ASYMPTOTIC BOUNDS FOR NORI'S CONNECTIVITY THEOREM

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Abstract. Let Y be a smooth complex projective variety. I study the cohomology of smooth families of hypersurfaces $\mathcal{X} \to B$ for $B \subset \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$ a codimension c subvariety. I give an asymptotically optimal bound on c and k when $d \to \infty$ for the space $\mathrm{H}^k(Y \times B, \mathcal{X}, \mathbb{Q})$ to vanish, thus extending the validity of Lefschetz Hyperplane section Theorem and Nori's Connectivity Theorem [No]. Next, I construct in the limit case explicit families of higher Chow groups whose class does not vanish in $\mathrm{H}^k(Y \times B, \mathcal{X}, \mathbb{Q})$. Some of them are indecomposable. This suggests that in the limit case the space $\mathrm{H}^k(Y \times B, \mathcal{X}, \mathbb{Q})$ should be spanned by higher Chow groups, containing [No] and [O] as special cases (*cf.* conjecture 2).

0. Introduction

0.1. **Nori's connectivity theorem.** Let Y be a smooth complex proper algebraic variety of dimension $N+1$ with an ample invertible sheaf $\mathcal{L}$ and $\pi\colon \mathcal{X} \to \mathbb{P}\mathrm{H}^0(Y, \mathcal{L})$ the universal family of hypersurfaces of class $c_1(\mathcal{L})$. For any $b \in \mathbb{P}\mathrm{H}^0(Y, \mathcal{L})$, let $X_b = \pi^{-1}(b)$ be the corresponding hypersurface of Y (throughout the paper, the notation $\mathbb{P}V = (V \setminus \{0\})/\mathbb{C}^*$ denotes the projectivisation of a vector space V).

The Lefschetz hyperplane section theorem asserts that for $k+1 \in \{0, \ldots, N\}$,

$$\mathrm{H}^{k+1}(Y, X_b, \mathbb{Z}) = 0.$$

Nori's connectivity theorem extends this result to generically defined cohomology classes of degree all the way up to $2N = \dim_{\mathbb{R}} X$. More precisely, let T be a smooth variety and $\phi\colon T \to \mathbb{P}\mathrm{H}^0(Y, \mathcal{L})$ a differentiable map. Let $\mathcal{X}_T = \mathcal{X} \times_{\mathbb{P}\mathrm{H}^0(Y,\mathcal{L})} T$ be the universal hypersurface over T and let $\mathcal{Y}_T = Y \times T$. By the Leray spectral sequence, the Lefschetz Hyperplane Section Theorem implies that for $k+1 \in \{0, \ldots, N\}$,

$$\mathrm{H}^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{Z}) = 0.$$

(*cf.* section 0.2.1).

With stronger assumptions one can say more. Assume that T is a smooth algebraic variety and that $\phi\colon T \to \mathbb{P}\mathrm{H}^0(Y, \mathcal{L})$ is a smooth (hence dominant) morphism. Let $\mathcal{O}(1)$ be a fixed very ample invertible sheaf on Y and assume $\mathcal{L} = \mathcal{O}(d)$ for some positive integer d.

Theorem (Nori's connectivity theorem [No])**.** *For d large enough and $k+1 \in \{0, \ldots, 2N\}$,*

$$\mathrm{H}^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{Q}) = 0.$$

Note that Nori's connectivity theorem holds more generally for families of complete intersections in Y (instead of only hypersurfaces). The more general setting is motivated by the construction of algebraic cycles which are homologically but not algebraically equivalent to zero. Since in this paper I do not consider applications to algebraic equivalence, I restrict myself to the case of hypersurfaces, which involves all the ideas and simplifies notations.

Despite its topological nature, the proof of Nori's connectivity theorem relies on Hodge theory (*cf.* [No], section 1). Namely, let $\Omega^p_{\mathcal{Y}_T,\mathcal{X}_T}$ be the coherent sheaf of $\mathcal{O}_{Y_T}$-modules defined by the short exact sequence

$$0 \to \Omega^p_{\mathcal{Y}_T,\mathcal{X}_T} \to \Omega^p_{\mathcal{Y}_T} \to j_{T*}\Omega^p_{\mathcal{X}_T} \to 0,$$

where $j_T\colon \mathcal{X}_T \to \mathcal{Y}_T$ is the closed immersion. Then $\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = \mathbb{H}^{p+q+1}(\mathcal{Y}_T, \Omega^\bullet_{\mathcal{Y}_T,\mathcal{X}_T})$. Following Nori, let

$$G^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = \mathrm{Im}\big(\mathbb{H}^{p+q+1}(\mathcal{Y}_T, \Omega^{\bullet\geq p}_{\mathcal{Y}_T,\mathcal{X}_T}) \to \mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C})\big)$$

denote the Hodge-Grothendick filtration. Nori shows that there is a natural inclusion

$$F^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) \subset G^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}),$$

where $F^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C})$ denotes the Hodge filtration of the mixed Hodge structure on $\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T)$. Hence if for some positive integer k one has $\mathrm{H}^{q+1}(\mathcal{Y}_T, \Omega^p_{\mathcal{Y}_T,\mathcal{X}_T}) = 0$ for all $p + q + 1 = k + 1$ and $q + 1 \leq k/2 + 1$, then by the Hodge–Fröhlicher spectral sequence $G^{k-[k/2]}\mathrm{H}^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0$, hence $F^{k-[k/2]}\mathrm{H}^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0$ and a mixed Hodge structures argument implies $\mathrm{H}^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{Q}) = 0$. Hence the connectivity theorem follows from the following more precise Hogde-theoretical statement

Theorem (Nori)**.** *For d large enough, $p + q + 1 \leq 2N$ and $q + 1 \leq N$,*

$$\mathrm{H}^{q+1}(\mathcal{Y}_T, \Omega^p_{\mathcal{Y}_T,\mathcal{X}_T}) = 0.$$

0.2. **Vanishing theorems.** The first goal of this paper is to investigate the minimal assumptions for Nori's connectivity theorem to extend. Natural questions are:

- how large does d need to be?
- what happens if the map $\phi\colon T \to \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$ fails to be dominant, i.e if one considers families of hypersurfaces which are not general, but close to being general?
- can the above assertions be made more precise for a fixed integer k (in Nori's connectivity theorem), and for fixed integers p and q (in the Hodge-theoretical version)?

The first question was studied in Nagel [N] and Voisin [V2]. In this paper I give the following asymptotic answer to the above questions.

Let T be a smooth algebraic variety with an algebraic morphism $\phi\colon T \to \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$, such that $\mathcal{X}_T$ is smooth. I assume that for all $t \in T$ the cokernel of the differential map $d\phi(t)\colon T_{T,t} \to T_{\mathbb{P}\mathrm{H}^0(Y,\mathcal{O}(d)),\phi(t)}$ is of dimension at most an integer c.

The notation $\mathbb{N}$ stands for nonnegative integers (hence $0 \in \mathbb{N}$), and $]0, 1[\subset \mathbb{R}$ denotes the open interval.

Theorem 1. *For all $\varepsilon \in]0,1[$ and for all $p \in \mathbb{N}$ and for $q+1 \in \{0,\dots,N\}$, there is an integer D depending only on ε, p, Y and $\mathcal{O}(1)$, such that for all $d \geq D$,*

$$\mathrm{H}^{q+1}(\mathcal{Y}_T, \Omega^p_{\mathcal{Y}_T,\mathcal{X}_T}) = 0$$

provided that

$$c \leq (1-\varepsilon)\frac{d^{N-q}}{(N-q)!}.$$

A straightforward adaptation of Nori's argument sketched above shows that in the numerical situation of Theorem 1 $G^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0$, hence $F^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0$, and the following topological statement holds.

Corollary. *For all $\varepsilon \in]0,1[$ and $k+1 \in \{0,\dots,2N\}$ there is an integer D depending only on ε, Y and $\mathcal{O}(1)$ such that for all $d \geq D$,*

$$\mathrm{H}^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{Q}) = 0$$

provided that

$$c \leq (1-\varepsilon)\frac{d^{N-[k/2]}}{(N-[k/2])!}.$$

The following remarks show that the bounds given by Theorem 1 and the Corollary are close to being asymptotically optimal:

0.2.1. *Case $k+1 \in \{0,\dots,N\}$.* By the Lefschetz hyperplane section theorem one has $\mathrm{H}^{k'+1}(Y, X_t, \mathbb{Q}) = 0$ for all $k' \in \{0,\dots,k\}$ and $t \in T$, hence $R^{k'+1}\pi_*(\mathbb{Q} \to j_{T*}\mathbb{Q}) = 0$ and the Leray spectral sequence implies $\mathrm{H}^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{Q}) = 0$ with no assumption on c. A similar argument for $p+q+1 \in \{0,\dots,N\}$ gives a more precise result: since $F^{p'}\mathrm{H}^{p'+q'+1}(Y, X_t, \mathbb{Q}) = 0$ for all $p' \in \{0,\dots,p\}$, $q' \in \{0,\dots,q\}$ and $t \in T$, one has $\mathbb{H}^{q'+1}(Y, \Omega^{\bullet \geq p'}_{Y,X_t}) = 0$, hence $R^{q'+1}\pi_*\Omega^{\bullet\geq p'}_{Y_T,X_T/T} \otimes \Omega^{p-p'}_T = 0$; the spectral sequence induced by the Leray filtration on $\Omega^{\bullet\geq p}_{Y_T,X_T}$ implies $R^{q'+1}\pi_*\Omega^{\bullet\geq p}_{Y_T,X_T} = 0$, then the holomorphic Leray spectral sequence implies $\mathrm{H}^{q+1}(\mathcal{Y}_T, \Omega^p_{\mathcal{Y}_T,\mathcal{X}_T}) = 0$ with no assumption on c.

0.2.2. *Case $k+1 \in \{N+1,\dots,2N\}$.* I show in the next section (see Theorem 3 below) that under a weak assumption on Y, the bound on c is asymptotically optimal for any couple of fixed integers p and q such that $N/2 \leq q \leq N-1$ and $p \geq q$. This implies that the bound given by the Corollary is asymptotically optimal.

0.2.3. *Case $k+1 \geq 2N+1$.* One has $\mathrm{H}^{2N+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{Q}) \neq 0$ for any positive integer c, $d > N$ and $T = \mathcal{X}_B$, where $B \subset \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(1))$ is any smooth codimension c subvariety contained in the locus of smooth hypersurfaces. Indeed, the condition $d > N$ implies $\mathrm{H}^{0,N}(X_b, \mathbb{Q}) \neq 0$ for all $b \in B$, hence (*cf.* [No], section 0.2) the cohomology class of the diagonal embedding of $\mathcal{X}_B$ in $\mathcal{X}_T = \mathcal{X}_B \times_B \mathcal{X}_B$ is an element of $\mathrm{H}^{2N}(\mathcal{X}_T, \mathbb{Q})$ which does not belong to $j_T^*\mathrm{H}^{2N}(\mathcal{Y}_T, \mathbb{Q})$, hence defines a non-zero element of $\mathrm{H}^{2N+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{Q})$.

0.3. **Some examples of algebraic classes.** The aim of this section is to give explicit constructions of *algebraic* classes in $F^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C})$ for $q+1 \in \{0,\dots N\}$, $p \geq q$ and c close to the bound given by theorem 1.

0.3.1. *The regulator map.* For any smooth algebraic variety X and positive integers $p \geq q$ the higher Chow groups $\mathrm{CH}^p(X, p-q)$ are defined in [B]. In this paper they are always implicitly assumed to be with $\mathbb{Q}$ coefficients. They come with the Deligne class map

$$\mathrm{cl}_{p,p-q}(\mathcal{D})\colon \mathrm{CH}^p(X, p-q) \to \mathrm{H}^{p+q}_{\mathcal{D}}(X, \mathbb{Q}(p)),$$

where $\mathrm{H}^{p+q}_{\mathcal{D}}(X, \mathbb{Q}(p))$ denotes Deligne cohomology. There is a natural map

$$r_{p,p-q}\colon \mathrm{H}^{p+q}_{\mathcal{D}}(X, \mathbb{Q}(p)) \to F^p\mathrm{H}^{p+q}(X, \mathbb{C}),$$

where $F^p\mathrm{H}^{p+q}(X, \mathbb{C})$ denotes the Hodge filtration. Composing $\mathrm{cl}_{p,p-q}(\mathcal{D})$ with $r_{p,p-q}$ one gets the regulator map

$$\mathrm{reg}_{p,p-q}\colon \mathrm{CH}^p(X, p-q) \to F^p\mathrm{H}^{p+q}(X, \mathbb{C}).$$

Note that this map is identically zero for $p > q$ whenever X is proper.

Consider the case $X = \mathcal{X}_T$ (note that $\mathcal{X}_T$ is *not* proper). Let

$$\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v = \mathrm{coker}\left(j_T^*\colon \mathrm{H}^{p+q}(\mathcal{Y}_T, \mathbb{C}) \to \mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})\right)$$

denote the *vanishing* cohomology, let $F^p\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v$ denote the image of $F^p\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})$ in $\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v$ and let

$$\overline{\mathrm{reg}}_{p,p-q}\colon \mathrm{CH}^p(\mathcal{X}_T, p-q) \to F^p\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v \subset F^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C})$$

be the map deduced from $\mathrm{reg}_{p,p-q}$ by projection on $\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v$.

An element of $F^p\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v$ (or $F^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C})$) will be called *algebraic* if it lies in the image of $\overline{\mathrm{reg}}_{p,p-q}$.

0.3.2. *Chow groups.* In the case $p = q$ higher Chow groups are Chow groups and the regulator is the class map. I then write $\mathrm{CH}^p(\mathcal{X}_T)$ for $\mathrm{CH}^p(\mathcal{X}_T, 0)$, cl_p for $\mathrm{reg}_{p,0}$ and $\overline{\mathrm{cl}}_p$ for $\overline{\mathrm{reg}}_{p,0}$

The following is a variation of my result in [O].

Theorem 2. *Let p, q, b and c be positive integers such that $p + q = N$, $q \in \{[(N+1)/2], \ldots, N-1\}$, $b \geq 1$ and $c \leq b\frac{d^{N-q}}{(N-q)!}$. Assume $\phi(T) \subset \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$ lies in the locus parametrizing smooth hypersurfaces. Then there is an integer D depending only on b, Y and $\mathcal{O}(1)$ such that for all $d \geq D$ the following holds.*

If N is even and $p = q = N/2$ then the space $F^p\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v$ is spanned via $\overline{\mathrm{cl}}_p$ by classes of flat families of algebraic cycles in $\mathcal{X}_T$ of codimension p and degree at most b.

If $q > N/2$ then $F^p\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v = 0$.

0.3.3. *Higher Chow groups.* I now fulfill the promise made in section 0.2.2. The notation $\sim_{d\to\infty}$ denotes asymptotic equivalence for $d \to \infty$.

Theorem 3. *Let p and q be positive integers such that $p \geq q$ and $q \in \{[(N+1)/2], \ldots, N-1\}$. Assume that $Y \subset \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(1))^\vee$ contains a linear subvariety $V \subset \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(1))^\vee$ of codimension $q+1$ in Y and let $T \subset \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$ be the subvariety parametrizing smooth hypersurfaces containing V. Then for $d \gg 0$ the variety T is of codimension $c \sim_{d\to\infty} \frac{d^{N-q}}{(N-q)!}$, and there is a higher cycle $Z \in \mathrm{CH}^p(\mathcal{X}_T, p-q)$ such that $\overline{\mathrm{reg}}_{p,p-q}(Z) \neq 0$.*

Hence in this situation the bound on c in Theorem 1 and its corollary are asymptotically optimal.

Note that the assumption on Y holds for $Y = \mathbb{P}^{N+1}_{\mathbb{C}}$, which should be regarded as the main example.

0.3.4. *Indecomposable Higher Chow groups.* For all $i \in \{0, \dots, p\}$ and $j \in \{0, \dots, p-q\}$ there are natural product maps

$$\psi_{i,j} \colon \mathrm{CH}^{p-i}(\mathcal{X}_T, p-q-j) \otimes \mathrm{CH}^i(\mathcal{X}_T, j) \to \mathrm{CH}^p(\mathcal{X}_T, p-q).$$

I say that a higher cycle in $\mathrm{CH}^p(\mathcal{X}_T, p-q)$ is *decomposable* if it belongs to the subgroup spanned by the images of the maps $\psi_{i,j}$ for $(i,j) \notin \{(0,0),(p,p-q)\}$, and that it is *indecomposable* otherwise.

The proof of Theorem 3 relies on an explicit construction of higher cycles in $\mathrm{CH}^p(\mathcal{X}_T, p-q)$. More precisely, since $\mathrm{CH}^1(\mathcal{X}_T, 1) \simeq \mathrm{H}^0(\mathcal{X}_T, \mathcal{O}^*_{\mathcal{X}_T}) \simeq \mathrm{H}^0(T, \mathcal{O}^*_T)$, iterating the above map $p-q$ times I obtain the map

$$\psi^{p-q}_{1,1} \colon \mathrm{CH}^q(\mathcal{X}_T) \otimes \mathrm{H}^0(T, \mathcal{O}^*_T)^{p-q} \to \mathrm{CH}^p(\mathcal{X}_T, p-q).$$

For $p > q$ the higher cycles in $\mathrm{CH}^p(\mathcal{X}_T, p-q)$ I construct are in the image of $\psi^{p-q}_{1,1}$. Since $\psi^{p-q}_{1,1}$ commutes to the restriction to a fiber, it follows that the restriction of these higher cycles to any fiber X_t, $t \in T$, is decomposable.

My result should be compared to Voisin's [V2], who studies the case $Y = \mathbb{P}^{N+1}_{\mathbb{C}}$, $p = N$, $q = N-1$, $d = 2N$ and $T \subset \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$ an open subset (this implies $c = 0$). She constructs a family of higher cycles in $\mathrm{CH}^N(\mathcal{X}_T, 1)$ that limits the validity of Nori's connectivity theorem and whose restriction to a very general fiber X_t for $t \in T$ is indecomposable. This suggests that for all p, q such that $p + q \geq N$ and $p \geq q$ there might be higher cycles in $\mathrm{CH}^p(\mathcal{X}_T, p-q)$ for $c \sim_{d\to\infty} \frac{d^{N-q}}{(N-q)!}$ whose restriction to a very general fiber X_t, $t \in T$, is indecomposable. Supporting this intuition, I prove the following slightly weaker result for $Y = \mathbb{P}^{N+1}$, N even, $q = N/2$ and $p = q+1$.

Theorem 4. *Assume N even, $q = N/2$ and that $Y \subset \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(1))^\vee$ contains a linear subvariety of $\mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(1))^\vee$ of codimension q in Y. Then for $d \gg 0$ there is a subvariety $U \subset \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$ contained in the locus parametrizing smooth hypersurfaces, of codimension $c \sim_{d\to\infty} 2\frac{d^{N-q}}{(N-q)!}$, and a higher cycle $Z_U \in \mathrm{CH}^{q+1}(\mathcal{X}_U, 1)$ such that*

- *$\overline{\mathrm{reg}}_{q+1,1}(Z_U) \neq 0$, and*
- *the restriction of Z_U to a very general fiber X_u for $u \in U$ is indecomposable.*

The cycle Z_U is constructed explicitly and is much simpler than Voisin's example. The proof of indecomposability relies on Theorems 1 and 2.

I believe that Theorem 4 should generalize as follows.

Hope. *Let p and q be positive integers such that $p > q$ and $q \in \{[(N+1)/2], \dots, N-1\}$. Assume $Y = \mathbb{P}^{N+1}$. Then for $d \gg 0$ there is a subvariety $U \subset \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$ of codimension $c \sim_{d\to\infty} (p-q+1)\frac{d^{N-q}}{(N-q)!}$ and a higher cycle $Z_U \in \mathrm{CH}^p(\mathcal{X}_U, p-q)$ such that*

- *$\overline{\mathrm{reg}}_{p,p-q}(Z_U) \neq 0$, and*
- *the restriction of Z_U to a very general fiber X_u, $u \in U$, is indecomposable.*

Theorem 4 is the case $p = N/2 + 1$, $q = N/2$.

Note that $c \sim_{d\to\infty} 2\frac{d^{N-q}}{(N-q)!}$ in Theorem 4 (and $c \sim_{d\to\infty} (p-q+1)\frac{d^{N-q}}{(N-q)!}$ in its conjectural generalization) while Theorem 3 gives examples of decomposable cycles in $\mathrm{CH}^p(\mathcal{X}_T, p-q)$ with a non zero vanishing cohomology class for $c \sim_{d\to\infty} \frac{d^{N-q}}{(N-q)!}$. This suggests that the bound on c in Theorem 4 may not be optimal.

Question . *Is there an indecomposable higher cycle $Z_U \in \mathrm{CH}^p(\mathcal{X}_U, p-q)$ such that $\overline{\mathrm{reg}}_{p,p-q}(Z_U) \neq 0$ for $c \sim_{d\to\infty} \frac{d^{N-q}}{(N-q)!}$?*

0.4. **Algebraicity conjecture.** The results of Section 0.3 suggest that for c slightly bigger than the bound $\dfrac{d^{N-q}}{(N-q)!}$ given by Theorem 1 the space $F^p\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{Q})_v$ should be algebraic.

More precisely, I believe the following should hold

Conjecture 1. *For all integers $b \in \mathbb{N}^*$ and $p \in \mathbb{N}$ there is an integer D depending only on b, p, Y and on $\mathcal{O}(1)$, such that for all $d \geq D$ and $0 \leq q \leq N-1$, if*

$$c \leq b\frac{d^{N-q}}{(N-q)!}$$

then the map $\overline{\mathrm{reg}}_{p,p-q} : \mathrm{CH}^p(\mathcal{X}_T, p-q) \otimes \mathbb{C} \to F^p\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v$ is surjective.

By convention, $\mathrm{CH}^p(\mathcal{X}_T, p-q) = 0$ for $p < q$.

0.4.1. *Case $p + q + 1 \in \{0, \ldots, N\}$.* Conjecture 1 is true since the target cohomology space vanishes (*cf.* section 0.2.1).

0.4.2. *Case $p + q = N$, $q \geq N/2$.* Conjecture 1 holds by Theorem 2 provided that $\phi(T) \subset \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$ lies in the locus parametrizing smooth hypersurfaces.

Mimicking Nori's argument, it is easy to see that Conjecture 1 implies the following purely topological statement.

Conjecture 2. *For every positive integer b there is an integer D depending only on b, Y and $\mathcal{O}(1)$ such that for all $d \geq D$ and $k \in \{N, \ldots, 2N-1\}$ and for*

$$c \leq b\frac{d^{N-[k/2]}}{(N-[k/2])!},$$

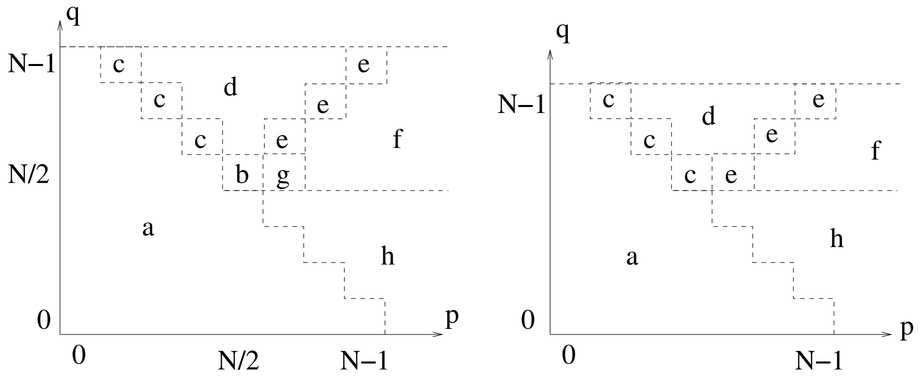
- *if k is even then the map $\overline{\mathrm{cl}}_{k/2} \colon \mathrm{CH}^{k/2}(\mathcal{X}_T) \otimes \mathbb{C} \to F^{k/2}\mathrm{H}^k(\mathcal{X}_T, \mathbb{C})_v$ is surjective;*
- *if k is odd then the maps*

$$\begin{aligned} \overline{\mathrm{reg}}_{(k+1)/2,1} \colon \mathrm{CH}^{(k+1)/2}(\mathcal{X}_T, 1) \otimes \mathbb{C} &\to \mathrm{H}^k(\mathcal{X}_T, \mathbb{C})_v, \quad \textit{and} \\ \psi \circ \overline{\mathrm{reg}}_{(k+1)/2,1} \colon \mathrm{CH}^{(k+1)/2}(\overline{\mathcal{X}_T}, 1) \otimes \mathbb{C} &\to \mathrm{H}^k(\overline{\mathcal{X}_T}, \mathbb{C})_v \to \mathrm{H}^k(\mathcal{X}_T, \mathbb{C})_v \end{aligned}$$

generate $\mathrm{H}^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C})_v$.

Here $\overline{\mathcal{X}_T}$ is the variety isomorphic to $\mathcal{X}_T$ as a real manifold and endowed with the opposite complex structure and ψ is the complex conjugation.

0.4.3. *Case $k \in \{0, \ldots, N\}$.* Conjecture 2 is true since the target cohomology spaces vanishes (*cf.* section 0.2.1).



N even N odd

0.4.4. *Case k even.* Theorem 1 implies that in the numerical situation of Conjecture 2, one has $F^{k/2+1}\mathrm{H}^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0$, hence by a mixed Hodge structures argument the space $\mathrm{H}^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C})$ is of pure Hodge type $(k/2, k/2)$ (like in [No], section 0.2), hence the subspace $\mathrm{H}^k(\mathcal{X}_T, \mathbb{C})_v$ is a sub–Hodge structure of pure Hodge type $(k/2, k/2)$. Hence for k even Conjecture 2 is implied by Hodge Conjecture.

0.5. **Summary.** The figures summarize the results. Assume $c \leq b\frac{d^{N-q}}{(N-q)!}$ and that $\phi(T) \subset \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$ lies in the locus parametrize smooth hypersurfaces.

a : $F^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0$ for any c by section 0.2.1.

b : $F^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0$ for $c \leq (1-\varepsilon)\frac{d^{N-q}}{(N-q)!}$ by Theorem 1 and $F^p\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v$ is spanned by classes of flat families of algebraic cycles of codimension $N/2$ and degree at most b for $c \leq b\frac{d^{N-q}}{(N-q)!}$ by Theorem 2, as predicted by Conjecture 1.

c : $F^p\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v = 0$ for $c \leq b\frac{d^{N-q}}{(N-q)!}$ by Theorem 2.

d : $F^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0$ for $c \leq (1-\varepsilon)\frac{d^{N-q}}{(N-q)!}$ by Theorem 1 and one should still have $F^p\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v = 0$ for $c \leq b\frac{d^{N-q}}{(N-q)!}$ by Conjecture 1.

e : $F^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0$ for $c \leq (1-\varepsilon)\frac{d^{N-q}}{(N-q)!}$ by Theorem 1 and $F^p\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v$ should be spanned by $\overline{\mathrm{cl}}_p(\mathrm{CH}^p(\mathcal{X}_T))$ for $c \leq b\frac{d^{N-q}}{(N-q)!}$ by Conjecture 1; by Theorem 3 there is a variety Y such that $\overline{\mathrm{cl}}_p(\mathrm{CH}^p(\mathcal{X}_T)) \neq 0$ for $c \sim \frac{d^{N-q}}{(N-q)!}$.

f : $F^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0$ for $c \leq (1-\varepsilon)\frac{d^{N-q}}{(N-q)!}$ by Theorem 1 and $F^p\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v$ should be spanned by $\overline{\mathrm{reg}}_{p,p-q}\mathrm{CH}^p(\mathcal{X}_T, p-q)$ for $c \leq b\frac{d^{N-q}}{(N-q)!}$ by Conjecture 1; by Theorem 3 there is a variety Y such that $\overline{\mathrm{reg}}_{p,p-q}\mathrm{CH}^p(\mathcal{X}_T, p-q) \neq 0$ for $c \sim \frac{d^{N-q}}{(N-q)!}$ and by Conjecture there should be an indecomposable higher cycle in $\mathrm{CH}^p(\mathcal{X}_T, p-q)$ with a non zero image under $\overline{\mathrm{reg}}_{p,p-q}$ for $c \sim (p-q+1)\frac{d^{N-q}}{(N-q)!}$.

g : as in f; by Theorem 4 there is a variety Y and an indecomposable higher cycle in $\mathrm{CH}^p(\mathcal{X}_T, p-q)$ with a non zero image under $\overline{\mathrm{reg}}_{p,p-q}$ for $c \sim 2\frac{d^{N-q}}{(N-q)!}$.

h : $F^p\mathrm{H}^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0$ for $c \leq (1-\varepsilon)\frac{d^{N-q}}{(N-q)!}$ by Theorem 1 and $F^p\mathrm{H}^{p+q}(\mathcal{X}_T, \mathbb{C})_v$ should be spanned by $\overline{\mathrm{reg}}_{p,p-q}(\mathrm{CH}^p(\mathcal{X}_T, p-q))$ for $c \leq b\frac{d^{N-q}}{(N-q)!}$ by Conjecture 1. More precisely, I believe that $F^p\mathrm{H}^{p+q}(\mathcal{X}_T,\mathbb{C})_v = 0$ for $c \le b\frac{d^{N-q}}{(N-q)!}$ in this case. If this is true one may further ask what is the (asymptotically) smallest c such that $\overline{\mathrm{reg}}_{p,p-q}(\mathrm{CH}^p(\mathcal{X}_T, p-q)) \neq 0$? This problem is linked to the existence of non trivial Hodge substructures in $\mathrm{H}^{p+q}(\mathcal{X}_T,\mathbb{C})_v$. Note that I do not have any example of a non-zero element in $\mathrm{CH}^p(\mathcal{X}_T, p-q)/j_T^*(\mathrm{CH}^p(\mathcal{Y}_T, p-q))$, for any c.

0.6. **Remark.** For $Y = \mathbb{P}^{N+1}$ the proofs below can easily be modified (and even simplified, since one avoids the technical Lemma 1.3.1) to give explicit bounds on the codimension c in Theorems 1, 3 and 4, which are very close to being optimal.

1. Proof of Theorem 1 and of its Corollary

1.1. **Reduction to the local case.** A straightforward adaptation of Nori's argument shows that if $\mathrm{H}^{q+1}(Y_t, \Omega^p_{\mathcal{Y}_T,\mathcal{X}_T|Y_t}) = 0$ for all $t \in T$ then $\mathrm{H}^{q+1}(\mathcal{Y}_T, \Omega^p_{\mathcal{Y}_T,\mathcal{X}_T}) = 0$ ([No], Lemma 2.1). This shows that Theorem 1 is implied by the following purely local-analytic statement.

Proposition 1.2. *For all $\varepsilon \in]0,1[$ there is an integer D depending only on p, ε, Y and on $\mathcal{O}(1)$ such that*

$$\mathrm{H}^{q+1}(Y_t, \Omega^p_{\mathcal{Y}_T,\mathcal{X}_T|Y_t}) = 0$$

for all integers $d \ge D$, $q \in \{-1, \dots, N-1\}$, for all smooth analytic varieties T endowed with a morphism $\phi\colon T \to \mathrm{H}^0(Y, \mathcal{O}(d))$ such that

- *for all $t \in T$ the cokernel of $d\phi(t)$ is of dimension at most $(1-\varepsilon)\frac{d^{N-q}}{(N-q)!}$ and*
- *$\mathcal{X}_T$ is smooth.*

1.3. **Proof of proposition 1.2.** Since the proof of proposition 1.2 proceeds by induction on several parameters, it is convenient to introduce the following notation. For a fixed positive integer d and a projective space $\mathbb{P}^A_{\mathbb{C}}$ I say that assertion $\mathcal{H}(Y,p,q,c,a)$ holds for the smooth projective variety $Y \subset \mathbb{P}^A_{\mathbb{C}}$ of dimension $N+1 > 0$ and for the integers $(p,q,c,a) \in \mathbb{N} \times \{-1,\dots,N-1\} \times \mathbb{N} \times \mathbb{Z}$ if

$$\mathrm{H}^{q+1}(Y_t, \Omega^p_{\mathcal{Y}_T,\mathcal{X}_T|Y_t} \otimes \mathcal{O}(a')) = 0$$

for all $a' \le a$, for all smooth analytic varieties T endowed with a morphism $\phi\colon T \to \mathrm{H}^0(Y, \mathcal{O}(d))$ such that $\mathcal{X}_T$ is smooth, and for all $t \in T$ such that the cokernel of $d\phi(t)$ is of dimension at most c.

The proof relies on the following four lemmas.

Recall (*cf.* [G1]) that for any positive integer c, a d-decomposition of c is the unique series $(c_d, \dots, c_r)$ such that $r \in \{1, \dots, d\}$, $c_d > \cdots > c_r \ge r$ and

$$c = \binom{c_d}{d} + \cdots + \binom{c_r}{r}.$$

Let

$$c_{<d>} = \binom{c_d - 1}{d} + \cdots + \binom{c_r - 1}{r},$$

with the convention $\binom{\alpha}{\beta} = 0$ for $\alpha < \beta$.

The asymptotic behavior of the function $c \mapsto c_{<d>}$ is described by the following lemma.

Lemma 1.3.1. *For all integers $M \geq 2$ there exists a function $f_M\colon]0,1[\times\mathbb{N} \to \mathbb{R}$, $(\varepsilon, d) \mapsto \varepsilon'$ such that for all $\varepsilon \in]0,1[$ $\varepsilon' \to \varepsilon^{\frac{M-1}{M}}$ for $d \to \infty$, and such that for all nonnegative integers*

$$c \leq (1-\varepsilon)\frac{d^M}{M!}$$

one has

$$c_{<d>} \leq (1-\varepsilon')\frac{d^{M-1}}{(M-1)!}.$$

The following lemma is the key ingredient of my proof. It relies on Green's hyperplane section theorem [G1].

Lemma 1.3.2. *Let Y' be a generic hyperplane section of Y. Then for $q \leq N-2$*

$$\left.\begin{array}{r}\mathcal{H}(Y', p, q, c_{<d>}, a)\\ \mathcal{H}(Y', p-1, q, c_{<d>}, a-1)\end{array}\right\} \Rightarrow \mathcal{H}(Y, p, q, c, a).$$

By convention, assumption $\mathcal{H}(Y', -1, q, c_{<d>}, a-1)$ holds.

The last two lemmas will only be used for $q = N-1$. Both rely on an idea of Green ([G2], p. 193-194).

Lemma 1.3.3. *Assume $c > 0$. Then*

$$\left.\begin{array}{r}\mathcal{H}(Y, p+1, q, c-1, a)\\ \mathcal{H}(Y, p+1, q-1, c, a)\end{array}\right\} \Rightarrow \mathcal{H}(Y, p, q, c, a).$$

By convention, assumption $\mathcal{H}(Y, p+1, -2, c, a)$ holds.

The last lemma has been proved by Nori for $a = 0$ ([No], remark 3.10); the general case involves no new ideas. See also [N] for computations of an explicit bound for s.

Lemma 1.3.4. *There is an integer s depending only on Y and $\mathcal{O}(1)$ such that for all $d \geq p+a+s$ the assertion $\mathcal{H}(Y, p, q, 0, a)$ holds for all $q \in \{-1, \ldots, N-1\}$.*

Lemma 1.3.4 will only be used for $q = N-1$.

1.3.5. *Proof of proposition 1.2.* Postponing the proofs of the above Lemmas, I now show how they imply proposition 1.2.

I show more generally that for all $\varepsilon \in]0,1[$ there is an integer D depending only on ε, p, a and Y such that assertion $\mathcal{H}(Y, p, q, c, a)$ holds for all $d \geq D$, $q \in \{-1, \ldots, N-1\}$ and $c \leq (1-\varepsilon)\frac{d^{N-q}}{(N-q)!}$. The case $a = 0$ is the proposition 1.2.

I proceed by induction on $N-q$, c and N.

Assume $N-q = 1$. The condition on c simplifies to $c \leq (1-\varepsilon)d$. Choose D such that $\varepsilon D \geq p+a+s$. Thus for $d \geq D$ one has $d \geq (1-\varepsilon)d + \varepsilon D \geq c+p+a+s$.

The case $c = 0$, $N-q = 1$ follows from Lemma 1.3.4.

The case $c > 0$, $N = 0$ and $q = -1$ follows from Lemma 1.3.3 by increasing induction on c. For the case $c > 0$, $N-q = 1$ let Y' be a generic hyperplane section of Y. Lemmas 1.3.2 and 1.3.3 provide successive implications

$$\left.\begin{array}{r}\\ \left.\begin{array}{r}\mathcal{H}(Y', p+1, q-1, c_{<d>}, a)\\ \mathcal{H}(Y', p, q-1, c_{<d>}, a-1)\end{array}\right\} \Rightarrow\end{array}\begin{array}{r}\mathcal{H}(Y, p+1, q, c-1, a)\\ \mathcal{H}(Y, p+1, q-1, c, a)\end{array}\right\} \Rightarrow \mathcal{H}(Y, p, q, c, a).$$

Assertion $\mathcal{H}(Y,p,q,c,a)$ follows by increasing induction on both c and N.

Assume $N-q>1$ and let Y' be a generic hyperplane section of Y. By induction on N and by Lemma 1.3.1 applied to $M=N-q$, assertions $\mathcal{H}(Y',p,q,c_{<d>},a)$ and $\mathcal{H}(Y',p-1,q,c_{<d>},a-1)$ hold. Hence by Lemma 1.3.2 assertion $\mathcal{H}(Y,p,q,c,a)$ holds. $\square$

1.4. **Proof of the Lemmas.**

1.4.1. *Proof of Lemma 1.3.1.* Let e be the largest positive integer such that

$$(1-\varepsilon)\frac{d^M}{M!}\leq\binom{M+d}{d}-\binom{M+e}{e}.$$

The identities

$$\begin{aligned}\binom{M+d}{d}-\binom{M+e}{e} &= \sum_{i=1}^{d-e}\binom{M+d-i}{d-i+1}\\ \binom{M-1+d}{d}-\binom{M-1+e}{e} &= \sum_{i=1}^{d-e}\binom{M-1+d-i}{d-i+1}\end{aligned}$$

imply that one has

$$\left[\binom{M+d}{d}-\binom{M+e}{e}\right]_{<d>} = \binom{M-1+d}{d}-\binom{M-1+e}{e}.$$

Since the map $c\mapsto c_{<d>}$ is increasing (*cf.* [G1]), and since $c\leq(1-\varepsilon)\frac{d^M}{M!}$, this implies

$$c_{<d>}\leq\binom{M-1+d}{d}-\binom{M-1+e}{e}.$$

Fix $\varepsilon\in]0,1[$ and consider e as a function of d. Then

$$\binom{M-1+d}{d}-\binom{M-1+e}{e}\sim_{d\to\infty}(1-\varepsilon^{\frac{M-1}{M}})\frac{d^{M-1}}{(M-1)!},$$

Indeed, since $\binom{M+d}{d}\sim_{d\to\infty}\frac{d^M}{M!}$, the definition of e implies $\binom{M+e}{e}\sim_{d\to\infty}\varepsilon\frac{d^M}{M!}$, hence $\frac{e^M}{M!}\sim_{d\to\infty}\varepsilon\frac{d^M}{M!}$ and $e\sim_{d\to\infty}\varepsilon^{1/M}d$. The assertion then follows from the equivalences $\binom{M-1+d}{d}\sim_{d\to\infty}\frac{d^{M-1}}{(M-1)!}$ and $\binom{M-1+e}{e}\sim_{d\to\infty}\frac{e^{M-1}}{(M-1)!}\sim_{d\to\infty}\varepsilon^{(M-1)/M}\frac{d^{M-1}}{(M-1)!}$.

Define $\varepsilon'\in\mathbb{R}$ by the formula

$$\binom{M-1+d}{d}-\binom{M-1+e}{e}=(1-\varepsilon')\frac{d^{M-1}}{(M-1)!}.$$

Then $c_{<d>}\leq(1-\varepsilon')\frac{d^{M-1}}{(M-1)!}$ and $\varepsilon'\sim_{d\to\infty}\varepsilon^{\frac{M-1}{M}}$. $\square$

1.4.2. *Proof of Lemma 1.3.2.* Let $i\colon Y'\to Y$ be the projective embedding, let $\mathcal{Y}'_T=Y'\times T$ and let $\mathcal{X}'_T=\mathcal{X}_T\times_{\mathcal{Y}_T}\mathcal{Y}'_T$. Fix $t\in T$. Since Y' is generic, one may assume that $\mathcal{X}'_T$ is smooth and that X_t meets Y'_t transversally in Y_t. Shrinking T if necessary, one may further assume that for all $u\in T$, X_u meets Y'_u transversally in Y_u.Then the image of the map $\phi\colon T\to\mathbb{P}\mathrm{H}^0(Y,\mathcal{O}(d))$ does not meet the base locus of the natural rational map $\mathbb{P}\mathrm{H}^0(Y,\mathcal{O}(d))\to\mathbb{P}\mathrm{H}^0(Y',\mathcal{O}(d))$, hence there is a well-defined composed map $\phi'\colon T\to\mathbb{P}\mathrm{H}^0(Y',\mathcal{O}(d))$.

Let

– $p_Y\colon \mathrm{H}^0(Y,\mathcal{O}(d))\setminus\{0\}\to \mathbb{P}\mathrm{H}^0(Y,\mathcal{O}(d))$ be the natural projection,
– $E = p_Y^*\mathrm{Im}\,(d\phi(t))$ and $c = \dim\mathrm{coker}\,(d\phi(t)) = \mathrm{codim}\,\big(E, \mathrm{H}^0(Y,\mathcal{O}(d))\big)$,
– $S = \bigoplus_{\delta\in\mathbb{N}} S_\delta$, $S_\delta = \mathrm{Sym}^\delta \mathrm{H}^0(Y,\mathcal{O}(1))$ be the homogeneous polynomial ring of the projective embedding $Y\to \mathbb{P}\mathrm{H}^0(Y,\mathcal{O}(1))^\vee$ and $I\subset S$ be the homogeneous ideal of Y.

Define the data $p_{Y'}, E', c', S', I'$ for Y' as p_Y, E, c, S, I for Y.

One has $c = \mathrm{codim}\,(E+I_d, S_d)$ and $c' = \mathrm{codim}\,(E'+I'_d, S'_d)$. Since $Y'\subset Y$ is generic, Green's hyperplane theorem [G1] asserts that

$$c' \le c_{<d>}.$$

Hence the cokernel of $d\phi'(t)$ is of dimension at most $c_{<d>}$.

The sheaf $\Omega^p_{\mathcal{Y}_T,\mathcal{X}_T|\mathcal{Y}'_T}$ fits into the short exact sequence

$$0\longrightarrow \Omega^{p-1}_{\mathcal{Y}'_T,\mathcal{X}'_T}\otimes \mathcal{O}'_Y(-1)\xrightarrow{dH\wedge} \Omega^p_{\mathcal{Y}_T,\mathcal{X}_T|\mathcal{Y}'_T}\longrightarrow \Omega^p_{\mathcal{Y}'_T,\mathcal{X}'_T}\longrightarrow 0$$

where $H\in \mathrm{H}^0(Y,\mathcal{O}(1))$ is an equation of Y'. Restricting this sequence to Y'_t and tensorising it by $\mathcal{O}'_Y(a)$ one gets the associated long exact sequence

$$\mathrm{H}^{q+1}(Y'_t,\Omega^{p-1}_{\mathcal{Y}'_T,\mathcal{X}'_T|Y'_t}\otimes \mathcal{O}'_Y(a-1))\xrightarrow{dH\wedge} \mathrm{H}^{q+1}(Y'_t,\Omega^p_{\mathcal{Y}_T,\mathcal{X}_T|Y'_t}\otimes\mathcal{O}'_Y(a))\longrightarrow$$
$$\longrightarrow \mathrm{H}^{q+1}(Y'_t,\Omega^p_{\mathcal{Y}'_T,\mathcal{X}'_T|Y'_t}\otimes\mathcal{O}'_Y(a)).$$

The term on the left vanishes by assumption $\mathcal{H}(Y',p-1,q,c_{<d>},a-1)$ and the term on the right vanishes by assumption $\mathcal{H}(Y',p,q,c_{<d>},a)$. Hence the middle term vanishes as well: one has

$$\mathrm{H}^{q+1}(Y'_t,\Omega^p_{\mathcal{Y}_T,\mathcal{X}_T|Y'_t}\otimes\mathcal{O}'_Y(a)) = 0. \tag{1}$$

For any locally free sheaf $\mathcal{F}$ on Y_t there is a short exact sequence

$$0\longrightarrow \mathcal{F}\otimes\mathcal{O}(-1)\xrightarrow{H}\mathcal{F}\longrightarrow i_*i^*\mathcal{F}\longrightarrow 0,$$

Take $\mathcal{F} = \Omega^p_{\mathcal{Y}_T,\mathcal{X}_T|Y_t}\otimes\mathcal{O}_Y(a)$ and consider the associated long exact sequence

$$\mathrm{H}^{q+1}(Y_t,\Omega^p_{\mathcal{Y}_T,\mathcal{X}_T|Y_t}\otimes\mathcal{O}_Y(a-1))\xrightarrow{H}\mathrm{H}^{q+1}(Y_t,\Omega^p_{\mathcal{Y}_T,\mathcal{X}_T|Y_t}\otimes\mathcal{O}_Y(a))$$
$$\longrightarrow \mathrm{H}^{q+1}(Y'_t,\Omega^p_{\mathcal{Y}_T,\mathcal{X}_T|Y'_t}\otimes\mathcal{O}'_Y(a)).$$

The term on the right vanishes by (1). Since $q\in\{0,\dots,N-1\}$, by increasing induction on a the term on the left vanishes as well, the case $a\ll 0$ following from Serre's vanishing theorem. Hence the term in the middle vanishes. □

1.4.3. *Proof of Lemma 1.3.3.* Since the assertion depends only on the tangent space of T at t, one may replace T by a small open neighborhood of t in the tangent space of T at t; hence one may assume $T = B\times S$, where $B\subset \mathbb{P}\mathrm{H}^0(Y,\mathcal{O}(d))$ is an open neighborhood of $\phi(t)$ of codimension at most c and S is an open ball.

Let $B'\subset\mathbb{P}\mathrm{H}^0(Y,\mathcal{O}(d))$ be a smooth analytic variety containing B as a codimension 1 subvariety, and let $T' = B'\times S$. Then $\mathcal{Y}_T\subset\mathcal{Y}_{T'}$ is a smooth codimension 1 subvariety and one has the following short exact sequence of sheaves on $\mathcal{Y}_T$:

$$0\longrightarrow\Omega^p_{\mathcal{Y}_T,\mathcal{X}_T}\longrightarrow\Omega^{p+1}_{\mathcal{Y}_{T'},\mathcal{X}_{T'}|\mathcal{Y}_T}\longrightarrow\Omega^{p+1}_{\mathcal{Y}_T,\mathcal{X}_T}\longrightarrow 0.$$

Restrict this sequence to Y_t, tensorise it with $\mathcal{O}(a)$ and consider the associated long exact sequence

$$\mathrm{H}^q(Y_t,\Omega^{p+1}_{\mathcal{Y}_T,\mathcal{X}_T|Y_t}\otimes\mathcal{O}(a)) \longrightarrow \mathrm{H}^{q+1}(Y_t,\Omega^{p}_{\mathcal{Y}_T,\mathcal{X}_T|Y_t}\otimes\mathcal{O}(a)) \longrightarrow \mathrm{H}^{q+1}(Y_t,\Omega^{p+1}_{\mathcal{Y}_{T'},\mathcal{X}_{T'}|Y_t}\otimes\mathcal{O}(a)).$$

The terms on the right and on the left vanish by assumption, hence the term in the middle vanishes as well. □

2. Proof of Theorem 2

Consider the exact sequence

$$R^k\pi_*\mathbb{C} \to R^k\pi_*(j_{T*}\mathbb{C}) \to R^k\pi_*(\mathbb{C}\to j_{T*}\mathbb{C}) \to R^{k+1}\pi_*\mathbb{C} \to R^{k+1}\pi_*(j_{T*}\mathbb{C})$$

and let $\mathcal{H}^k_{\mathcal{X}_T,v} = \operatorname{coker}\left(R^k\pi_*\mathbb{C}\to R^k\pi_*(j_{T*}\mathbb{C})\right)$ denote the locally trivial subsheaf of $R^k\pi_{Y*}(\mathbb{C}\to j_{T*}\mathbb{C})$ whose fiber at any point $t\in T$ is canonically isomorphic to $\mathrm{H}^k(X_t,\mathbb{C})_v$. Let $F^p\mathrm{H}^0\left(T,\mathcal{H}^k_{\mathcal{X}_T,v}\right)$ denote the space of sections of $\mathcal{H}^k_{\mathcal{X}_T,v}$ whose fiber at any point $t\in T$ belongs to $F^p\mathrm{H}^k(X_t,\mathbb{C})_v$.

In [O] I have shown

Theorem ([O]). *For all integers $b\in\mathbb{N}^*$ there is an integer D depending only on b, Y and on $\mathcal{O}(1)$, such that for all $d\geq D$, for all $q\in\{[(N+1)/2],\ldots,N-1\}$, for all*

$$c\leq b\frac{d^{N-q}}{(N-q)!}$$

for all codimension c analytic varieties $T\subset\mathbb{P}\mathrm{H}^0(Y,\mathcal{O}(d))$ lying in the locus parametrizing smooth hypersurfaces and for all $t\in T$, the following holds.

If N is even and $q=N/2$ then the space $F^{N-q}\mathrm{H}^0\left(T,\mathcal{H}^N_{\mathcal{X}_T,v}\right)$ is spanned by families of algebraic cycles of codimension q and degree at most b.

If $q>N/2$ then $F^{N-q}\mathrm{H}^0\left(T,\mathcal{H}^N_{\mathcal{X}_T,v}\right)=0$.

Since the statement is local-analytic, one can replace the analytic space $T\subset\mathbb{P}\mathrm{H}^0(Y,\mathcal{O}(d))$ by any algebraic variety T endowed with a morphism $\phi\colon T\to\mathbb{P}\mathrm{H}^0(Y,\mathcal{O}(d))$ such that

- $\phi(T)$ lies in the locus parametrizing smooth hypersurfaces and
- for all $t\in T$ the cokernel of the differential $d\phi(t)\colon T_{T,t}\to T_{\mathbb{P}\mathrm{H}^0(Y,\mathcal{O}(d)),\phi(t)}$ is of dimension at most c;

This stronger version of the theorem of [O] is equivalent to Theorem 2. Indeed, one only needs to show

$$F^{N-q}\mathrm{H}^N(\mathcal{X}_T,\mathbb{C})_v\simeq F^{N-q}\mathrm{H}^0\left(T,\mathcal{H}^N_{\mathcal{X}_T,v}\right).$$

But the Lefschetz hyperplane Theorem and the Leray spectral sequence give a canonical isomorphism

$$\mathrm{H}^N(\mathcal{X}_T,\mathbb{C})_v\simeq\mathrm{H}^0\left(T,\mathcal{H}^N_{\mathcal{X}_T,v}\right).$$

Since for any $t\in T$, the restriction map $\mathrm{H}^N(\mathcal{X}_T,\mathbb{C})_v\to\mathrm{H}^N(\mathcal{X}_t,\mathbb{C})_v$ is a homomorphism of Hodge structures, hence is strict, the result follows. □

3. Proof of Theorem 3

Let T be an open affine dense subset of the variety all smooth hypersurfaces containing V. The condition $q \geq N/2$ implies that for d large enough T is non–empty (*cf.* for example [K-A]). Hence its Zariski closure $\overline{T}$ in $\mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$ is the linear subspace of all hypersurfaces containing V, and is of codimension $\mathrm{h}^0(V, \mathcal{O}(d)) \sim_{d \to \infty} \frac{d^{N-q}}{(N-q)!}$. Hence $c \sim_{d \to \infty} \frac{d^{N-q}}{(N-q)!}$.

For all $i \in \{1, \ldots, p-q\}$ choose hyperplane sections $H_i \subset T$ in general position and and let $f_i \in \mathrm{H}^0(T, \mathcal{O}_T)$ be the equations of H_i; Let $P_i = \bigcap_{j>i} H_j$, $Q_i = \bigcup_{j \leq i} H_j$ and $T_i = P_i \setminus (P_i \cap Q_i)$. For $j \leq i$ the functions f_j do not vanish on T_i, hence belong to $\mathrm{H}^0(T_i, \mathcal{O}^*_{T_i})$ and there are well-defined higher cycles in $\mathrm{CH}^{q+i}(\mathcal{X}_{T_i}, i)$

$$Z_i = \psi^i_{1,1}([V \times T_i], f_1, \ldots, f_i).$$

I show by increasing induction on i that $\overline{\mathrm{reg}}_{q+i,i}(Z_i) \neq 0$. Case $i = p-q$ implies Theorem 3.

Case $i = 0$ a straightforward adaptation of [V2], Proposition 4.3: for $d \gg 0$ and for any open subset $U_0 \subset T_0$ the cohomology class of the cycle $[V \times U_0] \in \mathrm{CH}^q(\mathcal{X}_{U_0})$ in $\mathrm{H}^{2q}(\mathcal{X}_{U_0}, \mathbb{C})$ does not belong to $j^*_{U_0}(\mathrm{H}^{2q}(\mathcal{Y}_{U_0}, \mathbb{C}))$.

Assume $i > 0$. Then $T_{i-1} \cup T_i = P_i \setminus (P_i \cap Q_{i-1})$, hence T_{i-1} is a hyperplane section of $T_{i-1} \cup T_i$ and T_i is the complementary open set. Hence there is a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^{q+i}(\mathcal{X}_{T_i}, i) \arrow[r, "l_{q+i,i}"] \arrow[d, "\overline{\mathrm{reg}}_{q+i,i}"] & \mathrm{CH}^{q+i-1}(\mathcal{X}_{T_{i-1}}, i-1) \arrow[d, "\overline{\mathrm{reg}}_{q+i-1,i-1}"] \\
F^{q+i}\mathrm{H}^{2q+i+1}(\mathcal{Y}_{T_i}, \mathcal{X}_{T_i}, \mathbb{C}) \arrow[r, "\mathrm{res}"] & F^{q+i-1}\mathrm{H}^{2q+i}(\mathcal{Y}_{T_i}, \mathcal{X}_{T_{i-1}}, \mathbb{C})
\end{tikzcd}$$

where res is the residue map and $l_{q+i,i}$ is the linking map for the long exact sequence of higher Chow groups. One has

$$l_{q+i,i}(Z_i) = (l_{q,0} \circ \psi^i_{1,1})([V \times T_i], f_1, \ldots, f_i) = \psi^{i-1}_{1,1}([V \times T_{i-1}], f_1, \ldots, f_{i-1}) = Z_{i-1}.$$

The induction assumption $\overline{\mathrm{reg}}_{q+i-1,i-1}(Z_{i-1}) \neq 0$, and the commutativity of the diagram implies $\overline{\mathrm{reg}}_{q+i,i}(Z_i) \neq 0$. □

4. Proof of Theorem 4

4.1. **Construction of the cycle Z_U.** Let $P \subset Y$ be a codimension q linear subspace, let $R \subset Y$ be a codimension $q+2$ linear subspace such that $R \subset P$ and let S be the blow-up of P along R. There are natural maps $\psi \colon S \to \mathbb{P}^1_{\mathbb{C}}$ (where $\mathbb{P}^1_{\mathbb{C}}$ parametrizes hyperplanes of P containing R) and $s \colon S \to Y$. For any $x \in \mathbb{P}^1_{\mathbb{C}}$ let $L_x = s(\psi^{-1}(x))$ denote the codimension $q+1$ linear subspace of Y. For any subvariety $B \subset \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$ let $C_B = S \times_Y \mathcal{X}_B$ and let $\psi_B \colon C_B \to \mathbb{P}^1_{\mathbb{C}}$ and $s_B \colon C_B \to \mathcal{X}_B$ be the morphisms deduced from ψ and s by base change.

Let U, T, T' and T'' denote subvarieties of the locus of smooth hypersurfaces in $\mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$ defined as follows:

– $t \in U \iff [L_0 \cap X_t] = dR$ and $[L_\infty \cap X_t] = dR$;
– $t \in T \iff L_0 \subset X_t$ and $[L_\infty \cap X_t] = dR$;

– $t \in T' \iff L_\infty \subset X_t$ and $[L_0 \cap X_t] = dR$;
– $t \in T'' \iff L_0 \cup L_\infty \subset X_t$.

The condition $q \geq N/2$ implies that for d large enough these varieties are non–empty (*cf.* for example [K-A]). Let $\overline{U}$, $\overline{T}$, $\overline{T'}$ and $\overline{T''}$ denote their Zariski closures in $\mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$. Then $\overline{T}$ and $\overline{T'}$ are hyperplane sections of the projective space $\overline{U}$, and $\overline{T''} = \overline{T} \cap \overline{T'}$ is a linear subspace of codimension 2 in $\overline{U}$. Since $\operatorname{codim}(\overline{T''}, \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))) \sim_{d\to\infty} 2\frac{d^{N-q}}{(N-q)!}$ as in section 3, one has $\operatorname{codim}(U, \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))) \sim_{d\to\infty} 2\frac{d^{N-q}}{(N-q)!}$.

By construction, $s_U(\psi_U^{-1}(0)) = R \times U = s_U(\psi_U^{-1}(\infty))$, hence the couple (C_U, ψ_U) defines an element $Z_U \in \mathrm{CH}^{q+1}(\mathcal{X}_U, 1)$.

4.2. **Proof of $\overline{\mathrm{reg}}_{q+1,1}(Z_U) \neq 0$.** There is a natural linking homomorphism

$$l_{q+1,1} \colon \mathrm{CH}^{q+1}(\mathcal{X}_U, 1) \to \mathrm{CH}^q(\mathcal{X}_T).$$

Since C_T is the union of a variety dominating $\mathbb{P}^1_{\mathbb{C}}$ and of the variety $\psi_T^{-1}(0) \simeq L_0 \times T$, one has

$$l_{q+1,1}(Z_U) = d[s_T(\psi_T^{-1}(0))] = d[L_0 \times T].$$

I now proceed as in section 3. A straightforward adaptation of the argument of [V2], Proposition 4.2 shows that for $d \gg 0$ the class $\mathrm{cl}_q(d[L_0 \times T]) \in \mathrm{H}^{2q}(\mathcal{X}_T, \mathbb{C})$ does not belong to $j_T^* \mathrm{H}^{2q}(\mathcal{Y}_T, \mathbb{C})$, hence $\overline{\mathrm{cl}}_q(d[L_0 \times T]) \in \mathrm{H}^{2q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C})$ is non-zero. Since the diagram

$$\begin{array}{ccc}
\mathrm{CH}^{q+1}(\mathcal{X}_U, 1) & \xrightarrow{l_{q+1,1}} & \mathrm{CH}^q(\mathcal{X}_T) \\
\downarrow{\scriptstyle \overline{\mathrm{reg}}_{q+1,1}} & & \downarrow{\scriptstyle \overline{\mathrm{cl}}_q} \\
\mathrm{H}^{2q+2}(\mathcal{Y}_U, \mathcal{X}_U, \mathbb{C}) & \xrightarrow{\mathrm{res}} & \mathrm{H}^{2q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C})
\end{array}$$

is commutative, it follows that $\overline{\mathrm{reg}}_{q+1,1}(Z_U) \neq 0$.

4.3. **Indecomposability of the cycle Z_U.** Since $\overline{\mathrm{reg}}_{q+1,1}(Z_U) \neq 0$, it is enough to show that the space of decomposable cycles is contained in the kernel of $\overline{\mathrm{reg}}_{q+1,1}$.

Let $i \in \{1, \ldots, q\}$. One has to show

$$\mathrm{reg}_{q+1,1} \circ \phi_{i,0}(\mathrm{CH}^i(\mathcal{X}_U) \otimes \mathrm{CH}^{q+1-i}(\mathcal{X}_U, 1)) \subset j_U^* F^{q+1} \mathrm{H}^{2q+1}(\mathcal{Y}_U, \mathbb{C}).$$

Since the diagram

$$\begin{array}{ccc}
\mathrm{CH}^i(\mathcal{X}_U) \otimes \mathrm{CH}^{q+1-i}(\mathcal{X}_U, 1) & \xrightarrow{\phi_{i,0}} & \mathrm{CH}^{q+1}(\mathcal{X}_U, 1) \\
\downarrow{\scriptstyle \mathrm{cl}_i \otimes \mathrm{reg}_{q+1-i,1}} & & \downarrow{\scriptstyle \mathrm{reg}_{q+1,1}} \\
F^i \mathrm{H}^{2i}(\mathcal{X}_U, \mathbb{C}) \otimes F^{q-i+1} \mathrm{H}^{2q-2i+1}(\mathcal{X}_U, \mathbb{C}) & \xrightarrow{\smile} & F^{q+1} \mathrm{H}^{2q+1}(\mathcal{X}_U, \mathbb{C})
\end{array}$$

is commutative, it is enough to show

$$\begin{array}{rcl}
F^i \mathrm{H}^{2i}(\mathcal{X}_U, \mathbb{C}) & \simeq & F^i \mathrm{H}^{2i}(\mathcal{Y}_U, \mathbb{C}) \quad \text{and} \\
F^{q-i+1} \mathrm{H}^{2q-2i+1}(\mathcal{X}_U, \mathbb{C}) & \simeq & F^{q-i+1} \mathrm{H}^{2q-2i+1}(\mathcal{Y}_U, \mathbb{C}).
\end{array}$$

If $i < q$ these isomorphisms follow from Theorem 1.

If $i = q$ the second isomorphism is straightforward: one has

$$\mathrm{CH}^1(\mathcal{X}_U, 1) \simeq \mathrm{H}^0(U, \mathcal{O}^*(U)) \simeq \mathrm{CH}^1(\mathcal{Y}_U, 1).$$

It remains to show $F^q\mathrm{H}^{2q}(\mathcal{X}_U, \mathbb{C}) \simeq F^q\mathrm{H}^{2q}(\mathcal{Y}_U, \mathbb{C})$. By Theorem 2 it is enough to show that $\mathcal{X}_U$ does not contain any flat family of subvarieties of codimension q and degree 1 or 2.

Assume by contradiction that $\mathcal{X}_U$ contains such a family. Let $\mathcal{H}$ denote the space of all subvarieties of Y of pure codimension $q+1$ and degree 1 or 2 and, for a subvariety W of Y, let $T_W \subset \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))$ denote the family of degree d hypersurfaces of Y containing W. By assumption one has $U \subset \bigcup_{W\in\mathcal{H}} T_W$.

Choose $W \in \mathcal{H}$ such that T_W meets U and $\mathrm{codim}\,(T_W \cap U, U) \leq \dim \mathcal{H}$. One has $\mathrm{codim}\,(T_W \cap \overline{U}, \overline{U}) \leq \dim \mathcal{H}$, hence $\mathrm{codim}\,(T_W \cap \overline{T''}, \overline{U}) \leq \dim \mathcal{H} + 2$. But $\overline{T''} = T_{L_0 \cup L_\infty}$, hence $T_W \cap \overline{T''} = T_{W\cup L_0\cup L_\infty}$ and $\mathrm{codim}\,(T_{W\cup L_0\cup L_\infty}, \overline{U}) \leq \dim \mathcal{H} + 2$. Since $\dim \mathcal{H} + 2$ does not depend on d, this implies $\mathrm{codim}\,(T_{W\cup L_0\cup L_\infty}, \mathbb{P}\mathrm{H}^0(Y, \mathcal{O}(d))) \sim_{d\to\infty} 2\frac{d^{N-q}}{(N-q)!}$.

For d large enough this forces $\deg(W \cup L_0 \cup L_\infty) \leq 2$, hence $W \subset L_0 \cup L_\infty$. But the hypersurfaces parametrized by $T_W \cap U$ contain W and meet $L_0 \cup L_\infty$ at $R = L_0 \cap L_\infty$, hence $W \subset R$, contradicting $\mathrm{codim}\, R = q + 2$. □

4.4. **Indecomposability of the restriction of Z_U to a very general fiber $\mathcal{X}_u$, $u \in U$.** This follows from the indecomposability of Z_U by Proposition 4.5 of [V2].

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