NON-SIMPLICITY OF THE CREMONA GROUP, OVER ANY FIELD

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This is a translation of the v2 in arxiv of the paper "Non simplicité du groupe de Cremona sur tout corps" [Lon16], published in Annales de l'Institut Fourier (not the published version).

Abstract. Using a theorem of F. Dahmani, V. Guirardel and D. Osin we prove that the Cremona group in 2 dimension is not simple, over any field. More precisely, we show that some elements of this group satisfy a weakened WPD property which is equivalent in our particular context to the M. Bestvina and K. Fujiwara's one.

Introduction

Throughout the study of a group, an important question is to know if this group is simple or not. In the last case, the issue is to construct normal subgroups. These questions were asked at the end of the 19th century for the Cremona group $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$. It is the group of birational maps of the projective plane over a field k. Nevertheless, we had to wait 2013 so that S. Lamy and S. Cantat [CL13] answered it in the case where k is an algebraically closed field.

The Picard-Manin space associated to $\mathbb{P}^2_{\mathrm{k}}$ is the inductive limit of the Picard groups of surfaces obtained by blowing-up all finite sequences of points of $\mathbb{P}^2_{\mathrm{k}}$, infinitely near or not (see [Man86], [Can11] and paragraph §2.1). It is endowed with an intersection form of signature $(1, \infty)$. Considering an hyperboloid sheet, we can associate to it an infinite dimensional hyperbolic space, noted $\mathbb{H}_{\mathrm{k}}$. The S. Cantat and S. Lamy's strategy is to make the Cremona group acts by isometries on this hyperbolic space. In this way, they obtain a lot of normal subgroups. Their paper is divided in two parts.

In the first one, they define the notion of "tight element". We state it in the particular case of the Cremona group as follows: an element g in $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ is tight if the corresponding isometry is hyperbolic and satisfy two conditions:

(1) Its axis is rigid: for all $\varepsilon > 0$, there exists $C \geq 0$ such that if $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ satisfies $\mathrm{Diam}(\mathrm{Tube}_\varepsilon(\mathrm{Ax}(g)) \cap \mathrm{Tube}_\varepsilon(f\,\mathrm{Ax}(g))) \geq C$ then $f\,\mathrm{Ax}(g) = \mathrm{Ax}(g)$.
(2) For every $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$, if $f\,\mathrm{Ax}(g) = \mathrm{Ax}(g)$ then $fgf^{-1} = g^{\pm}$.

Their criterion permits them to establish a variant of the small cancellation property:

Theorem 1. [CL13] *Let* k *be an algebraically closed field. If* $g \in \mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ *is tight then there exists a non-zero integer* n *such that for all non-trivial element* h *belonging to the normal subgroup generated by* g^n*, the degree of* h *satisfies* $\deg(h) \geq \deg(g^n)$*. In particular, the normal subgroup* $\ll g^n \gg$ *of* $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ *is proper.*

The second part consists in showing that there exist tight elements in $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$. We come back to this at the end of this introduction.

Date: June 7, 2017.
2010 *Mathematics Subject Classification.* 14E07, 20F65.

The aim of this paper is to obtain a proof of the non-simplicity of the Cremona group which works for any field k:

Theorem 2. *The Cremona group* $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ *is not a simple group, over any field.*

To prove this theorem we will not use the fact that an element is tight but rather than it satisfies the WPD property ("weak proper discontinuity"). This property was proposed by M. Bestvina and K. Fujiwara [BF02] in 2002 in the context of the mapping class group. An element g satisfies the WPD property if for every $\varepsilon \geq 0$, it exists a point x and a positive integer n such that it exists only a finite number of elements in G moving x and $g^n(x)$ at most ε. The hyperbolic elements we are studying have an axis. Consequently we will use the R. Coulon's terminology introduced in his talk in the Bourbaki seminar [Cou14]: the group G acts discretely along the axis of g. The advantage of the R. Coulon's terminology is to make clear the role of the group G. Note that D. Osin [Osi16] includes this notion as well as other under the name of "acylindrical actions" [Osi16]. In this way, he unifies many works concerning different groups.

Recently, F. Dahmani, V. Guirardel and D. Osin [DGO14] also generalized the small cancellation theory for groups acting by isometries on δ-hyperbolic spaces. Recall that a geodesic metric space X is δ-hyperbolic if for each triangle of X, every edge is contained in the δ-neighborhood of the union of its two other edges. One of their motivations was to study the mapping class group of a hyperbolic Riemann surface. This group act on the curve complex which is a non locally compact δ-hyperbolic space (just like the space $\mathbb{H}^{\infty}_{\mathrm{k}}$ mentioned before). In this context, they built proper normal subgroups which are moreover free and purely pseudo-Anosov. The last property means that any non-trivial element is pseudo-Anosov. This answered two old open questions. We can see the link between the small cancellation theory and the WPD property through two statements from [Gui14, Theorem 1.3 and Corollary 2.9]. The first one says that in the normal group generated by a family satisfying the small cancellation property, elements have a large translation length. The second one says that if some element g satisfies the WPD property then the conjugates of $<g^n>$ form a family satisfying the small cancellation property. Combining these two statements (see also [DGO14, Theorem 5.3, Proposition 6.34]), we obtain:

Theorem 3 ([DGO14])**.** *Let C be a positive real number. Let G be a group acting by isometries on a δ-hyperbolic space X and let g be a hyperbolic element belonging to G. If G acts discretely along the axis of g then there exists $n \in \mathbb{N}$ such that for every non-trivial element h belonging to the normal subgroup generated by g^n, $L(h) > C$ where L is the translation length. In particular, for n big enough, the normal subgroup $\ll g^n \gg$ of G is proper. Moreover this subgroup is free.*

Using this theorem, the proof of the theorem 2 is reduced to exhibit some elements satisfying the WPD property. We find such elements in $\mathrm{Aut}(\mathbb{A}^2_{\mathrm{k}})$ which is identified to a subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ via the map sending $(x, y) \in \mathbb{A}^2_{\mathrm{k}}$ to $[x : y : 1] \in \mathbb{P}^2_{\mathrm{k}}$. We make the group $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ act on $\mathbb{H}_{\bar{\mathrm{k}}}$. The main result of this paper is:

Proposition 4. *Let $n \geq 2$ and let k be a field of characteristic which does not divide n. We consider the action of the group $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ on $\mathbb{H}_{\bar{\mathrm{k}}}$ where $\bar{\mathrm{k}}$ is the algebraic closure of k. The group $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ acts discretely along the axis of the map:*

$$h_n : \begin{array}{ccc} \mathbb{A}^2_{\mathrm{k}} & \longrightarrow & \mathbb{A}^2_{\mathrm{k}} \\ (x, y) & \longmapsto & (y, y^n - x) \end{array}.$$

Remark that if k is an algebraically closed field of characteristic $p > 0$, for any integer $k \geq 1$, the normal subgroup generated by h_p^k is the whole group $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$. Indeed, h_p normalizes translations:

$$(x^p - y, x) \circ (x + a, y + b) \circ (y, y^p - x) = (x + a^p - b, y + a).$$

We obtain that a non-trivial translation belongs to $\ll h_p^k \gg$. The Noether theorem permits us to show the equality $\ll h_p^k \gg = \mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$, see for example [CD13, Proposition 5.12]. More generally, over an infinite field of characteristic which does not divide n, the map h_n doesn't satisfy the WPD property. This explains the hypotheses of the statement of Proposition 4.

In fact, as a consequence of results of F. Dahmani, V. Guirardel et D. Osin, we get not only the non-simplicity of the group but also:

Theorem 5. *Let k be a field. The group $\mathrm{Bir}(\mathbb{P}^2_k)$ contains free normal subgroups, and it is SQ-universal.*

Recall that a group G is SQ-universal if every countable subgroup embeds in a quotient of G More details can be found in [Gui14, Theorem 2.14].

We finish this introduction by comparing our proof of the theorem 2 to recent results of [SB15] and to the strategy of [CL13].

In his article, N.I. Shepherd-Barron [SB15, Corollary 7.11] proved that any hyperbolic element in the Cremona group over any finite field k generates a proper normal subgroup. In particular, $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ is not a simple group over any finite field k. In the same paper (Theorem 7.6), he gives a criterion in terms of the translation length of a hyperbolic transformation g, to know if g is tight, and thus if the normal subgroup generated by one of its powers is proper.

Theorem 6. [SB15] *Let k be a field of characteristic zero or a field of characteristic $p > 0$ and algebraic over $\mathbb{F}_p$. If the translation lenght of a hyperbolic element g in $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ is not the logarithm of a quadratic unit neither of the form $\log p^n$ when the characteristic is strictly positive, then some power of g is tight.*

In order to avoid the above problem about transformations which normalize the translations subgroup, he makes some assumption on the field. It seems to be excessive in the case of the positive characteristic (k has to be isomorphic to a subfield of $\bar{F_p}$). However, even if we succeed to avoid this assumption and obtain in this way a different demonstration of the non-simplicity of $\mathrm{Bir}(\mathbb{P}^2)$ over any field, such a proof is not elementary because it's based on the papers [CL13] and [BC16].

In this last paper, J. Blanc et S. Cantat are interested in the dynamical degree of birational maps of projective surface. They prove in particular that there is no dynamical degree in $]1, \lambda_L[$ where λ_L is the Lehmer number ("gap property"). A corollary of this is that for any hyperbolic element g in the Cremona group, the index of $<g>$ in its centralizer is finite. Following a remark by R. Coulon [Cou14], this implies that if an element is tight then $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ acts discretely along this element's axis.

In [CL13] a relation between the integer n of the theorem 1 and the translation lenght of g is given. An other consequence of the "gap property" is the integer n can be choosen uniformely : $n \geq \max\{\frac{139347}{\lambda_L}, \frac{10795}{\lambda_L} + 374\}$ works.

In their paper [CL13], they exhibit, in two different ways, tight elements according whether the field is $\mathbb{C}$ or if it is only algebraically closed.

In the algebraically closed field case, they blow-up specific points in $\mathbb{P}^2_{\mathrm{k}}$ to obtain a surface S with large dynamical degree automorphisms. For example, blowing-up the 10 double-points of a rational sextic gives us such a surface, called Coble surface. If the field is not algebraically closed, the automorphisms' coefficients are in the algebraic closure but not in the ground field k. Consequently their proof can't be extended in this case.

If the field is $\mathbb{C}$, they consider a "general" element in the Cremona group writing $g = a \circ J$ with $a \in \mathrm{PGL}_3(\mathbb{C})$ and J a Jonquières transformation. They show that such elements are tight. The word "general" means that we can choose any element $a \in \mathrm{PGL}_3(\mathbb{C})$ after removing a countable number of Zariski closed sets in $\mathrm{PGL}_3(\mathbb{C})$. So, if the ground field is countable it's possible that no general element exists. However, it's this method which is generalized here. Indeed, in their proof, the birational map g has to satisfy two conditions. First, the base points of g and g^{-1} have to be in $\mathbb{P}^2_{\mathrm{k}}$. Second, the set of the base points of g's iterates have to be disjoint from the one of g^{-1}'s iterates. The birational maps h_n, which we are focused on, are the composition between a Jonquières involution and the linear involution which exchanges the coordinates, $h_n = (y, x) \circ (y^n - x, y)$. The first condition is not satisfied because h_n et h_n^{-1} have only one base point in $\mathbb{P}^2_{\mathrm{k}}$. But we will see that it's not a problem. Concerning the second one, to compose by the element $(y, x) \in \mathrm{PGL}_3(\mathrm{k})$ permit us to separate the base points of the h_n's iterates from the ones of the h_n^{-1}'s iterates. In this way, this specific element plays the same role than the above general element a. The proof given in this paper is not based on [BC16, CL13] with the exception of Lemma 18 in Section 2.3 which is a direct adaption of [CL13, Proposition 5.7].

The paper is organized as follows. In Section 1, we recall the construction of an infinite dimensional hyperbolic space. Then we introduce the notion of "geodesic tube". This permits us among other things to bypass [CL13, Lemma 3.1]. The main result of this section is the weakening of the hypotheses in the WPD property in the context of a group acting on an infinite dimensional hyperbolic space. In Section 2, we recall the Picard-Manin space definition together with the construction of the associated hyperbolic space $\mathbb{H}^{\infty}_{\bar{\mathrm{k}}}$. We use the action of $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ on $\mathbb{H}^{\infty}_{\bar{\mathrm{k}}}$ to exhibit elements satisfying the WPD property which we have just weakened. So that gives us a proof of Proposition 4 and consequently a proof of the non-simplicity of the Cremona group, over any field.

Acknowledgements

The author wishes to thank warmly her advisor, Stéphane Lamy, for suggesting the problem. The author wants also to thank him for his many re-readings. Thank you to te reviewer for his comments which permit to make more clear certain parts.

1. The WPD property in an infinite dimensional hyperbolic space

This section is devoted to show that in the context of a group acting on a hyperbolic space of infinite dimension, we can weaken the assertion of the WPD property.

1.1. Infinite hyperbolic space.

Here, we recall the construction of hyperbolic spaces of infinite dimension. Let H be an infinite real Hilbert space with a symmetric bilinear form $\mathcal{B}$ of signature $(1, \infty)$. Fix $u \in H$ such that $\mathcal{B}(u, u) = 1$. Let $\mathbb{H}^{\infty}$ denote the hyperboloid sheet given by:

$$\mathbb{H}^{\infty} := \{x \in H \mid \mathcal{B}(x, x) = 1 \text{ and } \mathcal{B}(u, x) > 0\}.$$

The space $\mathbb{H}^{\infty}$ endowed with the distance d defined as $\cosh \mathrm{d}(x, y) := \mathcal{B}(x, y)$ is a complete metric space of infinite dimension. Note that if the intersection between $\mathbb{H}^{\infty}$ and a $n+1$ vector subspace of H is non-empty then it is isometric to the usual hyperbolic space $\mathbb{H}^n$. In particular, there exists a unique geodesic between two points of $\mathbb{H}^{\infty}$. It's obtained as the intersection of $\mathbb{H}^{\infty}$ with the vector plane containing these two points. In practice, we often work on $\mathbb{H}^2$ by taking a hyperbolic plane of $\mathbb{H}^{\infty}$. Thus, each triangle of $\mathbb{H}^{\infty}$ is isometric to a triangle of $\mathbb{H}^2$. This implies that $\mathbb{H}^{\infty}$ is $\mathrm{CAT}(-1)$ and δ-hyperbolic for the same constant $\delta = \log(1 + \sqrt{2})$ than $\mathbb{H}^2$ (see [Cal07, Example 1.23]). From now on, when we will speak about a hyperbolic space $\mathbb{H}^{\infty}$, it will be always of this kind.

Now, let us introduce some definitions and some notation. Let f be an isometry of $\mathbb{H}^{\infty}$, its translation length is defined as $L(f) = \inf\limits_{x \in \mathbb{H}^{\infty}} \mathrm{d}(x, f(x))$. If the translation length of f is positive and achieved, we say that f is hyperbolic. In this case, it has an invariant axis given by points realizing the infimum:

$$\mathrm{Ax}(f) := \{x \in \mathbb{H}^\infty \mid \mathrm{d}(x, f(x)) = L(f)\}.$$

Moreover, f uniquely extends to the boundary $\partial\mathbb{H}^\infty$. For more details about the boundary, we refer to [BH99]. The hyperbolic isometry f has exactly two fixed points on $\partial\mathbb{H}^\infty$. One is repulsive, denoted by b^-, and the other one is attractive, denoted by b^+. They are the endpoints of the axis of f. We put an orientation on this axis from b^- to b^+. It provides an order relation on points of $\mathrm{Ax}(f)$. The point x is said to be smaller than y, denoted by $x < y$ if $x \in]b^-, y]$ (and by symmetry $y \in [x, b^+[$). For $x \in \mathrm{Ax}(f)$, we set $x - \varepsilon$ and $x + \varepsilon$ the two points on $\mathrm{Ax}(f)$ located at distance ε from x such that $x - \varepsilon < x + \varepsilon$. We remark that for any point $x \in \mathbb{H}^\infty$, the sequence $(f^{\pm n}(x))_{n\in\mathbb{N}}$ converges to $b^\pm$.

1.2. **The WPD property.** Let G be a group acting by isometries on a geodesic metric space (X, d). For any subset A of X and for any constant $\varepsilon \geq 0$, we set

$$\mathrm{Stab}_\varepsilon A := \{g \in G \mid \mathrm{d}(a, ga) \leq \varepsilon \quad \forall a \in A\},$$

the pointwise stabilizer of A by G up to ε. From now on, all group actions will be by isometries. The following lemma is well-known.

Lemma 7. *Let G be a group acting on a metric space X and g be an element of G. The two following properties are equivalent:*

(1) There exists $x \in X$ such that for all $\varepsilon \geq 0$, there exists $N \in \mathbb{N}$ such that $\mathrm{Stab}_\varepsilon\{x, g^N(x)\}$ is finite.
(2) For all $y \in X$, for all $\varepsilon \geq 0$, there exists $N \in \mathbb{N}$ such that $\mathrm{Stab}_\varepsilon\{y, g^N(y)\}$ is finite.

Proof. Let x satisfy (1), y an arbitrary point of X and $\varepsilon \geq 0$. Set $\varepsilon' = 2\,\mathrm{d}(x, y) + \varepsilon$. By the triangle inequality, we have the inclusion:

$$\mathrm{Stab}_\varepsilon\{y, g^N(y)\} \subset \mathrm{Stab}_{\varepsilon'}\{x, g^N(x)\}.$$

Thus, for N big enough, the set $\mathrm{Stab}_\varepsilon\{y, g^N(y)\}$ is finite. □

According to M. Bestvina and K. Fujiwara, we say in the lemma's situation that g *satisfies the WPD property* ("weak proper discontinuity"). As we mentioned in the introduction, we will rather use Coulon's terminology, introduced more recently, which is that the group G *acts discretely along the axis of* g.

In the following paragraphs we recall relations in hyperbolic quadrilaterals, then we introduce the notion of "geodesic tubes". This allows us to weaken the hypothesis in the WPD property in the case of a group acting on $\mathbb{H}^\infty$.

1.3. **Hyperbolic quadrilaterals.** Here, we prove a lemma about trigonometric relations for hyperbolic quadrilaterals with three right angles. First, we recall the hyperbolic relations in right triangles, see for example [Bea95, p.147, Theorem 7.11.2].

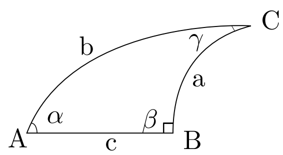

FIGURE 1. Triangle ABC.

Lemma 8. *Let ABC be a hyperbolic right triangle at B. We denote respectively α, β and γ the angles at A, B and C and a, b and c the respective lengths of opposite sides (see figure 1). We have the two following relations:*

$$\cos\gamma = \frac{\tanh a}{\tanh b} \tag{1}$$

$$\tan\gamma = \frac{\tanh c}{\sinh a}. \tag{2}$$

Lemma 9. *Let $ADCB$ be a quadrilateral of $\mathbb{H}^2$ with right angles at B, C and D. We have the relation:*

$$\tanh \mathrm{d}(A,B) = \tanh \mathrm{d}(D,C)\cosh \mathrm{d}(C,B).$$

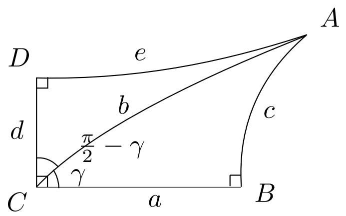

FIGURE 2. Quadrilateral ADCB.

Proof. In the right triangle ABC, we denote by a, b, c the opposite lengths from the vertices A, B, C, and γ the angle at C, and in the triangle ACD, d, b and e, the opposite lengths from vertices A, D and C, and ε the angle at C (see figure 2). Relation (1) of Lemma 8 applied to the right triangles ABC and ACD gives us:

$$\cos\gamma = \frac{\tanh a}{\tanh b} \quad \text{and} \quad \cos\varepsilon = \frac{\tanh d}{\tanh b}.$$

Moreover, we have $\cos\varepsilon = \sin\gamma$ because $\varepsilon+\gamma = \frac{\pi}{2}$. On the other hand, relation (2) of Lemma 8 applied to the triangle ABC implies that $\tan\gamma = \frac{\tanh c}{\sinh a}$. Finally, we obtain:

$$\tanh c = \tanh d \cosh a,$$

which is the desired equality. □

1.4. **Geodesic tubes.** Let us consider a group G acting on $(\mathbb{H}^\infty, \mathrm{d})$, and Γ a geodesic in $\mathbb{H}^\infty$. Now, we introduce a notion which will be central in the proof of the main proposition of this section (Proposition 11), the one of *geodesic tube* around Γ. The word "geodesic" is employed to emphasize the fact that if we consider the intersection between this tube and a hyperbolic plane $\mathcal{P}$ containing Γ, the edges of the resulting quadrilateral are geodesic segments. Fix x and y in $\mathbb{H}^\infty$ and let Γ be the geodesic joining these two points. We denote by pr_Γ the projection sending any point in $\mathbb{H}^\infty$ on its closest point in Γ. The orthogonal hyperplane to Γ at the point x is

$$\Gamma_x^\perp := \{z \in \mathbb{H}^\infty \mid \mathrm{pr}_\Gamma\, z = x\}$$

We define the *tube* $\mathrm{T}^\varepsilon_{x,x'}$ to be the convex hull of the union of the two closed sets $\bar{B}(x,\varepsilon)\cap\Gamma_x^\perp$ and $\bar{B}(x',\varepsilon)\cap\Gamma_{x'}^\perp$ (see figure 3, note that they are solid tubes). The *radius* of this tube at z, denoted by $\mathrm{r}^\varepsilon_{x,x'}(z)$, is given by

$$\mathrm{r}^\varepsilon_{x,x'}(z) := \sup\{\mathrm{d}(z,u) \mid u \in \Gamma_z^\perp \cap \mathrm{T}^\varepsilon_{x,x'}\}.$$

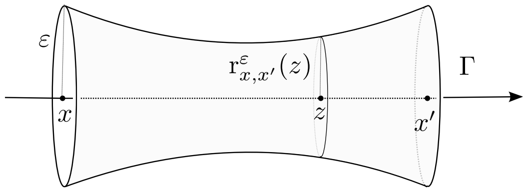

FIGURE 3. Tube $\mathrm{T}^{\varepsilon}_{x,x'}$.

Let $\mathrm{T}^{\varepsilon_1}_{x,x'}$ and $\mathrm{T}^{\varepsilon_2}_{y,y'}$ be two tubes centered around the same geodesic Γ. We say that the tube $\mathrm{T}^{\varepsilon_1}_{x,x'}$ *goes through* $\mathrm{T}^{\varepsilon_2}_{y,y'}$ if for all $z \in [x,x'] \cap [y,y']$, $r^{\varepsilon_1}_{x,x'}(z) \leq r^{\varepsilon_2}_{y,y'}(z)$ (see figure 4). Let us

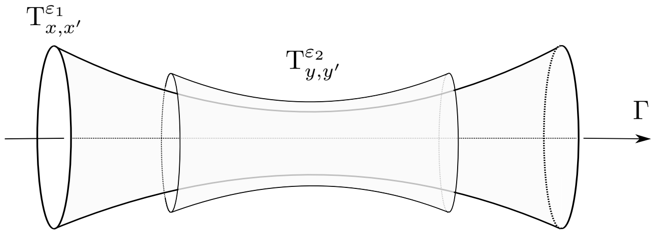

FIGURE 4. Tube $\mathrm{T}^{\varepsilon_1}_{x,x'}$ going through the tube $\mathrm{T}^{\varepsilon_2}_{y,y'}$.

remark that we only need to check that the previous inequality holds for the two endpoints b_1 and b_2 of $[x,x'] \cap [y,y']$. Indeed, assume that $r^{\varepsilon_1}_{x,x'}(b_1) \leq r^{\varepsilon_2}_{y,y'}(b_1)$ and $r^{\varepsilon_1}_{x,x'}(b_2) \leq r^{\varepsilon_2}_{y,y'}(b_2)$. Consider a plane $\mathcal{P}$ containing the geodesic Γ. We denote by $\Gamma_{x,x'}$ and $\Gamma_{y,y'}$ the geodesic segments parallel to Γ and obtained as the traces of the tubes $\mathrm{T}^{\varepsilon_1}_{x,x'}$ and $\mathrm{T}^{\varepsilon_2}_{y,y'}$ contained in a same half-plane delimited by Γ (two choices are possible). If the geodesics $\Gamma_{x,x'}$ and $\Gamma_{y,y'}$ have one point of intersection then they have at least two. By uniqueness of the geodesic between two given points, this is impossible. Consequently, the geodesic $\Gamma_{y,y'}$ lies above the geodesic $\Gamma_{x,x'}$.

The space $\mathbb{H}^\infty$ is CAT(-1) and so CAT(0). This implies that the distance is convex. This means (see [BH99, p.176]) that for all geodesic segments $[a,b]$ and $[c,d]$ and for all points $z \in [a,b]$ and $z' \in [c,d]$ with same barycentric coordinates, we have $\mathrm{d}(z,z') \leq (1-t)\,\mathrm{d}(a,c) + t\,\mathrm{d}(b,d)$. We recall that z and z' having same barycentric coordinates means that there exists $t \in [0,1]$ such that $\mathrm{d}(a,z) = t\,\mathrm{d}(a,b)$ and $\mathrm{d}(c,z') = t\,\mathrm{d}(c,d)$. Convexity of the distance will be used as follows.

Lemma 10. *Let h be a hyperbolic element of G acting on $\mathbb{H}^\infty$. Let $x, x', y, y' \in \mathrm{Ax}(h)$ satisfy $x \leq y < y' \leq x'$. For every $\varepsilon \geq 0$ we have:*

(1) $\mathrm{Stab}_\varepsilon\{x,x'\} \subset \mathrm{Stab}_\varepsilon\{y,y'\}$.
(2) The tube $\mathrm{T}^{\varepsilon}_{x,x'}$ goes through the tube $\mathrm{T}^{\varepsilon}_{y,y'}$.

Proof. Let us put $t := \frac{\mathrm{d}(x,y)}{\mathrm{d}(x,x')}$. The two points are proved by the same argument.

(1) Let $g \in \mathrm{Stab}_\varepsilon\{x,x'\}$. Then by convexity of the distance and by the isometric action of g, we have:

$$\mathrm{d}(y, g\cdot y) \leq t\,\mathrm{d}(x', g\cdot x') + (1-t)\,\mathrm{d}(x, g\cdot x) \leq \varepsilon.$$

Doing the same for y', we obtain the result.

(2) Using the remark after the definition of the tube's radius, it is sufficient to show that $r^{\varepsilon}_{x,x'}(y) \leq r^{\varepsilon}_{y,y'}(y) = \varepsilon$ and $r^{\varepsilon}_{x,x'}(y') \leq r^{\varepsilon}_{y,y'}(y') = \varepsilon$. Let us prove the first inequality, the

second one can be proven in the same way. Fix a plane containing the segment $[x,x']$. By the same argument given at point (1), we have:

$$\mathrm{r}^{\varepsilon}_{x,x'}(y) \leq t\, \mathrm{r}^{\varepsilon}_{x,x'}(x') + (1-t)\, \mathrm{r}^{\varepsilon}_{x,x'}(x) \leq \varepsilon. \qquad \square$$

1.5. **Weakening of the WPD property's assertion.** Now, we can weaken the assertion of the WPD property. The remainder of this subsection will be devoted to the proof of the next proposition.

Proposition 11. *Let G be a group acting on the hyperbolic space $\mathbb{H}^\infty$ and h be a hyperbolic element in G. The two following properties are equivalent:*

(1) There exist $x \in \mathrm{Ax}(h)$, $\varepsilon_0 > 0$ and $n, k \in \mathbb{N}$ such that the set $\mathrm{Stab}_{\varepsilon_0}\{h^{-k}(x), h^n(x)\}$ is finite.

(2) There exists $w \in \mathrm{Ax}(h)$ such that for all $\varepsilon \geq 0$, there exists $M \in \mathbb{N}$ such that the set $\mathrm{Stab}_{\varepsilon}\{w, h^M(w)\}$ is finite.

Point (2) corresponds to the definition given in Lemma 7.(1). Point (1) is the weaken version which will be used in Section 2. The proposition is proved by direct consequence of the two following lemmas.

Lemma 12. *Let G be a group acting on $\mathbb{H}^\infty$ and Γ be a geodesic in $\mathbb{H}^\infty$. For all constants $\varepsilon \geq 0$, $\eta > 0$ and for all $z, z' \in \Gamma$ with $z < z'$, there exist $x, x' \in \Gamma$ with $x \leq z < z' \leq x'$ such that for all $y, y' \in \Gamma$ with $y \leq x$ and $y' \geq x'$, the tube $\mathrm{T}^{\varepsilon}_{y,y'}$ goes through the tube $\mathrm{T}^{\eta}_{z,z'}$.*

Proof. If $\eta \geq \varepsilon$, we can set $x = z$ and $x' = z'$ (see Lemma 10.(2)).

Now, if $\eta < \varepsilon$, we want to find two points x and x' in Γ with $x \leq z$ and $x' \geq z'$ such that $\mathrm{r}^{\varepsilon}_{x,x'}(z) \leq \eta$ and $\mathrm{r}^{\varepsilon}_{x,x'}(z') \leq \eta$. For every $w \in [z,z']$, consider two points $x, x' \in \Gamma$ with $x \leq z$ and $x' \geq z'$, such that w is the midpoint of $[x,x']$. By symmetry, we can consider a plane $\mathcal{P}$ containing the geodesic Γ. In this plane, the trace of the tube $\mathrm{T}^{\varepsilon}_{x,x'}$ is a quadrilateral. We denote by x_1 and x'_1 two vertices of this quadrilateral located in a same half-plane delimited by Γ (see figure 5). Thus we have $\mathrm{d}(x_1,x) = \mathrm{d}(x'_1,x') = \varepsilon$.

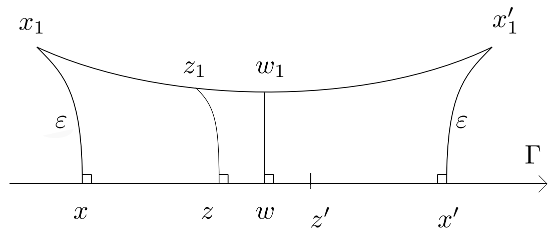

FIGURE 5. The trace of the tube $\mathrm{T}^{\varepsilon}_{x,x'}$ in the plane $\mathcal{P}$.

Let w_1 be the midpoint of $[x_1,x'_1]$. By symmetry, the geodesic going through the points w and w_1 is the geodesic that is orthogonal to the geodesics $[x_1,x'_1]$ and Γ. We call z_1 the point in $[x_1,w_1]$ satisfying $\mathrm{pr}_\Gamma\, z_1 = z$. The quadrilaterals z_1w_1wz and x_1w_1wx have three right angles respectively at w_1, w, z and at w_1, w, x. Lemma 9 gives us:

$$\begin{cases} \tanh \mathrm{d}(w,w_1) \cosh \mathrm{d}(w,z) = \tanh \mathrm{d}(z_1,z) \\ \tanh \mathrm{d}(w,w_1) \cosh \mathrm{d}(w,x) = \tanh \mathrm{d}(x_1,x) = \tanh \varepsilon. \end{cases}$$

Combining the previous equalities and doing the same for the point x' leads us to:

$$\frac{\tanh \varepsilon \cosh \mathrm{d}(w,z)}{\cosh \mathrm{d}(w,x)} = \tanh \mathrm{d}(z,z_1) \quad \text{and} \quad \frac{\tanh \varepsilon \cosh \mathrm{d}(w,z')}{\cosh \mathrm{d}(w,x')} = \tanh \mathrm{d}(z',z'_1).$$

We want to choose x and x' to obtain the upper-bounds $\mathrm{d}(z,z_1)\leq\eta$ and $\mathrm{d}(z',z'_1)\leq\eta$. For $K>0$, the map $t\mapsto\frac{K}{\cosh t}$ is decreasing on $\mathbb{R}_+$. Fix

$$K=\max(\tanh\varepsilon\cosh\mathrm{d}(w,z),\tanh\varepsilon\cosh\mathrm{d}(w,z')),$$

there exists t_0 such that $\frac{K}{\cosh t_0}\leq\tanh\eta$. We choose x and x' such that $\mathrm{d}(w,x)=\mathrm{d}(w,x')\geq t_0$. We obtain:

$$\frac{\tanh\varepsilon\cosh\mathrm{d}(w,z)}{\cosh\mathrm{d}(w,x)}\leq\tanh\eta\quad\text{and}\quad\frac{\tanh\varepsilon\cosh\mathrm{d}(w,z')}{\cosh\mathrm{d}(w,x')}\leq\tanh\eta,$$

that gives us the expected upper-bounds. □

Lemma 13. *Let G be a group acting on $\mathbb{H}^\infty$ and h be a hyperbolic isometry in G. For all constants $\varepsilon\geq0$, $\eta>0$, for all points $z,z'\in\mathrm{Ax}(h)$ with $z<z'$, for all $w\in[z,z']$, there exist $m,k\in\mathbb{N}$ such that if the set $\mathrm{Stab}_\eta\{z,z'\}$ is finite then $\mathrm{Stab}_\varepsilon\{h^{-k}(w),h^m(w)\}$ is finite too.*

Proof. Let $\varepsilon\geq0$, z and z' be two points on the axis of h and w be a point in $[z,z']$. Lemma 12 gives us two points x and x' on $\mathrm{Ax}(h)$ with $x\leq z<z'\leq x'$ such that for all $y,y'\in\mathrm{Ax}(h)$ satisfying $y\leq x$ and $y'\geq x'$, the tube $\mathrm{T}^\varepsilon_{y,y'}$ goes through the tube $\mathrm{T}^{\eta/3}_{z-\varepsilon,z'+\varepsilon}$. Choosing two non-negative integers m and k big enough such that $h^{-k}(w)+\varepsilon\leq x$ and $h^m(w)-\varepsilon\geq x'$, we obtain that the tube $\mathrm{T}^\varepsilon_{h^{-k}(w)+\varepsilon,h^m(w)-\varepsilon}$ goes through $\mathrm{T}^{\eta/3}_{z-\varepsilon,z'+\varepsilon}$(see figure 6).

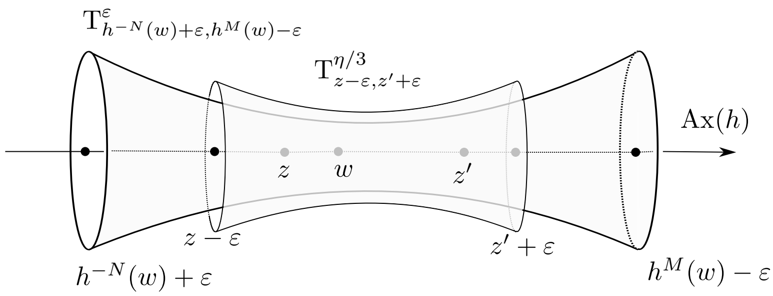

FIGURE 6. Tube $\mathrm{T}^\varepsilon_{h^{-k}(w)+\varepsilon,h^m(w)-\varepsilon}$ going through the tube $\mathrm{T}^{\eta/3}_{z-\varepsilon,z'+\varepsilon}$.

Suppose that the set $\mathrm{Stab}_\varepsilon\{h^{-k}(w),h^m(w)\}$ is infinite. There exists a sequence $(f_n)_{n\in\mathbb{N}}$ of pairwise different elements of $\mathrm{Stab}_\varepsilon\{h^{-k}(w),h^m(w)\}$. This implies that for every n, we have:

$$[f_n(h^{-k}(w)),f_n(h^m(w))]\subset\mathrm{T}^\varepsilon_{\mathrm{pr}_h f_n(h^{-k}(w)),\mathrm{pr}_h f_n(h^m(w))}.$$

Applying Lemma 10.(2) at the four points on the axis of h: $\mathrm{pr}_h f_n(h^{-k}(w))\leq h^{-k}(w)+\varepsilon$ and $\mathrm{pr}_h f_n(h^m(w))\geq h^m(w)-\varepsilon$, we get that the tube $\mathrm{T}^\varepsilon_{\mathrm{pr}_h f_n(h^{-k}(w)),\mathrm{pr}_h f_n(h^m(w))}$ goes through the tube $\mathrm{T}^\varepsilon_{h^{-k}(w)+\varepsilon,h^m(w)-\varepsilon}$ which, by construction, goes through the tube $\mathrm{T}^{\eta/3}_{z-\varepsilon,z'+\varepsilon}$. Moreover, Lemma 10.(1) applied to the points $h^{-k}(w)<z<z'<h^m(w)$, gives us the inclusion:

$$\mathrm{Stab}_\varepsilon\{h^{-k}(w),h^m(w)\}\subset\mathrm{Stab}_\varepsilon\{z,z'\}.$$

Consequently, for all integers n, the points $f_n(z)$ and $f_n(z')$ belong respectively to the closed balls $\bar{B}(z,\varepsilon)$ and $\bar{B}(z',\varepsilon)$. Denote by $z_n:=\mathrm{pr}_h f_n(z)$ and $z'_n:=\mathrm{pr}_h f_n(z')$, the respective projections on the $\mathrm{Ax}(h)$ of the points $f_n(z)$ and $f_n(z')$. In particular, $z_n,z'_n\in[z-\varepsilon,z'+\varepsilon]$ thus:

$$\mathrm{d}(z_n,f_n(z))\leq\frac{\eta}{3}\text{ and }\mathrm{d}(z'_n,f_n(z'))\leq\frac{\eta}{3}.$$

On another side, the sequences $(z_n)_{n\in\mathbb{N}}$ and $(z'_n)_{n\in\mathbb{N}}$ belong to the compact $[z-\varepsilon,z'+\varepsilon]$ so considering a subsequence, we can suppose that the two sequences are convergent Cauchy. So, there exists a non-negative integer N_0 such that for all $k \geq 0$, $\mathrm{d}(z_{N_0}, z_{N_0+k}) \leq \frac{\eta}{3}$ and $\mathrm{d}(z'_{N_0}, z'_{N_0+k}) \leq \frac{\eta}{3}$. Consequently, we have for all $n \geq N_0$:

$$\begin{cases} \mathrm{d}(f_{N_0}(z), f_n(z)) \leq \mathrm{d}(f_{N_0}(z), z_{N_0}) + \mathrm{d}(z_{N_0}, z_n) + \mathrm{d}(z_n, f_n(z)) \leq \eta \\ \mathrm{d}(f_{N_0}(z'), f_n(z')) \leq \mathrm{d}(f_{N_0}(z'), z'_{N_0}) + \mathrm{d}(z'_{N_0}, z'_n) + \mathrm{d}(z'_n, f_n(z)') \leq \eta \end{cases}.$$

Finally, the sequence $(f_{N_0}^{-1} f_n)_{n \geq N_0}$ is contained in $\mathrm{Stab}_\eta\{z, z'\}$. This allows us to conclude that the set $\mathrm{Stab}_\eta\{z, z'\}$ is infinite. □

2. Application to the Cremona group

Let k be a field and $\bar{\mathrm{k}}$ be its algebraic closure. In this section, we recall the Picard-Manin space construction over $\bar{\mathrm{k}}$. More precisely, the one of a Picard-Mainin's subspace which is isometric to hyperbolic spaces of infinite dimension constructed in Section 1. So using the weakened version of the WPD property established in Section 1.5, we will show that the group $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ acts discretely along the axis of some of its elements.

2.1. **The action of the Cremona group on the Picard-Manin space.** Now, we work over $\bar{\mathrm{k}}$. Let us recall briefly the construction of the Picard-Manin space and the action of the group $\mathrm{Bir}(\mathbb{P}^2_{\bar{\mathrm{k}}})$ on it. We can find more details in [CL13] and [Can11]. Let S be a smooth projective surface. The Néron-Severi group $\mathrm{N}^1(S)$ associated to S is the group of divisors on S with real coefficients up to numerical equivalence. Consider the inductive limit of the Néron-Severi groups of surfaces S' obtained as bow-up of S:

$$\mathcal{Z}_C(S) = \varinjlim_{S' \to S} \mathrm{N}^1(S'),$$

called sometimes the space of Cartier b-divisors. If we consider the projective limit rather than the inductive limit, we obtain the space of Weil b-divisors. We will not use this point of view.

We denote by $L \in \mathrm{N}^1(\mathbb{P}^2_{\bar{\mathrm{k}}})$ the class of a line in $\mathbb{P}^2_{\bar{\mathrm{k}}}$, S_p the surface obtained as the blow-up of p of an other surface S and $E_p \in \mathrm{N}^1(S_p)$ the exceptional divisor of this blow-up. By abuse of notation, L and E_p also denote respectively the strict transform in every surface which dominate respectively $\mathbb{P}^2_{\bar{\mathrm{k}}}$ and S_p. The total transforms correspond to classes in $\mathcal{Z}_C$. We denote them with lower cases. Thus, ℓ and e_p are respectively the classes of L and E_p. By example, on S_p, $e_p = E_p$ but on $(S_p)_q$ where $q \in E_p$, $e_p = E_p + E_q$. More generally, the inclusion $\mathrm{N}^1(S) \hookrightarrow \mathcal{Z}_C(S)$ makes the correspondence between a divisor D and its Picard-Manin class d. From now, we are interested by the L^2-completion of $\mathcal{Z}_C(S)$, denoted as $\mathcal{Z}(S)$ and called the Picard-Manin space. Remark that intersection form is compatible with Picard-Manin classes because it is stable by pull-back. It induces a quadratic form of signature $(1, \infty)$ on this space. We have the orthogonal decomposition:

$$\mathcal{Z}(S) = \mathrm{N}^1(S) \oplus \mathrm{E}(S),$$

where $\mathrm{E}(S) = \bigoplus_{p \in \mathcal{S}} \mathrm{Vect}(e_p)$ is called the exceptional part and $\mathcal{S}$ is the set of all points belonging to S and all of its blow-up. This allows us to decompose elements of $\mathcal{Z}(S)$ in this basis:

$$\mathcal{Z}(S) = \{c = d_0 + \sum_p \lambda_p e_p \mid \sum \lambda_p^2 < \infty \text{ and } D_0 \in \mathrm{N}^1(S)\}.$$

If the morphism $\pi : S' \to S$ is birational then the map $\pi_\#$ induced by π on the Picard-Manin spaces, which maps $c \in \mathcal{Z}(S')$ to $c \in \mathcal{Z}(S)$, is an isomorphism. This map modifies the basis of the orthogonal decomposition by moving elements in the basis of $\mathrm{N}^1(S')$ onto the exceptional part. We introduce this new notation for distinguish this isomorphism from the map induced by π on the Néron-Severi groups, $\pi^* : \mathrm{N}^1(S) \hookrightarrow \mathrm{N}^1(S')$ which is only an injective map.

Now, consider the hyperboloid sheet associated to the intersection form induced by the intersection number:

$$\mathbb{H}^\infty_{\bar{\mathrm{k}}}(S) = \{c \in \mathcal{Z}(S) \mid c \cdot c = 1 \text{ and } c \cdot d_0 > 0\}.$$

We can endow it with the distance defined as $\mathrm{d}(c, c') = \cosh^{-1}(c \cdot c')$ for all $c, c' \in \mathbb{H}^\infty_{\bar{\mathrm{k}}}(S)$. This space is isometric to the hyperbolic space $\mathbb{H}^\infty$ introduced in Section 1. The group $\mathrm{Bir}(\mathbb{P}^2_{\bar{\mathrm{k}}})$ act on $\mathbb{H}^\infty_{\bar{\mathrm{k}}}$ via the map $f \mapsto f_\#$ defined as follow. Consider a resolution of f:

$$\begin{tikzcd} & S \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ \mathbb{P}^2_{\bar{\mathrm{k}}} \arrow[rr, dashed, "f"] & & \mathbb{P}^2_{\bar{\mathrm{k}}} \end{tikzcd}$$

then $f_\# = \sigma_\# \circ (\pi_\#)^{-1}$. We remark that $(\pi_\#)^{-1} = (\pi^{-1})_\#$.

Remark 14. If f is an isomorphism from a neighborhood U of x to a neighborhood V of $f(x)$ then we have $f_\#(e_x) = e_{f(x)}$.

Look an example of the action of f on ℓ.

Example 15. Let f be a quadratic map with base points p_1, p_2, p_3 (infinitely near or not) and f^{-1} be its inverse map with base points q_1, q_2 and q_3. We denote by C the class of the conic obtained as the image of L by f. Let us compute $f_\#(\ell)$. First, we describe the action on Néron-Severi groups. The conic C passes through the points q_1, q_2 and q_3. Consequently $\sigma^*(C) = \bar{C} + E_{q_1} + E_{q_2} + E_{q_3}$, where $\bar{C}$ is the strict transform of C. If L is general, it doesn't pass through the base points of f. Therefore $\bar{C} = \bar{L} = \pi^*(L)$. A basis of $\mathrm{N}^1(S)$ is given by $\sigma^*(L)$, E_{q_1}, E_{q_2} and E_{q_3}. We can write $\pi^*(L)$ in this basis:

$$\begin{aligned} \pi^*(L) &= \sigma^*(C) - E_{q_1} - E_{q_2} - E_{q_3} \\ &= 2\sigma^*(L) - E_{q_1} - E_{q_2} - E_{q_3}. \end{aligned}$$

Now, we describe the action on Picard-Manin spaces. Considering the previous basis induced on $\mathcal{Z}(S)$, we have $\pi_\#^{-1}(\ell) = 2\sigma_\#^{-1}(\ell) - e_{q_1} - e_{q_2} - e_{q_3}$. Moreover, since σ is the birational morphism which contracts the exceptional divisors E_{q_1}, E_{q_2} and E_{q_3}, $\sigma_\#(e_{q_i}) = e_{q_i}$. Finally, we obtain:

$$f_\#(\ell) = 2\ell - e_{q_1} - e_{q_2} - e_{q_3}.$$

In general, for every $f \in \mathrm{Bir}(\mathbb{P}^2_{\bar{\mathrm{k}}})$,

$$f_\#(\ell) \cdot \ell = \deg(f).$$

2.2. **Action of the maps h_n on $\mathbb{H}^\infty_{\bar{\mathrm{k}}}$.** This subsection is devoted to the study of the action of the maps $h_n : (x, y) \mapsto (y, y^n - x)$, where $n \geq 2$, belonging to $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ which is included in $\mathrm{Bir}(\mathbb{P}^2_{\bar{\mathrm{k}}})$. So, we work over $\bar{\mathrm{k}}$. First, observe that these maps are Jonquières transformations. Consequently, they have a base point p_0 of multiplicity $n-1$ and $2n-2$ base points of multiplicity 1 denoted by p_k with $1 \leq k \leq 2n-2$. The homogeneous coordinates of the point p_0 are $[1 : 0 : 0]$. It is the unique base point of the maps h_n which is located in $\mathbb{P}^2_{\bar{\mathrm{k}}}$. Moreover, for $0 \leq k \leq 2n-3$, p_{k+1} is infinitely near to p_k. The base points of h_n^{-1}, denoted by $(q_k)_{0 \leq k \leq 2n-2}$, have the same properties as those of h_n. They form a tower above $q_0 = [0 : 1 : 0]$. For simplicity, we denote by e_n^+ (respectively e_n^-) the sum with multiplicity of the classes of the exceptional divisors obtained during the resolution of h_n (respectively of h_n^{-1}):

$$\begin{cases} e_n^+ = (n-1)e_{p_0} + e_{p_1} + \cdots + e_{p_{2n-2}} \\ e_n^- = (n-1)e_{q_0} + e_{q_1} + \cdots + e_{q_{2n-2}} \end{cases}.$$

The action of $h_{n\#}$ and of its iterates on ℓ is given by:

$$h_{n\#}(\ell) = n\ell - e_n^-,\ h_{n\#}^2(\ell) = n^2\ell - ne_n^- - h_{n\#}(e_n^-),\ \text{etc.}$$

The sequence $(\frac{1}{n^k}h_{n\#}^k(\ell))_{k\in\mathbb{N}}$ converges in the Picard-Manin space to an element b_n^+ with self-intersection 0. This point can be identified to a point on the boundary $\partial\mathbb{H}^\infty_{\bar{\mathrm{k}}}$. This element corresponds to an endpoint of $\mathrm{Ax}(h_{n\#})$. In the same way, the sequence $(\frac{1}{n^k}h_{n\#}^{-k}(\ell))_{k\in\mathbb{N}}$ converges to an element b_n^-. These two classes can be written:

$$b_n^+ = \ell - \sum_{i=0}^{\infty}\frac{h_{n\#}^i(e_n^-)}{n^{i+1}} \text{ and } b_n^- = \ell - \sum_{i=0}^{\infty}\frac{h_{n\#}^{-i}(e_n^+)}{n^{i+1}}.$$

Now, for any $n\geq 2$, consider the point w_n which is the projection of ℓ on $\mathrm{Ax}(h_{n\#})$ (see figure 7).

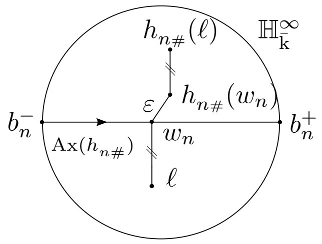

FIGURE 7. Projection of ℓ on the axis of $h_{n\#}$.

The axis of $h_{n\#}$ is uniquely determined by b_n^+ and b_n^-, so w_n is a linear combination of these two classes; $w_n = \alpha b_n^+ + \beta b_n^-$. We have $1 = w_n^2 = 2\alpha\beta$ because $(b_n^+)^2 = 0 = (b_n^-)^2$ and $b_n^+\cdot b_n^- = 1$. Moreover, $w_n\cdot\ell = \alpha+\beta$ has to be minimal because w_n is the projection of ℓ. Finally, we obtain:

$$w_n = \sqrt{2}\ell - \frac{1}{\sqrt{2}}r_n \quad\text{where}\quad r_n = \sum_{i=0}^{\infty}\frac{h_{n\#}^i(e_n^-)+h_{n\#}^{-i}(e_n^+)}{n^{i+1}}.$$

Lemma 16. *Consider the set of elements of the* $4n-2$ *sequences* $(h_{n\#}^k(e_{q_i}))_{k\in\mathbb{N}}$, $(h_{n\#}^{-k}(e_{p_i}))_{k\in\mathbb{N}}$ *with* $0\leq i\leq 2n-2$. *These elements are mutually orthogonal.*

Proof. By induction, we can extend the sequence of points $(q_i)_{i\geq 2n-1}$ such that $q_i\in E_{q_{i-1}}$ and $e_{q_i} = h_{n\#}(e_{q_{i-(2n-1)}})$.

Consider a resolution of h_n (see figure 8 with $n=2$). There exists a point q_{2n-1} on the exceptional divisor $E_{q_{2n-2}}$ such that the birational map $\pi^{-1}:\mathbb{P}^2_{\bar{\mathrm{k}}}\dashrightarrow S$ is a local isomorphism between a neighborhood of q_0 and of q_{2n-1}. According to Remark 14, we have $\pi_\#^{-1}(e_{q_0}) = e_{q_{2n-1}}$. This implies:

$$h_{n\#}(e_{q_0}) = \sigma_\#^{-1}(e_{q_{2n-1}}) = e_{q_{2n-1}}.$$

The first point q_{2n-1} is constructed. Now, suppose that the points q_i are constructed for $2n-1\leq i\leq m$, and we want to construct the point q_{m+1} in the same way. By blowing-up the points q_j pour $0\leq j\leq m-(2n-2)$, the point $q_{m+1-(2n-1)}\in E_{q_{m+1-(2n)}}$ appears. The birational map π^{-1} is a local isomorphism so its induced map $\tilde{\pi}^{-1}$ on the blow-up of $q_0\in\mathbb{P}^2_{\bar{\mathrm{k}}}$ and of $q_{2n-1}\in E_{q_{2n-2}}$ is a local isomorphism too. Thus, there exists a point $q_{m+1}\in E_{q_m}$ such that $\tilde{\pi}^{-1}$ is a local isomorphism between $q_{m+1-(2n-1)}$ and q_{m+1}. By the same argument as before, we have $\tilde{\pi}_\#^{-1}(e_{q_{m+1-(2n-1)}}) = e_{q_{m+1}}$. Finally we obtain $h_{n\#}(e_{q_{m+1-(2n-1)}}) = e_{q_{m+1}}$. In the same way, we can construct a sequence of points $(p_i)_{i\geq 2n-1}$ such that $p_i\in E_{p_{i-1}}$ and $e_{p_i} = h_{n\#}^{-1}(e_{p_{i-(2n-1)}})$. $\square$

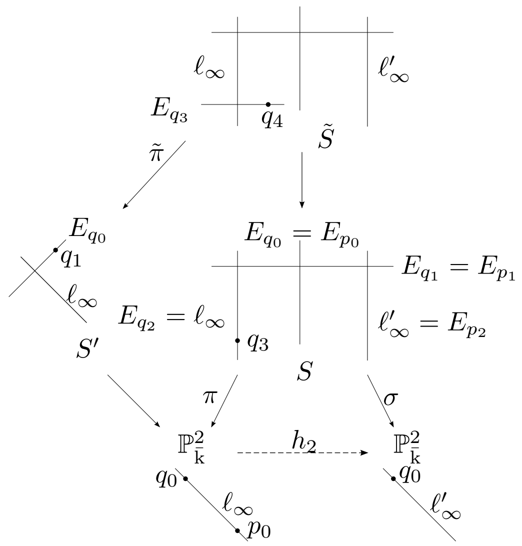

FIGURE 8. Resolution of h_2.

Remark 17. Let us write all the terms which appear in r_n:

$$r_n = (n-1)\frac{e_{q_0}}{n} + \frac{e_{q_1}}{n} + \cdots + \frac{e_{q_{2n-2}}}{n} + (n-1)\frac{e_{p_0}}{n} + \cdots + \frac{e_{p_{2n-2}}}{n} + (n-1)\frac{h_{n\#}(e_{q_0})}{n^2} + \cdots.$$

Lemma 16 implies that all terms of r_n are mutually orthogonal. So, the class of any exceptional divisor e_i has a non-zero intersection number with at most one term of r_n.

We can remark too, even if we don't use this fact, that the points p_i (respectively q_i), constructed in Lemma 16, are the base points of iterates of h_n (respectively h_n^{-1}). The following diagram gives an idea of proof in the case of h_n^2.

$$\begin{tikzcd}
& & \tilde{S} \arrow[dl, "\tilde{\pi}"'] \arrow[dr, "\tilde{\sigma}"] & & \\
& S \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & & S \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\
\mathbb{P}^2_{\mathrm{k}} \arrow[rr, dashed, "h_n"] & & \mathbb{P}^2_{\mathrm{k}} \arrow[rr, dashed, "h_n"] & & \mathbb{P}^2_{\mathrm{k}}
\end{tikzcd}$$

2.3. **Proof of the main result.** Let k be a field and $n \geq 2$ an integer such that the characteristic of k doesn't divide n. Consider the maps:

$$\begin{array}{rccc} h_n : & \mathbb{A}^2_{\mathrm{k}} & \longrightarrow & \mathbb{A}^2_{\mathrm{k}} \\ & (x,y) & \longmapsto & (y, y^n - x) \end{array}$$

belonging to the Cremona group. In this paragraph we prove Proposition 4 which says that the group $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ acts discretely along the axis of $h_{n\#}$. More precisely, we will show that the group $\mathrm{Bir}(\mathbb{P}^2_{\bar{\mathrm{k}}})$ acts discretely along the $h_{n\#}$'s axis. This immediately will imply the same property for the group $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$. To do it, we need a lemma from [CL13, Proposition 5.7]. They proved it in the case of general Jonquières map. We write the proof because we have to check that we can apply it to h_n which is not a general Jonquières. Before we state the lemma, let us recall that w_n is the projection of the class of ℓ on the axis of $h_{n\#}$.

Lemma 18. *There exists* $\varepsilon > 0$ *such that every element of* $\mathrm{Stab}_\varepsilon\{w_n\} \subset \mathrm{Bir}(\mathbb{P}^2_{\bar{\mathrm{k}}})$ *is an automorphism of* $\mathbb{P}^2_{\bar{\mathrm{k}}}$.

Proof. Fix $\varepsilon \in \left]0, \cosh^{-1}\frac{5}{2\sqrt{2}} - \cosh^{-1}\sqrt{2}\right]$. Thus this constant satisfies:

$$(*)\qquad \begin{cases} \cosh^{-1}(\sqrt{2}) + \varepsilon < \cosh^{-1}\dfrac{5}{2\sqrt{2}} < \cosh^{-1}\dfrac{3}{\sqrt{2}} \\ 2\cosh^{-1}(\sqrt{2}) + \varepsilon < \cosh^{-1} 4 \end{cases}.$$

Let $f \in \mathrm{Stab}_\varepsilon\{w_n\}$. We want to show that the degree of f is 1. According to the triangle inequality (see figure 7) and the fact that $\mathrm{d}(f_\#(w_n), w_n) \leq \varepsilon$, we obtain:

$$\begin{aligned}\mathrm{d}(f_\#(\ell), \ell) &\leq \mathrm{d}(f_\#(\ell), f_\#(w_n)) + \mathrm{d}(f_\#(w_n), w_n) + \mathrm{d}(w_n, \ell) \\ &\leq 2\,\mathrm{d}(w_n, \ell) + \varepsilon.\end{aligned}$$

We recall that $\cosh \mathrm{d}(f_\#(\ell), \ell) = f_\#(\ell)\cdot\ell = \deg f$, so $\deg f \leq \cosh(2\,\mathrm{d}(w_n,\ell) + \varepsilon)$. Compute the value of $\mathrm{d}(w_n, \ell) = \cosh^{-1}(w_n\cdot\ell) = \cosh^{-1}\sqrt{2}$. Finally, according to $(*)$, the degree of f is less than 4:

$$\deg f \leq \cosh(2\cosh^{-1}(\sqrt{2}) + \varepsilon) < 4.$$

Now, we have to prove that the degree of f can not be 2 or 3. We do it by contradiction and we obtain the inequality $\mathrm{d}(f_\#(\ell), w_n) > \cosh^{-1}(\sqrt{2}) + \varepsilon$. However, according to the triangle inequality, we have:

$$\begin{aligned}\mathrm{d}(f_\#(\ell), w_n) &\leq \mathrm{d}(f_\#(\ell), f_\#(w_n)) + \mathrm{d}(f_\#(w_n), w_n) \\ &\leq \mathrm{d}(w_n, \ell) + \varepsilon = \cosh^{-1}(\sqrt{2}) + \varepsilon,\end{aligned}$$

this leads us to a contradiction.

First, suppose that $\deg(f) = 3$. So f and f^{-1} have one base point of multiplicity 2 and four base points of multiplicity 1 according to Cremona relations (see [Ale16]). Consequently, the action of $f_\#$ on the Picard-Manin class ℓ is defined by $f_\#(\ell) = 3\ell - 2e_0 - e_1 - e_2 - e_3 - e_4$ where the e_i are the classes of exceptional divisors above $\mathbb{P}^2_{\bar{\mathrm{k}}}$. According to Remark 17, we obtain:

$$(2e_0 + e_1 + e_2 + e_3 + e_4)\cdot r_n \geq \frac{1}{n}(-2(n-1) - 1 - 1 - 1 - 1) = -2 - \frac{2}{n} \geq -3,$$

this implies $-f_\#(\ell)\cdot r_n \geq -3$. Moreover, by assumption we have $f_\#(\ell)\cdot\ell = 3$. This gives us the inequality:

$$f_\#(\ell)\cdot w_n = \sqrt{2} f_\#(\ell)\cdot\ell - \frac{1}{\sqrt{2}} f_\#(\ell)\cdot r_n \geq 3\sqrt{2} - \frac{3}{\sqrt{2}} = \frac{3}{\sqrt{2}}.$$

Taking the inverse of the hyperbolic cosine and using $(*)$ leads us to the expected contradiction:

$$\mathrm{d}(f_\#(\ell), w_n) \geq \cosh^{-1}\frac{3}{\sqrt{2}} > \cosh^{-1}(\sqrt{2}) + \varepsilon.$$

Now, suppose that $\deg f = 2$. In the same way that in the previous case, we consider the action of $f_\#$ on ℓ; $f_\#(\ell) = 2\ell - e_0 - e_1 - e_2$ where e_i are the classes of the exceptional divisors above $\mathbb{P}^2_{\bar{\mathrm{k}}}$. According to Remark 17, we have:

$$-f_\#(\ell)\cdot r_n = (e_0 + e_1 + e_2)\cdot r_n \geq -\frac{1}{n}(n - 1 + 1 + 1) = -1 - \frac{1}{n} \geq -\frac{3}{2}.$$

Moreover, $f_\#(\ell)\cdot\ell = 2$, this implies:

$$f_\#(\ell)\cdot w_n = \sqrt{2} f_\#(\ell)\cdot\ell - \frac{1}{\sqrt{2}} f_\#(\ell)\cdot r_n \geq 2\sqrt{2} - \frac{3}{2\sqrt{2}} = \frac{5}{2\sqrt{2}}.$$

By taking the inverse of the hyperbolic cosine and using ($*$), we obtain the contradiction:

$$\mathrm{d}(f_\#(\ell)\cdot w_n) \geq \cosh^{-1}\frac{5}{2\sqrt{2}} > \cosh^{-1}(\sqrt{2})+\varepsilon.$$

The degree of f is 1. Consequently, it's an automorphism of $\mathbb{P}^2_{\bar{\mathrm{k}}}$. □

Proof of Proposition 4. Since the group $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ is included in the group $\mathrm{Bir}(\mathbb{P}^2_{\bar{\mathrm{k}}})$, it acts also on the space $\mathbb{H}^\infty_{\bar{\mathrm{k}}}$. We want to prove that $\mathrm{Bir}(\mathbb{P}^2_{\mathrm{k}})$ acts discretely along the axis of $h_{n\#}$. As we have just said above, we will first show that $\mathrm{Bir}(\mathbb{P}^2_{\bar{\mathrm{k}}})$ acts discretely along the axis of $h_{n\#}$. This means that there exists $\varepsilon>0$ such that the set $\mathrm{Stab}_\varepsilon\{h^2_{n\#}(w_n), h^{-2}_{n\#}(w_n)\}\subset \mathrm{Bir}(\mathbb{P}^2_{\bar{\mathrm{k}}})$ is finite. Fix $\varepsilon>0$ satisfying ($*$), as on Lemma 18. Let f be a birational map of $\mathbb{P}^2_{\bar{\mathrm{k}}}$ belonging to $\mathrm{Stab}_\varepsilon\{h^2_{n\#}(w_n), h^{-2}_{n\#}(w_n)\}$. Lemma 10.(1), gives us the inclusion:

$$\mathrm{Stab}_\varepsilon\{h^2_{n\#}(w_n), h^{-2}_{n\#}(w_n)\}\subset \mathrm{Stab}_\varepsilon\{h_{n\#}(w_n), h^{-1}_{n\#}(w_n)\}\subset \mathrm{Stab}_\varepsilon\{w_n\}.$$

This implies that f belongs to $\mathrm{Stab}_\varepsilon\{w_n\}$. According to Lemma 18, we obtain that the degree of f is 1:

$$f\colon [x:y:z]\mapsto [ax+ky+bz : lx+cy+dz : hx+my+z].$$

Now, the aim is to find some constraints on coefficients of f to show that there is only a finite number of choices for such a map. The map f belongs to $\mathrm{Stab}_\varepsilon\{h_{n\#}(w_n), h^{-1}_{n\#}(w_n)\}$ too. So, $\mathrm{d}(f_\# h^{-1}_{n\#}(w_n), h^{-1}_{n\#}(w_n))\leq\varepsilon$. The map $h^{-1}_{n\#}$ being an isometry, we have:

$$\mathrm{d}(h_{n\#}f_\# h^{-1}_{n\#}(w_n), w_n)\leq\varepsilon.$$

Moreover, $h_{n\#}f_\#h^{-1}_{n\#}=(h_nfh_n^{-1})_\#$, so according to Lemma 18 the map $h_nfh_n^{-1}$ is an automorphism. By considering the curve C blown-down on p_0 by h_n^{-1} and by the fact that f is an automorphism, we have $f(p_0)=p_0$. This implies that $l=h=0$. Using the same argument and the fact that $\mathrm{d}(f_\#h_{n\#}(w_n), h_{n\#}(w_n))\leq\varepsilon$, we obtain $f(q_0)=q_0$ and so $k=m=0$. Thus, f is an affine automorphism:

$$f\colon (x,y)\mapsto(ax+b, cy+d).$$

Let us compute $h_n\circ f\circ h_n^{-1}$ and $h_n^{-1}\circ f\circ h_n$ to see under which conditions they are automorphisms of degree 1 of $\mathbb{A}^2_{\bar{\mathrm{k}}}$.

$$\begin{aligned}
h_n\circ f\circ h_n^{-1} &= (y, y^n-x)\circ(ax+b,cy+d)\circ(x^n-y,x)\\
&=(cx+d,(cx+d)^n+ay-ax^n-b)\\
&=(cx+d, x^n(c^n-a)+nc^{n-1}dx^{n-1}+\cdots+ncd^{n-1}x+ay+d^n-b).
\end{aligned}$$

In the same way, we have:

$$h_n^{-1}\circ f\circ h_n=(y^n(a^n-c)+na^{n-1}by^{n-1}+\cdots+nab^{n-1}y+cx+b^n-d, ay+b).$$

The maps $h_n\circ f\circ h_n^{-1}$ and $h_n^{-1}\circ f\circ h_n$ are affine automorphisms of $\mathbb{A}^2_{\bar{\mathrm{k}}}$, so the coefficients a and c of f satisfy the relations:

$$c^n=a \text{ and } c=a^n$$

This means that a and c are (n^2-1)th roots of unity because f is an automorphism so a and c are non-zeros. Now, we distinguish two cases.

First, assume $n\geq 3$. By considering the coefficients of x^{n-1} and y^{n-1} we have: $nc^{n-1}d=na^{n-1}b=0$. Since the characteristic of the field doesn't divide n and a and c are non-zeros, we conclude that

$$d=b=0.$$

In short, if $n\geq3$, f belongs to the set of maps $(x,y)\mapsto(ax,cy)$ with $a,c\in\mathbb{U}_{n^2-1}$ and $a^n=c$. Thus the set $\mathrm{Stab}_\varepsilon\{h^2_{n\#}(w_n), h^{-2}_{n\#}(w_n)\}$ is finite.

If $n = 2$, we can't conclude directly. By assumption, f belongs to $\mathrm{Stab}_\varepsilon\{h^2_{2\#}(w_2), h^{-2}_{2\#}(w_2)\}$. This means that

$$\mathrm{d}(f_\# h^{-2}_{2\#}(w_2), h^{-2}_{2\#}(w_2)) \leq \varepsilon \text{ and } \mathrm{d}(f_\# h^{2}_{2\#}(w_2), h^{2}_{2\#}(w_2)) \leq \varepsilon.$$

Since $h^{-2}_{2\#}$ and $h^2_{2\#}$ are isometries we have actually:

$$\mathrm{d}(h^2_{2\#} f_\# h^{-2}_{2\#}(w_2), w_2) \leq \varepsilon \text{ and } \mathrm{d}(h^{-2}_{2\#} f_\# h^{2}_{2\#}(w_2), w_2) \leq \varepsilon.$$

According to Lemma 18, the degree of the maps $h_2^2 \circ f \circ h_2^{-2}$ and $h_2^{-2} \circ f \circ h_2^2$ is 1. Moreover, the point p_0 (respectively the point q_0) is the unique base point of h_2^2 (respectively of h_2^{-2}) belonging to $\mathbb{P}^2_{\Bbbk}$. With the same argument as previously, $h_2 \circ f \circ h_2^{-1}$ preserves p_0 and $h_2^{-1} \circ f \circ h_2$ preserves q_0. We obtain that $2cd = 0$ and $2ab = 0$. Since the characteristic of the field is not 2 and that f is an automorphism we must have a and c be non-zeros so $b = d = 0$. Finally, when $n = 2$, f is written $(x, y) \mapsto (ax, cy)$ with $a, c \in \mathbb{U}_3$ and $a^2 = c$. The set $\mathrm{Stab}_\varepsilon\{h^2_{2\#}(w_2), h^{-2}_{2\#}(w_2)\}$ is finite as expected.

We have proved that for any $n \geq 2$, the set $\mathrm{Stab}_\varepsilon\{h^2_{n\#}(w_n), h^{-2}_{n\#}(w_n)\}$ is finite. So the set $\mathrm{Stab}_\varepsilon\{h^2_{n\#}(w_n), h^{-2}_{n\#}(w_n)\} \cap \mathrm{Bir}(\mathbb{P}^2_{\Bbbk})$ is finite too. This means that the group $\mathrm{Bir}(\mathbb{P}^2_{\Bbbk})$ acts discretely along the $h_{n\#}$'s axis.

□

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