NILPOTENT ORBITS AND FINITE W-ALGEBRAS

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Abstract. A finite W-algebra is a certain associative algebra associated with a pair $(\mathfrak{g}, e)$ where $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra and e is a nilpotent element of $\mathfrak{g}$. It can be viewed as a generalization of the enveloping algebra of $\mathfrak{g}$. The study of finite W-algebras was initiated by Kostant in 1978 who dealt with the regular case. The general definition was given by Premet in 2002. For mathematicians, one of the main motivations for their study is the connections between their representations and that of $\mathfrak{g}$, as it was illustrated by the famous Skryabin's equivalence.

In this lecture, I will present the Premet's definition, and some of its variations, in the framework of admissible gradings associated with a nilpotent element, following the presentation of the recent Guilnard Sadaka's thesis (2013). These gradings are generalizations of good gradings, and so of Dynkin gradings. I will also describe various properties and a few applications of the finite W-algebras.

The lecture will be organized as follows:

(1) I will start with basic results on nilpotent orbits and nilpotent elements in a complex semisimple Lie algebra. Properties of Dynkin gradings, good gradings and admissible gradings associated with nilpotent elements will be also discussed.
(2) Then, I will focus on the finite W-algebras corresponding to admissible gradings and will study some of their remarkable properties, like the Skryabin's equivalence or their interpretation as a quantization of a certain transversal slice.
(3) At last, I will address some isomorphism problems. Namely, I will be interested in the question of wether the finite W-algebras constructed from a given admissible grading are isomorphic to the one introduced by Premet. This will include results of Gan-Ginzburg (2002), Brundan-Goodwin (2007) and Sadaka (2013) which concern the Dynkin gradings, good gradings and admissible gradings respectively.

Contents

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1. Introduction

The ground field is the field of complex numbers $\mathbb{C}$.

1.1. **Short historic.** A finite W-algebra is a certain associative algebra associated with a pair $(\mathfrak{g}, e)$ where $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, and e is a nilpotent element of $\mathfrak{g}$. A finite W-algebra can be viewed as a generalization of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. If $e = 0$, then the corresponding finite W-algebra is $U(\mathfrak{g})$. If e is a regular nilpotent element of $\mathfrak{g}$, i.e., its nilpotent orbit is dense in the nilpotent cone of $\mathfrak{g}$, we get the center of $U(\mathfrak{g})$.

The study of finite W-algebras started with the celebrated Kostant's paper, [K78], which concerns the case where e is regular. Kostant's motivations came from the study of Whittaker vectors and of Whittaker models. Shortly after, Lynch generalized Kostant's construction to arbitrary even nilpotent elements, [L79].

Finite W-algebras also attracted attention from mathematical physicists, to whom one owes the name *W-algebras*[1]; see for example [BT93, RS99, VD96]. One of their main motivations was a deep link between finite W-algebras and affine W-algebras. The later are certain vertex algebras (cf. e.g., [FF90, KRW03, KWa04]) and the finite W-algebra associated with $(\mathfrak{g}, e)$ is the twisted Zhu algebra of the affine W-algebra associated with $(\mathfrak{g}, e)$; [DK06, Ar07]. The most recent important developments in representation theory of affine W-algebras were done by Arakawa; cf. e.g., [Ar05, Ar07].

For mathematicians, the general definition of finite W-algebras goes back to Alexander Premet in 2002, [P02]. Premet's motivations was the study of non-restricted representations of semisimple Lie algebras in positive characteristic. To this, he first considered analogs of finite W-algebras over an algebraically closed field of positive characteristic, [P95]. He used them to settle the famous Kac-Weisfeiller conjecture [KWe71] (see also [J97] for a more recent review). Since [P02], there has been a great deal of research interest in finite W-algebras and their representation theory; see for example [BGK08, BK06, L10b, L11a, L11b, L12, P07]. The reason comes from close

[1] The name "W-algebra" comes from Zamolodchikov who used the letter W in one of his examples, [Z85]. The connexion with Whittaker models is another good reason to call these algebras "W-algebras".

connections between the representations of the finite W-algebra associated with a pair $(\mathfrak{g}, e)$ and that of $U(\mathfrak{g})$, as it was illustrated by the famous Skryabin's equivalence, [Sk02]. There is also an interesting connection between the primitive ideals of $U(\mathfrak{g})$ whose associated variety contains the nilpotent orbit of e, and the primitive ideals of the finite W-algebra of $(\mathfrak{g}, e)$; see e.g., [P07, P10, P14, PT14, L12].

1.2. **Premet's definition.** There are actually several different equivalent definitions for the finite W-algebras; see for instance [D³HK06, BGK08, W10, L10a]. In particular, it was proved in [D³HK06] (see also [Ar07] for the regular case) that the definition in the mathematical physics literature via BRST cohomology agrees with Premet's definition [P02]. All definitions provide interesting points of view. In this lecture, we will only consider the Whittaker model definition (close to Premet's definition), and some of its variations; we refer the reader to Section 7.

The definition given by Premet starts with a pair $(\mathfrak{g}, e)$ consisted of a complex semisimple Lie algebra $\mathfrak{g}$ and a nilpotent element e. By the Jacobson-Morozov theorem, e is part of an $\mathfrak{sl}_2$-triple (e, h, f). The eigenspace decomposition of $\operatorname{ad} h$ induces a $\mathbb{Z}$-grading on $\mathfrak{g}$:

$$\mathfrak{g} := \bigoplus \mathfrak{g}(i).$$

Define a linear form χ by setting $\chi(x) = \langle e, x\rangle$ for any $x \in \mathfrak{g}$, where $\langle\, ,\rangle$ denotes a nonzero multiple of the Killing form of $\mathfrak{g}$ such that $\langle e, f\rangle = 1$. This induces a nondegenerate anti-symmetric bilinear form on $\mathfrak{g}(-1)$ defined by:

$$\omega(x, y) := \chi([x, y]), \qquad x, y \in \mathfrak{g}(-1).$$

Choose a Lagrangian subspace $\mathfrak{l}$ in $\mathfrak{g}(-1)$ and define a nilpotent subalgebra by

$$\mathfrak{m}_{\chi,\mathfrak{l}} := \mathfrak{l} \oplus \bigoplus_{j \leqslant -2} \mathfrak{g}(j). \tag{1}$$

It acts on $U(\mathfrak{g})$ by the adjoint action, and the left ideal I_χ of $U(\mathfrak{g})$ generated by the element $x - \chi(x)$, for $x \in \mathfrak{m}_{\chi,\mathfrak{l}}$, is invariant under this action. The invariant quotient

$$\mathcal{W}_\chi := (U(\mathfrak{g})/I_\chi)^{\operatorname{ad} \mathfrak{m}_{\chi,\mathfrak{l}}}$$

inherits the associative algebra structure from $U(\mathfrak{g})$. This algebra is called the ***finite W-algebra associated with*** $(\mathfrak{g}, e)$ and only depends, up to isomorphism, on the nilpotent orbit of e.

Gan and Ginzburg proved that the above definition does not depend on the choice of a Lagrangian subspace $\mathfrak{l}$ in $\mathfrak{g}(-1)$, [GG02]; this is the reason why we do not refer in our notation to this Lagragian subspace. More recently, Brundan and Goodwin extended the definition to any ***good grading for e*** (cf. Definition 5.1) and proved that the so-obtained algebra is isomorphic to $\mathcal{W}_\chi$, [BG05]. In her recent thesis [Sa13, Sa14], Sadaka have considered the more general framework of ***admissible grading for e*** (cf. Definition 6.1). Here, the isomorphism problem is still open in general. Some particular cases (including Brundan-Goodwin case) are solved.

All the above facts we will discussed in more details in Parts 2 and 3.

1.3. **Plan of the lecture.** The lectures will be organized as follows.

Part 1 starts with well-known facts on nilpotent orbits and nilpotent elements in a complex semisimple Lie algebra $\mathfrak{g}$. Then we will study some properties of Dynkin gradings, good gradings and admissible gradings associated with a nilpotent element (cf. Definitions 5.1 and 6.1).

These ingredients will be used in Part 2 where the finite W-algebras, and some of their ramifications, will be introduced. In this lecture, we consider the following setting: to any admissible pair $(\mathfrak{m}, \mathfrak{n})$ for a nilpotent element e of $\mathfrak{g}$ (cf. Definition 6.1), we attach an endomorphism algebra $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ that we call the ***finite W-algebra associated with*** $(\mathfrak{m}, \mathfrak{n})$. These algebras have nice properties similar to those verified by the finite W-algebra $\mathcal{W}_{\chi}$. In particular, there is a filtration on $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ whose associated graded algebra is isomorphic to the algebra of regular functions over a certain affine transversal variety $\mathcal{S}$ to the adjoint orbits in $\mathfrak{g}^*$; see Theorem 10.3. This result was proved by Gan-Ginzburg for the algebra $\mathcal{W}_{\chi}$, [GG02]. Furthermore, the Skryabin's equivalence, [Sk02], holds for $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$, with $\mathfrak{m} = \mathfrak{n}$; see Theorem 11.2.

Part 3 is devoted to isomorphism problems. More precisely, we address the question of whether $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ is isomorphic to $\mathcal{W}_{\chi}$. This part will include results of Gan-Ginzburg (2002), Brundan-Goodwin (2007) and Sadaka (2013) which concern the Dynkin gradings, good gradings and admissible gradings respectively.

1.4. **References.** Our main references concerning the finite W-algebras are [P02, GG02]; see also [L10a] for a review on the topic. The present lecture partly follows the lecture series given by W. Wang, entitled "Nilpotent orbits and finite W-algebras" too, [W10]. In addition, our presentation of the topic mainly follows Sadaka's thesis, [Sa13] (see also [Sa14] for a shorter and more recent version of her work).

Our basic reference for algebraic groups and Lie algebras is [TY05]. For basics on nilpotent orbits in a semisimple Lie algebra, we refer to [CM93] or [J04]. At last, for Poisson structures, we refer to [V94], [LPV13] or [CG97, Chap. 1].

Part 1. Good gradings and admissible gradings

2. Main notations

Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra, i.e., $\{0\}$ is the only abelian ideal of $\mathfrak{g}$, with adjoint group[2] G, and equipped with a nondegenerate G-invariant symmetric bilinear form $\langle\, , \rangle$ which induces an isomorphism,

$$\kappa : \mathfrak{g} \to \mathfrak{g}^*, \quad x \mapsto \langle x, . \rangle.$$

[2] The ***algebraic adjoint group*** A of a Lie algebra $\mathfrak{a}$ is the smallest algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$ whose Lie algebra contains $\mathrm{ad}\,\mathfrak{a}$. If $\mathrm{ad}\,\mathfrak{a}$ is an algebraic Lie subalgebra of $\mathfrak{gl}(\mathfrak{a})$, then we say that A is the ***adjoint group*** of $\mathfrak{a}$ and that $\mathfrak{a}$ is **ad** ***-algebraic***. We always have

$$\mathrm{Aut}_e(\mathfrak{a}) \subseteq A \subseteq \mathrm{Aut}(\mathfrak{a})$$

where $\mathrm{Aut}_e(\mathfrak{a})$ is the subgroup of elementary elements, that is the elements $\exp(\mathrm{ad}\, x)$ with x a nilpotent element of $\mathfrak{a}$. If $\mathfrak{a}$ is semisimple, then $A = \mathrm{Aut}_e(\mathfrak{a})$. We refer to [TY05, §24.8.2] for more details.

Remark 2.1. *Since $\mathfrak{g}$ is semisimple, any such a bilinear form $\langle\, ,\rangle$ is a nonzero multiple of the Killing form of $\mathfrak{g}$,*

$$(x, y) \mapsto \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y).$$

In the sequel, it will be convenient to choose for $\langle\, ,\rangle$ a suitable nonzero multiple of the Killing form of $\mathfrak{g}$, not necessarily the Killing form itself.

Example 2.2. *Our typical example of simple*[3] *Lie algebra will be $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, the set of complex n-size square matrices with zero trace, whose Killing form of $\mathfrak{g}$ is given by*

$$(A, B) \mapsto 2n \operatorname{tr}(AB).$$

If U is a subspace of $\mathfrak{g}$, we denote by $U^\perp$ its orthogonal complement with respect to $\langle\, ,\rangle$. We say that U and V are in ***pairing with respect to $\langle\, ,\rangle$*** if the restriction of $\langle\, ,\rangle$ to $U \times V$ or $V \times U$ is nondegenerate, i.e., $U \cap V^\perp = \{0\} = V \cap U^\perp$. If U and V are in pairing, then $\dim U = \dim V$. Moreover, U is in pairing with any complement subspace of $U^\perp$ in $\mathfrak{g}$.

2.1. For $\mathfrak{a}$ a subalgebra of $\mathfrak{g}$, we will denote by $U(\mathfrak{a})$ the enveloping algebra of $\mathfrak{a}$ and by $S(\mathfrak{a})$ its symmetric algebra which are the quotient of the tensor algebra of $\mathfrak{a}$ by the bilateral ideal generated by the elements $x \otimes y - y \otimes x$ and the bilateral ideal generated by the elements $x \otimes y - y \otimes x - [x, y]$ respectively, with $x, y \in \mathfrak{a}$.

For $x \in \mathfrak{g}$, we denote by $\mathfrak{a}^x$ the centralizer of x in $\mathfrak{a}$, that is

$$\mathfrak{a}^x = \{y \in \mathfrak{a} \mid [x, y] = 0\},$$

which is also the intersection of $\mathfrak{a}$ with the kernel of the map

$$\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}, \quad y \mapsto [x, y].$$

2.2. A ***$\mathbb{Z}$-grading*** of the Lie algebra $\mathfrak{g}$ is a decomposition $\Gamma : \mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ which verifies $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all i, j.

Lemma 2.3. *If Γ is a $\mathbb{Z}$-grading of $\mathfrak{g}$, then for some semisimple element h_Γ of $\mathfrak{g}$,*

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \mid [h_\Gamma, x] = jx\}.$$

Proof. The operator $\partial : \mathfrak{g} \to \mathfrak{g}$ which maps x to jx, for $x \in \mathfrak{g}_j$, is a derivation of the semisimple Lie algebra $\mathfrak{g}$. Hence, it is an inner derivation of $\mathfrak{g}$ given by $\operatorname{ad} h_\Gamma$ for some semisimple element h_Γ of $\mathfrak{g}$, (cf. e.g., [TY05, Prop. 20.1.5]). □

Since the Killing form of $\mathfrak{g}$ is $\operatorname{ad} h_\Gamma$-invariant and nondegenerate, we get

$$\langle \mathfrak{g}_i, \mathfrak{g}_j\rangle = 0 \iff i + j \neq 0.$$

Hence, $\mathfrak{g}_j$ and $\mathfrak{g}_{-j}$ are in pairing. In particular, they have the same dimension.

In the sequel, we will use the following notations:

$$\mathfrak{g}_{\leqslant k} := \oplus_{j\leqslant k}\mathfrak{g}_j, \quad \mathfrak{g}_{<k} := \oplus_{j<k}\mathfrak{g}_j, \quad \mathfrak{g}_{\geqslant k} := \oplus_{j\geqslant k}\mathfrak{g}_j, \quad \mathfrak{g}_{>k} := \oplus_{j>k}\mathfrak{g}_j.$$

Note that $\mathfrak{g}_0 = \mathfrak{g}^{h_\Gamma}$ is a Levi subalgebra of $\mathfrak{g}$ since h_Γ is semisimple. Moreover, $\mathfrak{g}_{\geqslant 0}$ is a parabolic subalgebra of $\mathfrak{g}$ with $\mathfrak{g}_0$ as a Levi factor and $\mathfrak{g}_{>0}$ as nilpotent radical.

[3] i.e., $\{0\}$ and $\mathfrak{g}$ are the only ideals of $\mathfrak{g}$ and $\dim \mathfrak{g} \geqslant 3$.

3. Nilpotent orbits and nilpotent elements

3.1. Let $\mathcal{N}$ be the ***nilpotent cone of*** $\mathfrak{g}$, that is the set of all nilpotent elements of $\mathfrak{g}$. If $\mathfrak{g}$ is a simple Lie algebra of matrices, note that $\mathcal{N}$ coincides with the set of nilpotent matrices. If $e \in \mathfrak{g}$, we denote by $G.e$ its adjoint G-orbit. There is a unique nilpotent orbit, denoted by $\mathcal{O}_{\text{reg}}$ and called the ***regular nilpotent orbit of*** $\mathfrak{g}$, which is a dense open subset of $\mathcal{N}$. An element $x \in \mathfrak{g}$ is ***regular*** if $\dim \mathfrak{g}^x$ has the minimal dimension, that is the rank of $\mathfrak{g}$. Thus, $\mathcal{O}_{\text{reg}}$ is the set of all regular nilpotent elements of $\mathfrak{g}$.

Example 3.1. *If $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then the rank of $\mathfrak{g}$ is $n-1$ and $\mathcal{O}_{\text{reg}}$ is the conjugacy class of the n-size Jordan block J_n, i.e., $\mathcal{O}_{\text{reg}} = G.J_n$ with*

$$J_n := \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} = \sum_{i=1}^{n-1} E_{i,i+1},$$

where $E_{i,j}$ denotes the elementary matrix whose entries are all zero, except the one in position (i, j) which equals 1.

Next, there is a unique dense open orbit in $\mathcal{N} \setminus \mathcal{O}_{\text{reg}}$ which is called the ***subregular nilpotent orbit of*** $\mathfrak{g}$, and denoted by $\mathcal{O}_{\text{subreg}}$. Its codimension in $\mathfrak{g}$ is the rank of $\mathfrak{g}$ plus two. At the extreme opposite, there is a unique nilpotent orbit of smallest positive dimension called the ***minimal nilpotent orbit of*** $\mathfrak{g}$, and denoted by $\mathcal{O}_{\text{min}}$.

Example 3.2. *Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Every nilpotent matrix in $\mathfrak{g}$ is conjugate to a Jordan block diagonal matrix. Therefore, the nilpotent orbits in $\mathfrak{g}$ are parameterized by the partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ of n, with $\lambda_i \geqslant 0$. We will denote by $\mathcal{O}_\lambda$ the nilpotent orbit of $\mathfrak{sl}_n(\mathbb{C})$ corresponding to the partition λ of n. Then $\mathcal{O}_\lambda$ is represented by the standard Jordan form* $\operatorname{diag}(J_{\lambda_1}, \ldots, J_{\lambda_n})$*, where J_k is the k-size Jordan block, for $k \in \mathbb{N}^*$. Thus, the regular, subregular, minimal and zero nilpotent orbits of $\mathfrak{sl}_n(\mathbb{C})$ correspond to the partitions (n), $(n-1, 1)$, $(2, 1^{n-2})$ and (1^n) of n respectively (here, we do not write the zeroes in the partition).*

The nilpotent orbits in the other classical simple Lie algebras are also associated with some partitions, but the correspondence is more sophisticated; see for instance [J04, Chap. 1]. These correspondences show that the set of nilpotent orbits in the simple Lie algebras of classical type is finite. For the simple Lie algebras of exceptional type, the statement is still true but one needs a different argument (see further below).

The set of nilpotent orbits in $\mathfrak{g}$ is naturally a poset $\mathcal{P}$ with partial order $\leqslant$ defined as follows: $\mathcal{O}' \leqslant \mathcal{O}$ if and only if $\mathcal{O}' \subseteq \overline{\mathcal{O}}$. The regular nilpotent orbit $\mathcal{O}_{\text{reg}}$ is maximal and the zero orbit is the minimal with respect to this order. Moreover, $\mathcal{O}_{\text{subreg}}$ is maximal in the poset $\mathcal{P} \setminus \mathcal{O}_{\text{reg}}$ and $\mathcal{O}_{\text{min}}$ is minimal in the poset $\mathcal{P} \setminus \{0\}$.

Example 3.3. *Let again $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. The partial order on $\mathcal{P}$ corresponds to a partial order on the set $\mathcal{P}(n)$ of partitions of n, first described by Gerstenhaber. Let us give the description of the poset*

$\mathcal{P}$ or, equivalently, the poset $\mathcal{P}(n)$ for $n = 6$. The column on the right indicates the dimension of the orbits appearing in the same row:

$$
\begin{tikzcd}[column sep=small]
& \mathcal{O}_{\mathrm{reg}} \leftrightarrow (6) \arrow[d] & & 30 \\
& \mathcal{O}_{\mathrm{subreg}} \leftrightarrow (5,1) \arrow[d] & & 28 \\
& (4,2) \arrow[dl] \arrow[dr] & & 26 \\
(4,1^2) \arrow[dr] & & (3^2) \arrow[dl] & 24 \\
& (3,2,1) \arrow[dl] \arrow[dr] & & 22 \\
(2^3) \arrow[dr] & & (3,1^3) \arrow[dl] & 18 \\
& (2^2,1^2) \arrow[d] & & 16 \\
& \mathcal{O}_{\mathrm{min}} \leftrightarrow (2,1^4) \arrow[d] & & 10 \\
& 0 & & 0.
\end{tikzcd}
$$

Remark. The dimension of $\mathcal{O}_\lambda$ is easy to compute, [CM93, Thm. 6.1.3]. *Let $\lambda^* = (\lambda_1^*, \dots, \lambda_t^*)$ be the dual partition of $\lambda = (\lambda_1, \dots, \lambda_s)$, with $\lambda_1 \geqslant \cdots \geqslant \lambda_s \geqslant 1$. This means that $t = \lambda_1$ and for any $j \in \{1, \dots, t\}$, $\lambda_j^* := \operatorname{card}\{i \mid \lambda_i \geqslant j\}$; see Figure 1.*

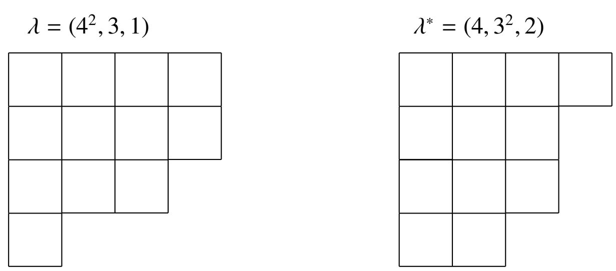


Figure 1. Example of dual partitions

We have

$$\dim \mathcal{O}_\lambda = n^2 - 1 - (\sum_{i=1}^{t} (\lambda_i^*)^2 - 1) = n^2 - \sum_{i=1}^{t} (\lambda_i^*)^2.$$

Actually, $\sum_{i=1}^t (\lambda_i^*)^2 - 1$ is the dimension of $\mathfrak{g}^e$ for $e \in \mathcal{O}_\lambda$ while $n^2 - 1$ is the dimension of $\mathfrak{g}$.

4. Jacobson-Morosov theorem and Dynkin grading

From now on, we fix a nonzero nilpotent element $e \in \mathfrak{g}$.

4.1. By the Jacobson-Morosov Theorem (cf. e.g., [CM93, §3.3]), there exist $h, f \in \mathfrak{g}$ such that the triple (e, h, f) verifies the so-called ***$\mathfrak{sl}_2$-triple relations***:

$$[h, e] = 2e, \qquad [e, f] = h, \qquad [h, f] = -2f.$$

In particular, h is semisimple and the eigenvalues of $\operatorname{ad} h$ are integers. Moreover, e and f belongs to the same nilpotent G-orbit.

Example 4.1. *Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $e = J_n$ as in Example 3.1. Set $h := \operatorname{diag}(n-1, n-3, \dots, -n+3, -n+1)$ and*

$$f := \begin{pmatrix} 0 & & & 0 \\ \mu_1 & \ddots & & \\ 0 & \ddots & \ddots & \\ 0 & & \mu_{n-1} & 0 \end{pmatrix} = \sum_{i=1}^{n-1} \mu_i E_{i+1,i}$$

with $\mu_i := i(n-i)$ for $i \in \{1, \dots, n-1\}$. Then (e, h, f) forms an $\mathfrak{sl}_2$-triple. From this observation, we readily construct $\mathfrak{sl}_2$-triples for any standard Jordan form $\operatorname{diag}(J_{\lambda_1}, \dots, J_{\lambda_n})$ with $(\lambda_1, \dots, \lambda_n) \in \mathcal{P}(n)$.

4.2. The group G acts on the collection of $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ by simultaneous conjugation. This defines a natural map:

$$\Omega : \{\mathfrak{sl}_2\text{-triples}\}/G \longrightarrow \{\text{nonzero nilpotent orbits}\}, \quad (e, h, f) \mapsto G.e.$$

Theorem 4.2 (cf. e.g., [CM93, Thm. 3.2.10])**.** *The map Ω is bijective.*

The map Ω is surjective according to Jacobson-Morosov Theorem. The injectivityy is a result of Kostant, [CM93, Thm. 3.4.10] (see [W10, §2.6] for a sketch of proof).

4.3. Since h is semisimple and since the eigenvalues of $\operatorname{ad} h$ are integers, we get a $\mathbb{Z}$-grading on $\mathfrak{g}$ defined by h, called the ***Dynkin grading associated with h***:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j), \qquad \mathfrak{g}(j) := \{x \in \mathfrak{g} \mid [h, x] = jx\}.$$

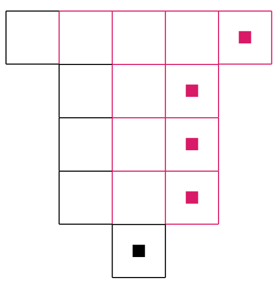
We have $e \in \mathfrak{g}(2)$. Moreover, it follows from the representation theory of $\mathfrak{sl}_2$ that $\mathfrak{g}^e \subset \oplus_{j \geqslant 0} \mathfrak{g}(j)$ and that $\dim \mathfrak{g}^e = \dim \mathfrak{g}(0) + \dim \mathfrak{g}(1)$.

Remark 4.3. *One can draw a picture to visualize the above properties. Decompose $\mathfrak{g}$ into simple $\mathfrak{sl}_2$-modules $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$ and denote by d_k the dimension of V_k for $k \in \{1, \dots, r\}$. We can assume that $d_1 \geqslant \cdots \geqslant d_r \geqslant 1$. We have $\dim V_k \cap \mathfrak{g}(j) \leqslant 1$ for any $j \in \mathbb{Z}$. We represent the module V_k on the kth row with d_k boxes, each box corresponding to a nonzero element of $V_k \cap \mathfrak{g}(j)$ for j such that $V_k \cap \mathfrak{g}(j) \neq \{0\}$. We organize the rows so that the jth column corresponds to a generator of $V_k \cap \mathfrak{g}(j)$. Then the boxes appearing on the right position of each row lie in $\mathfrak{g}^e$.*

Examples. 1) *Let us consider the element* $e = \mathrm{diag}(J_3, J_1)$ *of* $\mathfrak{sl}_4(\mathbb{C})$. *Here, we get* $\dim \mathfrak{g}(0) = 5$, $\dim \mathfrak{g}(1) = 0$, $\dim \mathfrak{g}(2) = 4$ *and* $\dim \mathfrak{g}(4) = 1$.[4]

2) *Let us consider the element* $e = \mathrm{diag}(J_2, J_1, J_1)$ *of* $\mathfrak{sl}_4(\mathbb{C})$ *which lies in the minimal nilpotent orbit of* $\mathfrak{sl}_4$. *Here, we get* $\dim \mathfrak{g}(0) = 5$, $\dim \mathfrak{g}(1) = 4$, $\dim \mathfrak{g}(2) = 1$.[5]

We represent in Figure 2 the corresponding pictures for the above examples. In this figure, the boxes marked with a ■ *correspond to nonzero elements lying in* $\mathfrak{g}^e$*; the red boxes correspond to nonzero elements lying in* $[e, \mathfrak{g}]$.

1)


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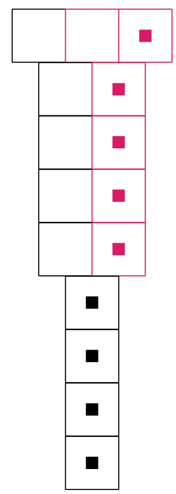


Figure 2. Examples of decompositions into $\mathbb{C}\langle e, h, f\rangle$-modules in $\mathfrak{sl}_n(\mathbb{C})$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}(0)$ which is also a Cartan subalgebra of $\mathfrak{g}$, and let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

be the corresponding root decomposition of $(\mathfrak{g}, \mathfrak{h})$ where Δ is the root system of $(\mathfrak{g}, \mathfrak{h})$.

Lemma 4.4. (i) *For any* $\alpha \in \Delta$, $\mathfrak{g}_\alpha$ *is contained in* $\mathfrak{g}(j)$ *for some* $j \in \mathbb{Z}$.

(ii) *Fix a positive root system* Δ_0^+ *of* $(\mathfrak{g}(0), \mathfrak{h})$. *Then* $\Delta^+ := \Delta_0^+ \cup \{\alpha \mid \mathfrak{g}_\alpha \subseteq \mathfrak{g}(> 0)\}$ *is a positive root system of* $(\mathfrak{g}, \mathfrak{h})$.

Denoting by Π the set of simple roots of Δ^+, we get

$$\Pi = \bigcup_{j\in\mathbb{Z}} \Pi_j \quad \text{with} \quad \Pi_j := \{\alpha \in \Pi \mid \mathfrak{g}_\alpha \subseteq \mathfrak{g}(j)\}.$$

Lemma 4.5. *We have* $\Pi = \Pi_0 \cup \Pi_1 \cup \Pi_2$.

[4] This is an example of ***even nilpotent element***, which means that $\mathfrak{g}(i) = \{0\}$ for all odd i. The nilpotent orbit of an even nilpotent element is called an ***even nilpotent orbit***. Note that the regular nilpotent orbit is always even.

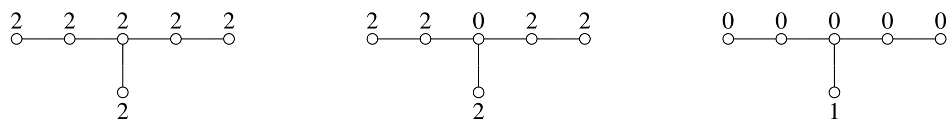
[5] We observe that $\oplus_{i\geqslant 2}\mathfrak{g}(i)$ equals $\mathfrak{g}(2)$ and has dimension 1. This is actually a general fact: if e lies in the minimal nilpotent orbit of any simple $\mathfrak{g}$, then $\oplus_{i\geqslant 2}\mathfrak{g}(i) = \mathfrak{g}(2) = \mathbb{C}e$ and $\oplus_{i\geqslant 2}\mathfrak{g}(i)$ has thus dimension 1.

Proof. Assume that there exists $\beta \in \Pi_s$ for $s > 2$. A contradiction is expected. Since $e \in \mathfrak{g}(2)$ and since $\mathfrak{g}(2)$ is contained in the subalgebra generated by the root spaces $\mathfrak{g}_\alpha$ with $\alpha \in \Pi_0 \cup \Pi_1 \cup \Pi_2$, we get $[e, \mathfrak{g}_{-\beta}] = \{0\}$. In other words, $\mathfrak{g}_{-\beta} \subseteq \mathfrak{g}^e$. This contradicts the fact that $\mathfrak{g}^e \subseteq \mathfrak{g}(\geqslant 0)$. □

From Lemma 4.5 we define the ***weighted Dynkin diagram***, or ***characteristic***, of the nilpotent orbit $G.e$ when $\mathfrak{g}$ is simple as follows. Consider the Dynkin diagram of the simple Lie algebra $\mathfrak{g}$. Each node of this diagram corresponds to a simple root $\alpha \in \Pi$. Then the weighted Dynkin diagram is obtained by labeling the node corresponding to α with the value $\alpha(h) \in \{0, 1, 2\}$.

By convention, the zero orbit has a weighted Dynkin diagram with every node labeled with 0.

Example 4.6. *In type E_6, the characteristics of the regular, subregular and minimal nilpotent orbits are respectively:*



An important consequence of Lemma 4.5 is that there are only finitely many nilpotent orbits, namely at most $3^{\operatorname{rank}\mathfrak{g}}$. Also, the weighted Dynkin diagram is a complete invariant, i.e., two such diagrams are equal if and only if the corresponding nilpotent orbits are equal, [CM93, Thm. 3.5.4].

The regular nilpotent orbit always corresponds to the weighted Dynkin diagram with only 2's (this result is not obvious, cf. e.g., [CM93, Thm. 4.1.6]). More generally, a nilpotent orbit is even if and only if the weighted Dynkin diagram have only 2's or 0's (see Remark 4.3 for the definition of *even*).

5. Good gradings

Let $\Gamma : \mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ be a $\mathbb{Z}$-grading.

5.1. The following definition is due to Brundan and Goodwin, [BG05, §5], and generalizes the notion of good gradings of [KRW03] (see also [FORTW92, §3.3]):

Definition 5.1. *Let $a \in \mathbb{N}$, with $a \geqslant 1$. The $\mathbb{Z}$-gradation Γ is said to be* ***a*-good for *e*** *if $e \in \mathfrak{g}_a$ and if the map $\operatorname{ad} e : \mathfrak{g}_j \to \mathfrak{g}_{j+a}$ is injective for any $j \leqslant -\frac{1}{2}a$ and surjective for any $j \geqslant -\frac{1}{2}a$. Our convention is that $\mathfrak{g}_r = \{0\}$ if $r \in \mathbb{R} \setminus \mathbb{Z}$.*

For the 2-good gradings, such a grading is simpler called a **good grading for *e***.

A classification of good gradings can be found in [EK05]; see also [BG05]. In particular, for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, there is a nice combinatorial description of good gradings in term of some diagrams in the plane called *pyramids*. From this, one sees for instance that if the nilpotent orbit of $e \in \mathfrak{sl}_n(\mathbb{C})$ is associated with a "rectangular partition", i.e., of the form $\lambda = (p^r)$ with $pr = n$, then the only good gradings are the Dynkin gradings.

By [EK05, Thm. 2.1], even good gradings, i.e., whose odd terms are zero, correspond to nice parabolic subalgebras that have been independently classified by Baur and Wallach, [BW05]. In [BG05], the authors also consider the notion of $\mathbb{R}$-***good gradings***.

Example 5.2. *Verify that the Dynkin gradings are good gradings.*

Hint. The "injectivity" results from the inclusion $\mathfrak{g}^e \subset \mathfrak{g}(\geqslant 0)$; the "surjectivity" can be seen using the picture as in Remark 4.3. □

Example 5.3. *Let* $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ *and* $e = E_{1,3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Consider the* $\mathbb{Z}$*-grading* Γ *defined by the semisimple* $h_\Gamma = \frac{1}{3}\mathrm{diag}\,(2, 2, -4)$. *The elementary matrices are homogeneous and their degrees are given in the following matrix:*

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix}.$$

The grading Γ *is good for* e*, but it is not Dynkin. Indeed,* e *is in the minimal nilpotent orbit of* $\mathfrak{g}$ *but* $\dim \mathfrak{g}_2 > 1$ *whence a contradiction (see the footnote 5 in Example 2 of Remark 4.3).*

Remark 5.4. *Assume that* Γ *is good for* e*. Then* $\mathfrak{g}^e \subset \mathfrak{g}_{\geqslant 0}$ *and* $\dim \mathfrak{g}^e = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1$*; see for instance* [W10, Prop. 5] *or* [EK05, Thm. 1.4].

5.2. Assume for the rest of the section that Γ is a-good for e for $a \in \mathbb{Z}$, with $a \geqslant 2$. Choose $\langle\, ,\rangle$ so that

$$\langle e, f\rangle = 1,$$

set

$$\chi := \kappa(e) = \langle e, .\rangle$$

and consider the antisymmetric bilinear form,

$$\omega_\chi \;:\; \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \qquad (x, y) \mapsto \langle e, [x, y]\rangle.$$

Exercise 1. *Show that the restriction to* $\mathfrak{g}_{-\frac{1}{2}a} \times \mathfrak{g}_{-\frac{1}{2}a}$ *of* ω_χ *is nondegenerate.*

Hint. Use the paring between $\mathfrak{g}_{\frac{1}{2}a}$ and $\mathfrak{g}_{-\frac{1}{2}a}$ and the injectivity of the map $\mathrm{ad}\, e : \mathfrak{g}_{-\frac{1}{2}a} \to \mathfrak{g}_{\frac{1}{2}a}$. □

Let $\mathfrak{l}$ be a Lagrangian subspace of $\mathfrak{g}_{-\frac{1}{2}a}$, that is $\omega_\chi(\mathfrak{l}, \mathfrak{l}) = \{0\}$ and $\dim \mathfrak{l} = \frac{1}{2} \dim \mathfrak{g}_{-\frac{1}{2}a}$, and set

$$\mathfrak{m}_{\chi,\mathfrak{l}} := \mathfrak{l} \oplus \bigoplus_{j < -\frac{1}{2}a} \mathfrak{g}_j.$$

Then $\mathfrak{m}_{\chi,\mathfrak{l}}$ is an ad-nilpotent[6] Γ-graded subalgebra of $\mathfrak{g}$. Moreover, the algebra $\mathfrak{m} := \mathfrak{m}_{\chi,\mathfrak{l}}$ verifies the following properties (see Exercise 2 for more general properties):

(χ1) $\chi([\mathfrak{m}, \mathfrak{m}]) = \langle e, [\mathfrak{m}, \mathfrak{m}]\rangle = \{0\}$;
(χ2) $\mathfrak{m} \cap \mathfrak{g}^e = \{0\}$;
(χ3) $\dim \mathfrak{m} = \frac{1}{2} \dim G.e$.

[6] i.e., $\mathfrak{m}_{\chi,\mathfrak{l}}$ only consists in nilpotent elements of $\mathfrak{g}$.

Consider a slightly more general situation: let $\mathfrak{l}$ be an isotropic subspace of $\mathfrak{g}_{-\frac{1}{2}a}$, that is $\omega_\chi(\mathfrak{l},\mathfrak{l}) = \{0\}$, and set

$$\mathfrak{m}_{\chi,\mathfrak{l}} := \mathfrak{l} \oplus \bigoplus_{j<-\frac{1}{2}a} \mathfrak{g}_j, \qquad \mathfrak{n}_{\chi,\mathfrak{l}} := \mathfrak{l}^{\perp_{\omega_\chi}} \oplus \bigoplus_{j<-\frac{1}{2}a} \mathfrak{g}_j$$

where

$$\mathfrak{l}^{\perp_{\omega_\chi}} = \{x \in \mathfrak{g}_{-\frac{1}{2}a} \mid \omega_\chi(x,\mathfrak{l}) = \{0\}\}$$

is the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{g}_{-\frac{1}{2}a}$ with respect to the bilinear form ω_χ. Then the pair $(\mathfrak{m},\mathfrak{n})$, with $\mathfrak{m} := \mathfrak{m}_{\chi,\mathfrak{l}}$ and $\mathfrak{n} := \mathfrak{n}_{\chi,\mathfrak{l}}$, satisfies the following properties:

(A1) $e \in \mathfrak{g}_a$;
(A2) $\mathfrak{m}$ and $\mathfrak{n}$ are Γ-graded and $\mathfrak{g}_{\leqslant -a} \subseteq \mathfrak{m} \subseteq \mathfrak{n} \subseteq \mathfrak{g}_{<0}$;
(A3) $\mathfrak{m}^\perp \cap [\mathfrak{g}, e] = [\mathfrak{n}, e]$;
(A4) $\mathfrak{n} \cap \mathfrak{g}^e = \{0\}$;
(A5) $[\mathfrak{n},\mathfrak{m}] \subseteq \mathfrak{m}$;
(A6) $\dim \mathfrak{m} + \dim \mathfrak{n} = \dim \mathfrak{g} - \dim \mathfrak{g}^e$.

If $\mathfrak{l} = 0$, for instance if e is even, we will simply write $\mathfrak{m}_\chi$ for $\mathfrak{m}_{\chi,0}$.

Exercise 2. *Verify the above properties.*

Correction. See [Sa13, Prop. 1.2.9 and Rem. 1.2.10] or [Sa14, Prop. 2.12]. □

We rediscover that for $\mathfrak{l}$ Lagrangian in $\mathfrak{g}_{-\frac{1}{2}a}$, $\mathfrak{m}_{\chi,\mathfrak{l}}$ verifies the above properties $(\chi 1)$,$(\chi 2)$,$(\chi 3)$.

6. Admissible gradings

6.1. The above remarks leads us to the following definition which will be our basic setting in Part 2:

Definition 6.1 ([Sa14, Def. 2.2])**.** *Let $\mathfrak{m}$ and $\mathfrak{n}$ be two subalgebras of $\mathfrak{g}$. We say that the pair $(\mathfrak{m},\mathfrak{n})$ is* **admissible for *e*** *if there exist a $\mathbb{Z}$-grading $\Gamma : \mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ and an integer $a > 1$ such that the above properties (A1),–,(A6) are satisfied. In the particular case where $\mathfrak{m} = \mathfrak{n}$, we will say that the algebra $\mathfrak{m}$ is* **admissible for *e***.

We say that a $\mathbb{Z}$-grading $\Gamma : \mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ is **admissible for *e*** *if there exists an integer $a > 1$ such that $e \in \mathfrak{g}_a$ and if there exists an admissible pair with respect to this grading.*

Since e is fixed for the whole lecture, we will often quickly write *good*, *a-good*, *admissible*... omitting "for e".

We have already noticed that a-good gradings are admissible (see Exercise 2). Namely, the pair $(\mathfrak{m}_{\chi,\mathfrak{l}}, \mathfrak{n}_{\chi,\mathfrak{l}})$ constructed as in Exercise 2 from an isotropic subspace $\mathfrak{l} \subseteq \mathfrak{g}_{-\frac{1}{2}a}$ is admissible. Such a pair will be called ***a-good*** (or ***good, Dynkin*** if Γ is good, Dynkin). When $\mathfrak{l}$ is Lagrangian so that $\mathfrak{m}_{\chi,\mathfrak{l}} = \mathfrak{n}_{\chi,\mathfrak{l}}$ in such a pair, we will call $\mathfrak{m}_{\chi,\mathfrak{l}}$ an ***a-good subalgebra*** (or a ***good, Dynkin subalgebra*** if Γ is good, Dynkin).

Remark 6.2. *If $(\mathfrak{m},\mathfrak{n})$ is admissible, then the subalgebras $\mathfrak{m}$ and $\mathfrak{n}$ are* ad *-nilpotent and we have the following properties:*

(1) $\mathfrak{m}^\perp \subset \mathfrak{g}_{\leqslant a-1}$;

(2) $\chi([\mathfrak{n}, \mathfrak{m}]) = \{0\}$;
(3) $\dim \mathfrak{m} + \dim \mathfrak{n}$ *and* $\dim \mathfrak{m} - \dim \mathfrak{n}$ *are even integers.*

In particular, if $\mathfrak{m}$ *is admissible, then* $\mathfrak{m}$ *verifies the properties* ($\chi 1$), ($\chi 2$), ($\chi 3$) *of* §5.2.

Exercise 3. *Assume that* $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$ *and that* $e = E_{1,3} + E_{2,4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. *Consider the* $\mathbb{Z}$*-grading* Γ *with* $h_\Gamma = \frac{1}{3}\mathrm{diag}\,(3, 1, -1, -3)$ *and set*

$$\mathfrak{m} = \mathfrak{g}_{\leqslant -2}, \qquad \mathfrak{n} = \mathfrak{m} \oplus \mathbb{C}E_{2,1} + \mathbb{C}E_{3,2}.$$

1) *Verify that the pair* $(\mathfrak{m}, \mathfrak{n})$ *is admissible for* e *with* $a = 2$.
2) *Show that* Γ *is not good for* e.

Correction. (Ref.: [Sa14, Ex. 2.8] *or* [Sa13, Ex. 1.2.11] *for the details.)*

The elementary matrices are homogenous and their degrees are given by the following matrix:

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{pmatrix}.$$

In particular, $e \in \mathfrak{g}_2$. Then we verify by a direct computation that $(\mathfrak{m}, \mathfrak{n})$ is admissible with $a = 2$. Since the map $\mathrm{ad}\, e : \mathfrak{g}_{-1} \to \mathfrak{g}_1$ is not injective, Γ is not good for e. □

In the above example, the partition associated to e is (2^2). We have noticed that in the rectangular case, only the Dynkin gradings are good. Thus, we see that the situation is very different for admissible gradings.

It is not always possible to find an admissible pair $(\mathfrak{m}, \mathfrak{n})$ with $\mathfrak{m} = \mathfrak{g}_{\leqslant -a}$ as in Exercise 3; see for instance [Sa13, Ex. 1.3.9]. In addition, when it exists, the algebra $\mathfrak{n}$ such that $(\mathfrak{g}_{\leqslant -a}, \mathfrak{n})$ is admissible is not unique.

Lemma 6.3 ([TY05, Prop. 32.1.7])**.** *Let* $\Gamma : \mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ *be a* $\mathbb{Z}$*-grading of* $\mathfrak{g}$ *such that* $e \in \mathfrak{g}_a$*, with* $a > 1$. *There exist* $h \in \mathfrak{g}_0$ *and* $f \in \mathfrak{g}_{-a}$ *such that* (e, h, f) *is an* $\mathfrak{sl}_2$*-triple.*

Proof. Write

$$h = \sum_{j \in \mathbb{Z}} h_j \quad \text{and} \quad f = \sum_{j \in \mathbb{Z}} f_j,$$

with $h_j, f_j \in \mathfrak{g}_j$. Since $[h, e] = 2e$ we get $[h_0, e] = 2e$ and $[h_j, e] = 0$ for any $j \neq 0$. Moreover, since $[e, f] = h = \sum_{j \in \mathbb{Z}} [e, f_j]$, we get that $h_0 = [e, f_{-a}]$. Thus, $[h_0, e] = 2e$ and $h_0 \in [e, \mathfrak{g}]$. By Morosov's lemma (cf. e.g., [TY05, Lem. 32.1.3],

> *"if* $h, e \in \mathfrak{g}$ *are such that* $[h, e] = 2e$ *and* $h \in [e, \mathfrak{g}]$*, then there exists* $f \in \mathfrak{g}$ *such that* $[h, f] = -2f$ *and* $[e, f] = h$*",*

we deduce that for some $f' \in \mathfrak{g}$, (e, h_0, f') is an $\mathfrak{sl}_2$-triple. Writing $f' = \sum_{j\in\mathbb{Z}} f'_j$, with $f'_j \in \mathbb{Z}$, we obtain that

$$[h_0, f'_{-a}] = -2f'_a \quad \text{and} \quad [e, f'_a] = h_0,$$

and the the triple (e, h_0, f'_{-a}) does the job. $\square$

From now on, we let $\Gamma : \mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ be a $\mathbb{Z}$-grading of $\mathfrak{g}$ such that $e \in \mathfrak{g}_a$, with $a > 1$, and we let (e, h, f) be an $\mathfrak{sl}_2$-triple as in Lemma 6.3.

Exercise 4. 1) *Assume that e is* **distinguished**[7]. *Show that the Dynkin gradings are, up to homothety, the only admissible gradings for e. In other words, show that for admissible Γ, $h_\Gamma = kh$ for some $k \in \mathbb{N}^*$.*

2) *Assume that Γ is admissible for e (e arbitrary). Prove that the map $\operatorname{ad} e : \mathfrak{g}_{\geqslant 0} \to \mathfrak{g}_{\geqslant a}$ is surjective.*

Correction. 1) Let (e, h, f) be an $\mathfrak{sl}_2$-triple as in Lemma 6.3. The element $t := \frac{a}{2}h - h_\Gamma$ centralizes e and t is semsimple. Since e is distinguished, we get $t = 0$ and so $h_\Gamma = \frac{a}{2}h$ whence the statement.

2) As a first step, we show the following: let $k \in \mathbb{Z}$ such that the map $\operatorname{ad} e : \mathfrak{g}_k \to \mathfrak{g}_{k+a}$ is injective, then the map $\operatorname{ad} e : \mathfrak{g}_{-(k+a)} \to \mathfrak{g}_{-k}$ is surjective (see [Sa14, Lem. 2.10]). To this see, observe that $\mathfrak{g}_{k+a}$ and $\mathfrak{g}_{-(k+a)}$ are in pairing. So, for some $V \in \mathfrak{g}_{-(k+a)}$, $[e, \mathfrak{g}_k] \subset \mathfrak{g}_{k+a}$ and V are in paring. In particular, $\dim V = \dim[e, \mathfrak{g}_k] = \dim \mathfrak{g}_k$. By the invariance of the Killing form, we get a pairing between $[e, V]$ and $\mathfrak{g}_k$. We deduce that $[e, V] = \mathfrak{g}_{-k}$ for dimension reasons. Hence the map $\operatorname{ad} e : \mathfrak{g}_{-(k+a)} \to \mathfrak{g}_{-k}$ is surjective.

Next, since Γ is admissible $\mathfrak{g}_{\leqslant -a} \cap \mathfrak{g}^e = \{0\}$ and so the map $\operatorname{ad} e : \mathfrak{g}_k \to \mathfrak{g}_{k+a}$ is injective for any $k \leqslant -a$. By the first step, the map $\operatorname{ad} e : \mathfrak{g}_{-(k+a)} \to \mathfrak{g}_{-k}$ is then surjective for any $k \leqslant -a$, whence the statement. $\square$

6.2. The following result, obtained by Sadaka, gives a nice characterization of admissible gradings.

Theorem 6.4 ([Sa14, Thm. 2.14])**.** *The grading Γ is admissible for e if and only if $\mathfrak{g}_{\leqslant -a} \cap \mathfrak{g}^e = \{0\}$.*

Note that the condition for that a $\mathbb{Z}$-grading is admissible is much less restrictive than the condition for being a good grading.

Sketch of proof. The direct application is straightforward according to Definition 6.1. For the converse implication assume that $\mathfrak{g}_{\leqslant -a} \cap \mathfrak{g}^e = \{0\}$ and let

$$t := h_\Gamma - \frac{a}{2}h.$$

Then t is a semisimple of $\mathfrak{g}$ which centralizes the $\mathfrak{sl}_2$-triple (e, h, f). Furthermore, its eigenvalues are rational numbers. The idea is to construct an admissible pair $(\mathfrak{m}, \mathfrak{n})$ as follows. Consider the decomposition of $\mathfrak{g}$ in isotypic $\mathbb{C}\langle e, h, f\rangle$-modules

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_r.$$

[7] i.e., $\mathfrak{g}^e$ is only consisted of nilpotent elements of $\mathfrak{g}$ or, equivalently, $\mathfrak{g}^e \cap \mathfrak{g}^f = \{0\}$.

According to the Schur's lemma, this decomposition is orthogonal with respect to the Killing form. Since t commutes with $\mathbb{C}\langle e, h, f\rangle$, still according to the Schur's lemma, each isotypic component V_i is t-stable. Hence, for each i, the isotypic component V_i decomposes into $\operatorname{ad} t$-eigenspaces,

$$V_i = \bigoplus_{\lambda \in \mathbb{Q}} V_{i,\lambda}$$

such that $\langle\, , \rangle|_{V_{i,\lambda} \times V_{i,\mu}} = 0$ if $\lambda + \mu \neq 0$ and $\langle\, , \rangle|_{V_{i,\lambda} \times V_{i,-\lambda}}$ is nondegenerate. Set for any $\lambda \in \mathbb{Q}_{\geqslant 0}$,

$$W_{i,\lambda} := V_{i,\lambda} + V_{i,-\lambda}.$$

Then we get an orthogonal decomposition in t-stable subspaces of $\mathfrak{g}$,

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \bigoplus_{\lambda \in \mathbb{Q}_{\geqslant 0}} W_{i,\lambda}.$$

Then the idea is to construct for any $i \in \{1, \dots, r\}$ and any $\lambda \in \mathbb{Q}$, a pair $(\mathfrak{m}_{i,\lambda}, \mathfrak{n}_{i,\lambda})$ such that

(1) $W_{i,\lambda} \cap \mathfrak{g}_{\leqslant -a} \subseteq \mathfrak{m}_{i,\lambda} \subseteq \mathfrak{n}_{i,\lambda} \subseteq W_{i,\lambda} \cap \mathfrak{g}_{<0}$;
(2) $\mathfrak{m}_{i,\lambda}^{\perp} \cap [e, W_{i,\lambda}] = [e, \mathfrak{n}_{i,\lambda}]$;
(3) $\mathfrak{n}_{i,\lambda} \cap \mathfrak{g}^e = \{0\}$;
(4) $[\mathfrak{n}_{i,\lambda}, \mathfrak{m}_{i,\lambda}] \subseteq \mathfrak{m}_{i,\lambda}$;
(5) $\dim \mathfrak{m}_{i,\lambda} + \dim \mathfrak{n}_{i,\lambda} = \dim W_{i,\lambda} - \dim(W_{i,\lambda} \cap \mathfrak{g}^e)$.

Then we verify that the pair $(\mathfrak{m}, \mathfrak{n})$ is admissible for e, with

$$\mathfrak{m} := \bigoplus_{i=1}^{r} \bigoplus_{\lambda \in \mathbb{Q}_{\geqslant 0}} \mathfrak{m}_{i,\lambda} \quad \text{and} \quad \mathfrak{n} := \bigoplus_{i=1}^{r} \bigoplus_{\lambda \in \mathbb{Q}_{\geqslant 0}} \mathfrak{n}_{i,\lambda}.$$

The construction of the pairs $(\mathfrak{m}_{i,\lambda}, \mathfrak{n}_{i,\lambda})$ and the verifications are quite technical. We refer to [Sa14, Proof of Thm. 2.14] for the details. □

Remark 6.5. *If Γ is admissible for e, then one can adapt the arguments used for h to h_Γ and show that*

$$\Pi = \Pi_0 \cup \cdots \cup \Pi_a$$

if Π denotes a suitable simple root system with respect to $\mathfrak{g}^{h_\Gamma} = \mathfrak{g}_0$.

In this part, we have assumed that e is nonzero. If e is zero, our convention is that any $\mathbb{Z}$-grading with $\mathfrak{g}_0 = \mathfrak{g}$ is admissible for e.

Part 2. Finite W-algebras

We present in this part a family of algebras constructed from admissible pairs for a given nilpotent element $e \in \mathfrak{g}$, which includes the finite W-algebra $\mathcal{W}_\chi$ associated with e as introduced by Premet (cf. Introduction).

7. Definition(s)

Let e be a nilpotent element and fix an admissible pair $(\mathfrak{m},\mathfrak{n})$ of $\mathfrak{g}$ with respect to an admissible $\mathbb{Z}$-grading Γ. Set, as in §5.2,

$$\chi := \kappa(e) = \langle e, . \rangle.$$

Since $\chi([\mathfrak{m},\mathfrak{m}]) \subset \chi([\mathfrak{n},\mathfrak{m}]) = \{0\}$, the restriction to $\mathfrak{m}$ of χ is a character of $\mathfrak{m}$. Hence, it extends to a representation

$$\chi : U(\mathfrak{m}) \longrightarrow \mathbb{C}$$

and we denote by $\mathbb{C}_\chi$ the corresponding left $U(\mathfrak{m})$-module, $x.z := \chi(x)z$ for$x \in \mathfrak{m}$ and $z \in \mathbb{C}$. On the other hand, the right multiplication by an element of $\mathfrak{m}$ induces a right $U(\mathfrak{m})$-module on $U(\mathfrak{g})$. Denote by $I_\mathfrak{m}$ the left ideal of $U(\mathfrak{g})$ generated by the elements $x - \chi(x)$, for $x \in \mathfrak{m}$, and set

$$Q_\mathfrak{m} := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi \simeq U(\mathfrak{g})/I_\mathfrak{m}.$$

As an exercise, one can verify that indeed $Q_\mathfrak{m} \simeq U(\mathfrak{g})/I_\mathfrak{m}$ as an $U(\mathfrak{g})$-module.

In the case where $\mathfrak{m} = \mathfrak{m}_{\chi,\mathfrak{l}}$ is a Dynkin subalgebra, with $\mathfrak{l}$ Lagrangian, $Q_\mathfrak{m}$ is called a ***generalized Gelfand-Graev module*** (cf. e.g., [Ka85, Ya85, Mo86, Ma90]).

7.1. **Definition via Whittaker models.** The adjoint action of $\mathfrak{n}$ in $\mathfrak{g}$ uniquely extends to an action, still denoted by ad, of $\mathfrak{n}$ in $U(\mathfrak{g})$ and the ideal $I_\mathfrak{m}$ is $\mathfrak{n}$-stable. Indeed, for any $x \in \mathfrak{m}$, $y \in \mathfrak{n}$ and $u \in U(\mathfrak{g})$,

$$[y, u(x - \chi(x))] = [y,u](x - \chi(x)) + u[y, x - \chi(x)] = [y,u](x - \chi(x)) + u[y,x]$$

is in $I_\mathfrak{m}$ since $\chi([\mathfrak{n},\mathfrak{m}]) = \{0\}$. Thus $Q_\mathfrak{m}$ can be endowed with a $\mathfrak{n}$-module structure by setting:

$$\forall\, y \in \mathfrak{n},\ \forall\, u \in U(\mathfrak{g}), \quad y.(u + I_\mathfrak{m}) := [y,u] + I_\mathfrak{m}.$$

We set

(2) $$\mathcal{W}_{\mathfrak{m},\mathfrak{n}} := Q_\mathfrak{m}^{\mathrm{ad}\,\mathfrak{n}} \quad = \quad \{\bar{u} \in Q_\mathfrak{m} \mid [y,u] \in I_\mathfrak{m} \text{ for any } y \in \mathfrak{n}\}$$

where $\bar{u}$ denotes the coset $u + I_\mathfrak{m}$ of $u \in U(\mathfrak{g})$.

Definition 7.1. *The algebra $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$, whose algebra structure is given by*

$$\forall\, u, v \in U(\mathfrak{g}), \qquad \bar{u}\bar{v} = \overline{uv},$$

is called the **finite W-algebra associated with $(\mathfrak{m},\mathfrak{n})$**.

We refer the above definition of $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ as the ***Whittaker model realization*** of $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$.

Remark 7.2. *When $\mathfrak{m} = \mathfrak{n}$, the algebra $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ is actually the space of* Whittaker vectors *(see further Definition 11.1) of $Q_\mathfrak{m}$,*

$$\begin{aligned} \mathcal{W}_{\mathfrak{m},\mathfrak{m}} = \mathrm{Wh}(Q_\mathfrak{m}) &= \{\bar{u} \in Q_\mathfrak{m} \mid (x - \chi(x))\bar{u} = 0 \textit{ for any } x \in \mathfrak{m}\} \\ &= \{\bar{u} \in Q_\mathfrak{m} \mid x\bar{u} = \chi(x)\bar{u} \textit{ for any } x \in \mathfrak{m}\} \end{aligned}$$

since $[x,u] = xu - ux = (x - \chi(x))u + u(\chi(x) - x) \in (x - \chi(x))u + I_\mathfrak{m}$ for any $u \in U(\mathfrak{g})$ and $x \in \mathfrak{m}$.

Example 7.3. *Assume $e = 0$. Then $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{m} = \mathfrak{n} = 0$, $Q_\mathfrak{m} = U(\mathfrak{g})$ and $\mathcal{W}_{\mathfrak{m},\mathfrak{n}} = U(\mathfrak{g})$.*

If $(\mathfrak{m},\mathfrak{n}) = (\mathfrak{m}_{\chi,\mathfrak{l}},\mathfrak{n}_{\chi,\mathfrak{l}})$ is a good pair attached to some isotropic subspace $\mathfrak{l}\subset\mathfrak{g}_{-1}$, then we set following usual notations,

$$\mathcal{W}_\chi := \mathcal{W}_{\mathfrak{m}_{\chi,\mathfrak{l}},\mathfrak{n}_{\chi,\mathfrak{l}}}.$$

The notation should refer to the good grading Γ and the isotropic space $\mathfrak{l}$. By [GG02], the algebra $\mathcal{W}_\chi$ does not depend, up to isomorphism, on the choice of the isotropic subspace $\mathfrak{l}$ in $\mathfrak{g}_{-1}$ if Γ is Dynkin. Furthermore, according to the main result of [BG05], the algebra $\mathcal{W}_\chi$ does not depend, up to isomorphism, on the choice of the good grading Γ.

For an admissible pair $(\mathfrak{m},\mathfrak{n})$, we have the following statement that we will discuss in Part 3 (see [Sa14, Thm. 5] or here Theorem 14.9). Let $b\in\mathbb{Q}_{>0}$. We say that the admissible grading Γ is ***b-optimal***[8] if $\mathfrak{g}_{<-\frac{1}{2}b}\cap\mathfrak{g}^e=\{0\}$ and if $e\in\mathfrak{g}_a$ for some $a\in\mathbb{N}$, with $a\geqslant 2$ and $a\geqslant b$.

Theorem 7.4 (Sadaka, 2013). *Let $b\in\mathbb{Q}_{>0}$ and let $(\mathfrak{m},\mathfrak{n})$ be an admissible pair with respect to a b-optimal admissible grading. Then $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ is isomorphic to $\mathcal{W}_\chi$.*

As a consequence, the a-good pairs constructed from an isotropic subspace in $\mathfrak{g}_{-\frac{1}{2}a}$ (see §5.2) lead to the finite W-algebra $\mathcal{W}_\chi$ up to isomorphism. This provides a new proof of the main result of [BG05]. But there does not always exist b-optimal gradings; see e.g., [Sa14, Ex. 2.14].

The problem for arbitrary admissible pairs is still open. More precisely, we do not know so far if for an arbitrary admissible pair $(\mathfrak{m},\mathfrak{n})$, the algebras $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ and $\mathcal{W}_\chi$ are isomorphic, except for some particular cases. These "isomorphism problems" will be addressed in more detail in Part 3.

7.2. **Premet's definition.** We assume in this paragraph that $\mathfrak{m}$ is an admissible subalgebra, i.e., $\mathfrak{m}=\mathfrak{n}$.

Definition 7.5. *The* **finite *W*-algebra associated with *m*** *is defined to be the endomorphism algebra*[9]

$$\mathcal{W}_{\mathfrak{m}} := \mathrm{End}_{U(\mathfrak{g})}(Q_{\mathfrak{m}})^{\mathrm{op}}. \tag{3}$$

Exercise 5. *Show that for $\mathfrak{m}=\mathfrak{n}$, we get $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}\simeq\mathcal{W}_{\mathfrak{m}}$. In other words, show that the equalities (2) and (3) lead to the same definition.*

Correction. Any endomorphism of the $\mathfrak{g}$-module $Q_{\mathfrak{m}}$ is determined by the image of $\bar{1}=1+I_{\mathfrak{m}}$. Indeed, if $f(\bar{1})=\bar{u}$, for $f\in\mathrm{End}_{\mathfrak{g}}(Q_{\mathfrak{m}})$, then $f(\bar{v})=f(v\bar{1})=v\bar{u}=\bar{v}\bar{u}$ for any $v\in U(\mathfrak{g})$, since $I_{\mathfrak{m}}u$ must be zero.

Since a representative of the image of $\bar{1}$ must be annihilated by the ideal $I_{\mathfrak{m}}$, we obtain the following identification

$$\begin{aligned}\mathcal{W}_{\mathfrak{m}} &\simeq \{\bar{u}\in U(\mathfrak{g})/I_{\mathfrak{m}} \mid (x-\chi(x))u\in I_{\mathfrak{m}} \text{ for any } x\in\mathfrak{m}\}\\ &= \{\bar{u}\in Q_{\mathfrak{m}} \mid [x,u]\in I_{\mathfrak{m}} \text{ for any } x\in\mathfrak{m}\}.\end{aligned}$$

The algebra isomorphism if given by

$$\phi\ :\ Q_{\mathfrak{m}}^{\mathrm{ad}\,\mathfrak{m}} \longrightarrow \mathrm{End}_{\mathfrak{g}}(Q_{\mathfrak{m}})^{\mathrm{op}},\qquad \bar{u}\longmapsto(\bar{y}\to\bar{y}\bar{u}=\overline{yu})$$

[8] Our definition corresponds to $\frac{b}{2}$-optimal gradings in [Sa13, Sa14].

[9] The symbol "op" means that we consider the ring $\mathrm{End}_{U(\mathfrak{g})}(Q_{\mathfrak{m}})$ with "reversed" composition operation $u.v := v\circ u$.

so that the op operation is respected. Note that the inverse of this isomorphism is given by

$$\mathrm{End}_{\mathfrak{g}}(Q_{\mathfrak{m}})^{\mathrm{op}} \longrightarrow Q_{\mathfrak{m}}^{\mathrm{ad}\,\mathfrak{m}}, \quad h \longmapsto h(1 \otimes 1).$$

□

According to Exercise 5, one is legitimated to denote by $\mathcal{W}_{\mathfrak{m}}$ the algebra $\mathcal{W}_{\mathfrak{m},\mathfrak{m}}$ if $\mathfrak{m}$ is admissible.

Exercise 6. *Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Show that the restriction to $Z(\mathfrak{g})$ of the representation*

$$\varrho_{\mathfrak{m}} \;:\; U(\mathfrak{g}) \longrightarrow \mathrm{End}_{\mathbb{C}}(Q_{\mathfrak{m}}).$$

is injective.

Correction. (Ref.: [P02, §6.1].*)*

Consider a triangular decomposition,

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are opposite radical nilpotents of opposite Borel subalgebras containing $\mathfrak{h}$ such that $\mathfrak{n}_+ \supseteq \mathfrak{m}$. Then consider the Harish-Chandra projection

$$\mathrm{hc} \;:\; U(\mathfrak{g}) \longrightarrow U(\mathfrak{h}) \simeq S(\mathfrak{h})$$

with respect to the decomposition

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+).$$

Its restriction to $U(\mathfrak{g})^{\mathfrak{h}}$ is an algebra homomorphism and its restriction to $Z(\mathfrak{g})$ is injective. From this, one can show that the restriction of $\varrho_{\mathfrak{m}}$ to $Z(\mathfrak{g})$ is injective: if an element $z \in Z(\mathfrak{g})$ is in $\ker \varrho_{\mathfrak{m}}$, then $z \in I_{\mathfrak{m}} \cap Z(\mathfrak{g})$ and must be 0. □

According to above exercise, we get an inclusion map,

$$Z(\mathfrak{g}) \hookrightarrow \mathcal{W}_{\mathfrak{m}}.$$

As explained in the footnote of [P07, Quest. 5.1], the above map is surjective onto the center $Z(\mathcal{W}_\chi)$ of $\mathcal{W}_\chi$ so that we get an algebra isomorphism

$$Z(\mathfrak{g}) \simeq Z(\mathcal{W}_\chi).$$

According to a result of Kostant, if e is regular then $\mathcal{W}_\chi$ is isomorphic to $Z(\mathfrak{g})$, which is known to be a polynomial algebra in rank of $\mathfrak{g}$ variables; see §12.

7.3. **Alternative description for even good gradings.** *(Ref.: cf.* [W10, §3.3]. *See also* [BK06, §8] *for more details.)*

We assume in this paragraph that Γ is good and ***even***, i.e., $\mathfrak{g}_j = \{0\}$ for any odd j, and that

$$\mathfrak{m} = \mathfrak{n} = \oplus_{i \leqslant -2} \mathfrak{g}_i.$$

We give here a simplified description of $\mathcal{W}_\chi$ in this case. Set

$$\mathfrak{p} := \oplus_{i \geqslant 0} \mathfrak{g}_i.$$

Then

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{m}$$

and $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. It follows from the PBW theorem that

$$U(\mathfrak{g}) = U(\mathfrak{p}) \oplus I_{\mathfrak{m}}.$$

The projection $U(\mathfrak{g}) \to U(\mathfrak{p})$ with respect to this decomposition induces an isomorphism

$$\overline{\mathrm{pr}}_{\mathfrak{m}} : Q_{\mathfrak{m}} = U(\mathfrak{g})/I_{\mathfrak{m}} \longrightarrow U(\mathfrak{p}).$$

Thereby, since $\mathcal{W}_\chi \simeq Q_{\mathfrak{m}}^{\mathrm{ad}\,\mathfrak{n}} \subset Q_{\mathfrak{m}}$, one can view $\mathcal{W}_\chi$ as a subalgebra of $U(\mathfrak{p})$. Define a action of $\mathfrak{m}$ on $U(\mathfrak{p})$, called the ***χ-twisted adjoint action of $\mathfrak{m}$***, by

$$\forall\, y \in U(\mathfrak{p}),\ \forall\, x \in \mathfrak{m}, \quad x.y := \overline{\mathrm{pr}}_{\mathfrak{m}}([x, y]).$$

Then for any $y \in U(\mathfrak{p})$ and any $x \in \mathfrak{m}$,

$$\overline{\mathrm{pr}}_{\mathfrak{m}}(x.(y + I_{\mathfrak{m}})) = \overline{\mathrm{pr}}_{\chi}([x, y]) = x.y$$

so that $\overline{\mathrm{pr}}_{\mathfrak{m}}$ is an isomorphism of $\mathfrak{m}$-modules, and we get by (2),

$$\begin{aligned} \mathcal{W}_\chi &= U(\mathfrak{p})^{\mathrm{ad}\,\mathfrak{m}} := \{y \in U(\mathfrak{p}) \mid x.y = 0 \text{ for any } x \in \mathfrak{m}\} \\ &= \{y \in U(\mathfrak{p}) \mid [x, y] \in I_{\mathfrak{m}} \text{ for any } x \in \mathfrak{m}\}. \end{aligned} \tag{4}$$

Thus, in this special case, on can take the equality (4) as a third definition for $\mathcal{W}_\chi$. This was the original definition of Kostant and Lynch [K78, L79], for Γ a Dynkin grading. It was observed by Brundan and Kleshchev, [BK06], that the definition works as well for any even good gradings. There is also a similar version to the definition (4) for any good grading, not necessarily even, given by Brundan-Goodwin-Kleshchev [BGK08, Sec. 1] which is more complicated.

Exercise 7. *Let* $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *which is a regular nilpotent element of* $\mathfrak{sl}_2$. *Show that*

$$\mathcal{W}_\chi \simeq \mathbb{C}[e + \frac{1}{4}h^2 - \frac{1}{2}h].$$

Correction. Setting $f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, (e, h, f) is an $\mathfrak{sl}_2$-triple of $\mathfrak{sl}_2$ and we get

$$\mathfrak{m} = \mathbb{C}f, \qquad \chi(f), \qquad \mathfrak{p} = \mathbb{C}h + \mathbb{C}e.$$

Remind also that $\langle e, f\rangle = 1$ by our choice of $\langle\,,\rangle$. A direct computation shows that $e + \frac{1}{4}h^2 - \frac{1}{2}h$ lies in $U(\mathfrak{p})^{\mathrm{ad}\,\mathfrak{m}}$.

Since $Z(\mathfrak{sl}_2)$ is a polynomial algebra of dimension one by Kostant's theorem, we deduce from §12 that

$$\mathcal{W}_\chi \simeq \mathbb{C}[e + \frac{1}{4}h^2 - \frac{1}{2}h].$$

Remark: how to think of the above element as a good candidate? It is well-known that the ***Casimir element*** Ω lies in the center $Z(\mathfrak{g})$, [Hu08, §0.5]. This element is defined as follows: let $\{x_i\}_{i=1}^n$ and $\{x^i\}_{i=1}^n$ be dual basis of $\mathfrak{g}$ with respect to the Killing form, then $\Omega := \sum_{i=}^n x_i x^i$. Here, we get $\Omega = h^2 - 2h + 4ef$. Now, observe that

$$\Omega = h^2 - 2h + 4ef = h^2 - 2h + 4e + 4e(f - \chi(f))$$

since $\chi(f) = \langle e, f\rangle = 1$. Hence, its image by the projection $\overline{\mathrm{pr}}_{\mathfrak{m}}$ is

$$h^2 - 2h + 4e = 4(e + \frac{1}{4}h^2 - \frac{1}{2}h).$$

$\square$

7.4. **Remarks on the other definitions.** There are yet other definitions that we just mention here without explanation:

$*$ The BRST definition (from the names Becchi, Rouet, Stora and Tyutin), or quantum Hamiltonian reduction definition, [D^{3}HK06] (or [W10, §3.4] for a simplified presentation) uses the BRST complex. The equivalence of this definition with the Whittaker model definition was proved in [D^{3}HK06]. This result was independently proved in [Ar07]. Actually, in [Ar07], the result is stated only for the regular case, but the proof can be adapted to the general case.

$*$ There is also a definition due to Losev, [L10b], via deformation quantization.

$*$ In type **A**, we have a pleasant description of the finite algebra $\mathcal{W}_\chi$ as a quotient of shifted Yangians, [BK06]. The rectangular case in type **A** was first obtained by Ragoucy and Sorba, [RS99]; see also [Br09] for a generalization of [RS99] to the other classical Lie algebras. In the rectangular case in type **A**, note that there is also analogue description for the affine W-algebras, recently obtained by Arakawa and Molev, [AM14].

We refer to [BGK08] for a nice presentation of three equivalent definitions of finite W-algebras.

8. Transvsersal slices

We follow here the arguments of [GG02] to prove that $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ is a quantization of some transversal slice. In [GG02], the authors consider Dynkin gradings. Here, we state the results in the more general context of admissible gradings, [Sa14, §3].

For the rest of the section, we fix an integer $a \geqslant 2$ and an admissible grading $\Gamma : \mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$, with $e \in \mathfrak{g}_a$.

8.1. Since $\chi([\mathfrak{n}, \mathfrak{m}]) = \{0\}$, we have $[\mathfrak{n}, e] \subseteq \mathfrak{m}^\perp$ and we can choose a complement subspace $\mathfrak{s}$ of $[\mathfrak{n}, e]$ in $\mathfrak{m}^\perp$,

$$\mathfrak{m}^\perp = [\mathfrak{n}, e] \oplus \mathfrak{s}.$$

From now on, we fix such a space $\mathfrak{s}$.

Exercise 8. 1) *Show that* $\mathfrak{g} = [\mathfrak{g}, e] \oplus \mathfrak{s}$.

2) *In the case where* Γ *is a good grading, show that* $\mathfrak{s} = \mathfrak{g}^f$ *is suitable.*

Remark 8.1. *In general,* $\mathfrak{g}^f$ *is not contained in* $\mathfrak{m}^\perp$. *For example, let* $e = E_{1,3} + E_{3,4}$ *be a nilpotent element of* $\mathfrak{sl}_4$ *associated with the partition* (2^2). *Then the algebra*

$$\mathfrak{m} = \mathrm{span}(E_{2,1}, E_{2,3}, E_{2,4}, E_{3,1}, E_{4,1}, E_{4,3})$$

is admissible (this must be verified!) for e *but* $\mathfrak{g}^f$ *is not contained in* $\mathfrak{m}^\perp$.

Let us introduce a $\mathbb{C}^*$-action on $\mathfrak{g}$ which stabilizes $e+\mathfrak{s}$. Let $\gamma : \mathbb{C}^* \to G$ be the one-parameter subgroup associated with h_Γ. Then for any $x \in \mathfrak{g}_j$, $\gamma(t)x = t^j x$. In particular, $\gamma(t)e = t^a x$. Define a $\mathbb{C}^*$-action on $\mathfrak{g}$ as follows: for any $t \in \mathbb{C}^*$ and $x \in \mathfrak{g}$, we set

$$\rho(t)x := t^a \gamma(t^{-1})(x). \tag{5}$$

So, for any $x \in \mathfrak{g}_j$, $\rho(t)x = t^{a-j}x$. In particular,

$$\rho(t)e = e.$$

Since $\mathfrak{s}$ is Γ-graded, it is ρ-stable and the action ρ stabilizes $e+\mathfrak{s}$. Moreover, it is contracting on $e+\mathfrak{s}$, i.e., $\lim_{t\to 0}\rho(t)(e+x) = e$ for any $x \in \mathfrak{s}$, because $\mathfrak{s} \subset \mathfrak{m}^\perp \subseteq \mathfrak{g}_{\leqslant a-1}$.

The same lines of arguments show that the action ρ stabilizes $e+\mathfrak{m}^\perp$ and it is contracting on $e+\mathfrak{m}^\perp$, too.

8.2. Set

$$\mathcal{S} := \chi + \kappa(\mathfrak{s}) \subset \mathfrak{g}^*.$$

In the case where $\mathfrak{s} = \mathfrak{g}^f$, we call the affine variety $\mathcal{S}$ *the* ***Slodowy slice associated with e***. The affine space $\mathcal{S}$ is indeed a "slice" according to the following result:

Theorem 8.2. *The affine space $\mathcal{S}$ is transversal to the coadjoint orbits of $\mathfrak{g}^*$. More precisely, for any $\xi \in \mathcal{S}$, one has $\mathrm{T}_\xi(G.\xi) + \mathrm{T}_\xi(\mathcal{S}) = \mathfrak{g}^*$. An analogue statement holds for the affine variety $\chi + \kappa(\mathfrak{m}^\perp)$.*

Sketch of proof. (Ref.: [GG02, §2.2] *or* [Sa14, Thm. 3.5].*)*

Identify $\mathfrak{g}$ and $\mathfrak{g}^*$ through κ. Then we have to prove that $[\mathfrak{g}, x] + \mathfrak{s} = \mathfrak{g}$ for any $x \in e+\mathfrak{s}$ since $\mathrm{T}_x(G.x) = [\mathfrak{g}, x]$ and $\mathrm{T}_x(e+\mathfrak{s}) = \mathfrak{s}$. First, we verify that the map

$$\eta : G \times (e+\mathfrak{s}) \to \mathfrak{g}$$

is a submersion[10] at any point of $G\times\Omega$ where Ω is an open neighborhood of e in $e+\mathfrak{s}$. In particular, for any $x \in \Omega$,

$$\mathfrak{g} = [\mathfrak{g}, x] + \mathfrak{s}$$

Next, we use the contracting $\mathbb{C}^*$-action ρ on $e+\mathfrak{s}$ to show that η is actually a submersion at any point of $G \times (e+\mathfrak{s})$. □

8.3. Let N be the closed connected subgroup of G with Lie algebra $\mathfrak{n}$, and consider the adjoint map

$$N \times (e+\mathfrak{m}^\perp) \to \mathfrak{g}, \quad (g, x) \mapsto g.x$$

It image is contained in $e+\mathfrak{m}^\perp$. Indeed, for any $x \in \mathfrak{n}$ and any $y \in \mathfrak{m}^\perp$, $\exp(\operatorname{ad} x)(e+y) \in e+\mathfrak{m}^\perp$ since $[\mathfrak{n}, \mathfrak{m}] \subseteq \mathfrak{m}$ and $\chi([\mathfrak{n}, \mathfrak{m}]) = \{0\}$, and this is enough to conclude because, $\mathfrak{n}$ being ad-nilpotent, N is generated by the elements $\exp(\operatorname{ad} x)$ for x running through $\mathfrak{n}$.

As a result, by restriction, we get a map

$$\alpha \ : \ N \times (e+\mathfrak{s}) \to e+\mathfrak{m}^\perp.$$

[10] η is a submersion at a point $(g, x) \in G \times (e+\mathfrak{s})$ if the differential of η at (g, x), that is the linear map $\mathfrak{g} \times \mathfrak{s} \to \mathfrak{g}$, $(v, w) \mapsto g([v, x]) + g(w)$, is surjective.

Theorem 8.3. *The map α is an isomorphism of affine varieties.*

Sketch of proof. (Ref.: [GG02, §2.3] *or* [Sa14, Thm. 3.8]*.)*

We have a contracting $\mathbb{C}^*$-action on $N \times (e + \mathfrak{s})$ defined by:

$$\forall\, t \in \mathbb{C}^*,\ \forall\, g \in N, \forall\, x \in e + \mathfrak{s}, \qquad t.(g, x) := (\gamma(t^{-1})g\gamma(t), \rho(t)x)$$

and α is $\mathbb{C}^*$-equivariant with respect to this action and the preceding $\mathbb{C}^*$-action on $e + \mathfrak{m}^\perp$.

Then we conclude thanks to the following result, formulated in [GG02, Proof of Lem. 2.1]:

> *A $\mathbb{C}^*$-equivariant morphism $\alpha : X_1 \to X_2$ of smooth affine $\mathbb{C}^*$-varieties with contracting $\mathbb{C}^*$-actions which induces an isomorphism between the tangent spaces of the $\mathbb{C}^*$-fixed points is an isomorphism.*

□

9. Poisson structure on transversal slices

Our goal is to show that $\mathcal{S}$ has a Poisson structure. We start by some recalls on Poisson algebras and Poisson structures. We mainly follow [CG97, Chap. 1].

9.1. **Poisson algebras and Poisson structures.** Let $\mathcal{A}$ be a commutative associative with unit $\mathbb{C}$-algebra.

Definition 9.1. *Suppose that $\mathcal{A}$ is endowed with an additional $\mathbb{C}$-bilinear bracket $\{\ ,\ \} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. Then $\mathcal{A}$ is called a* **Poisson algebra** *if the following conditions holds:*

(1) *$\mathcal{A}$ is a Lie algebra with respect to $\{\ ,\ \}$,*
(2) *Leibniz rule: $\{a, b \cdot c\} = \{a, b\} \cdot c + b \cdot \{a, c\}$, for all $a, b, c \in \mathcal{A}$.*

The Lie bracket $\{\ ,\ \}$ is called a **Poisson bracket** *on $\mathcal{A}$.*

Example 9.2. *Let (M, ω) be a symplectic varieties. Then the algebra $(\mathcal{O}(M), \{\ ,\ \})$ of regular functions, with pointwise multiplication, is a Poisson algebra.*

As an example, let $\mathbb{O} = G.\xi$ be a coadjoint orbit of $\mathfrak{g}^*$. Then $\mathbb{O}$ has a natural structure of symplectic structure, see e.g. [CG97, Prop. 1.1.5]*; for $\xi \in \mathfrak{g}^*$, we have*

$$\mathrm{T}_\xi(\mathbb{O}) = \mathrm{T}_\xi(G/G^\xi) \simeq \mathfrak{g}/\mathfrak{g}^\xi$$

and the bilinear form $\omega_\xi : (x, y) \mapsto \xi([x, y])$ descends to $\mathfrak{g}/\mathfrak{g}_\xi$. This gives the symplectic structure. Hence, together with a coadjoint orbit in $\mathfrak{g}^*$, we have a natural Poisson algebra.

In another direction, we have examples of Poisson algebras coming from some noncommutative algebras. Let $\mathcal{B}$ be an associative filtered (noncommutative) algebra with unit,

$$\mathbb{C} \subset \mathcal{B}_0 \subset \mathcal{B}_1 \subset \cdots,\ \bigcup_{i=0}^{\infty} \mathcal{B}_i = \mathcal{B},$$

such that $\mathcal{B}_i.\mathcal{B}_j \subset \mathcal{B}_{i+j}$ for any $i, j \geqslant 0$. Let

$$\mathcal{A} := \operatorname{gr} \mathcal{B} = \oplus_i(\mathcal{B}_i/\mathcal{B}_{i+1})$$

be its graded algebra (the multiplication in $\mathcal{B}$ gives rise a well-defined product $\mathcal{B}_i/\mathcal{B}_{i-1}\times\mathcal{B}_j/\mathcal{B}_{j-1} \to \mathcal{B}_{i+j}/\mathcal{B}_{i+j-1}$, making $\mathcal{A}$ an associative algebra). We said that $\mathcal{B}$ is ***almost commutative*** if $\mathcal{A}$ is commutative: this means that $a_ib_j - b_ja_i \in \mathcal{B}_{i+j-1}$ for $a_i \in \mathcal{B}_i, b_j \in \mathcal{B}_j$.

Assume that $\mathcal{B}$ is almost commutative. Then $\operatorname{gr}\mathcal{B}$ has a natural structure of Poisson algebra. We define the Poisson bracket

$$\{\ ,\ \} : \mathcal{B}_i/\mathcal{B}_{i-1} \times \mathcal{B}_j/\mathcal{B}_{j-1} \to \mathcal{B}_{i+j-1}/\mathcal{B}_{i+j-2}$$

as follows: for $a_1 \in \mathcal{B}_i/\mathcal{B}_{i-1}$ and $a_2 \in \mathcal{B}_j/\mathcal{B}_{j-1}$, let b_1 (resp. b_2) be a representative of a_1 in $\mathcal{B}_i$ (resp. $\mathcal{B}_j$) and set

$$\{a_1, a_2\} := b_1b_2 - b_2b_1 \mod \mathcal{B}_{i+j-2} .$$

Then we can check the required properties.

Example 9.3. *By the PBW theorem, there are canonical isomorphisms:* $\operatorname{gr} U(\mathfrak{g}) \simeq S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$. *Thus $U(\mathfrak{g})$ is almost commutative. Hence there is a canonical bracket $\{\ ,\ \}$ on* $\mathbb{C}[\mathfrak{g}^*]$. *Let us describe it explicitly (see* [CG97, Prop. 1.3.18]*). Let $(e_1, \dots, e_n)$ be a basis of $\mathfrak{g}$ and let $x_1, \dots, x_n$ be the coordinate functions on $\mathfrak{g}^*$ corresponding to the basis $(e_1, \dots, e_n)$. Define the structure constants c_{ij}^k by $[e_i, e_j] = \sum_k c_{ij}^k e_k$. Then, for $f, g \in \mathbb{C}[\mathfrak{g}^*]$,*

$$\{f, g\} = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}.$$

In a more concise way, we have:

$$\{f, g\} : \mathfrak{g}^* \to \mathbb{C}, \quad \xi \longmapsto \xi([\mathrm{d}_\xi f, \mathrm{d}_\xi g])$$

where $\mathrm{d}_\xi f, \mathrm{d}_\xi g \in (\mathfrak{g}^*)^* \simeq \mathfrak{g}$ denote the differentials of f and g at ξ. Moreover, if $\mathcal{O}$ is a coadjoint orbit of $\mathfrak{g}^*$,

$$\{f, g\}|_{\mathcal{O}} = \{f|_{\mathcal{O}}, g|_{\mathcal{O}}\}_{\text{symplectic}}.$$

9.2. A ***Poisson variety*** is a variety V such that the algebra $\mathcal{O}(V)$ is a Poisson algebra. For the definition of ***Poisson manifold***, see [LPV13, §1.3.2]. Every symplectic variety is a Poisson variety, but the converse is not true. According to Weinstein's Splitting Theorem, a Poisson manifold can be split into a collection of ***symplectic leaves***: each ***leaf*** is a submanifold of the Poisson manifold and is a symplectic manifold itself. In fact, the symplectic leaves are equivalence classes defined as follows: two points $v, v' \in V$ are in the same symplectic leaf if there is a piecewise Hamiltonian path[11] in V from v to v'.

Example 9.4. *For example, the space $\mathfrak{g}^*$ is a Poisson variety (and a Poisson manifold) and the symplectic leaves of $\mathfrak{g}^*$ are the coadjoint orbits of $\mathfrak{g}^*$, cf.* [V94, Prop. 3.1]. *The Poisson structure on the coadjoint orbits of $\mathfrak{g}^*$ is known as the* **Kirillov-Kostant Poisson structure***.*

Recall a result of Weinstein; see [V94, Prop. 3.10, p 39]:

Proposition 9.5 (Weinstein, 83)**.** *Let W be a submanifold of a Poisson manifold V such that:*

[11] By a *Hamiltonian path* in V from v to v' we mean a curve γ defined on an open neighborhood of $[0, 1]$ in $\mathbb{C}$, with $\gamma(0) = v$ and $\gamma(1) = v'$, which is an integral curve of a Hamiltonian vector field ξ_f, for some $f \in \mathcal{O}(V)$, defined on an open neighborhood of $\gamma([0, 1])$. See for example [LPV13, Chap. 1] for more details.

(1) *W is transversal to the symplectic leaves, i.e., for any symplectic leaf L and any $x \in W \cap L$, $\mathrm{T}_x W + \mathrm{T}_x L = \mathrm{T}_x V$;*
(2) *For any $x \in W$, $\mathrm{T}_x W \cap \mathrm{T}_x L$ is a symplectic subspace of $\mathrm{T}_x L$, where L is the leaf of V containing x.*

Then, there is a natural induced Poisson structure on W and the symplectic leaf of W through $x \in W$ is $W \cap L$ if L is the symplectic leaf through x in V.

We aim to apply the result to $\mathcal{S} \subset \mathfrak{g}^*$. Part (1) is known (cf. Theorem 8.2). For the part (2), it suffices to prove that for any coadjoint orbit $\mathcal{O}$ in $\mathfrak{g}^*$ and any $\xi \in \mathcal{O} \cap \mathcal{S}$, the restriction of the symplectic form on $\mathrm{T}_\xi(\mathcal{O})$ to $\mathrm{T}_\xi(\mathcal{S}) \cap \mathrm{T}_\xi(\mathcal{O})$ is nondegenerate. Remember that the symplectic form on $\mathrm{T}_\xi(\mathcal{O})$ was described in Example 9.2. Since the annihilator of $\mathrm{T}_\xi(\mathcal{S}) \simeq \kappa(\mathfrak{s})$ in $\mathfrak{g}$ is $\mathfrak{s}^\perp$, it suffices to verify that for any $\xi \in \mathcal{S}$,

$$\kappa([\kappa^{-1}(\xi), \mathfrak{s}^\perp]) \cap \mathrm{T}_\xi(\mathcal{S}) = \kappa([\kappa^{-1}(\xi), \mathfrak{s}^\perp] \cap \mathfrak{s}) = \{0\}.$$

The result is a consequence of:

Lemma 9.6. *Let $\xi \in \mathcal{S}$. Then $[\kappa^{-1}(\xi), \mathfrak{s}^\perp] \cap \mathfrak{s} = \{0\}$.*

Proof. Let Y be the set of $y \in e + \mathfrak{s}$ such that $[y, \mathfrak{s}^\perp] \cap \mathfrak{s} \neq \{0\}$. Since $\mathfrak{s}$ and $\mathfrak{s}^\perp$ are $\mathrm{ad}\, h_\Gamma$-stable, we have for any $t \in \mathbb{C}^*$,

$$\gamma(t^{-1})([y, \mathfrak{s}^\perp]) \cap \mathfrak{s} = [\gamma(t^{-1})y, \mathfrak{s}^\perp] \cap \mathfrak{s}$$

whence

$$\rho(t)([y, \mathfrak{s}^\perp] \cap \mathfrak{s}) = [\rho(t)y, \mathfrak{s}^\perp] \cap \mathfrak{s}.$$

Therefore, ρ stabilizes Y. In addition, since $\mathfrak{g} = [\mathfrak{g}, e] \oplus \mathfrak{s}$ (cf. Exercise 8),

$$e \in (e + \mathfrak{s}) \setminus Y$$

Hence, for any y is an open neighborhood U of e in $e + \mathfrak{s}$, $y \in (e + \mathfrak{s}) \setminus Y$. Assume that $Y \neq \varnothing$ and let $y \in Y$. Since ρ stabilizes Y, we get $\rho(t)y \in Y$ for any $t \in \mathbb{C}^*$. But for t sufficiently small, $\rho(t)y$ lies in U because ρ is contracting, whence the contradiction. □

Remark 9.7. 1) *If $\mathfrak{s} = \mathfrak{g}^f$, then $(\mathfrak{g}^f)^\perp = [f, \mathfrak{g}]$.*

2) *In the case where $(\mathfrak{m}, \mathfrak{n})$ is a Dynkin pair and where $\mathfrak{s} = \mathfrak{g}^f$, then a proof of Lemma 9.6 in given a new version of* [GG02] *(see the appendix of* `http://arxiv.org/pdf/math/0105225v3.pdf`*). The above proof is slightly different from that of* [GG02].

In conclusion, according to Proposition 9.5, $\mathcal{S} \subset \mathfrak{g}^*$ is a Poisson subvariety of $\mathfrak{g}^*$, i.e., is has a Poisson structure induced by the Kirillov-Kostant structure on $\mathfrak{g}^*$ (see Example 9.4). In other words, the Poisson bracket $\{\,,\,\}_\mathcal{S}$ on $\mathbb{C}[\mathcal{S}]$ is given by,

$$\{f, g\}_\mathcal{S}(\xi) = \{f|_\mathcal{O}, g|_\mathcal{O}\}_{\text{symplectic}}(\xi),$$

for any $f, g \in \mathbb{C}[\mathcal{S}]$ and $\xi \in \mathcal{S}$, if $\mathcal{O}$ denotes the coadjoint orbit through ξ is $\mathfrak{g}^*$.

9.3. **Hamiltonian reduction.** The Poisson structure on $\mathcal{S}$ can also be described via Hamiltonian reduction in the case where $\mathfrak{m} = \mathfrak{n}$ is an admissible subalgebra of $\mathfrak{g}$. Let in this case M be the unipotent subgroup of G whose Lie algebra is $\mathfrak{m}$.

Let us first recall the classical Hamiltonian reduction in a more general setting. Let A be a Lie group, with Lie algebra $\mathfrak{a}$, acting on a Poisson variety $(V, \{\, , \,\})$.

Definition 9.8. *The action of A in V is said to be* **Hamiltonian** *if there is a Lie algebra homomorphism*

$$\tilde{\mu} \,:\, \mathfrak{a} \longrightarrow \mathcal{O}(V), \quad x \mapsto \tilde{\mu}_x$$

such that the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{a} \arrow[r] \arrow[dr, dashed, "\tilde{\mu}"'] & \mathscr{X}(V) \\ & \mathcal{O}(V) \arrow[u] \end{tikzcd}$$

where $\mathscr{X}(V)$ is the Lie algebra of (symplectic) vector fields on V and where the vertical map is the natural map from $\mathcal{O}(V)$ to $\mathscr{X}(V)$. As for the horizontal map, it comes from the A-action on V. Namely, it is the map

$$\mathfrak{a} \to \mathscr{X}(V), \quad x \mapsto (v \mapsto \frac{\mathrm{d}}{\mathrm{d}t}(\exp(t \operatorname{ad} x).v)|_{t=0} \in \mathrm{T}_v V).$$

We call the map $\tilde{\mu}$ the ***co-moment map*** of the action, or the ***Hamiltonian*** of the action. Its dual map

$$\mu \,:\, V \longrightarrow \mathfrak{a}^*, \quad v \longmapsto \mu(v),$$

with $\mu(v) \in \mathfrak{a}^*$ the linear map $x \mapsto \tilde{\mu}_x(v)$, is called the ***moment map*** of the action.

Remark 9.9. *If the group A is connected, then μ is A-equivariant with respect to the coadjoint action on $\mathfrak{a}^*$.*

We refer the reader to [V94] or [LPV13, Prop. 5.39 and Def. 5.9] for the following result.

Theorem 9.10 (Marsden-Weinstein)**.** *Assume that A is connected and that the action of A in V is Hamiltonian. Let $\gamma \in \mathfrak{a}^*$. Assume that γ is a regular value*[12] *of μ, that $\mu^{-1}(\gamma)$ is A-stable and that $\mu^{-1}(\gamma)/A$ is a variety. Let $\iota : \mu^{-1}(\gamma) \hookrightarrow V$ and $\pi : \mu^{-1}(\gamma) \twoheadrightarrow \mu^{-1}(\gamma)/A$ be the natural maps: ι is the inclusion and π is the quotient map. Then the triple*

$$(V, \mu^{-1}(\gamma), \mu^{-1}(\gamma)/A)$$

is **Poisson-reducible***, i.e., there exists a Poisson structure $\{\, , \}'$ on $\mu^{-1}(\gamma)/A$ such that for all open subset $U \subset V$ and for all $f, g \in \mathcal{O}(\pi(U \cap \mu^{-1}(\gamma))$, on has*

$$\{f, g\}' \circ \pi(w) = \{\tilde{f}, \tilde{g}\} \circ \iota(w)$$

at any point $w \in U \cap \mu^{-1}(\gamma)$, where $\tilde{f}, \tilde{g} \in \mathcal{O}(U)$ are arbitrary extensions of $f \circ \pi|_{U\cap\mu^{-1}(\gamma)}, g \circ \pi|_{U\cap\mu^{-1}(\gamma)}$ to U.

[12] If $f : X \to Y$ is a smooth map between varieties, we way that a point y is a *regular value* of f if for all $x \in f^{-1}(y)$, the map $\mathrm{d}_x f : \mathrm{T}_x(X) \to \mathrm{T}_y(Y)$ is surjective. If so, then $f^{-1}(y)$ is a subvariety of X and the codimension of this variety in X is equal to the dimension of Y.

We intend to apply the Theorem to the connected Lie group M acting on the Poisson variety $\mathfrak{g}^*$ by the coadjoint action. The action is Hamiltonian and the moment map $\mu : \mathfrak{g}^* \to \mathfrak{m}^*$ is just the restriction of functions from $\mathfrak{g}$ to $\mathfrak{m}$. Recall that χ is the element $\kappa(e)$ of $\mathfrak{g}^*$. Since $\chi|_{\mathfrak{m}}$ is a character on $\mathfrak{m}$, it is fixed by the coadjoint action of M. As a consequence, the set

$$\mu^{-1}(\chi|_{\mathfrak{m}}) = \{\xi \in \mathfrak{g}^* \mid \mu(\xi) = \chi|_{\mathfrak{m}}\}$$

is M-stable. Moreover, we have the following lemma:

Lemma 9.11. *$\chi|_{\mathfrak{m}}$ is a regular value for the restriction of μ to each symplectic leaf of $\mathfrak{g}^*$.*

Proof. Note that $\mu^{-1}(\chi|_{\mathfrak{m}}) = \chi + \kappa(\mathfrak{m}^\perp)$. Then we have to prove that for any $\xi \in \chi + \kappa(\mathfrak{m}^\perp)$, the map

$$\mathrm{d}_\xi\mu : \mathrm{T}_\xi(G.\xi) \to \mathrm{T}_{\chi|_{\mathfrak{m}}}(\mathfrak{m}^*)$$

is surjective. But $\mathrm{T}_\xi(G.\xi) \simeq [\mathfrak{g}, \kappa^{-1}(\xi)]$ while $\mathrm{T}_{\chi|_{\mathfrak{m}}}(\mathfrak{m}^*) = \mathfrak{m}^*$. Since $\chi + \kappa(\mathfrak{m}^\perp)$ is transversal to the coadjoint orbits in $\mathfrak{g}^*$ (cf. Theorem 8.2), we have

$$\mathfrak{g} = [\mathfrak{g}, \kappa^{-1}(\xi)] + \mathfrak{m}^\perp.$$

Let $\gamma \in \mathfrak{m}^*$ and write $\kappa^{-1}(\gamma) = x + x'$, with $x \in [\mathfrak{g}, \kappa^{-1}(\xi)]$ and $x' \in \mathfrak{m}^\perp$, according to the above decomposition of $\mathfrak{g}$. Then $\mu(\kappa(x)) = \gamma$. □

Since the map

$$M \times \mathcal{S} \longrightarrow \chi + \kappa(\mathfrak{m}^\perp)$$

is an isomorphism of affine varieties (cf. Theorem 8.3), $\mathcal{S}$ identifies with $(\chi + \kappa(\mathfrak{m}^\perp))/M$. Therefore, the conditions of Theorem 9.10 are fulfilled and we get a symplectic structure on $\mathcal{S}$.

In fact, thanks to Lemma 9.11, we have shown that the symplectic form on each leaf on $\mathcal{S}$ is obtained by symplectic reduction from the symplectic form of the corresponding leaf of $\mathfrak{g}^*$.

From this, one can see that the latter Poisson structure defined on $\mathcal{S}$ is the same as that defined in §9.2. It is described as follows. Let $\pi : (\chi + \kappa(\mathfrak{m}^\perp)) \twoheadrightarrow (\chi + \kappa(\mathfrak{m}^\perp))/M \simeq \mathcal{S}$ be the natural projection map, and $\iota : (\chi + \kappa(\mathfrak{m}^\perp)) \hookrightarrow \mathfrak{g}^*$ be the natural inclusion. Then for any $f, g \in \mathbb{C}[\mathcal{S}]$,

$$\{f, g\}_{\mathcal{S}} \circ \pi = \{\tilde{f}, \tilde{g}\} \circ \iota$$

where $\tilde{f}, \tilde{g}$ are arbitrary extensions of $f \circ \pi, g \circ \pi$ to $\mathfrak{g}^*$.

10. Quantization of Slodowy slices

We obtain in this section that $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ is a quantization of the transversal slice $\chi + \kappa(\mathfrak{s})$.

10.1. **Kazhdan filtrations.** Let $(U^j(\mathfrak{g}))$ be the standard filtration on $U(\mathfrak{g})$. The adjoint action $\mathrm{ad}\, h_\Gamma$ uniquely extends to a derivation on $U(\mathfrak{g})$ and we set for any $i \in \mathbb{Z}$,

$$U_i(\mathfrak{g}) := \{x \in U(\mathfrak{g}) \mid (\mathrm{ad}\, h_\Gamma)x = ix\}.$$

Let $\mathcal{F}$ be the filtration defined by:

$$\mathcal{F}_k U(\mathfrak{g}) := \sum_{i + aj \leqslant k} U_i(\mathfrak{g}) \cap U^j(\mathfrak{g}) \qquad (k \in \mathbb{Z})$$

and let $\mathrm{gr}_{\mathcal{F}} U(\mathfrak{g})$ be the corresponding graded algebra, i.e.,

$$\mathrm{gr}_{\mathcal{F}} U(\mathfrak{g}) = \bigoplus_{k\in\mathbb{Z}} \mathcal{F}_k U(\mathfrak{g})/\mathcal{F}_{k-1} U(\mathfrak{g}).$$

Definition 10.1. *We will refer $\mathcal{F}$ as the* **(generalized) Kazhdan filtration** *on $U(\mathfrak{g})$.*

For $x \in \mathcal{F}_r U(\mathfrak{g})$ and $y \in \mathcal{F}_s U(\mathfrak{g})$, $[x, y] \in \mathcal{F}_{r+s-a} U(\mathfrak{g})$. Hence $\mathrm{gr}_{\mathcal{F}} U(\mathfrak{g}) \simeq S(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$ is commutative. In particular, it inherits a Poisson algebra structure (cf. §9.1).

10.2. **Slodowy gradings.** Let $k \in \mathbb{Z}$ and let us denote by $\mathbb{C}[\mathfrak{g}^*](k)$ the k-degree component of $\mathbb{C}[\mathfrak{g}^*]$ corresponding to $\mathrm{gr}_{\mathcal{F},k} U(\mathfrak{g})$. Then $\mathbb{C}[\mathfrak{g}^*](k)$ is the subspace of $\mathbb{C}[\mathfrak{g}^*]$ generated by the monomials $x = x_1 \dots x_j$ such that $(\mathrm{ad}\, h_\Gamma)x = ix$ and $i + aj = k$.

One can also describe $\mathbb{C}[\mathfrak{g}^*](k)$ using the $\mathbb{C}^*$-action ρ defined by the equality (5). Let $\rho^\#$ be the contragredient action of ρ, i.e.,

$$\forall\, t \in \mathbb{C}^*,\ \forall\, \xi \in \mathfrak{g}^*, \qquad \rho^\#(t)\xi = t^{-a}\gamma(t)\xi.$$

It induces a $\mathbb{C}^*$-action on $\mathbb{C}[\mathfrak{g}^*]$, still denoted by $\rho^\#$. Then

$$\mathbb{C}[\mathfrak{g}^*](k) = \{f \in \mathbb{C}[\mathfrak{g}^*] \mid \rho^\#(t)f = t^k f \text{ for any } t \in \mathbb{C}^*\}. \tag{6}$$

Exercise 9. 1) *Verify the equality (6).*
2) *Prove that: $\rho^\#\chi = \chi$, that the spaces $\kappa(\mathfrak{s})$ and $\kappa(\mathfrak{m}^\perp)$ are $\rho^\#$-stable and that the $\rho^\#$-weights on $\kappa(\mathfrak{s})$ and $\kappa(\mathfrak{m}^\perp)$ are strictly negative integers.*

Correction. See [Sa13, Lem. 2.3.1 and Lem. 2.3.2]. □

According to Exercise 9, the algebras $\mathbb{C}[\mathcal{S}]$ and $\mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)]$ are $\rho^\#$-graded.

10.3. Recall that $Q_\mathfrak{m} = U(\mathfrak{g})/I_\mathfrak{m}$. Let $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_\mathfrak{m} = Q_\mathfrak{m}$ be the quotient morphism and set

$$\mathcal{F}_k Q_\mathfrak{m} := \pi(\mathcal{F}_k U(\mathfrak{g})) \qquad (k \in \mathbb{Z}).$$

Exercise 10. *For any $k < 0$, prove that $\mathcal{F}_k Q_\mathfrak{m} = \{0\}$.*

Correction. See [Sa13, Lem. 2.3.5]. Once Proposition 10.2 will be proved, one can alternatively deduce Exercise 10 from Exercise 9. □

Let

$$\mathrm{gr}(\pi) : \mathrm{gr}_{\mathcal{F}} U(\mathfrak{g}) \to \mathrm{gr}_{\mathcal{F}} Q_\mathfrak{m}$$

be the surjective graded morphism associated with π, i.e.,

$$\forall\, u \in \mathcal{F}_k U(\mathfrak{g}), \qquad \mathrm{gr}(\pi)(u + \mathcal{F}_{k-1} U(\mathfrak{g})) = \pi(u) + \mathcal{F}_{k-1} U(\mathfrak{g}).$$

We have an exact sequence of $\mathrm{gr}_{\mathcal{F}} U(\mathfrak{g})$-modules,

$$0 \to \mathrm{gr}_{\mathcal{F}} I_\mathfrak{m} \to \mathrm{gr}_{\mathcal{F}} U(\mathfrak{g}) \to \mathrm{gr}_{\mathcal{F}} Q_\mathfrak{m} \to 0$$

so that $\mathrm{gr}_{\mathcal{F}} I_\mathfrak{m}$ is an ideal of $\mathrm{gr}_{\mathcal{F}} U(\mathfrak{g})$ and

$$\mathrm{gr}_{\mathcal{F}} Q_\mathfrak{m} \simeq \mathrm{gr}_{\mathcal{F}} U(\mathfrak{g})/\mathrm{gr}_{\mathcal{F}} I_\mathfrak{m}.$$

Moreover, $\mathrm{gr}_{\mathcal{F}} I_\mathfrak{m}$ is the kernel of $\mathrm{gr}(\pi)$.

Let $\mathcal{J}_{\chi+\kappa(\mathfrak{m}^\perp)}$ be the defining ideal of the affine variety $\chi + \mathfrak{m}^\perp$ in $\mathbb{C}[\mathfrak{g}^*]$, that is the ideal of $\mathbb{C}[\mathfrak{g}^*]$ of the polynomials vanishing on $\chi + \kappa(\mathfrak{m}^\perp)$.

Exercise 11. *Prove that the image of* $\mathrm{gr}_{\mathcal{F}} I_{\mathfrak{m}}$ *by the isomorphism* $\mathrm{gr}_{\mathcal{F}} U(\mathfrak{g}) \to S(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$ *is* $\mathcal{J}_{\chi+\kappa(\mathfrak{m}^\perp)}$.

Correction. (Ref.: cf. [Sa14, Lem .3.15].*)*

Let J be the image of $\mathrm{gr}_{\mathcal{F}} I_{\mathfrak{m}}$ by the isomorphism $\mathrm{gr}_{\mathcal{F}} U(\mathfrak{g}) \to \mathbb{C}[\mathfrak{g}^*]$. Since $\mathfrak{m}$ is Γ-graded, there is a basis

$$(y_1, \dots, y_m, y_{m+1}, \dots, y_n)$$

of $\mathfrak{g}$ with y_i of $\mathrm{ad}\, h_\Gamma$-weight d_i and such that $(y_1, \dots, y_m)$ is a basis of $\mathfrak{m}$. Set for any $i \in \{1, \dots, n\}$,

$$\tilde{y}_i := y_i - \chi(y_i) \in \mathcal{F}_{a+d_i} U(\mathfrak{g}).$$

By definition of $I_{\mathfrak{m}}$, the ideal $\mathrm{gr}_{\mathcal{F}} I_{\mathfrak{m}}$ is generated by the elements $\tilde{y}_i + \mathcal{F}_{a+d_i-1} U(\mathfrak{g})$ for $i = 1, \dots, m$. So, J is generated by the elements $\tilde{y}_i$, for $i = 1, \dots, m$ viewed as elements of $\mathbb{C}[\mathfrak{g}^*]$.

Let $(y_1^*, \dots, y_n^*)$ be the dual basis of $(y_1, \dots, y_n)$. Since $\mathfrak{g}_{\leqslant -a} \subseteq \mathfrak{m}$ and $e \in \mathfrak{g}_a$, $\tilde{y}_i = y_i$ for any $i > m+1$ and we get,

$$\chi = \chi(y_i) y_i^*.$$

On the other hand, $(y_{m+1}^*, \dots, y_n^*)$ is a basis of $\kappa(\mathfrak{m}^\perp)$ and we deduce that the set of common zeros of J in $\mathbb{C}[\mathfrak{g}^*]$ is $\chi + \kappa(\mathfrak{m}^\perp)$. Since J is generated by affine functions, it is radical, whence $J = \mathcal{J}_{\chi+\kappa(\mathfrak{m}^\perp)}$. ☐

Proposition 10.2. *We have a graded* $\mathfrak{n}$*-equivariant isomorphism,*

$$\mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}} \to \mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)].$$

Proof. Consider the following isomorphisms (see Exercises 11):

$$\mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}} \simeq \mathrm{gr}_{\mathcal{F}} U(\mathfrak{g})/\mathrm{gr}_{\mathcal{F}} I_{\mathfrak{m}} \simeq \mathbb{C}[\mathfrak{g}^*]/\mathcal{J}_{\chi+\kappa(\mathfrak{m}^\perp)} \simeq \mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)].$$

Since $\mathrm{gr}_{\mathcal{F}} I_{\mathfrak{m}}$ and $\mathcal{J}_{\chi+\kappa(\mathfrak{m}^\perp)}$ are $\mathfrak{n}$-stable, the isomorphism $\mathrm{gr}_{\mathcal{F}} I_{\mathfrak{m}} \simeq \mathcal{J}_{\chi+\kappa(\mathfrak{m}^\perp)}$ is $\mathfrak{n}$-equivariant and so is the map of the proposition. ☐

The filtration $(\mathcal{F}_k Q_{\mathfrak{m}})_k$ induces a filtration on $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ and we get an injective graded algebra morphism

$$\mathrm{gr}_{\mathcal{F}} \mathcal{W}_{\mathfrak{m},\mathfrak{n}} \hookrightarrow \mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}}.$$

Finally, we get a graded algebra morphism $\nu : \mathrm{gr}_{\mathcal{F}} \mathcal{W}_{\mathfrak{m},\mathfrak{n}} \to \mathbb{C}[\mathcal{S}]$ as the compound map:

$$\mathrm{gr}_{\mathcal{F}} \mathcal{W}_{\mathfrak{m},\mathfrak{n}} \hookrightarrow \mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}} \longrightarrow \mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)] \longrightarrow \mathbb{C}[\mathcal{S}].$$

The later map is the comorphism corresponding to the inclusion $\mathcal{S} \hookrightarrow \chi + \kappa(\mathfrak{m}^\perp)$, the second is the isomorphism given by Proposition 10.2.

Theorem 10.3. *The morphism*

$$\nu \ : \ \mathrm{gr}_{\mathcal{F}} \mathcal{W}_{\mathfrak{m},\mathfrak{n}} \to \mathbb{C}[\mathcal{S}]$$

is an isomorphism of graded Poisson algebras.

The Poisson structure on $\mathrm{gr}_{\mathcal{F}}\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ comes from the almost commutative algebra $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ with respect to $\mathcal{F}$ while the Poisson structure on $\mathbb{C}[\mathcal{S}]$ was obtained by Hamiltonian reduction (see §9.3).

Sketch of proof. We follow the proof of Gan-Ginzburg (cf. [GG02, §5] or [W10, Thm. 30]), which can be adapted to the setting of admissible pairs as shown in [Sa14, Proof of Thm. 3.8].

Regard $U(\mathfrak{g})$ and $Q_{\mathfrak{m}}$ as $\mathfrak{n}$-modules via the adjoint $\mathfrak{n}$-action; $\mathfrak{n}$ indeed acts in $Q_{\mathfrak{m}}$ because $I_{\mathfrak{m}}$ is $\mathfrak{n}$-stable (cf. §7.1). Consider the standard Chevalley-Eilenberg cochain complex of the $\mathfrak{n}$-module $Q_{\mathfrak{m}}$:

$$0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^{i-1} \xrightarrow{\partial} C^i \longrightarrow \cdots$$

where

$$C^i := \mathrm{Hom}(\bigwedge^i \mathfrak{n}, Q_{\mathfrak{m}}) \simeq (\bigwedge^i \mathfrak{n})^* \otimes Q_{\mathfrak{m}} \simeq \bigwedge^i \mathfrak{n}^* \otimes Q_{\mathfrak{m}}.$$

Then $C^0 = Q_{\mathfrak{m}}$ and we have

$$H^0(\mathfrak{n}, Q_{\mathfrak{m}}) = Q_{\mathfrak{m}}^{\mathrm{ad}\,\mathfrak{n}} = \mathcal{W}_{\mathfrak{m},\mathfrak{n}}. \tag{7}$$

Define an increasing filtration $\mathscr{F}$ on the complex $C^\bullet$, which agrees with the Kazhdan filtration $\mathcal{F}$ at zero degree, as follows:

$$\mathscr{F}_k C^i := \sum_{q+j\leqslant k} (\bigwedge^i \mathfrak{n}^*)_q \otimes \mathcal{F}_j Q_{\mathfrak{m}} \qquad (k \in \mathbb{Z}),$$

with

$$(\bigwedge^i \mathfrak{n}^*)_q := \bigoplus_{j_1+\cdots+j_i=q} \mathfrak{n}^*(j_1) \wedge \ldots \wedge \mathfrak{n}^*(j_i), \qquad \mathfrak{n}^*(j) := \{\xi \in \mathfrak{n}^* \; ; \; (\mathrm{ad}\,^* h_\Gamma)\xi = j\xi\}.$$

Observe that $\mathscr{F}_k C^0 = \mathcal{F}_k Q_{\mathfrak{m}}$ for any k and that $\mathscr{F}_k C^i = 0$ for k sufficiently (negatively) small. Furthermore, the filtration is compatible with the differential corresponding to the cochain, that is $\partial(\mathscr{F}_k C^i) \subset \mathscr{F}_k(C^{i+1})$ (to be checked!) and so induces a filtration on $H^i(\mathfrak{n}, Q_{\mathfrak{m}})$.

We have

$$\mathrm{gr}_{\mathscr{F}} H^0(\mathfrak{n}, Q_{\mathfrak{m}}) = \mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}}^{\mathrm{ad}\,\mathfrak{n}} = \mathrm{gr}_{\mathcal{F}} \mathcal{W}_{\mathfrak{m},\mathfrak{n}}. \tag{8}$$

The $\mathfrak{n}$-module structure on $Q_{\mathfrak{m}}$ induces a $\mathfrak{n}$-module structure on $\mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}}$, and we can consider the corresponding Chevalley-Eilenberg cochain complex.

Claim 10.4. (i) *We have an algebras isomorphism $H^0(\mathfrak{n}, \mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}}) \simeq \mathbb{C}[\mathcal{S}]$.*
(ii) *For any $i > 0$, $H^i(\mathfrak{n}, \mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}}) = 0$.*

Proof. (i) The isomorphism $\alpha : N \times \mathcal{S} \to \chi + \kappa(\mathfrak{m}^\perp)$ of Theorem 8.3 induces an isomorphism

$$\alpha^* : \mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)] \to \mathbb{C}[N] \otimes \mathbb{C}[\mathcal{S}].$$

The group N acts on $N \times \mathcal{S}$ be left multiplication on the first factor and on $\chi + \kappa(\mathfrak{m}^\perp)$ by the coadjoint action, which induces actions of N on $\mathbb{C}[N] \otimes \mathbb{C}[\mathcal{S}]$ and on $\mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)]$. The maps α and α^* are N-equivariant for these actions and

$$\mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)]^N \simeq (\mathbb{C}[N] \otimes \mathbb{C}[\mathcal{S}])^N \simeq \mathbb{C}[N]^N \otimes \mathbb{C}[\mathcal{S}] \simeq \mathbb{C}[\mathcal{S}]$$

because $\mathbb{C}[N]^N \simeq \mathbb{C}$. On the other hand, by Proposition 10.2, $\mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}} \simeq \mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)]$ and we have

$$\mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)]^N = \mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)]^{\mathrm{ad}\,\mathfrak{n}} = H^0(\mathfrak{n}, \mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)]) \simeq H^0(\mathfrak{n}, \mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}}),$$

whence the expected isomorphism.

(ii) The isomorphisms $\mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}} \simeq \mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)]$ and $\mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)] \simeq \mathbb{C}[N] \otimes \mathbb{C}[\mathcal{S}]$ yield for any $i \geqslant 0$,

$$H^i(\mathfrak{n}, \mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}}) \simeq H^i(\mathfrak{n}, \mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)]) \simeq H^i(\mathfrak{n}, \mathbb{C}[N] \otimes \mathbb{C}[\mathcal{S}]).$$

But $H^i(\mathfrak{n}, \mathbb{C}[N] \otimes \mathbb{C}[\mathcal{S}]) = H^i(\mathfrak{n}, \mathbb{C}[N]) \otimes \mathbb{C}[\mathcal{S}]$ and by [ChE48, Thm. 10.1], for $i > 0$, $H^i(\mathfrak{n}, \mathbb{C}[N])$ is the i-th De Rham cohomology group for N which is zero by [Ha77, Ch. III, Thm. 3.7]. In conclusion,

$$H^i(\mathfrak{n}, \mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}}) \simeq H^i(\mathfrak{n}, \mathbb{C}[N]) \otimes \mathbb{C}[\mathcal{S}]) = 0,$$

whence (ii). □

According to Claim 10.4,(i), and Equality (8), it remains to prove that

$$\mathrm{gr}_{\mathscr{F}} H^0(\mathfrak{n}, Q_{\mathfrak{m}}) \simeq H^0(\mathfrak{n}, \mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}}).$$

To do this, we introduce a judicious spectral sequence. A reference for spectral sequences is for instance [CaE56]. Consider the spectral sequence with

$$E_0^{p,q} := \mathscr{F}_p(C^{p+q})/\mathscr{F}_{p-1}(C^{p+q}).$$

Then $E_1^{p,q} \simeq H^{p+q}(\mathfrak{n}, \mathrm{gr}_{\mathcal{F},p} Q_{\mathfrak{m}})$ for any $p, q \in \mathbb{Z}$. Using Claim 10.4,(ii), we show that the spectral sequence is stationary and converges to $E_\infty^{p,q} \simeq \mathrm{gr}_{\mathscr{F},p} H^{p+q}(\mathfrak{n}, Q_{\mathfrak{m}})$. With integers p, q such that $p + q = 0$, we get the expected result,

$$\mathrm{gr}_{\mathscr{F}} H^0(\mathfrak{n}, Q_{\mathfrak{m}}) \simeq H^0(\mathfrak{n}, \mathrm{gr}_{\mathcal{F}} Q_{\mathfrak{m}}).$$

□

11. Skryabin equivalence

Let $\mathfrak{m}$ be an admissible subalgebra for e. We establish in this section the Skryabin's equivalence for $\mathcal{W}_{\mathfrak{m}}$ (cf. Theorem 11.2). The original Skryabin's equivalence concerns $\mathcal{W}_\chi$, [Sk02] (see also [GG02, §6] for an alternative proof over $\mathbb{C}$). All arguments of [GG02, §6] can be adapted to the context of admissible pairs as it is shown in [Sa13, Thm. 4.4.9].

Definition 11.1. *A $\mathfrak{g}$-module E is called a* **Whittaker module** *if for all $x \in \mathfrak{m}$, $x - \chi(x)$ acts on E locally nilpotently. A* **Whittaker vector** *in a Whittaker $\mathfrak{g}$-module E is a vector $v \in E$ which satisfies $(x - \chi(x)v = 0$ for any $x \in \mathfrak{m}$, i.e., $xv = \chi(x)v$ for any $x \in \mathfrak{m}$.*

Let $\mathfrak{g} - \mathtt{Wmod}$ be the category of finitely generated Whittaker $\mathfrak{g}$-modules and set for E an object of this category,

$$\mathrm{Wh}(E) := \{v \in E \mid (x - \chi(x))v = 0 \text{ for any } x \in \mathfrak{m}\}.$$

Observe that $\mathrm{Wh}(E) = 0$ implies that $E = 0$.[13] Let $\mathcal{W}_\mathfrak{m}$−mod be the category of finitely generated $\mathcal{W}_\mathfrak{m}$-modules and introduce the ***Whittaker functor***:

$$\mathrm{Wh} \ : \ \mathfrak{g}\text{−}\mathtt{Wmod} \longrightarrow \mathcal{W}_\mathfrak{m}\text{−}\mathtt{mod}, \qquad E \longmapsto \mathrm{Wh}(E)$$

with $\mathrm{Wh}(\psi)(x) = \psi(x)$ for $E, F \in \mathrm{Ob}(\mathfrak{g}\text{−}\mathtt{Wmod})$, $\psi \in \mathrm{Hom}_{\mathfrak{g}\text{−}\mathtt{Wmod}}(E, F)$ and $x \in \mathrm{Wh}(E)$. Given a Whittaker $\mathfrak{g}$-module E with an action map ϱ, $\mathrm{Wh}(E)$ is indeed naturally a $\mathcal{W}_\mathfrak{m}$-module by setting

$$\bar{y}.v = \varrho(y)v, \qquad \text{for } v \in \mathrm{Wh}(E) \text{ and } \bar{y} \in \mathcal{W}_\mathfrak{m} = (U(\mathfrak{g})/I_\mathfrak{m})^{\mathrm{ad}\,\mathfrak{n}};$$

see Exercise 12.

We define another functor by:

$$Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} - : \mathcal{W}_\mathfrak{m}\text{−}\mathtt{mod} \longrightarrow \mathfrak{g}\text{−}\mathtt{Wmod}, \qquad V \longmapsto Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} V$$

with $Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} (\varphi)(q \otimes v) = q \otimes \varphi(v)$ for $V, W \in \mathrm{Ob}(\mathcal{W}_\mathfrak{m}\text{−}\mathtt{mod})$, $\varphi \in \mathrm{Hom}_{\mathcal{W}_\mathfrak{m}\text{−}\mathtt{mod}}(V, W)$, $q \in Q_\mathfrak{m}$ and $v \in V$. For $V \in \mathcal{W}_\mathfrak{m}$−mod, $Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} V$ is a Whittaker $\mathfrak{g}$-module by setting

$$y.(q \otimes v) = (y.q) \otimes v, \qquad \text{for } y \in U(\mathfrak{g}),\ q \in Q_\mathfrak{m} = U(\mathfrak{g})/I_\mathfrak{m},\ v \in V.$$

Exercise 12. *Verify that the functors* Wh *and* $Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} -$ *are well-defined.*

Correction. See for instance [W10, Lem. 35]. □

The Skryabin's equivalence establishes an equivalence of categories between the categories $\mathfrak{g} - \mathtt{Wmod}$ and $\mathcal{W}_\mathfrak{m} - \mathtt{mod}$ whenever Γ is a Dynkin grading. We state here, [Sa13, Thm. 2.4.9]:

Theorem 11.2. *The functor* $Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} - : \mathcal{W}_\mathfrak{m}\text{−}\mathtt{mod} \longrightarrow \mathfrak{g}\text{−}\mathtt{Wmod}$ *is an equivalence of categories, with* $\mathrm{Wh} : \mathfrak{g}\text{−}\mathtt{Wmod} \longrightarrow \mathcal{W}_\mathfrak{m}\text{−}\mathtt{mod}$ *as inverse.*

Proof. (Ref.: [GG02, Thm. 6.1]*; see also* [W10, Thm. 36] *or* [Sa13, Thm. 2.4.9].*)*

Let V be a $\mathcal{W}_\mathfrak{m}$-module generated by a finite-dimensional vector space V_0. First, we prove that $\mathrm{Wh}(Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} V) = V$. Define an increasing filtration on V by setting for any i,

$$\mathcal{F}_i V := (\mathcal{F}_i \mathcal{W}_\mathfrak{m}).V_0.$$

Let M be the unipotent subgroup of G with Lie algebra $\mathfrak{m}$. According to Proposition 10.2 and Theorem 8.3,

$$\mathrm{gr}_\mathcal{F} Q_\mathfrak{m} = \mathbb{C}[M] \otimes \mathrm{gr}_\mathcal{F} \mathcal{W}_\mathfrak{m}.$$

Hence, according to Claim 10.4, we get

$$H^0(\mathfrak{m}, \mathrm{gr}_\mathcal{F} Q_\mathfrak{m} \otimes_{\mathrm{gr}_\mathcal{F} \mathcal{W}_\mathfrak{m}} \mathrm{gr}_\mathcal{F} V) = \mathrm{gr}_\mathcal{F} V$$

and for any $i > 0$,

$$H^i(\mathfrak{m}, \mathrm{gr}_\mathcal{F} Q_\mathfrak{m} \otimes_{\mathrm{gr}_\mathcal{F} \mathcal{W}_\mathfrak{m}} \mathrm{gr}_\mathcal{F} V) = 0.$$

The filtrations on $Q_\mathfrak{m}$ and V enables to define a filtration on $Q_\mathfrak{m} \otimes_\mathbb{C} V$ as follows:

$$\mathcal{F}_k(Q_\mathfrak{m} \otimes_\mathbb{C} V) := \sum_i \mathcal{F}_i Q_\mathfrak{m} \otimes_\mathbb{C} \mathcal{F}_{k-i} V.$$

[13] If E is nonzero, then for some nonzero finite-dimensional subspace E' of E, the map $\rho : \mathfrak{m} \to (E' \to E')$, $x \mapsto (v \mapsto (x - \chi(x)).v)$ is a representation of $\mathfrak{m}$ such that $\rho(x)$ is nilpotent for any $x \in \mathfrak{m}$. Then by Engel's theorem, we get a nonzero *Whittaker vector*, i.e., a nonzero element of $\mathrm{Wh}(E)$, [TY05, Thm. 19.3.6].

Remember that $Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} V$ is the quotient of $Q_\mathfrak{m} \otimes_\mathbb{C} V$ by the subspace generated by the elements $qh \otimes v - q \otimes hv$, for $q \in Q_\mathfrak{m}$, $h \in \mathcal{W}_\mathfrak{m}$ and $v \in V$. Let $\pi : Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} V \to Q_\mathfrak{m} \otimes_\mathbb{C} V$ the quotient map, and set:

$$\mathcal{F}_k(Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} V) := \pi(Q_\mathfrak{m} \otimes_\mathbb{C} V).$$

Since $\mathrm{gr}_\mathcal{F} Q_\mathfrak{m} \simeq \mathbb{C}[\chi + \kappa(\mathfrak{m}^\perp)]$ is a free module over $\mathrm{gr}_\mathcal{F} \mathcal{W}_\mathfrak{m} \simeq \mathbb{C}[\mathcal{S}]$, we deduce an isomorphism

$$\mathrm{gr}_\mathcal{F}(Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} V) \simeq \mathrm{gr}_\mathcal{F} Q_\mathfrak{m} \otimes_{\mathrm{gr}_\mathcal{F} \mathcal{W}_\mathfrak{m}} \mathrm{gr}_\mathcal{F} V.$$

Arguing as in the proof of Theorem 10.3, we obtain that

$$H^0(\mathfrak{m}, Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} V) = V \quad \text{and} \quad H^i(\mathfrak{m}, Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} V) = 0, \quad i > 0. \tag{9}$$

But $H^0(\mathfrak{m}, Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} V) = V$ means that $\mathrm{Wh}(Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} V) = V$.

It remains to show that $Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} -$ is surjective. It suffices to show that any object $E \in \mathfrak{g}\text{-}\mathtt{Wmod}$, the map $f : Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} \mathrm{Wh}(E) \to E$ which sends $\bar{u} \otimes x \in Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} \mathrm{Wh}(E)$ to ux, is an isomorphism. Let E' be the kernel of f, and E'' its cokernel. We have to show that $E' = 0$ and $E'' = 0$. Observe that $\mathrm{Wh}(E') = E' \cap \mathrm{Wh}(Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} \mathrm{Wh}(E))$ which is equal to $E' \cap \mathrm{Wh}(E)$ according to (9). But $\mathrm{Wh}(E)$ has a zero intersection with the kernel of f, whence $\mathrm{Wh}(E') = E' \cap \mathrm{Wh}(E) = 0$. Since E' is an object of $\mathfrak{g}\text{-}\mathtt{Wmod}$, we get $E' = 0$.

Write the long exact sequence of cohomology groups associated with the short exact sequence $0 \to Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} \mathrm{Wh}(E) \xrightarrow{f} E \to E'' \to E$:

$$0 \longrightarrow H^0(\mathfrak{m}, Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} \mathrm{Wh}(E)) \xrightarrow{H^0(f)} H^0(\mathfrak{m}, E) \longrightarrow H^0(\mathfrak{m}, E'') \longrightarrow H^1(\mathfrak{m}, Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} \mathrm{Wh}(E)) \longrightarrow \cdots$$

In the sequence the term $H^1(\mathfrak{m}, Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} \mathrm{Wh}(E))$ vanishes by (9). The map $H^0(f)$ is bijective since, by (9), $H^0(\mathfrak{m}, Q_\mathfrak{m} \otimes_{\mathcal{W}_\mathfrak{m}} \mathrm{Wh}(E)) = \mathrm{Wh}(E)$ and $H^0(\mathfrak{m}, E) = E$. The long exact sequence gives thus $\mathrm{Wh}(E'') = H^0(\mathfrak{m}, E'') = 0$, which implies $E'' = 0$. The theorem follows. □

The above theorem shows that Skryabin's equivalence still holds for the generalized finite W-algebras $\mathcal{W}_\mathfrak{m}$. If one can find an admissible subalgebra $\mathfrak{m}$ for which $\mathcal{W}_\mathfrak{m}$ is not isomorphic to $\mathcal{W}_\chi$, then it would be very interesting to see what kind of $\mathfrak{g}$-modules we get through the Skryabin's equivalence (see §15 for more details).

12. Regular nilpotent elements

Assume in this paragraph that e is regular. Then e is distinguished and the Dynkin gradings are the unique admissible gradings which are then even (see Exercise 4). So, one can assume that $\mathfrak{m} = \mathfrak{m}_\chi$ and that $\mathcal{W}_{\mathfrak{m},\mathfrak{n}} = \mathcal{W}_\chi$.

Remind that the restriction to $Z(\mathfrak{g})$ of the representation

$$\varrho_\mathfrak{m} \ : \ U(\mathfrak{g}) \longrightarrow \mathrm{End}_\mathbb{C}(Q_\mathfrak{m})$$

is injective, [P02, §6.1] (see Exercise 6). In particular, we get an inclusion map,

$$Z(\mathfrak{g}) \hookrightarrow \mathcal{W}_\chi.$$

As it has been already noticed just after Exercise 6, we have an algebra isomorphism

$$Z(\mathfrak{g}) \simeq Z(\mathcal{W}_\chi),$$

where $Z(\mathcal{W}_\chi)$ is the center of $\mathcal{W}_\chi$. The following result is due to Premet, [P02, §7.2]:

Theorem 12.1 (Premet, 2002)**.** *Let e be a regular nilpotent element. Then $\mathcal{W}_\chi \simeq Z(\mathfrak{g})$. In particular, in that event, $\mathcal{W}_\chi$ is commutative.*

Proof. (Ref.: See [P02, §7.2] *or* [W10, §4.8].*)*

Let ℓ be the rank of $\mathfrak{g}$. Then $\dim \mathfrak{g}^e = \ell$. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through κ. By a result of Kostant, [K63], the algebra of invariants

$$S(\mathfrak{g})^G \simeq \mathbb{C}[\mathfrak{g}^*] \simeq \mathbb{C}[\mathfrak{g}]$$

is a polynomial algebra in ℓ variables. Let $f_1, \dots, f_\ell$ be algebraically independent homogeneous generators of $S(\mathfrak{g})^G$ of degrees $m_1+1, \dots, m_\ell+1$ respectively where $m_1, \dots, m_\ell$ are the *exponents*[14] of $\mathfrak{g}$.

By Kostant (see e.g., [Di74, §7.4]), there exist algebraically independent $\tilde{f}_i \in Z(\mathfrak{g}) \cap U^{m_i+1}$, for $i \in \{1, \dots, \ell\}$, such that

$$Z(\mathfrak{g}) = \mathbb{C}[\tilde{f}_1, \dots, \tilde{f}_\ell] \quad \text{and} \quad \mathrm{gr}\,\tilde{f}_i = f_i.$$

Let us consider the adjoint quotient map

$$\varphi \;:\; \mathfrak{g} \longrightarrow \mathbb{C}^\ell, \quad x \longmapsto (f_1(x), \dots, f_\ell(x)).$$

The morphism φ is faithfully flat, i.e., φ is surjective, all fibers have the same dimension $\dim \mathfrak{g} - \ell$, all fibers are normal and consist of finitely many G-orbits.

Now, consider the restriction $\varphi|_{\mathcal{S}}$ to $\mathcal{S} = e + \mathfrak{g}^f$ of φ. By [Sl80, Cor. 7.4.1], the morphism $\varphi|_{\mathcal{S}}$ is still faithfully flat. Moreover, it is a $\mathbb{C}^*$-equivariant isomorphism of affine varieties, where the $\mathbb{C}^*$-action on $\mathcal{S}$ is ρ. Hence, by Theorem 10.3, $\mathrm{gr}_{\mathcal{F}} \mathcal{W}_\chi$ is generated by the restrictions to $\mathcal{S}$ of $f_1, \dots, f_\ell$. Since the restriction to $Z(\mathfrak{g})$ of the representation $\varrho_{\mathfrak{m}}$ is injective, we obtain of morphism $Z(\mathfrak{g}) \to \mathcal{W}_\chi$ whose associated graded map is an isomorphism. By a classical filtered algebras argument, it is an isomorphism. □

Part 3. Isomorphism problems

In this part, we are interested in the following question:

Question 12.2. *Given two admissible pairs $(\mathfrak{m}, \mathfrak{n})$ and $(\mathfrak{m}', \mathfrak{n}')$ for e, are the algebras $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ and $\mathcal{W}_{\mathfrak{m}',\mathfrak{n}'}$ isomorphic? In particular, are they isomorphic to $\mathcal{W}_\chi$?*

We have already mentioned (without any proof for the moment) some particular cases: e.g., $(\mathfrak{m}, \mathfrak{n})$ and $(\mathfrak{m}', \mathfrak{n}')$ are both good admissible pairs; [GG02, BG05]. In this part, we give some explanations and (sketchy) proofs of slightly more general statements following [Sa14].

[14] The **exponents** $(m_1, \dots, m_\ell)$ of the Weyl group of $\mathfrak{g}$ are the dual partition to the partition of the root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ formed by the sets $\Delta(i)$ of positive roots of height i. The exponents of $\mathfrak{g}$ can also be defined from the Poincaré polynomial of G: $p_G(t) = \prod_{i=1}^{\ell}(1 + t^{2m_i+1})$, [CM93, §4.4].

12.1. Actually, we can consider a slightly different problem. To each ad-nilpotent subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ verifying the following conditions (already considered in 5.2):

(χ1) $\chi([\mathfrak{m},\mathfrak{m}]) = \langle e, [\mathfrak{m},\mathfrak{m}]\rangle = \{0\}$;
(χ2) $\mathfrak{m} \cap \mathfrak{g}^e = \{0\}$;
(χ3) $\dim \mathfrak{m} = \frac{1}{2} \dim G.e$,

one can attach an endomorphism algebra, that we still denote by $\mathcal{W}_\mathfrak{m}$, setting

$$\mathcal{W}_\mathfrak{m} := Q_\mathfrak{m}^{\mathrm{ad}\,\mathfrak{m}} = \mathrm{End}_\mathfrak{g}(Q_\mathfrak{m})^{\mathrm{op}}$$

where $Q_\mathfrak{m}$ is, as before, the generalized Gelfand-Graev module,

$$Q_\mathfrak{m} = U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi.$$

The following question was first raised by Premet:

Question 12.3. *Let $\mathfrak{m}$ be an* ad*-nilpotent subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ verifying the condition (χ1),(χ2),(χ3). Are the algebras $\mathcal{W}_\mathfrak{m}$ and $\mathcal{W}_\chi$ isomorphic?*

The question is natural since its analogue in positive characteristic is positive, as we briefly explain below. For the zero characteristic, the problem is still open. In the definition of admissible pairs, we have considered a graded version of this problem. The reason for considering pairs is that it is more convenient in some arguments (following Gan-Ginzburg ideas). Even in this context, the problem is still open, but much better understood.

Exercise 13. *Assume that e is regular. Show that Question 12.3 has a positive answer. More precisely, show that the* ad*-nilpotent subalgebras of $\mathfrak{g}$ verifying the conditions (χ1),(χ2),(χ3) are exactly the G_e^0-conjugate to $\bigoplus_{i\leqslant -2} \mathfrak{g}(i)$, with G_e^0 the neutral component of the stabilizer G_e of e in G.*

Correction. (Ref.: See [BGM10]*.)*

Assume that e is regular. Then $\mathfrak{b} := \bigoplus_{i\geqslant 0} \mathfrak{g}(i)$ is a Borel subalgebra of $\mathfrak{g}$, and $\mathfrak{b}_- := \bigoplus_{i\leqslant 0} \mathfrak{g}(i)$ is the opposite Borel subalgebra. Let $\mathfrak{n}$ and $\mathfrak{n}^-$ be the nilpotent radicals of $\mathfrak{b}$ and $\mathfrak{b}^-$ respectively. Note that $\mathfrak{n} := \bigoplus_{i\geqslant 2} \mathfrak{g}(i)$ and $\mathfrak{n}_- := \bigoplus_{i\leqslant -2} \mathfrak{g}(i)$.

Let now $\mathfrak{m}$ be an ad-nilpotent subalgebra of $\mathfrak{g}$ verifying the conditions (χ1),(χ2),(χ3). Then $\mathfrak{m}$ is G-conjugated to $\mathfrak{n}^-$. Then, there is $g \in G$ such that $\mathfrak{m} = g(\mathfrak{n}^-)$. First, we show that g belongs to the set

$$X_e := \{g \in G \; ; \; g^{-1}(e) \in \sum_{i\leqslant 2} \mathfrak{g}(i)\}.$$

By condition (χ1), we get

$$0 = \langle e, [g(\mathfrak{n}^-), g(\mathfrak{n}^-)]\rangle = \langle g^{-1}(e), [\mathfrak{n}^-, \mathfrak{n}^-]\rangle.$$

Since $[\mathfrak{n}^-, \mathfrak{n}^-] = \sum\limits_{i\leqslant -4} \mathfrak{g}(i)$ and since the orthogonal complement of $\sum\limits_{i\leqslant -4} \mathfrak{g}(i)$ in $\mathfrak{g}$ is $\sum\limits_{i\leqslant 2} \mathfrak{g}(i)$, the assertion follows.

Let B^- be the connected subgroup of G with Lie algebra $\mathfrak{b}^-$. Denote by φ the map

$$\mathfrak{g} \times B^- \longrightarrow G, \quad (v, b) \longmapsto \exp(\mathrm{ad}\, v) \circ b.$$

We claim that X_e is equal to the image of $\mathfrak{g}^e \times B^-$ by φ. The image of $\mathfrak{g}^e \times B^-$ by φ is clearly contained in X_e.

Let us show the other one. Denote by U the unipotent radical of B. It is well-known, [Pu67, Ch. I, Part. II,§3] or [Hu75, Prop. 8.5], that the image of $\mathfrak{n} \times B^-$ by φ contains a dense open subset of G. Hence, $\varphi(\mathfrak{n} \times B^-) \cap X_e$ contains a dense open subset of X since it is nonempty; in particular $\varphi(\mathfrak{n} \times B^-)$ meets any irreducible component of X_e. Assume that X_e strictly contains $\varphi(\mathfrak{g}^e \times B^-)$. Thus we can choose $g = \exp(\operatorname{ad} v) \circ b$ in $X_e \cap \varphi(\mathfrak{n} \times B^-)$ which does not belong to $\varphi(\mathfrak{g}^e \times B^-)$. Let l be the maximal integer satisfying $(\operatorname{ad} v)^l(e) \neq 0$. Since v does not belong to $\mathfrak{g}^e$ by assumption, l is at least 1. Let now j be the maximal integer satisfying $(\operatorname{ad} v_j)^l(e) \neq 0$ where v_j is the component of v over $\mathfrak{g}(j)$ in the decomposition $\mathfrak{g} = \sum_{j\in\mathbb{Z}} \mathfrak{g}(j)$. Then $(\operatorname{ad} v_j)^l(e)$ is the highest weight term of $\exp(-\operatorname{ad} v)(e)$ and its degree is $2 + lj$. The fact that $g = \exp(\operatorname{ad} v) \circ b$ belongs to X_e implies that $\exp(-\operatorname{ad} v)(e) \in \sum_{i\leqslant 2} \mathfrak{g}(i)$. In turn, since v belongs to $\mathfrak{n}$, j is at least 2 and the degree of the nonzero term $(\operatorname{ad} v_j)^l(e)$ is $2 + lj \geqslant 4$; this contradicts $\exp(-\operatorname{ad} v)(e) \in \sum_{i\leqslant 2} \mathfrak{g}(i)$.

We are now in a position to conclude. By what foregoes, $\mathfrak{m} = g(\mathfrak{n}^-)$ with $g \in G_e^0 \times B^-$. Since B^- stabilizes $\mathfrak{n}^-$, $\mathfrak{m}$ is thus G_e^0-conjugated to $\mathfrak{n}^-$. Conversely, for all $g \in G_e^0$, $g(\mathfrak{n}^-)$ satisfies the conditions $(\chi 1)$,$(\chi 2)$,$(\chi 3)$. □

13. Digression to the positive characteristic

(Ref.: See [P02, §2] *or* [Sa13, Intro.].*)*

Let K be an algebraically closed field of characteristic $p > 0$ and let $\mathfrak{g}_K$ be a finite-dimensional simple Lie algebra over K. It is a restricted Lie algebra whose p-structure is denoted by $x \mapsto x^{[p]}$. We assume that p is big enough for that the Killing form of $\mathfrak{g}_K$, that we still denote by $\langle\, ,\rangle$, is nondegenerate and that p is good[15] for the root system of $\mathfrak{g}_K$. Let G_K be a simple and simply connected Lie group such that $\operatorname{Lie}(G_K) = \mathfrak{g}_K$ and let $\mathcal{N}_p(\mathfrak{g}_K)$ be the set of $x \in \mathfrak{g}_K$ such that $x^{[p]} = 0$. Fix $e \in \mathcal{N}_p(\mathfrak{g}_K)$ and set $\chi := \langle e, .\rangle$. Since $\mathfrak{g}_K$ is simple, the p-structure of an ad-nilpotent subalgebra of $\mathfrak{g}_K$ is zero.

Definition 13.1 ([P02, Def. 2.3])**.** *A restricted* ad-*nilpotent subalgebra* $\mathfrak{m}$ *of* $\mathfrak{g}_K$ *is said to be* **χ-admissible** *if it verifies the following conditions:*

$(\chi 1)_p$ $\chi([\mathfrak{m}, \mathfrak{m}]) = \{0\}$;
$(\chi 2)_p$ $\mathfrak{m} \cap \mathfrak{g}_K^e = \{0\}$;
$(\chi 2)_p$ $\dim \mathfrak{m} = (\dim G_K.\chi)/2$.

A similar construction to (1) shows that there exists such χ-admissible subalgebras; see [P02, §2.6]. To such a subalgebra, we attach an endomorphism algebra by setting

$$\mathcal{W}_{K,\mathfrak{m}} := \operatorname{End}_{\mathfrak{g}_K}(Q_{K,\mathfrak{m}})^{\mathrm{op}}$$

[15] p is ***good*** if and only if p does not appear as the coefficient of a simple root in the decomposition of the highest positive root as an integral linear combination of simple roots. The good primes for the simple types are easily determined; see [Bo68, Plates I–IX].

where $Q_{K,\mathfrak{m}}$ is the induced $U_\chi(\mathfrak{g}_K)$-module

$$U_\chi(\mathfrak{g}_K) \otimes_{U_\chi(\mathfrak{m})} K_\chi,$$

with K_χ the 1-dimensional module defined by the character $\chi|_{\mathfrak{m}}$, and $U_\chi(\mathfrak{g}_K)$ the *restricted enveloping algebra associated with* χ, i.e., the quotient of the enveloping algebra $U(\mathfrak{g}_K)$ by the bilateral ideal generated by the elements $x^p - x^{[p]} - \chi(x)^p$, for $x \in \mathfrak{g}_K$. Proposition 2.6 of [P02], and the comments which follow it, show that $\mathcal{W}_{K,\mathfrak{m}}$ and $Q_{K,\mathfrak{m}}$ do not depend on the choice of the restricted ad-nilpotent χ-admissible $\mathfrak{m}$. This positively answers Question 12.3 in this context[16].

14. Main results

14.1. **Comparable and equivalent pairs.** We follow in this section the approach of [Sa14, §4].

Definition 14.1. *Let* $(\mathfrak{m}, \mathfrak{n})$ *and* $(\mathfrak{m}', \mathfrak{n}')$ *be two admissible pairs with respect to a common admissible grading* Γ. *If* $\mathfrak{m} \subseteq \mathfrak{m}' \subseteq \mathfrak{n}' \subseteq \mathfrak{n}$, *we write*

$$(\mathfrak{m}', \mathfrak{n}') \preccurlyeq_\Gamma (\mathfrak{m}, \mathfrak{n}).$$

This defines a partial order on the set of Γ*-admissible pairs. We say that the admissible pairs* $(\mathfrak{m}, \mathfrak{n})$ *and* $(\mathfrak{m}', \mathfrak{n}')$ *are* **comparable** *if*

$$(\mathfrak{m}', \mathfrak{n}') \preccurlyeq_\Gamma (\mathfrak{m}, \mathfrak{n}) \quad \text{or} \quad (\mathfrak{m}, \mathfrak{n}) \preccurlyeq_\Gamma (\mathfrak{m}', \mathfrak{n}').$$

Example 14.2. *1) If* Γ *is good, then the pair* $(\mathfrak{g}_{\leqslant -2}, \mathfrak{g}_{<0})$ *is admissible and for any* Γ*-admissible pair* $(\mathfrak{m}, \mathfrak{n})$*, we have*

$$(\mathfrak{m}, \mathfrak{n}) \preccurlyeq_\Gamma (\mathfrak{g}_{\leqslant -2}, \mathfrak{g}_{<0})$$

2) For an arbitrary admissible grading Γ*, the pair* $(\mathfrak{g}_{\leqslant -2}, \mathfrak{g}_{<0})$ *is not always admissible,* [Sa14, Ex. 2.13]. *If there exists a* Γ*-admissible pair* $(\mathfrak{m}, \mathfrak{n})$*, with* $\mathfrak{m} = \mathfrak{g}_{\leqslant -a}$*, then such a pair is maximal for the partial order* $\preccurlyeq_\Gamma$.

Two Γ*-admissible pairs are not always comparable (even for good gradings),* [Sa14, Ex .4.5].

Definition 14.3. *We say that two admissible pairs* $(\mathfrak{m}, \mathfrak{n})$ *and* $(\mathfrak{m}', \mathfrak{n}')$ *are* **equivalent***, and we write* $(\mathfrak{m}, \mathfrak{n}) \sim (\mathfrak{m}', \mathfrak{n}')$*, if there is finitely many admissible pairs* $(\mathfrak{m}_1, \mathfrak{n}_1), \dots, (\mathfrak{m}_s, \mathfrak{n}_s)$ *such that*

(1) $(\mathfrak{m}_1, \mathfrak{n}_1) = (\mathfrak{m}, \mathfrak{n})$;
(2) *the pairs* $(\mathfrak{m}_i, \mathfrak{n}_i)$ *and* $(\mathfrak{m}_{i+1}, \mathfrak{n}_{i+1})$ *are comparable for* $i = 1, \dots, s-1$;
(3) $(\mathfrak{m}_s, \mathfrak{n}_s) = (\mathfrak{m}', \mathfrak{n}')$.

The relation $\sim$ is an equivalence relation on the set of admissible pairs for e.

Proposition 14.4. *If* $(\mathfrak{m}_1, \mathfrak{n}_1)$ *and* $(\mathfrak{m}_2, \mathfrak{n}_2)$ *are two equivalent admissible pairs, then the algebras* $\mathcal{W}_{\mathfrak{m}_1,\mathfrak{n}_2}$ *and* $\mathcal{W}_{\mathfrak{m}_2,\mathfrak{n}_2}$ *are isomorphic.*

In particular, if $(\mathfrak{m}, \mathfrak{n})$ *is a good admissible pair, then* $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ *is isomorphic to* $\mathcal{W}_\chi$.

[16] Definition 2.3 of [P02] is more general.

Proof. We follow the proof of [Sa14, Prop. 4.6]; see also [GG02, §5.5].

It is enough to prove the statement in the case where $(\mathfrak{m}_1, \mathfrak{n}_1)$ and $(\mathfrak{m}_2, \mathfrak{n}_2)$ are Γ-comparable for some admissible grading Γ and $(\mathfrak{m}_2, \mathfrak{n}_2) \preccurlyeq_\Gamma (\mathfrak{m}_1, \mathfrak{n}_1)$, i.e.,

$$\mathfrak{g}_{\leqslant -a} \subseteq \mathfrak{m}_1 \subseteq \mathfrak{m}_2 \subseteq \mathfrak{n}_2 \subseteq \mathfrak{n}_1 \subseteq \mathfrak{g}_{<0}.$$

Since $I_{\mathfrak{m}_1} \subset I_{\mathfrak{m}_2}$, we get a short exact sequence

$$0 \longrightarrow \ker\phi \longrightarrow Q_{\mathfrak{m}_1} \xrightarrow{\phi} Q_{\mathfrak{m}_2} \longrightarrow 0$$

Since $\mathfrak{n}_2 \subset \mathfrak{n}_1$, the image of $\mathcal{W}_{\mathfrak{m}_1,\mathfrak{n}_1}$ by ϕ is contained in $\mathcal{W}_{\mathfrak{m}_2,\mathfrak{n}_2}$. Let us denote by $\operatorname{gr}\phi$ the graded morphism associated with ϕ with respect to the increasing Kazhdan filtration $\mathcal{F}$. Our goal is to show that the morphism

$$(\operatorname{gr}\phi)|_{\operatorname{gr}_{\mathcal{F}}\mathcal{W}_{\mathfrak{m}_1,\mathfrak{n}_1}} \ : \ \operatorname{gr}_{\mathcal{F}}\mathcal{W}_{\mathfrak{m}_1,\mathfrak{n}_1} \longrightarrow \operatorname{gr}_{\mathcal{F}}\mathcal{W}_{\mathfrak{m}_2,\mathfrak{n}_2}$$

is an isomorphism of graded algebras. Then, by a classical filtered algebras argument [TY05, Prop. 7.5.7 and 7.5.9], we will conclude that the restriction to $\mathcal{W}_{\mathfrak{m}_1,\mathfrak{n}_1}$ of ϕ is an isomorphism of algebras onto $\mathcal{W}_{\mathfrak{m}_2,\mathfrak{n}_2}$.

Let $\mathfrak{s}$ be graded complement subspace of $\mathfrak{g}$ to $[\mathfrak{n}_2, e]$ in $\mathfrak{m}_2^\perp$. Since $\mathfrak{s} \cap [\mathfrak{g}, e] = \{0\}$, we get $\mathfrak{s} \oplus [\mathfrak{n}_1, e] = \mathfrak{m}_1^\perp$ as well for dimension reasons; see the condition (A6) of Definition 6.1. In other words, one can choose a space $\mathfrak{s}$ both "adapted" to the pairs $(\mathfrak{m}_1, \mathfrak{n}_1)$ and $(\mathfrak{m}_2, \mathfrak{n}_2)$. Set

$$\mathcal{S} = \chi + \kappa(\mathfrak{s})$$

and let for $i = 1, 2$, $\mu_i : \mathbb{C}[\chi + \kappa(\mathfrak{m}_i^\perp)] \to \mathbb{C}[\mathcal{S}]$ be the morphism whose comorphism is the natural inclusion $\mathcal{S} \hookrightarrow \mathbb{C}[\chi + \kappa(\mathfrak{m}_i^\perp)]$. According to Theorem 10.3 and Proposition 10.2, the following diagram is commutative:

$$\begin{tikzcd}
\operatorname{gr}_{\mathcal{F}}\mathcal{W}_{\mathfrak{m}_2,\mathfrak{n}_2} \arrow[r, hook] \arrow[drr, bend left, "\nu_2"] & \operatorname{gr}_{\mathcal{F}} Q_{\mathfrak{m}_2} \simeq \mathbb{C}[\chi + \kappa(\mathfrak{m}_2^\perp)] \arrow[dr, "\mu_2"] & \\
 & & \mathbb{C}[\mathcal{S}] \\
\operatorname{gr}_{\mathcal{F}}\mathcal{W}_{\mathfrak{m}_1,\mathfrak{n}_1} \arrow[r, hook] \arrow[uu, "(\operatorname{gr}\phi)|_{\operatorname{gr}_{\mathcal{F}}\mathcal{W}_{\mathfrak{m}_1,\mathfrak{n}_1}}"] \arrow[urr, bend right, "\nu_1"'] & \operatorname{gr}_{\mathcal{F}} Q_{\mathfrak{m}_1} \simeq \mathbb{C}[\chi + \kappa(\mathfrak{m}_1^\perp)] \arrow[ur, "\mu_1"'] \arrow[uu, "\operatorname{gr}\phi"] &
\end{tikzcd}$$

whose we deduce that $\nu_2 \circ (\operatorname{gr}\phi)|_{\operatorname{gr}_{\mathcal{F}}\mathcal{W}_{\mathfrak{m}_1,\mathfrak{n}_1}} = \nu_1$ where for $i = 1, 2$, $\nu_i : \operatorname{gr}_{\mathcal{F}}\mathcal{W}_{\mathfrak{m}_i,\nu_i} \to \mathbb{C}[\mathcal{S}]$ is the isomorphism of Theorem 10.3. Hence, $(\operatorname{gr}\phi)|_{\operatorname{gr}_{\mathcal{F}}\mathcal{W}_{\mathfrak{m}_1,\mathfrak{n}_1}}$ is an isomorphism too. $\square$

Remark 14.5. *In the above proof, the isomorphism* $(\operatorname{gr}\phi)|_{\operatorname{gr}_{\mathcal{F}}\mathcal{W}_{\mathfrak{m}_1,\mathfrak{n}_1}}$ *is not canonical since it depends on the choice of a complement subspace* $\mathfrak{s}$ *of* $\mathfrak{g}$ *to* $[\mathfrak{n}_2, e]$ *in* $\mathfrak{m}_2^\perp$.

14.2. **Connected gradings.** The definition of admissible gradings can be easily extended to $\mathbb{Q}$-graduations. If Γ is $\mathbb{Q}$-admissible, then $\lambda\Gamma$ is $\mathbb{Z}$-admissible for some $\lambda \in \mathbb{Q}_+^*$.

Definition 14.6. *Two admissible $\mathbb{Q}$-gradings Γ and Γ' are said to be* **adjacent** *if they share a common admissible pair, and they are said to be* **connected** *if there is a sequence $\Gamma_1, \dots, \Gamma_s$ such that*

(1) $\Gamma_1 = \Gamma$*;*
(2) *the gradings Γ_i and Γ_{i+1} are adjacent for $i = 1, \dots, s-1$;*
(3) $\Gamma_s = \Gamma'$.

Theorem 14.7 ([Sa14, Prop. 5.6])**.** *The admissible $\mathbb{Q}$-gradings are all connected to each others. In particular, they are connected to the Dynkin grading associated with h.*

Idea of the proof. Let $\Gamma : \mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ be an admissible grading such that $e \in \mathfrak{g}_a$ for some $a > 1$. Then we set

$$t := h_\Gamma - \frac{a}{2}h.$$

The element t is semisimple and lies in $\mathfrak{g}^h \cap \mathfrak{g}^e$, i.e., centralizes $\mathbb{C}\langle e, h, f\rangle$. For any $\varepsilon \in [0,1] \cap \mathbb{Q}$, we introduce the semisimple element

$$h_\Gamma^{(\varepsilon)} := \frac{a}{2}h + \varepsilon t$$

and we consider the $\mathbb{Q}$-admissible grading[17]

$$\Gamma^{(\varepsilon)} : \mathfrak{g} = \bigoplus_{j\in\mathbb{Q}} \mathfrak{g}_j^{(\varepsilon)}, \quad \mathfrak{g}_j^{(\varepsilon)} := \{x \in \mathfrak{g} \mid (\operatorname{ad} h_\Gamma^{(\varepsilon)})x = jx\}.$$

Observe that $\Gamma^{(0)}$ is adjacent to the Dynkin grading associated with h, and that $\Gamma^{(s)} = \Gamma$. Then the idea is to construct to a sequence of rational numbers

$$0 = \varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_s = 1$$

such that for any $i \in \{1, \dots, s-1\}$, $\Gamma^{(\varepsilon_i)}$ and $\Gamma^{(\varepsilon_{i+1})}$ are adjacent.

The end of the proof broadly resumes the ideas of the proof of Theorem 6.4... □

14.3. **Optimal pairs.** According to Theorem 14.7, to answer Question 12.2, it suffices to study the equivalence relation $\sim$ on the set of admissible pairs with respect to a given admissible grading.

Definition 14.8. *Let $b \in \mathbb{Q}_{>0}$. We say that the admissible grading Γ is* ***b*-optimal** *if $\mathfrak{g}_{<-\frac{1}{2}b} \cap \mathfrak{g}^e = \{0\}$ and if $e \in \mathfrak{g}_a$ for some $a \in \mathbb{N}$, with $a \geqslant 2$ and $a \geqslant b$.*

For example, a-good gradings are a-optimal.

Theorem 14.9 (Sadaka, 2013)**.** *Let $b \in \mathbb{Q}_{>0}$ and let $(\mathfrak{m}, \mathfrak{n})$ be an admissible pair with respect to a b-optimal admissible grading. Then $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ is isomorphic to $\mathcal{W}_\chi$.*

[17] The fact that $\Gamma^{(\varepsilon)}$ is indeed $\mathbb{Q}$-admissible must be checked!

Idea of the proof. Fix a b-optimal grading Γ. The first step is to show that the Γ-admissible pairs are all equivalent. The second step is to introduce a sequence of rational numbers $0 = \varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_s = 1$, as in the proof Theorem 14.7, and show that the gradings $\Gamma^{(\varepsilon_i)}$ are all b-optimal. The conclusion then follows from Proposition 14.4. □

15. Other results and conjectures

One can positively answer Question 12.2 in some particular cases. We have already seen the case where e is distinguished, that is $\mathfrak{g}^e \cap \mathfrak{g}^f = \{0\}$. The next case we consider is when the reductive Lie algebra $\mathfrak{g}^e \cap \mathfrak{g}^f$ has rank 1.

One can describe such nilpotent elements, and the corresponding nilpotent orbits; see [J04, §3.7] for the classical cases. For the exceptional cases, one can use the computer program `GAP4` to determinate them.

Exercise 14. *If* $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ *show that* $\mathfrak{g}^e \cap \mathfrak{g}^f$ *has rank 1 if and only the partition associated with the nilpotent orbit* $G.e$ *has only two parts, and show that* $\mathfrak{g}^e \cap \mathfrak{g}^f \simeq \mathfrak{sl}_2$ *if the two parts are equal, and* $\mathfrak{g}^e \cap \mathfrak{g}^f \simeq \mathbb{C}$ *otherwise.*

Theorem 15.1 ([Sa14, §7])**.** *Assume that* $\mathfrak{g}$ *is simple either of classical type, or of exceptional types* $\mathbf{G}_2$*,* $\mathbf{F}_4$ *or* $\mathbf{E}_6$*, and assume that the reductive Lie algebra* $\mathfrak{g}^e \cap \mathfrak{g}^f$ *has rank 1. Then all admissible pairs for e are equivalent. In particular, for any such a pair* $(\mathfrak{m}, \mathfrak{n})$*, the algebras* $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ *and* $\mathcal{W}_\chi$ *are isomorphic.*

Conjecture 15.2. *If* $(\mathfrak{m}, \mathfrak{n})$ *is a graded admissible pair for e and if the reductive Lie algebra* $\mathfrak{g}^e \cap \mathfrak{g}^f$ *has rank 1, then* $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$.

We could also propose a stronger conjecture without the condition "$\mathfrak{g}^e \cap \mathfrak{g}^f$ has rank 1". Actually, a positive or a negative answer to this stronger conjecture would thus be both interesting, as we explain now.

$*$ In the positive case, then it would be legitimate to formulate the following still stronger conjecture which would positively answer Premet's question in zero characteristic:

Conjecture 15.3. *If* $\mathfrak{m}$ *is any* ad*-nilpotent algebra satisfying the conditions (*χ*1),(*χ*3),(*χ*3), then* $\mathcal{W}_{\mathfrak{m},\mathfrak{n}} \simeq \mathcal{W}_\chi$.

As far as I know, Conjecture 2 is known only for the regular case in zero characteristic (see Exercise 13).

$*$ In the negative case, it would be then interesting to see what kind of Whittaker $\mathfrak{g}$-modules we obtained though the generalized Skryabin equivalence (Theorem 11.2). Good examples to consider in first is when e is associated with a rectangular partition in type $\mathbf{A}$. In this case, the Dynkin gradings are the only good gradings while there are non-Dynkin admissible gradings. On the other hand, we have an explicit description of $\mathcal{W}_\chi$ in term of shifted Yangians, with generators and relations, [BK06]. We can expect a similar description for $\mathcal{W}_{\mathfrak{m},\mathfrak{n}}$ and thus check directly with these generators and relations if the conjecture holds, or not.

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