M \eta^{2n} = cq(\eta,\eta)^n$, we obtain **the polarized form** of

[1] By Calabi-Yau theorem, such an M admits a unique hyperkähler metric in each Kähler class, hence the names "hyperkähler" and "holomorphic symplectic" are used interchangeably. See [Bes] for details.

Fujiki's relation, used below in section 5.1:

$$\int_M \eta_1 \wedge \cdots \wedge \eta_{2n} = K \sum_{\sigma \in S_{2n}} q(\eta_{\sigma(1)}, \eta_{\sigma(2)}) \dots q(\eta_{\sigma(2n-1)}, \eta_{\sigma(2n)})$$

with $K = \frac{c}{(2n)!}$.

2.2 MBM classes and existence of automorphisms

Let M be a projective maximal holonomy hyperkähler manifold. We denote by $\mathrm{Pos}_{\mathbb{Q}}(M)$[2] the connected component of the cone of classes of positive square in $\mathsf{NS}(M) \otimes \mathbb{R}$ which contains the ample classes. The projectivization $\mathbb{P}(\mathrm{Pos}_{\mathbb{Q}}(M))$ shall be viewed as a model for the hyperbolic space $\mathbb{H}^k$, where $k = \rho - 1$ and $\rho = rk(\mathsf{NS}(M))$ is the Picard number.

Definition 2.6: ([Mar]) The **monodromy group** Mon is the subgroup of $GL(H^2(M, \mathbb{Z}))$ generated by the parallel transports along the Gauss-Manin connection in families. The **Hodge monodromy group**, which we denote by Γ here, is the image in $GL(\mathsf{NS}(M))$ of the subgroup of Mon preserving the Hodge decomposition.

Theorem 2.7: ([V2]) The group Mon is a finite index subgroup of the group $O(H^2(M, \mathbb{Z}), q)$, and Γ is a finite index subgroup of $O(\mathsf{NS}(M), q)$. In particular, Γ acts on $\mathbb{H}^k = \mathbb{P}(\mathrm{Pos}_{\mathbb{Q}}(M))$ with finite covolume. ■

Definition 2.8: A class $z \in \mathsf{NS}(M)$ is MBM if for some $\gamma \in \Gamma$, the orthogonal to γz contains a face of the ample cone of some IHS birational model of M. A class $z \in H^2(M, \mathbb{Z})$ is MBM if it is MBM in some complex structure where it is of type $(1, 1)$.

An MBM class z always satisfies $q(z) < 0$, and the dual homology class to z, up to a scalar multiplier, is represented by a rational curve on some deformation of M ([AV1]).

The following theorem is a restatement of results proven in [Mar], [AV5] and [AV3] (cf. [AV5], Theorem 3.8).

Theorem 2.9: Let M be a projective hyperkähler manifold, and $\mathbb{H}_{\mathbb{Q}} := \mathbb{P}(\mathrm{Pos}_{\mathbb{Q}}(M))$ the hyperbolic space associated with its Picard lattice (which

[2]The notation $\mathrm{Pos}(M)$ usually refers to the positive cone in $H^{1,1}(M)$, so that $\mathrm{Pos}_{\mathbb{Q}}(M)$ is its "algebraic version".

we assume to be of rank at least three). Denote by $S_\alpha \subset \mathbb{H}_\mathbb{Q}$ the hyperplane obtained as orthogonal complement to an MBM class α of type (1,1), and consider the set $\{S_\alpha\}$ of all such hyperplanes. Let $\Gamma \subset O(\mathsf{NS}(M))$ be the Hodge monodromy group of M. Then Γ acts on the set $\{S_\alpha\}$ with finitely many orbits. Moreover, denote by $K_1, ..., K_n$ the connected components of $\mathfrak{K} := \frac{H_\mathbb{Q} \setminus \bigcup_\alpha S_\alpha}{\Gamma}$ with the orbifold structure given by the quotient map.

Then

(i) For each K_i, the universal orbifold covering of K_i is a locally finite convex hyperbolic polyhedron in $\mathbb{P}\operatorname{Pos}_\mathbb{Q}(M)$, obtained by projectivization of the ample cone of a hyperkähler birational model of M.

(ii) The set of isomorphism classes of birational models of M is in bijective correspondence with the set $\{K_i\}$.

(iii) Let M_i be a birational model of M such that K_i is the image by the quotient map of the projectivization of its ample cone. Denote by $\pi_1(K_i)$ the orbifold fundamental group, that is, the deck group of the universal cover of K_i. Then $\pi_1(K_i) = \operatorname{Aut}(M_i)/G$, where G is the group of automorphisms of M_i acting trivially on the Neron-Severi group. Moreover, the group G is finite.

Proof of Theorem 2.9 (i): The set of orthogonal complements to MBM classes is locally finite, and its Γ-quotient is finite, as shown in [AV3]. In [AV1] it was proven that the ample[3] cones of birational models of M are locally polyhedral, and that the orthogonal hyperplanes to the MBM classes are precisely unions of codimension one faces of these cones. It goes back to Markman [Mar] that there is a wall-and-chamber decomposition of $\operatorname{Pos}_\mathbb{Q}(M)$ into the union of all Γ-images of ample cones of all birational models of M. This takes care of Theorem 2.9 (i).

Proof of Theorem 2.9 (ii): In [Mar] it was shown that the set $U := \frac{\text{The set of Kähler chambers in Pos}(M)}{\Gamma}$ is in bijective correspondence with the set of isomorphism classes of birational models of M. However, the set U coincides by construction with the set of connected components of $\mathfrak{K}$.

Proof of Theorem 2.9 (iii): To see that $\pi_1(K_i) = \operatorname{Aut}(M_i)/G$, let $\widetilde{K}_i \subset H^{1,1}(M, \mathbb{Q})$ be the ample cone of M_i. Then $\widetilde{K}_i \longrightarrow K_i$ is the orbifold universal covering. As shown in [Mar], an element $u \in \Gamma$ is induced by an

[3] More precisely, the results of [AV1] and [Mar] are formulated in the Kähler context, but they carry over the ample setting without changes.

automorphism of M_i if and only if u maps its Kähler chamber to itself. This gives $\pi_1(K_i) = \mathrm{Aut}(M_i)/G$. To see that G is finite we use the Calabi-Yau theorem, which gives a bijection between the set of hyperkähler metrics on M_i and the set of Kähler classes in $H^{1,1}(M_i)$. If $u \in \mathrm{Aut}(M_i)$ fixes a Kähler class $\omega \in H^{1,1}(M, \mathbb{Q})$, it preserves the corresponding hyperkähler metric. Since $\mathrm{Aut}(M_i)$ is a Lie group by Bochner-Montgomery theorem, discrete in our case, and the group of isometries of a compact manifold is compact, this implies that G is finite (compact and discrete).[4]. ■

2.3 Cusps of a hyperbolic quotient of the Kähler cone and parabolic automorphisms

Theorem 2.9 associates an orbifold hyperbolic polyhedron K with each Kähler chamber of a hyperkähler manifold, in such a way that $\pi_1(K)$ is the automorphism group of the manifold with this Kähler chamber.

In this subsection, we interpret the parabolic automorphisms in terms of the cusp points of K, which leads to a much simplified construction of hyperkähler manifolds admitting parabolic automorphisms.

The following result is used to describe the cusps geometrically. We use the "thick-thin decomposition theorem", well-known in hyperbolic geometry and also valid in the orbifold context, see [S].

Theorem 2.10: (Thick-thin decomposition)
Any n-dimensional complete hyperbolic manifold $\mathbb{H}/\Gamma$ of finite volume can be represented as a union of a "thick part", which is a compact manifold (with a boundary), and a "thin part", which is a finite union of quotients of form $B/\mathbb{Z}^{n-1}$, where B is a horoball tangent to the boundary at a cusp point, and $\mathbb{Z}^{n-1} = \mathrm{St}_\Gamma(B)$.

Proof: See [Th, Section 5.10] or [Ka1, page 491]. ■

A hyperbolic orbifold $\mathbb{H}/\Gamma$ is smooth if and only if Γ is torsion-free. By Selberg lemma ([A]), any finitely-generated matrix subgroup contains a finite index torsion-free subgroup. Applying this result to Γ, we obtain that any hyperbolic orbifold $\mathbb{H}/\Gamma$ has a finite cover which is a hyperbolic manifold.

This gives the following version of thick-thin decomposition theorem.

[4] This also follows from Fujiki-Liebermann theorem, see Proposition 3.8

Theorem 2.11: (Thick-thin decomposition for orbifolds)
Consider n-dimensional complete hyperbolic orbifold $M := \mathbb{H}/\Gamma$ of finite volume. Then M can be represented as a union of a "thick part", which is a compact manifold (with a boundary), and a "thin part", which is a finite union of quotients of form B/Γ_i, where B is a horoball tangent to the boundary at a cusp point, and $\Gamma_i = \mathrm{St}_\Gamma(B)$ contains $\mathbb{Z}^n$ as a finite index subgroup. ■

We are now in position to deduce the following corollary from Theorem 2.9.

Corollary 2.12: (see also [Mats3])
Let M be a projective hyperkähler manifold with the Picard rank $\rho \geqslant 3$, carrying a nef isotropic line bundle L. Assume that no MBM class is Beauville-Bogomolov orthogonal to L. Then there exists an automorphism of M preserving L, acting on the corresponding hyperbolic space $\mathbb{H} = \mathbb{P}\,\mathrm{Pos}_{\mathbb{Q}}(M)$ parabolically.

Proof: Let $H := \mathbb{P}\,\mathrm{Pos}_{\mathbb{Q}}(M)/\Gamma$ be the hyperbolic orbifold associated with M, and $K \subset H$ a hyperbolic polyhedron obtained as the image of the Kähler cone. As shown e. g. in [Ka2], the cusps of H are in bijective correspondence with the set R/Γ of orbits of Γ on R, where R is the set of rational points on the absolute. The statement "no MBM class is orthogonal to L" is translated, in this language, as "the hyperplanes which cut H into $K_1, ..., K_n$ don't pass through the cusp associated with the isotropic vector $c_1(L) \in H^{1,1}(M, \mathbb{Z})$". This means that the hyperbolic polyhedron K contains an open neighbourhood of a cusp.

Now, consider the thick-thin decomposition of K near the cusp. The map $\pi_1(B/\Gamma_i) \longrightarrow \pi_1(K)$ is injective, because $\pi_1(K)$ acts on $\widetilde{K} \subset \mathbb{H}$ in a neighbourhood of a cusp, and the quotient is isometric to $B/\mathbb{Z}^{n-1}$. By construction, the group Γ_i contains $\mathbb{Z}^{n-1}$ as a finite index subgroup. All elements in $O(NS(M)) \supset \Gamma$ corresponding to $\mathbb{Z}^{n-1} \subset \Gamma_i \subset \Gamma$ are parabolic by construction. Since $\mathrm{Aut}(M)/G = \pi_1(K)$, this implies that $\mathrm{Aut}(M)$ contains parabolic automorphisms. ■

To finish this section, we give a simple method to construct such hyperkähler manifolds, inspired by [AV2] but substantially simpler. The proof uses (also in [AV2]) the following result from [AV4].

Theorem 2.13: ([AV4], Corollary 1.4) Let M be a hyperkähler manifold with $b_2 \geqslant 5$. There exists a positive integer N depending only on the deformation class of M such that $q(z) \geqslant -N$ for any primitive MBM class z in $H^2(M, \mathbb{Z})$ (the smallest such number is called the MBM bound for M).

Theorem 2.14: Let M be a hyperkähler manifold with $b_2 \geqslant 5$. Then M admits a deformation which has a parabolic automorphism.

Proof: When $b_2 \geqslant 5$, the lattice $H^2(M, \mathbb{Z})$ contains an isotropic vector by Meyer's theorem [Me]. Let y be a primitive element of $H^2(M, \mathbb{Z})$ of square zero. Let N be the MBM bound for M. We first show that y can be embedded into a primitive sublattice Λ' of signature $(0, -)$[5], not representing integers strictly between $-N$ and 0. Indeed one can find a rank-two sublattice of signature $(+, -)$ of $H^2(M, \mathbb{Z})$ orthogonal to y. Let L be the smallest primitive sublattice containing that one and y: it has signature $(+, 0, -)$. The quotient Λ of L by the sublattice spanned by y is again of signature $(+, -)$. As such, Λ has primitive vectors of arbitrarily negative length[6]. Take a primitive $u \in \Lambda$ with square less than $-N$ and consider the sublattice Λ' which is the inverse image of the one generated by u under the projection map. Then Λ' is as required.

Now take any $a \in H^2(M, \mathbb{Z})$ with positive square and not orthogonal to y. By the surjectivity of the period map, there exists a deformation M' of M such that the smallest primitive sublattice containing a and Λ' is the Neron-Severi lattice of M'.

Moreover, by Theorem 2.9, the positive cone of M' is decomposed by the orthogonal hyperplanes to the MBM classes into the union of Kähler chambers, that is, the Kähler cones of the hyperkähler birational models of M' and their monodromy images. The hyperkähler birational models of M' are still deformations of M by [H]; in fact such a birational model is unseparable from M' in the Teichmüller space for hyperkähler manifolds, which is not Hausdorff. So replacing M' by an appropriate birational model, we may assume that y is the class of a nef line bundle L on M'.

Now the statement follows from Corollary 2.12. Indeed $NS(M')$ does not contain MBM classes orthogonal to y, since all integer negative classes

[5]This notation means that Λ' is of rank 2 as a $\mathbb{Z}$-module and that its quadratic form is negative semi-definite with 1-dimensional kernel

[6]This elementary lemma can be checked as follows: if true for a lattice Λ, it is also true for $\Lambda' \subset \Lambda \otimes \mathbb{Q}$ commensurable to Λ; any Λ is commensurable to a diagonal lattice, for which the statement is obvious.

orthogonal to y are in Λ' (which is primitive by construction) and hence have square lower than $-N$. ■

2.4 Holomorphic Lagrangian fibrations on hyperkähler manifolds

Theorem 2.15: (Matsushita, [Mats1])
Let $\pi : M \longrightarrow X$ be a proper surjective holomorphic map with connected fibers from a projective hyperkähler manifold M to a normal projective variety X, with $0 < \dim X < \dim M$. Then $\dim X = \frac{1}{2}\dim M$, and the fibers of π are Lagrangian (this means that the holomorphic symplectic form vanishes on the fibers $\pi^{-1}(x)$). Moreover, X has $\mathbb{Q}$-factorial log-terminal singularities and Picard number 1. ■

Definition 2.16: A proper surjective holomorphic map $\pi : M \to X$ with connected fibers is called a fibration; if its fibers are Lagrangian, it is **a holomorphic Lagrangian fibration**. When M is normal, X may be assumed normal (taking the normalization). We assume it throughout the paper.

Remark 2.17: It is well-known that a smooth fiber F of a Lagrangian fibration is a torus, indeed it has trivial tangent bundle (the symplectic form gives an isomorphism between the tangent and the conormal bundle). Matsushita formulates his theorem in the projective setting but the proof is valid in the non-projective case as well. When M is hyperkähler, F is projective even if M itself is not (see Proposition 2.1 of [Cam], attributed to Voisin by the author; see also [SV, Corollary 3.4]). It is also true that X is projective even if M is not: the reason is that X is compact Kähler and Moishezon with rational singularities, and such varieties are projective (see [KL], Theorem 2.8, and [N, Corollary 1.7]).

Remark 2.18: The base X of a Lagrangian fibration on a hyperkähler manifold is conjectured to be rational. Hwang ([Hw]) proved that $X \cong \mathbb{P}^n$ whenever X is smooth. Previously, Matsushita ([Mats2]) proved that X has the same Dolbeault cohomology as $\mathbb{P}^n$ if it is smooth.

The inverse image of the ample generator of the Picard group of the base X is clearly semiample with zero Beauville-Bogomolov square. **The hyperkähler SYZ** Conjecture 1.2, when satisfied, implies the converse: any nef line bundle L with $q(L) = 0$ should be semiample, so that the linear

system of sections of $L^{\otimes N}$ for suitable N defines a Lagrangian fibration by Matsushita's theorem.

Remark 2.19: The hyperkähler SYZ conjecture has been proved for all known examples of hyperkähler manifolds. See [BM] for Hilbert schemes of K3 surfaces, and implicitly [Yo] for generalized Kummer varieties, as well as [MR], [MO] for the two sporadic O'Grady's examples.

Proposition 2.20: Let $\pi_1 : M \longrightarrow X_1$, $\pi_2 : M \longrightarrow X_2$ be two distinct Lagrangian fibrations. Then the intersection of general fibers of π_1, π_2 is finite. Moreover, the restriction of π_2 on any fiber of π_1 is surjective.

Proof: Let π_1, π_2 be two distinct Lagrangian fibrations given by the sections of nef line bundles L_1 resp. L_2. The Hodge index theorem implies $q(L_1, L_2) > 0$. By Fujiki formula this implies that the intersection number $L_1^n L_2^n > 0$. This intersection number is also equal to the top self-intersection number of the restriction of L_2 to any fiber of π_1. The restriction of L_1 to a smooth fiber of π_2 is therefore ample, because a nef and big line bundle on an abelian variety is ample. Hence the restriction of π_2 to a smooth fiber of π_1 is a finite map.[7] The restriction to any fiber remains surjective, because it is proper and dominant. ■

3 Parabolic automorphisms of hyperkahler manifolds

3.1 Hyperkähler manifolds with ergodic automorphism groups

Definition 3.1: An automorphism of a hyperkähler manifold M is called **elliptic (parabolic, hyperbolic)** if it is elliptic (parabolic, hyperbolic) as an element of $SO(H^{1,1}(M, \mathbb{R}))$ (cf. Theorem-definition from the Introduction and the remarks following it).

Remark 3.2: Let p be a parabolic automorphism of a hyperkähler manifold, and η be a fixed point in $H^{1,1}(M)$ associated with the fixed point in

[7] Note however that π_2 can possibly contract an irreducible component of some singular fiber of π_1.

the absolute. Then η is proportional to an integral cohomology class (as in the lines following the definition of a parabolic automorphism in the introduction) which lies on the boundary of the Kähler cone, that is, the class of a nef line bundle L. Indeed the Kähler cone is invariant under the action of p, and η can be obtained as a limit $p^i(w)$ for any Kähler class w on M.

One of the main results of this paper can be stated as follows (see Remark 3.12 for the proof).

Theorem 3.3: Let M be a hyperkähler manifold admitting two parabolic automorphisms p_1, p_2 which preserve nef line bundles L_1, L_2, with $c_1(L_1)$ not proportional to $c_1(L_2)$.[1] Assume the SYZ conjecture holds for L_1, L_2. Then p_1, p_2 generate a group acting on M ergodically.

Remark 3.4: If M admits a parabolic automorphism p and $\operatorname{Aut}(M)$ is not virtually abelian,[2] M admits parabolic automorphisms which have different fixed points on the absolute. Indeed it is well-known that a discrete subgroup in the stabilizer of a point on the absolute in the group of isometries of a hyperbolic space is virtually abelian (see e.g. [R], chapter 5, Theorem 5.5.9). So if $\operatorname{Aut}(M)$ is not virtually abelian, then not all automorphisms of M fix the same point x of the absolute as p does. Conjugating p by an automorphism g which does not fix x we obtain a parabolic automorphism p' with a different fixed point x'. Note that x' is also on the boundary of the Kähler cone, since the latter is preserved by g. In particular, x and x' are classes of two different nef line bundles L, L'. Also, if SYZ conjecture holds for L, it holds for L', since L' is the image of L by an automorphism. Hence Theorem 3.3 implies Theorem B from the introduction (if $\operatorname{Aut}(M)$ is virtually abelian and contains a parabolic automorphism, all of its elements fix the same point x on the absolute, as follows from [R], Theorem 5.5.9).

Remark 3.5: In [AV2] it was shown that that any hyperkähler manifold with $b_2 \geqslant 14$ admits a projective deformation M_p with a parabolic automorphism; moreover, the automorphism group of M_p is arithmetic, hence M_p admits parabolic automorphisms with different fixed points in the absolute (note that the construction of [AV2] gives manifolds with Picard number $\rho \geqslant 5 > 2$). If we are looking for a single automorphism, rather than for two

[1] Equivalently, the fixed points of p_1, p_2 on the absolute of the hyperbolic space $\mathbb{P}\operatorname{Pos}(M)$ are not equal.

[2] A group is called **virtually abelian** if it contains an abelian subgroup of finite index.

automorphisms with different fixed points on the absolute, we now have a simpler construction in Subsection 2.2.

3.2 Parabolic automorphisms and Lagrangian fibrations on hyperkähler manifolds

Let $p : M \to M$ preserve a nef line bundle L defining a Lagrangian fibration $\pi : M \to X$. Then p takes fibers to fibers, inducing an automorphism of the base B. It turns out that this automorphism is of finite order.

This fact is due to Federico Lo Bianco, see [LoB1, Theorem B] for the case when $X = \mathbb{P}^n$, [LoB2] for the general case.[3]

Theorem 3.6: (Lo Bianco) Let p be a parabolic automorphism of a hyperkähler manifold M, and $\pi : M \longrightarrow X$ a Lagrangian fibration such that for a Kähler class ω on X, its pulback $\pi^*\omega$ is the class on the boundary of the Kähler cone fixed by p. Then a certain positive iterate of p preserves the fibers of π.

Remark 3.7: If p satisfies $\pi \circ p = \pi$ (which by Theorem 3.6 is always the case after replacing p by a power), we say that p **preserves the Lagrangian fibration π.** The fibers (connected by definition of a Lagrangian fibration) are uniquely determined by p, because they are uniquely determined by the cohomology class $\pi^*\omega$, and p fixes one and only one point on the absolute.

Proposition 3.8: Let p be a parabolic automorphism preserving a Lagrangian fibration $\pi : M \to X$. Then some power of p acts as a translation on a general fiber.

Proof: Let T be a smooth fiber of π; this is a complex torus, algebraic by Remark 2.17 (or by Proposition 5.1 below). Recall that an automorphism of a compact torus $A = \mathbb{C}^g/\Lambda$ is induced by an affine transformation of $\mathbb{C}^g$ with linear part preserving Λ, and $\mathrm{Aut}(A)^0$ is the group of translations, in general not of finite index in $\mathrm{Aut}(A)$. If we denote by $\mathrm{Aut}^\omega(A)$ the subgroup of automorphisms preserving a Kähler class ω, then by Fujiki-Liebermann theorem (see e.g. [Fuj1]) $\mathrm{Aut}(A)^0$ is of finite index in $\mathrm{Aut}^\omega(A)$. It turns out that the restriction of p to T must indeed preserve a polarization. Indeed it is well-known ([O1], see also Proposition 5.1 here) that the rank of the restriction map $H^2(M,\mathbb{Z}) \xrightarrow{r} H^2(T,\mathbb{Z})$ is one. The map induced by p

[3] Lo Bianco formulates his theorem in the projective context, but looking at the proof one sees that only the projectivity of the base X is relevant, cf. Remark 2.17.

must preserve the image of r. This implies that the restriction of a Kähler class from M to T is also preserved by p, hence the conclusion. ■

Remark 3.9: A complex torus T is not a group, unless you fix the origin. However, its translation group $\mathrm{Aut}(T)^0$ is a complex, commutative Lie group, and it is isomorphic to T as a manifold. We denote it by T^0 when we want to stress the group manifold structure.

Claim 3.10: A translation τ_x of a torus T by an element $x \in T^0$ has all its orbits dense if and only if x is not contained in a real subtorus $T' \subset T^0$. In this case, τ_x is ergodic (Remark 1.5).

3.3 Parabolic automorphisms are fiberwise ergodic on Lagrangian fibrations

The following theorem is the main result of this paper, used to obtain ergodicity of the parabolic actions.

Theorem 3.11: Let p be a parabolic automorphism of a hyperkähler manifold preserving a Lagrangian fibration $\pi: M \longrightarrow X$. Then there exists a full measure, Baire second category subset $R \subset X$, such that for all $r \in R$ the fibers $\pi^{-1}(r)$ are tori, and the automorphism p acts on $\pi^{-1}(r)$ with dense orbits.

Remark 3.12: Theorem 3.11 immediately implies Theorem 3.3, in the same way as for K3 surfaces in [C]. Indeed, let Γ be the group generated by two parabolic automorphisms p_1, p_2, and $\pi_1, \pi_2: M \longrightarrow X_i$ the Lagrangian fibrations associated with p_1, p_2. Ergodicity of the action of the group Γ means that any Γ-invariant measurable set U is either full measure or zero measure. By Fubini theorem, the intersection of U and the fibers $\pi_i^{-1}(x)$ is measurable with respect to Haar measure for almost all $x \in X_i$ (where "almost all" is meant with respect to the pushforward of the canonical probability measure on M). Since U is Γ-invariant, and the parabolic action on the fibers of π_i is ergodic, the intersection $\pi_i^{-1}(x) \cap U$ is full measure or zero measure for almost all $x \in X_i$.

Assume that U is not of measure zero. Let $U_i \subset X_i$ be the set of all $x \in X_i$ such that $\pi_i^{-1}(x) \cap U$ is measurable and of full measure in $\pi_i^{-1}(x)$. Applying Fubini theorem, we obtain that the symmetric difference of U and $\pi_1^{-1}(U_1)$, as well as that of U and $\pi_2^{-1}(U_2)$, is of measure zero.

Hence the symmetric difference of $\pi_2^{-1}(U_2)$ and $\pi_1^{-1}(U_1)$ is of measure zero, and applying Fubini again we see that this is only possible if both $\pi_2^{-1}(U_2)$ and $\pi_1^{-1}(U_1)$ are of full measure. Hence so is U.

The proof of Theorem 3.11 is based on Hodge theory, and we give it in Section 6 after the relevant results about variations of Hodge structures are introduced.

4 Variations of Hodge structures and Deligne's semisimplicity theorem

We give a brief introduction to the variations of Hodge structures. For more detail, see [Voi1, Grf, PS].

4.1 Hodge structures

Definition 4.1: Let $V_{\mathbb{R}}$ be a real vector space. **A (real) Hodge structure of weight** w on a vector space $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is a decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$. It is called an **integral** or **rational** Hodge structure if one fixes an integral or rational lattice $V_{\mathbb{Z}}$ or $V_{\mathbb{Q}}$ in $V_{\mathbb{R}}$. A Hodge structure is equipped with an $U(1)$-action[1] with $u \in U(1)$ acting as u^{p-q} on $V^{p,q}$. **A morphism** of integral/rational Hodge structures is an integral/rational linear map which is $U(1)$-invariant.

Definition 4.2: A polarization on a rational Hodge structrure of weight w is a $U(1)$-invariant non-degenerate 2-form $h \in V_{\mathbb{Q}}^* \otimes V_{\mathbb{Q}}^*$ (symmetric or antisymmetric depending on parity of w) which satisfies

$$(-1)^{w(w-1)/2}\sqrt{-1}^{p-q} h(x, \overline{x}) > 0 \tag{4.1}$$

("Riemann-Hodge relations") for each non-zero $x \in V^{p,q}$.

Definition 4.3: Two complex tori T_1, T_2 are called **isogeneous** if there exists a surjective finite holomorphic map $T_1 \longrightarrow T_2$.

Remark 4.4: The category of complex tori with a group structure (that is, the zero point fixed) is equivalent to the category of integral Hodge structures of weight 1. The category of complex tori with a group structure

[1] $U(1)$ denotes the unit circle in $\mathbb{C}$.

up to isogeny is equivalent to the category of rational Hodge structures of weight 1. Under this correspondence, abelian varieties correspond to Hodge structures admitting a polarization. See e.g. [Voi1, §7.2.2].

Remark 4.5: The category $\mathcal{C}$ of rational Hodge structures admitting a polarization is semisimple, that is, any object of $\mathcal{C}$ is a direct sum of irreducible ones. In the case of weight one (the one relevant for this paper), an equivalent statement is that the category of abelian varieties up to isogeny is semisimple (Poincaré reducibility theorem).

4.2 Variations of Hodge structures

Definition 4.6: Let X be a complex manifold. A **(polarized) real variation of Hodge structures (VHS)** on X is a complex vector bundle (B, ∇) with a flat connection equipped with a parallel anti-complex involution (defining a fiberwise real structure) and (polarized) Hodge structures at each point, $B = \bigoplus_{p+q=w} B^{p,q}$ which satisfy the following conditions:

(a) the polarization, the rational lattice and the real structure are preserved by ∇.

(b) ("Griffiths transversality condition") $\nabla^{1,0}(B^{p,q}) \subset B^{p,q} \oplus B^{p+1,q-1}$.

Example 4.7: ([Voi1], 10.1, 10.2.1) Let $\pi : M \longrightarrow X$ be a proper holomorphic submersion. Consider the bundle $V := R^k\pi_*(\mathbb{C}_M)$ with k-th cohomology of $\pi^{-1}(x)$ as the fiber at x, the Hodge decomposition coming from the complex structure on $\pi^{-1}(x)$, and the Gauss-Manin connection. This defines a variation of Hodge structures on X. Assume that there is a class ω in $H^2(M, \mathbb{Z})$ such that its restriction on each fiber is ample. Then the cup product with ω induces a Lefschetz operator L, and the polarization is produced by setting $h(a, b)$ to be the product of $L^{n-k}a$ and b.In general one must restrict to the primitive cohomology to get the Hodge-Riemann condition. In this paper we deal with the special case $k = 1$, so that the whole cohomology is primitive.

Example 4.8: More precisely, we shall consider the following situation. Take a lagrangian fibration $\pi : M \to X$ on a hyperkähler manifold M. Let $X_0 \subset X$ be the locus of non-critical values of π (by definition, the critical locus includes the singularities of X, hence X_0 is smooth). We denote by the same letter $\pi : M_0 \to X_0$ the restriction of the fibration to $\pi^{-1}(X_0)$. This

is a fibration in projective complex tori. Then $R^1\pi_*(\mathbb{C}_{M_0})$ is a polarized variation of Hodge structures even if M is not projective. Indeed, since by Oguiso's observation (also Proposition 5.1 here) the rank of the restriction of $H^2(M,\mathbb{Z})$ to the smooth fiber is 1, there is a class ω in $H^2(M,\mathbb{Z})$ with ample restriction to the fibers.

Claim 4.9: Polarized integral variations of Hodge structures of weight 1 are in a functorial bijection with holomorphic Abelian fibrations. More precisely, polarized variations of Hodge structures of weight 1 correspond to fibrations in complex tori $Y \longrightarrow X$ with a holomorphic section, giving the fiberwise choice of zero, and a line bundle L on Y which is ample on all fibers, inducing a polarization. Given a fibration in abelian varieties without a section, one can associate to it the "relative Albanese", or "translation fibration", replacing each fiber T by T^0. The variation of Hodge structures defined as in Example 4.7 with $k = 1$ is the same for a fibration and its relative Albanese.

4.3 Deligne's semisimplicity theorem

The following important theorem is one of the main tools of our argument.

Theorem 4.10: (Deligne's semisimplicity theorem, [D, Wr])[2]
Let V be a polarized rational variation of Hodge structures over a quasiprojective base M. Then the underlying flat bundle can be decomposed as $V = \bigoplus_i W_i \otimes L_i$, where the L_i correspond to pairwise non-isomorphic irreducible representations of $\pi_1(M)$, and W_i are trivial representations. Moreover, each W_i is equipped with a Hodge structure, each L_i is equipped with a variation of Hodge structures, and the decomposition $V = \bigoplus_i W_i \otimes L_i$ is compatible with the Hodge structures on W_i, L_i.

Applying this result in the case of weight 1, we notice that one of two terms in $W_i \otimes L_i$ has weight 0, and the other has weight 1. Separating the sum in two parts according to whether W_i or L_i is of weight 0, we obtain the following corollary. Recall that a fibration in abelian varieties is called **isotrivial** if all its fibers are isomorphic.

[2]The theorem as stated by Deligne is valid for so-called complex variations of Hodge structures. This is actually a stronger statement which we shall not need in this paper. The version for real or rational VHS is obtained by putting together the summands conjugate by Galois action.

Corollary 4.11: Let V be a quasiprojective manifold fibered in abelian varieties. Then, after passing to an isogeneous fibration V_1, we can decompose V_1 into a product of abelian fibrations with irreducible monodromy of the Gauss-Manin connection and an isotrivial abelian fibration. The isotrivial fibration corresponds to $\bigoplus_i W_i \otimes L_i$ with weight of W_i equal to 1. ■

4.4 Deligne's global invariant cycle theorem

Further on, we shall use the following important theorem, also due to Deligne.

Theorem 4.12: (see for example [Voi1], II, Theorem 4.24, also Proposition 4.23 and 4.18). Let $\pi : M_0 \longrightarrow X_0$ be a smooth family of projective manifolds , and $V = R^i\pi_*(\mathbb{C}_{M_0})$ the corresponding Gauss-Manin local system. Consider a monodromy invariant vector $\alpha \in R^i\pi_*(\mathbb{C}_{M_0})\Big|_x$, and let $[\alpha] \in H^i(\pi^{-1}(x))$ be the corresponding cohomology class in the fiber. Then $[\alpha]$ belongs to the image of the restriction map $H^i(M) \longrightarrow H^i(\pi^{-1}(x))$, where M is any compactification of M_0.[3] ■

Remark 4.13: This theorem is usually formulated in the algebraic context, so that M is projective and M_0 is quasiprojective. But the proof as given in [Voi1] adapts to our setting where $\pi : M \to X$ is a Lagrangian fibration on a hyperkähler manifold, see [Voi1] II, remark 4.16 and compare to Example 4.8.

5 Variations of Hodge structures and Lagrangian fibrations

In this section, we prove some preliminary results about the Hodge structures associated with a Lagrangian fibration.

5.1 AM-GM inequality, products of Hermitian forms and Lagrangian fibrations

In this section we give an elementary proof of the following useful observation by Oguiso ([O1], p.3) who combined results by Voisin [Voi2] and Matsushita

[3] $H^i(M)$ and $H^i(M_0)$ actually have the same image in $H^i(\pi^{-1}(x))$, [Voi1], II, Proposition 4.23.

[Mats2].

Proposition 5.1: Let $\pi : M \longrightarrow X$ be a Lagrangian fibration on a hyperkähler manifold and T a smooth fiber[1] of π. Then the natural restriction map $H^2(M) \longrightarrow H^2(T)$ has rank 1.

We use the following basic lemma, found in Cauchy's *"Cours d'analyse de l'École Royale Polytechnique, premiére partie, Analyse algébrique"* (1821). In high school, it is usually called **the AM-GM[2] inequality**.

Lemma 5.2: Let $\alpha_1, ..., \alpha_n$ be positive real numbers. Then

$$\frac{\sum \alpha_i}{n} \geqslant \sqrt[n]{\prod_i \alpha_i},$$

and the equality holds if and only if all α_i are equal.

Corollary 5.3: Let $\alpha_1, ..., \alpha_n$ be positive numbers such that $\sum \alpha_i = n$ and $\prod \alpha_i = 1$. Then all $\alpha_i = 1$. ■

Proposition 5.4: Let ω_1, ω_2 be positive Hermitian forms on a vector space $V = \mathbb{C}^n$, and h_1, h_2 the corresponding Hermitian metrics. Suppose that $\omega_1 \wedge \omega_2^{n-1} = \omega_1^n = \omega_2^n$. Then $\omega_1 = \omega_2$.

Proof: Simultaneous diagonalization theorem implies that h_1 can be diagonalized in an orthonormal basis for h_2. Let $A = \omega_1 \omega_2^{-1}$ be the corresponding diagonal matrix. Clearly, $\frac{\omega_1 \wedge \omega_2^{n-1}}{\omega_2^n} = \frac{1}{n} \operatorname{Tr}(A)$ and $\frac{\omega_1^n}{\omega_2^n} = \det A$. In other words, $\frac{\omega_1^n}{\omega_2^n} = \det A$ is the product $\prod_{i=1}^n \alpha_i$ of all eigenvalues of A, and $\frac{\omega_1 \wedge \omega_2^{n-1}}{\omega_2^n}$ the arithmetic mean $\frac{1}{n} \sum_{i=1}^n \alpha_i$ of all these eigenvalues. On the other hand, $\omega_1^n = \omega_2^n$ implies that the product of all eigenvalues of A is 1, hence the geometric mean of these eigenvalues is also 1. Then $\omega_1 \wedge \omega_2^{n-1} = \omega_1^n = \omega_2^n$ implies that

$$\frac{1}{n} \sum_{i=1}^n \alpha_i = \sqrt[n]{\prod_{i=1}^n \alpha_i} = 1.$$

The AM-GM inequality implies that all α_i are equal to 1, which gives $\omega_1 = \omega_2$. ■

[1] A projective torus, see Remark 2.17

[2] Short for "arithmetic mean - geometric mean"

Proof of Proposition 5.1: Let $\eta_0 \in H^2(X, \mathbb{R})$ be a class in the singular cohomology of X satisfying $\eta_0^n = 1$, where $n = \dim_{\mathbb{C}} X$, $\omega_1, \omega_2 \in H^2(M)$ any cohomology classes, and $\eta := \pi^*\eta_0$. We view η as a de Rham cohomology class. We need to show that $\omega_1\Big|_T$ is proportional to $\omega_2\Big|_T$. Since T is Lagrangian, the (2,0)-forms are restricted to zero, and we may assume that $\omega_1, \omega_2 \in H^{1,1}(M)$. Picking ω_1 Kähler and replacing ω_2 by an appropriate linear combination of ω_1 and ω_2, we may assume that ω_2 is also Kähler. Clearly, $q(\eta, \eta) = 0$. On the other hand, η^n is cohomologous to the fundamental class $[T]$ of the fiber of π, because $\eta_0^n = 1$. By Fujiki formula in the polarized form, $\int_T \omega_i^n = \int_M \omega_i^n \wedge \eta^n = Cq(\omega_i, \eta)^n$, where C is the product of $K = c/(2n)!$ and of the number of permutations $\sigma \in S_{2n}$ such that for every pair of consequent indices $(2k-1, 2k)$ exactly one of the values $\sigma(2k-1)$ and $\sigma(2k)$ is greater than n. Indeed apply Remark 2.5 with $\eta_1 = \cdots = \eta_n = \omega_i$ and $\eta_{n+1} = \cdots = \eta_{2n} = \eta$ and use $q(\eta, \eta) = 0$. In the same way and for the same C,

$$\int_T \omega_2 \wedge \omega_1^{n-1} = \int_M \omega_2 \wedge \omega_1^{n-1} \wedge \eta^n = Cq(\omega_2, \eta)q(\omega_1, \eta)^{n-1}.$$

Rescaling ω_i in such a way that $q(\omega_1, \eta) = q(\omega_2, \eta) = C^{-1/n}$, we obtain that

$$\omega_1 \wedge \omega_2^{n-1}\Big|_T = \omega_1^n\Big|_T = \omega_2^n\Big|_T$$

for the two Kähler classes restricted to the fiber. Representing these Kähler classes by translation-invariant Hermitian forms on T, we obtain two Hermitian forms on $\mathbb{C}^n$ satisfying $\omega_1 \wedge \omega_2^{n-1} = \omega_1^n = \omega_2^n$. By Proposition 5.4, these Hermitian forms are equal, hence $\omega_1\Big|_T$ is cohomologous to $\omega_2\Big|_T$. ■

5.2 Irreducibility of the VHS associated with a Lagrangian fibration

From now on, we denote by $X_0 \subset X$ the locus of non-critical values of $\pi : M \to X$, i.e. the maximal smooth Zariski-open subset over which π is smooth.

Lemma 5.5: Let $\pi : M \longrightarrow X$ be a Lagrangian fibration on a hyperkähler manifold of maximal holonomy. Let V be the variation of Hodge structures over X_0 associated with $R^1\pi_*(\mathbb{C}_M)$. Then the space of parallel sections of $\bigwedge^2 V$ is 1-dimensional.

Proof: Let ω be the polarization on V (see Example 4.8). It is a parallel section of $\bigwedge^2 V^*$, corresponding to the restriction of the Riemann-Hodge form to the fibers of π. Since the smooth fibers of π are tori, the bundle of second cohomologies of the fibers of π over X_0 is $\bigwedge^2 V$, as a variation of Hodge structures. By Deligne's global invariant cycle theorem (Theorem 4.12), any monodromy invariant element in a fiber of $R^2\pi_*(\mathbb{C}_M)$ is obtained as a restriction of a globally defined cohomology class in $H^2(M)$. Applying this to $\bigwedge^2 V = R^2\pi_*(\mathbb{C}_M)$, we obtain that any antisymmetric, monodromy-invariant 2-tensor on V is obtained as a restriction of a globally defined cohomology class in $H^2(M)$.

By Proposition 5.1, the restriction map from $H^2(M)$ to a smooth fiber of π has rank 1, hence the conclusion. Dually, any parallel 2-form on V is equal to the polarization. ■

Corollary 5.6: Let $\pi : M \longrightarrow X$ be a Lagrangian fibration on a hyperkähler manifold. Then the Gauss-Manin local system $V := R^1\pi_*(\mathbb{C}_M)$ over X_0 is irreducible as a variation of Hodge structures. Moreover, the corresponding monodromy representation is irreducible, unless π is isotrivial.

Proof: Suppose that V is not irreducible as a rational variation of Hodge structures: $V = V_1 \oplus V_2$. Restricting the polarization ω to V_1 and V_2[3], we obtain that the space of parallel sections of $\bigwedge^2 V^* \cong \bigwedge^2 V$ is at least 2-dimensional, which is impossible by Lemma 5.5.

To prove the second part, apply Deligne's semisimplicity Theorem 4.10 to obtain the decomposition $V = \bigoplus_i W_i \otimes L_i$. Since V is irreducible as a VHS, only one of the summands is non-trivial: $V = W \otimes L$. Then either W has weight 0, that is, $W = \mathbb{Q}$, and L is irreducible of weight 1, or W has weight 1 and L has weight 0. In the second case π is isotrivial. ■

Remark 5.7: More geometrically, one can argue that a Hodge substructure of V yields a fibration in subtori on each smooth fiber of π. This in turn gives a rational map φ from M to the component of the Chow variety of M which parameterizes such subtori. Considering the inverse image by φ of an ample line bundle (or a Kähler class in the non-algebraic case, indeed cycle

[3]The restriction of a polarization to a Hodge substructure is itself a polarization, in particular it is non-zero.

spaces on compact Kähler manifolds are compact and Kähler by [Var]) on this component, one again sees that the rank of the restriction of $H^2(M)$ to a smooth fiber of π must be greater than one: indeed this inverse image is semiample but not ample on a smooth fiber of π.

6 Parabolic automorphisms and their orbits

In this section and the next one, we prove Theorem 3.11 which claims that a generic orbit of a parabolic automorphism of a hyperkähler manifold is dense in the corresponding fiber of a Lagrangian fibration.

6.1 Parabolic automorphisms and monodromy-invariant subtori

Let p be an automorphism of a hyperkähler manifold, preserving a Lagrangian fibration $\pi : M \longrightarrow X$ and acting as a translation on smooth fibers: we recall that for a regular value $x \in X$, the fiber $\pi^{-1}(x)$ is a compact complex (in fact algebraic) torus. Consider the group of its translations $\operatorname{Aut}(\pi^{-1}(x))^0 = \frac{H^1(\pi^{-1}(x),\mathbb{R})}{H^1(\pi^{-1}(x),\mathbb{Z})}$ understood as the connected subgroup of its group of automorphisms. Consider the element $p_x \in \operatorname{Aut}(\pi^{-1}(x))^0$ induced by p, and let P_x be the connected component of the unity of the closure of the group it generates in $\operatorname{Aut}(\pi^{-1}(x))^0$. This is a real subtorus. Clearly the orbits of p are dense in the fibers of π if and only if P_x is equal to $\operatorname{Aut}(\pi^{-1}(x))^0$, and otherwise their closures are finite unions of translates of subtori of the same dimension as that of P_x.

Since π is differentiably locally trivial over X_0, the bundle of all translations of the fibers of π is equipped with a natural connection, induced by the Gauss-Manin connection on $R^1\pi_*(\mathbb{C}_M)$. Denote by Γ its monodromy group.

Proposition 6.1: In the setting described above, let $r := \max_{z\in X_0} \dim P_z$, and assume that the dimension of $\dim P_x$ is equal to r. Then P_x is a Γ-invariant subtorus of $\operatorname{Aut}_0(\pi^{-1}(x))$.

Proof: The map $x \mapsto p_x$ gives a section of the relative Albanese fibration of π. This fibration is locally trivial in the real analytic category. Then the map $x \mapsto p_x$ can be interpreted as a map from a neighbourhood U of some x_0 to a fixed real torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$.

The dimension of P_x can be expressed through the smallest number of relations with integer coefficients between the coordinate components of $p_x \in T_n$. For each of these integer relations, denoted by A, the set Y_A of x for which the coordinate components of p_x satisfy the relation A is real analytic in U. Now let $\mathcal{P}$ be the set of all possible integer relations. For some $A \in \mathcal{P}$, $Y_A = U$, so that the relation is satisfied everywhere on U. For other $A \in \mathcal{P}$ this set is a proper analytic subset. Let $Y_U \subset U$ be the union of all $Y_A \subsetneq U$. Since Y_U is a countable union of measure zero sets, its complement is of full measure, and outside of Y_U the torus P_x is constant (indeed it is defined by all relations A such that $Y_A = U$). Taking the union of Y_U's we obtain a countable union of proper analytic subsets $Y \subset X_0$. The torus P_x is locally constant of maximal dimension outside Y, and over Y some extra relations are satisfied, so that $\dim P_y$ is lower for $y \in Y$.

Now by definition of the Gauss-Manin connection this means that P_x is monodromy invariant. Indeed, a closed real analytic path which is not contained in Y intersects Y in a countable set. Clearly, it suffices to show that P_x stays constant along such a real analytic path.

Passing to the universal cover of the fiber, we can consider p as a real analytic map from γ to $\mathbb{R}^n$. Under this interpretation, P_x is given by the smallest rational subspace V of $\mathbb{R}^n$ containing $p(x)$. This subspace is constant outside of Y, indeed it is defined by all integer relations A satisfied over X_0, and more relations are satisfied over Y. This means that $p(\gamma) \subset V$, and $P_x = V$ for all $x \in \gamma \backslash Y$. ■

Remark 6.2: A priori one could have $r = 0$, however this means that p is of finite order and therefore not parabolic. Indeed $r = 0$ means that all translations in the fibers over X_0 are by torsion elements. But unless p is of finite order, the fixed point set of p^m is a proper analytic subvariety of M, and M_0 is not a countable union of proper analytic subvarieties.

6.2 Ergodicity of parabolic automorphisms

Now we can prove the following corollary, which immediately implies Theorem 3.11 when π is not isotrivial.

Corollary 6.3: Let $\pi : M \longrightarrow X$ be a non-isotrivial Lagrangian fibration on a hyperkähler manifold of maximal holonomy, and p a parabolic automorphism preserving π. Then p acts with dense orbits on a general fiber of π.

Proof: The closure of an orbit of maximal dimension is given by a monodromy-invariant subspace in $H^1(\pi^{-1}(x))$, non-zero by Remark 6.2. However, for a non-isotrivial Lagrangian fibration, the monodromy is irreducible by Corollary 5.6. ■

7 Isotrivial Lagrangian fibrations

Up to this point, to prove non-existence of subtori in a holomorphic Lagrangian fibration, we have used the irreducibility of the corresponding Gauss-Manin local system. This works when the Lagrangian fibration is non-isotrivial; for isotrivial Lagrangian fibrations, an extra effort is needed. Indeed, though it is irreducible as a variation of Hodge structure, the bundle of the first cohomologies is not necessarily irreducible as a local system in this case (Remark 7.4).

The monodromy of the weight 1 variation of Hodge structures V associated with an isotrivial Lagrangian fibrations is unitary, and therefore finite. Indeed, when the complex structure is constant in the family, the complex structure operator I acting on $V^{0,1}$ as $-\sqrt{-1}$ and on $V^{1,0}$ as $\sqrt{-1}$ is parallel, and it defines a positive definite Hermitian metric (4.1), which is also parallel, hence preserved by the monodromy.

Lemma 7.1: Let $V_{\mathbb{R}}$ be a real local system equipped with a lattice $V_{\mathbb{Z}}$, such that $V_{\mathbb{R}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$, and suppose that the monodromy of the torus bundle $V_{\mathbb{R}}/V_{\mathbb{Z}}$ is trivial. Then the monodromy of $V_{\mathbb{R}}$ is also trivial.

Proof: The group of automorphisms of a fiber of the lattice is isomorphic to $GL(n,\mathbb{Z})$, and this group acts faithfully on the torus, indeed, already the action on its tangent space is faithful. ■

Lemma 7.2: Let $\pi : M \longrightarrow X$ be an isotrivial Lagrangian fibration on a hyperkähler manifold of maximal holonomy, p a parabolic automorphism, $M_0 := \pi^{-1}(X_0)$, and $P \subset M_0$ the family of subtori obtained as components of the closure of orbits of p (Proposition 6.1). Let $V := R^1\pi_*(\mathbb{C}_{M_0})$ be the variation of Hodge structures associated with π and $\mathbb{P} \subset V_{\mathbb{R}}$ a parallel sub-bundle associated with $R^1\pi_*(\mathbb{R}_P)$. Then $\mathbb{P}$ is locally generated by the holomorphic sections of $V^{1,0}$ projected to $V_{\mathbb{R}}$ along $V^{0,1}$.[1]

[1] In other words, each fiber P_x is generated, as a real vector space, by the images of

Proof: $V^{1,0}/V_{\mathbb{Z}} \cong V_{\mathbb{R}}/V_{\mathbb{Z}}$ is the relative Albanese space Alb_π of $\pi:\ M_0 \longrightarrow X_0$. Therefore, p can be considered as a section of the relative Albanese map mapping X_0 to the total space of $\mathbb{P}/\mathbb{P}_{\mathbb{Z}}$. This section, considered as a map $X_0 \longrightarrow V^{1,0}/V_{\mathbb{Z}}$, is holomorphic by definition. On the other hand, the iterations of p are dense in P, hence after projection generate $\mathbb{P} \subset V_{\mathbb{R}}$. ■

The following proposition takes care of isotrivial Lagrangian fibrations, finishing the proof of Theorem 3.11. Indeed, any parabolic automorphism whose orbits are not dense in the fibers of the Lagrangian projection defines a parallel subbundle $\mathbb{P} = (V_1)_{\mathbb{R}} \subset V_{\mathbb{R}} = R^1\pi_*(\mathbb{R}_{M_0})$ (Proposition 6.1), and the corresponding subbundle in $V^{1,0}$ is generated by holomorphic sections (Lemma 7.2).

Proposition 7.3: Let $\pi:\ M \longrightarrow X$ be an isotrivial Lagrangian fibration on a hyperkähler manifold of maximal holonomy. Suppose that the corresponding flat bundle $V := R^1\pi_*(\mathbb{C}_{M_0})$ contains a parallel subbundle $V_1 \neq 0$, a complexification of $(V_1)_{\mathbb{R}} \in V_{\mathbb{R}}$. If $(V_1)_{\mathbb{R}}$ is generated by real parts of holomorphic sections of $V^{1,0}$, then $V_1 = V$.

Proof. Step 1: We prove that any proper parallel subbundle of V is totally real (that is, contains no holomorphic subspaces) and Lagrangian with respect to the polarization.

Since π is isotrivial, the complex structure operator on V is parallel with respect to the flat connection. Let ω be the polarization on V. It is a parallel section of $\bigwedge^2 V^*$, induced by the restriction of the Riemann-Hodge form to the fibers of π. Then the bilinear symmetric form $g(x,y) = \omega(I(x),y)$ is positive definite and parallel on V.

Any parallel sub-bundle $V_1 \subset V$ is Lagrangian with respect to ω. To see this, consider the orthogonal decomposition $V = V_1 \oplus V_1^\perp$ with respect to the positive definite form g on V. This decomposition is parallel, because g is parallel. Restricting ω to V_1, we obtain a parallel 2-form on V_1; using the decomposition $V = V_1 \oplus V_1^\perp$, we can consider this 2-form as a parallel section $\omega_1 \in \bigwedge^2(V)$. Since the rank of ω_0 is strictly less than $\operatorname{rk}\omega$, we have $\omega_1 = 0$ by Lemma 5.5. Therefore, V_1 is isotropic. Similarly, the restriction of ω to $V_1^\perp$ also vanishes. This implies that these two sub-bundles are Lagrangian.

The fact that $V_1 \subset V$ is Lagrangian with respect to the polarization implies that V_1 is totally real, indeed the polarization would be positive on

local holomorphic sections at x.

a holomorphic subspace.

Step 2:
Let $U \subset X$ be an open subset, and $s:\ U \longrightarrow V^{1,0}$ a holomorphic section such that $\operatorname{Re}(s) \subset (V_1)_{\mathbb{R}}$. Since s is holomorphic, $\overline{\partial} s = 0$ and $\nabla(s) = \nabla^{1,0}(s)$. Unless $\nabla(s) = 0$, this would imply that $V_1 \cap V^{1,0} \neq 0$; indeed, $\nabla^{1,0}_\xi(s) \subset V^{1,0}$ for any $\xi \in TX$. However, Step 1 implies that $V_1 \cap V^{1,0} = 0$, implying that $\nabla(s) = 0$. Then V_1 is generated by parallel sections, which is impossible by Deligne's global invariant cycle theorem (Theorem 4.12), because $H^1(M) = 0$ whenever M is a hyperkähler manifold of maximal holonomy. ■

Remark 7.4: Consider the Kummer K3 surface associated to a product of elliptic curves. The projection to each factor induces an isotrivial Lagrangian fibration on the Kummer surface, and it is easy to see that the monodromy of its local system of the first cohomologies is $\{\pm 1\}$. Hence this local system of real rank 2 is not irreducible. In the context of Deligne's theorem Theorem 4.10, this example illustrates the case when an irreducible real variation of Hodge structures is obtained as a tensor product $L \otimes W$ of a rank 1 real local system L, understood as a variation of Hodge structures of weight 0, and a weight 1 Hodge structure on a two-dimensional vector space W.

Remark 7.5: After this article has been released, Ljudmila Kamenova has remarked to us that similar results can also be obtained in the singular context, that is, for primitive symplectic varieties, where the analogues of the main theorems of hyperkähler geometry have recently been obtained by Bakker and Lehn. See [KL] for basic results on fibrations in this case (remarkably, the analogue of Matsushita's theorem holds, that is, any fibration is lagrangian with almost all of the fibers smooth projective tori, and projective base).

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