

Representation theory of Lie algebras

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Marius Sophus Lie *(17 December 1842 – 18 February 1899) was a Norwegian mathematician. He largely created the theory of continuous symmetry and applied it to the study of geometry and differential equations. He also made substantial contributions to the development of algebra.*

Contents

Introduction

The course is an introduction to the theory of Lie algebras, emphasizing their representations.

A Lie algebra is a vector space $\mathfrak{g}$, defined on a field $\Bbbk$, equipped with an antisymmetric bilinear application $[-,-]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, the *Lie bracket*, which verifies the Jacobi relation:

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 \quad \text{for all } x,y,z\in\mathfrak{g}.$$

For example, any vector space with the zero bracket is a Lie algebra, the space $\mathscr{M}_n(\Bbbk)$ of n-size square matrices with the bracket $[x,y]=xy-yx$ is a Lie algebra, denoted $\mathfrak{gl}_n(\Bbbk)$, the Euclidean space $\mathbb{R}^3$ with the wedge product as Lie bracket is a Lie algebra, etc. Another fundamental example is the following: if M a differential manifold, then the vector space of all vector fields on M has a natural structure of Lie algebra, without being an algebra.

Lie algebras are naturally associated with *Lie groups* and *algebraic groups*, who play an important role in both mathematics and physics, where they describe the *continuous symmetry*. The study of Lie groups and Lie algebras was initiated in the 19th century with the work of mathematicians Sophus Lie, Wilhelm Killing, Elie Cartan and Hermann Weyl, among others. The classification of Lie algebras is crucial for the study of Lie groups, algebraic groups and their representations.

A *representation* of a Lie algebra $\mathfrak{g}$ is a linear map $\rho\colon\mathfrak{g}\to\operatorname{End}(V)$, where V is a vector space (which we will often assume to be of finite dimension), which preserves the Lie bracket, that is,

$$\rho([x,y])=[\rho(x),\rho(y)]:=\rho(x)\circ\rho(y)-\rho(y)\circ\rho(x) \quad \text{for all } x,y\in\mathfrak{g}.$$

Representation theory of Lie algebras occupy a central place in many branches of mathematics (algebraic geometry, number theory, combinatorics, topology, ...) and theoretical physics (integrable systems, conformal field theory, gauge theory, string theory, ...).

There are several important families of Lie algebras which allow their classification: *solvable* Lie algebras (the typical example is the set of upper triangular matrices), *nilpotent* Lie algebras (the typical example is the set of strictly upper triangular matrices), and the *semisimple* Lie algebras (an important example is the Lie algebra $\mathfrak{sl}_n(\Bbbk)$ of traceless square matrices, which is even simple!). The above families of Lie algebras will be studied in Part 1 about the structure of Lie algebras.

The structure of semisimple Lie algebras is particularly rich, and these Lie algebras have remarkable properties. The finite-dimensional representations of such Lie algebras are *completely reducible* (if char($\Bbbk$) $\neq 0$), that is to say they are direct sums of *irreducible* representations. Remarkably, the structure of semisimple Lie algebras and their representations is governed by combinatorial tools such as *root systems* and *highest weights*. In some sense, understanding the simple representations of semisimple Lie algebras is sufficient to understand the simple representations of any finite-dimensional Lie algebra. Part 2 is devoted the the representation theory of semisimple Lie algebras.

The course will also address some more geometric aspects. Part 4 is about Borel–Weil–Bott Theorem. This theorem allows to describe geometrically any irreducible representation of a (connected) semisimple, or even reductive, group G over an algebraically closed field of zero characteristic.

Another interesting aspect of semisimple Lie algebras is the study of *nilpotent orbits*. Surprisingly, there are interesting links between the representations and the nilpotent orbits associated with a semisimple Lie algebra. These links are subtle, related to invariant theory, and allow to understand certain aspects of infinite-dimensional representations. This is the main purpose of Part 3.

Part 1

Structure of Lie algebras

This part is about generalities on Lie algebras. We introduce standard definitions about Lie algebras in Chapter 1, and about their representations in Chapter 3. The part explores some classes of Lie algebras: the nilpotent Lie algebras and solvable Lie algebras (cf. Chapter 4) and the large class of algebraic Lie algebras, that is, those coming from linear algebraic groups (cf. Chapter 2). We also introduce an important tool for the representation theory: the enveloping algebra (cf. Chapter 5).

1

Definitions and examples

Let $\Bbbk$ be any commutative field (e.g., $\Bbbk = \mathbb{R}, \mathbb{C}, \mathbb{F}_q, \ldots$).

***Carl Gustav Jacob Jacobi**, 1804 – 1851, was a German mathematician who made fundamental contributions to elliptic functions, dynamics, differential equations, determinants, and number theory.*

1.1. Lie algebras

Definition 1.1 – Lie algebra

A ***Lie algebra*** is a $\Bbbk$ vector space equipped with an bilinear application

$$[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (x,y) \mapsto [x,y],$$

called the ***Lie bracket***, satisfying the following conditions:

(i) $[x,x] = 0$ for all $x \in \mathfrak{g}$ (antisymmetry),

(ii) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x,y,z \in \mathfrak{g}$ (Jacobi identity).

Note that (i) implies that $[x,y] = -[y,x]$ for all $x,y \in \mathfrak{g}$. The two conditions are equivalent if $\mathrm{char}(\Bbbk) \neq 2$.

EXAMPLE 1.1. Any $\Bbbk$-vector space $\mathfrak{g}$ equipped with the zero bracket, i.e., $[x,y] = 0$ for all $x,y \in \mathfrak{g}$, is a Lie algebra.

A Lie algebra $\mathfrak{g}$ such that $[x,y] = 0$ for all $x,y \in \mathfrak{g}$ is called ***abelian*** or ***commutative***.

EXAMPLE 1.2 (associative algebra). Let $(A,+,\cdot)$ be a $\Bbbk$-associative algebra. Then A has naturally a structure a Lie algebra by setting:

$$\forall\,(x,y) \in A \times A, \quad [x,y] = x \cdot y - y \cdot x.$$

If V is $\Bbbk$-vector space, then $(\mathrm{End}(V),+,\circ)$ endowed with the above bracket is a Lie algebra, denoted by $\mathfrak{gl}(V)$. For $n \in \mathbb{N}^*$, $(\mathscr{M}_n(\Bbbk),+,\times)$, the set of n-size square matrices with coefficients in $\Bbbk$, endowed with the above bracket is a Lie algebra, denoted by $\mathfrak{gl}_n(\Bbbk)$.

EXAMPLE 1.3 (the Lie algebra of vector fields). Let M be a smooth manifold (i.e., differentiable of class $\mathcal{C}^\infty$). We recall that a vector field is a smooth section $\xi\colon M \to TM$ of the tangent bundle of M. Such a vector field defines a derivation of the algebra $\mathcal{C}^\infty(M,\mathbb{R})$ of smooth functions on M sending a function f to the function $x \mapsto d_x f(\xi(x))$.

Denoting by $\mathcal{V}(M)$ the set of all vector fields over M, we show by a local calculation that the application $\mathcal{V}(M) \to \mathrm{Der}(\mathcal{C}^\infty(M,\mathbb{R}))$ thus obtained is bijective. It follows that the $\mathbb{R}$-vector space (of infinite dimension) $\mathcal{V}(M)$ of all vector fields over M is equipped with a structure of a $\mathbb{R}$-Lie algebra.

EXERCISE 1.1 (Heisenberg Lie algebra). Let $\widehat{\mathfrak{h}} = \Bbbk[t,t^{-1}] \oplus \mathbb{C}\mathbf{1}$. Show that the following bracket

$$\begin{cases} [\widehat{\mathfrak{h}}, \mathbf{1}] = 0, \\ [f(t), g(t)] = \mathrm{Res}_{t=0}(f'(t)g(t))\mathbf{1}, \end{cases}$$

gives to $\widehat{\mathfrak{h}}$ a Lie algebra structure. The infinite dimensional Lie algebra $\widehat{\mathfrak{h}}$ is called the ***Heisenberg Lie algebra***.

1.2. Lie subalgebras, ideals and morphisms

Definition 1.2 – Lie subalgebra and ideal

Let $\mathfrak{g}$ be a Lie algebra.

(i) A subset $\mathfrak{h}$ of $\mathfrak{g}$ is a ***Lie subalgebra*** of $\mathfrak{g}$ if $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ and if for all $x, y \in \mathfrak{h}$, we have $[x,y] \in \mathfrak{h}$.

(ii) A subset I of $\mathfrak{g}$ is an ***ideal*** of $\mathfrak{g}$ if I is a subspace of $\mathfrak{g}$ and if for any $x \in I$ and any $y \in \mathfrak{g}$, we have $[x,y] \in I$.

We easily verify that if $\mathfrak{h}, \mathfrak{k}$ are Lie subalgebras (resp. ideals) of $\mathfrak{g}$, then $\mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{h} + \mathfrak{k}$ are Lie subalgebras (resp. ideals) of $\mathfrak{g}$. Moreover, if $\mathfrak{h}, \mathfrak{k}$ are ideals, notice that

$$[\mathfrak{h}, \mathfrak{k}] = \mathrm{Span}_{\Bbbk}\{[x,y] \colon x \in \mathfrak{h},\ y \in \mathfrak{k}\}$$

is an ideal of $\mathfrak{g}$, and that $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{h} \cap \mathfrak{k}$.

EXAMPLE 1.4. Let $n \in \mathbb{N}^*$. The following sets $\mathfrak{gl}_n(\Bbbk)$ are Lie subalgebras of $\mathfrak{gl}_n(\Bbbk)$:

1) $\mathfrak{sl}_n(\Bbbk)$, the set of traceless matrices of $\mathfrak{gl}_n(\Bbbk)$; it is an ideal of $\mathfrak{gl}_n(\Bbbk)$,
2) $\mathfrak{b}_n^+(\Bbbk)$, the set of upper triangular matrices of $\mathfrak{gl}_n(\Bbbk)$,
3) $\mathfrak{u}_n^+(\Bbbk)$, the set of strictly upper triangular matrices of $\mathfrak{gl}_n(\Bbbk)$; it is an ideal of $\mathfrak{b}_n^+(\Bbbk)$,
4) $\mathfrak{d}_n(\Bbbk)$, the set of diagonal matrices of $\mathfrak{gl}_n(\Bbbk)$ (it is an abelian Lie algebra).

EXERCISE 1.2 (Lie algebra of the symplectic group). Let $n \geqslant 1$. Consider the matrix $J \in \mathscr{M}_{2n}(\Bbbk)$ defined by:

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Define

$$\mathfrak{sp}_{2n}(\Bbbk) = \{x \in \mathfrak{gl}_{2n}(\Bbbk) \colon Jx = -x^T J\},$$

where x^T is the transpose matrix of x. Show that $\mathfrak{sp}_{2n}(\Bbbk)$ is a Lie subalgebra of $\mathfrak{gl}_{2n}(\Bbbk)$. What is its dimension?

EXERCISE 1.3 (Lie algebra quotient). Let I be an ideal of a Lie algebra $\mathfrak{g}$. Show that the quotient $\mathfrak{g}/I$ equipped with the bracket:

$$[x+I, y+I] = [x,y] + I, \quad \forall (x,y) \in \mathfrak{g} \times \mathfrak{g},$$

is a Lie algebra.

A Lie algebra $\mathfrak{g}$ has always at least two ideals, $\{0\}$ and $\mathfrak{g}$.

Definition 1.3 – simple Lie algebra

A Lie algebra $\mathfrak{g}$ is called ***simple*** if $\mathfrak{g}$ is not abelian and its only ideals are $\{0\}$ and $\mathfrak{g}$.

REMARK 1.1. If $\mathfrak{g}$ is Abelian, then any subspace of $\mathfrak{g}$ is an ideal. So the above definition excludes only the case of abelian Lie algebras of dimension zero or one.

EXERCISE 1.4 ($\mathfrak{sl}_2(\Bbbk)$ is simple if $\operatorname{char}(\Bbbk) \neq 2$). Show that $\mathfrak{sl}_2(\Bbbk)$ is simple if $\operatorname{char}(\Bbbk) \neq 2$.

Hint: setting,

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

verify that

$$[h, e] = 2e, \quad [e, f] = h, \quad [h, f] = -2f.$$

EXERCISE 1.5 ($\mathfrak{b}_n^+$ and $\mathfrak{u}_n^+$ are not simple). Show that for any $n \geqslant 2$, $\mathfrak{b}_n^+(\Bbbk)$ and $\mathfrak{u}_n^+(\Bbbk)$ are not simple.

Definition 1.4 – morphism of a Lie algebra

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie Lie algebras, and $\theta \colon \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ a linear map. We say that θ is a ***Lie algebra morphism*** if for all $(x, y) \in \mathfrak{g}_1 \times \mathfrak{g}_1$,

$$\theta([x, y]) = [\theta(x), \theta(y)].$$

The map θ is a ***Lie algebra isomorphism*** if moreover θ if bijective.
The Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ***isomorphic*** if there exists an isomorphism of Lie algebras $\theta \colon \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$.

EXERCISE 1.6 (kernel and image of a Lie algebra morphism). Let $\theta \colon \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ be a Lie algebra morphism.

(1) Show that $\operatorname{Ker}\theta$ is an ideal of $\mathfrak{g}_1$ and that $\operatorname{Im}\theta$ is a Lie subalgebra of $\mathfrak{g}_2$.
(2) Show that the quotient $\mathfrak{g}_1 / \operatorname{Ker}\theta$ is isomorphic to $\operatorname{Im}\theta$.

EXERCISE 1.7 (a natural isomorphism). Consider the n-size square matrix

$$P = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Show that the map

$$\begin{array}{ccc} \mathfrak{gl}_n(\Bbbk) & \longmapsto & \mathfrak{gl}_n(\Bbbk) \\ x & \longmapsto & PxP. \end{array}$$

is an isomorphism of Lie algebras, which induces the Lie algebra isomorphisms:

$$\mathfrak{b}_n^+(\Bbbk) \xrightarrow{\sim} \mathfrak{b}_n^-(\Bbbk), \qquad \mathfrak{u}_n^+(\Bbbk) \xrightarrow{\sim} \mathfrak{u}_n^-(\Bbbk),$$

where $\mathfrak{b}_n^-(\Bbbk)$ (resp. $\mathfrak{u}_n^-(\Bbbk)$) is the set of lower triangular matrices (resp. strictly lower triangular matrices).

1.3. Derivations

Let A be an associative algebra. A ***derivation*** δ of A is an endomorphism of the vector space A satisfying the Leibniz rule: $\delta(ab) = (\delta a)b + a(\delta b)$ for all $a, b \in A$. The set $\operatorname{Der}(A) = \operatorname{Der}_{\Bbbk}(A)$ of all derivations of A is a subspace of $\operatorname{End}(A) = \operatorname{End}_{\Bbbk}(A)$.

The composition $\delta \circ \delta'$ of two derivations δ, δ' is not a derivation in general.

But $[\delta, \delta'] = \delta \circ \delta' - \delta' \circ \delta$ is still a derivation of A. Thus, $\mathrm{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$. More generally, we have the following definition.

Definition 1.5 – derivation

If A is a vector space equipped with a bilinear map

$$A \times A \to A, \quad (a, b) \mapsto a \star b,$$

call a ***derivation*** δ of A an endomorphism of the vector space A satisfying the Leibniz rule:

$$\delta(a \star b) = (\delta a) \star b + a \star (\delta b)$$

for all $a, b \in A$.

We denote by $\mathrm{Der}(A)$ or $\mathrm{Der}_{\Bbbk}(A)$ the subspace of all derivations of A.

EXERCISE 1.8 (Lie algebra of derivations). Show that $\mathrm{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

For example, if $\mathfrak{g}$ is a Lie algebra, let $A = \mathfrak{g}$ with $- \star - = [-, -]$. Thus a derivation of the Lie algebra $\mathfrak{g}$ is an element f of $\mathrm{End}(\mathfrak{g})$ satisfying $f([x, y]) = [f(x), y] + [x, f(y)]$ for all $x, y \in \mathfrak{g}$. Here is a very important example of derivation of a Lie algebra $\mathfrak{g}$. For $x \in \mathfrak{g}$, set

$$\begin{aligned} \mathrm{ad}\, x \colon \mathfrak{g} &\longmapsto \mathfrak{g} \\ y &\longmapsto [x, y]. \end{aligned}$$

The antisymmetry (i) and the Jacobi identity (ii) ensures that $\mathrm{ad}\, x$ is a derivation of $\mathfrak{g}$.

2 Linear algebraic group

We consider in this chapter (linear) algebraic groups only as examples of interesting structures which allow to produce all *algebraic Lie algebras*. They will be also needed in Part 4.

We do not deal with the general structure of linear algebraic groups, and most of substantial results about algebraic groups will be admitted in this course.

Our basic reference for this chapter is [8], and we refer to this book or [12] for more details.

We assume in this chapter that $\Bbbk$ is an algebraically closed field of characteristic zero. For us, a variety is a reduced separated scheme of finite type over $\Bbbk$.

Definition 2.1 – algebraic group

An ***algebraic group*** is a variety G which is also a group and such that the maps defining the group structure, $\mu\colon G \times G \to G$, $(x, y) \mapsto xy$ and $\iota\colon G \to G$, $x \mapsto x^{-1}$ are morphisms of algebraic varieties.

EXAMPLE 2.1. Here are examples of algebraic groups of dimension one.

1) The ***additive group*** is the affine line $\mathbb{A}^1 = \Bbbk$ with group low $\mu(x, y) = x + y$.

2) The ***multiplicative group*** is the affine open set $A^1 \setminus \{0\} = \Bbbk^*$ with group low $\mu(x, y) = xy$.

EXERCISE 2.1 (general linear group). Show that the general linear group

$$\mathrm{GL}_n(\Bbbk) = \{x \in \mathscr{M}_n(\Bbbk)\colon\ \det(x) \neq 0\}$$

is an algebraic group.

A connected algebraic group whose underlying variety is complete is called an ***Abelian variety***. The terminology comes from the fact that such groups are indeed equipped with a structure of abelian groups.

An algebraic group whose underlying variety if affine is called a ***linear algebraic group***. For example, the general linear group $\mathrm{GL}_n(\Bbbk)$ is linear.

The term *linear* actually comes from the following theorem (that we admit in this course).

Theorem 2.2 – Chevalley

Any affine algebraic group is a closed subgroup of $\mathrm{GL}_n(\Bbbk)$ for some $n \in \mathbb{N}^*$.

***Claude Chevalley**, born February 11, 1909 in Johannesburg in South Africa and died on June 28, 1984 in Paris, was a French mathematician who made important contributions to number theory, algebraic geometry, class field theory, finite group theory and the theory of algebraic groups. He was a founding member of the Bourbaki group.*

Conversely, it is clear from Exercise 2.1 that any closed subgroup of $\mathrm{GL}_n(\Bbbk)$ is a linear algebraic group.

EXAMPLE 2.2. Here are some standard examples of linear algebraic groups:

1) finite subgroups of $\mathrm{GL}_n(\Bbbk)$,
2) $\mathrm{SL}_n(\Bbbk) = \{x \in \mathscr{M}_n(\Bbbk)\colon \det(x) = 1\}$,
3) $\mathrm{D}_n(\Bbbk)$, the subgroup of diagonal matrices of $\mathrm{GL}_n(\Bbbk)$,
4) $\mathrm{T}_n(\Bbbk)$, the subgroup of upper triangular matrices of $\mathrm{GL}_n(\Bbbk)$,
5) $\mathrm{U}_n(\Bbbk)$, the subgroup of strictly upper triangular matrices of $\mathrm{GL}_n(\Bbbk)$,
6) $\mathrm{O}_n(\Bbbk) = \{x \in \mathscr{M}_n(\Bbbk)\colon xx^T = I_n\}$, $\mathrm{SO}_n(\Bbbk) = \mathrm{O}_n(\Bbbk) \cap \mathrm{SL}_n(\Bbbk)$,
7) $\mathrm{Sp}_{2n}(\Bbbk) = \{x \in \mathscr{M}_n(\Bbbk)\colon xJx^T = J\}$, with J as in Exercise 1.2.

EXERCISE 2.2 (the projective linear group $\mathrm{PGL}_n(\Bbbk)$ is linear!). Observe that the center of $\mathrm{GL}_n(\Bbbk)$ is reduced to the set of scalar matrices, that is, $Z(\mathrm{GL}_n(\Bbbk)) \cong \mathbb{G}_m$, and prove that $\mathrm{PGL}_n(\Bbbk) = \mathrm{GL}_n(\Bbbk)/Z(\mathrm{GL}_n(\Bbbk))$ is a linear algebraic group.

Hint: consider the morphism of algebraic groups $\mathrm{GL}_n(\Bbbk) \to \mathrm{Aut}(\mathfrak{gl}_n) = \{f \in \mathrm{End}(\mathfrak{gl}_n)\colon f([a,b]) = [f(a), f(b)] \text{ for all } a, b \in \mathfrak{gl}_n\}$, sending g to the map $x \mapsto gxg^{-1}$.

2.1. Recalls on tangent spaces and derivations

Let R be a commutative ring, A an algebra over R, and M an A-module. We set

$$\mathrm{Der}_R(A, M) = \{R\text{-linear map } \delta\colon A \to M,\ \delta(ab) = a\delta(b) + b\delta(a) \text{ for all } a, b \in A\}.$$

If $\delta \in \mathrm{Der}_R(A, M)$, then $\delta(1) = 0$ and so $\delta(r) = 0$ for all $r \in R$. If S is a multiplicative set of A, then

$$\mathrm{Der}_R(S^{-1}A, M) \cong \mathrm{Der}_R(A, M). \tag{1}$$

Let now X be an affine variety, and $x \in X$. Let $\mathcal{O}_X$ be the structure sheaf of X so that $\Bbbk[X] = \mathcal{O}_X(X)$ is the coordinate ring of X, and $\mathcal{O}_{X,x}$ the local ring of x at X. We view $\Bbbk(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ as a module over $\mathcal{O}_{X,x}$, where $\mathfrak{m}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$ of functions vanishing at x. Identifying $\Bbbk(x)$ with $\Bbbk$, we thus have

$$\mathrm{Der}_\Bbbk(\mathcal{O}_{X,x}, \Bbbk(x)) = \{\Bbbk\text{-linear map } \delta\colon \mathcal{O}_{X,x} \to \Bbbk,\ \delta(fg) = f(x)\delta(g) + \delta(f)g(x) \text{ for all } a, b \in \mathcal{O}_{X,x}\}.$$

Definition 2.3 – tangent space at a point

The tangent space of X at x is defined by

$$T_x(X) = \mathrm{Der}_\Bbbk(\mathcal{O}_{X,x}, \Bbbk(x)) = \mathrm{Der}_\Bbbk(\Bbbk[X], \Bbbk(x)).$$

The second equality holds by (1).

If $\varphi\colon X \to Y$ is a morphism of affine varieties, then we define its differential at the point x to be the linear map

$$\mathrm{d}_x\varphi\colon T_x(X) \to T_{\varphi(x)}(Y), \quad \delta \mapsto \delta \circ \varphi^*,$$

where $\varphi^*\colon \Bbbk[Y] \to \Bbbk[X]$ is the comorphism. Note that for $\delta \in \mathrm{Der}_\Bbbk(\Bbbk[X], \Bbbk(x))$, then $\delta \circ \varphi^*$ is indeed a derivation of $\mathrm{Der}_\Bbbk(\Bbbk[Y], \Bbbk(y))$.

EXERCISE 2.3 (some properties of the differential). Let $\varphi\colon X \to Y$, $\psi\colon Y \to Z$ be two morphisms of affine varieties, and $x \in X$. Show the that the following statements hold:

(1) $\mathrm{d}_x(\psi \circ \varphi) = \mathrm{d}_{\varphi(x)}\psi \circ \mathrm{d}_x\varphi$,

(2) φ is an isomorphism or the identity, then so is $\mathrm{d}_x\varphi$,

(3) if φ is a constant map, then $\mathrm{d}_x\varphi = 0$ for all $x \in X$.

Lemma 2.4

We have $T_x(X) = \mathrm{Der}_{\Bbbk}(\mathcal{O}_{X,x}, \Bbbk(x)) = \left(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2\right)^*$.

PROOF. We define a linear map $\pi\colon T_x(X) \to \left(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2\right)^*$ by

$$\pi(\delta)(m) = \delta(m)$$

for $\delta \in \mathrm{Der}_{\Bbbk}(\mathcal{O}_{X,x}, \Bbbk(x))$ and $m \in \mathfrak{m}_{X,x}$. To verify it is well-defined, we need to check that $\delta(\mathfrak{m}_{X,x}^2) = 0$, which is easy.

Conversely, if $f \in \left(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2\right)^*$, let us define $\delta_f \in \mathrm{Der}_{\Bbbk}(\mathcal{O}_{X,x}, \Bbbk(x))$ by

$$\delta_f(a) = f(a - \bar{a}),$$

where $\bar{a}$ is the image of $a \in \mathcal{O}_{X,x}$ in $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = \Bbbk(x)$ so that $a - \bar{a} = a - a(x) \in \mathfrak{m}_{X,x}$.

We observe that

$$\pi(\delta_f)(m) = \delta_f(m) = f(m - \bar{m}) = f(m),$$

whence $\pi(\delta_f) = f$, and that

$$\delta_{\pi(\delta)}(a) = \pi(\delta)(a - \bar{a}) = \delta(a - \bar{a}) = \delta(a),$$

because $\delta(\Bbbk(x)) = 0$ for $a \in \mathcal{O}_{X,x}$. □

If $\varphi\colon X \to Y$ is a morphism of affine varieties, then $\mathrm{d}_x\varphi\colon T_x(X) \to T_{\varphi(x)}(Y)$ is given, through this identification, by the transpose map (in the sense of linear algebra) of the comorphism

$$\varphi^*\colon \mathfrak{m}_{Y,\varphi(x)}/\mathfrak{m}_{Y,\varphi(x)}^2 \to \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2.$$

2.2. Lie algebra of a linear algebraic group

For simplicity, we assume in this section that G is a linear algebraic group.

The aim is to show that the tangent space

$$T_e(G) = \mathrm{Der}_{\Bbbk}(\mathcal{O}_{G,e}, \Bbbk)$$

of G at the neutral element e has naturally a structure of a Lie algebra. To do this, we intend to identify $\mathrm{Der}_{\Bbbk}(\mathcal{O}_{G,e}, \Bbbk)$ with a Lie subalgebra of the Lie algebra $\mathrm{Der}_{\Bbbk}(\Bbbk[G])$ (see Section 1.3).

The group G acts on itself by left multiplication $(g, h) \mapsto gh$. This induces an action of G on $\Bbbk[G]$: for all $g \in G$ and $f \in \Bbbk[G]$, define $\lambda_g(f) \in \Bbbk[G]$ by

$$\lambda_g(f)(h) = f(g^{-1}h) \quad \text{for all} \quad h \in G.$$

Similarly, the right action $(g, h) \mapsto hg^{-1}$ induces an action of G on $\Bbbk[G]$ setting for all $g \in G$ and $f \in \Bbbk[G]$,

$$\rho_g(f)(h) = f(hg) \quad \text{for all} \quad h \in G.$$

Note that λ_g and ρ_g are both automorphisms of $\Bbbk[G]$. They yield two G-actions on $\mathfrak{gl}(\Bbbk[G]) = \mathrm{End}_{\Bbbk}(\Bbbk[G])$ by conjugation: for all $g \in G$ and $F \in \mathfrak{gl}(\Bbbk[G])$,

$$\lambda_g(F) = \lambda_g F \lambda_g^{-1}, \quad \rho_g(F) = \rho_g F \rho_g^{-1}.$$

EXERCISE 2.4 (the space of derivations is preserved by the left and right actions). Show that the left and right actions of G on $\mathfrak{gl}(\Bbbk[G])$ preserve the space of derivations $\mathrm{Der}_{\Bbbk}(\Bbbk[G])$.

The subspace

$$\mathrm{Der}_{\Bbbk}(\Bbbk[G])^{\lambda(G)} = \{\delta \in \mathrm{Der}_{\Bbbk}(\Bbbk[G])\colon \lambda_g\delta = \delta\lambda_g \text{ for all } g \in G\},$$

of invariant derivations for the left action is a Lie subalgebra of $\mathrm{Der}_{\Bbbk}(\Bbbk[G])$. Indeed, δ, δ' are left invariant derivations, for all $g \in G$, we have

$$\lambda_g[\delta, \delta'] = \lambda_g(\delta\delta' - \delta'\delta) = \delta\delta'\lambda_g - \delta'\delta\lambda_g = [\delta, \delta']\lambda_g.$$

Definition 2.5 – Lie algebra of a linear algebraic group

The Lie algebra $\mathscr{L}(G)$ of G is the Lie algebra $\mathrm{Der}_{\Bbbk}(\Bbbk[G])^{\lambda(G)}$.

EXERCISE 2.5. We have

$$\mathscr{L}(G) = \{\delta \in \mathrm{Der}_{\Bbbk}(\Bbbk[G])\colon \Delta \circ \delta = (1 \otimes \delta) \circ \Delta\},$$

where

$$\Delta\colon \Bbbk[G] \longmapsto \Bbbk[G] \otimes \Bbbk[G]$$

is the *comultiplication* μ^* (the comorphism of μ).

Let

$$\mathrm{ev}_e\colon \Bbbk[G] \to \Bbbk$$

be the ***evaluation map*** at the neutral element e; this is the comorphism

$$\varepsilon = e^*\colon \Bbbk[G] \to \Bbbk, \quad f \mapsto f(e)$$

of the constant morphism $e\colon \mathrm{Spec}(\Bbbk) \to G$ whose image is e.

Theorem 2.6 – the tangent space at the neutral element is a Lie algebra

The map $\mathscr{L}(G) \to T_e(G)$ sending δ to $\varepsilon \circ \delta$ is an isomorphism of vector spaces. Hence, one can endow $T_e(G)$ with a Lie algebra structure.

REMARK 2.1. By Exercise 2.5, a derivation $\delta \in \mathscr{L}(G)$ is entirely determined by $\varepsilon \circ \delta$ since

$$\delta = (1 \otimes \varepsilon) \circ \Delta \circ \delta = (1 \otimes \varepsilon) \circ (1 \otimes \delta) \circ \Delta = (1 \otimes (\varepsilon \circ \delta)) \circ \Delta.$$

REMARK 2.2. We can also consider the right-invariant action, and we have an isomorphism of Lie algebras

$$\mathrm{Der}_{\Bbbk}(\Bbbk[G])^{\lambda(G)} \cong \mathrm{Der}_{\Bbbk}(\Bbbk[G])^{\rho(G)}.$$

EXERCISE 2.6 (proof of Theorem 2.6). Define a map from $T_e(G) = \mathrm{Der}_{\Bbbk}(\Bbbk[G], \Bbbk(e))$ to $\mathscr{L}(G) = \mathrm{Der}_{\Bbbk}(\Bbbk[G])^{\lambda(G)}$ as follows. For $\delta \in T_e(G)$, let $D_\delta \in \mathscr{L}(G)$ defined by

$$D_\delta(f)(g) = \delta(\lambda_{g^{-1}} f),$$

that is, $D_\delta = (1 \otimes \delta) \circ \Delta$.

(1) Verify that D_δ is a derivation, and that D_δ is left invariant.
(2) Check that $D \mapsto \varepsilon \circ D$ and $\delta \mapsto D_\delta$ are inverse to each others.

We write $\mathrm{Lie}(G)$ for the Lie algebra $\mathscr{L}(G) \cong T_e(G)$, and we call it the ***Lie algebra of*** G. By the theorem,

$$\dim \mathrm{Lie}(G) = \dim G.$$

EXAMPLE 2.3. The Lie algebras of the additive group $\mathbb{G}_a$ and the multiplicative group $\mathbb{G}_m$ are one-dimensional so they are commutative. More precisely, $\Bbbk[\mathbb{G}_a] = \Bbbk[T]$ and the left-invariant derivation $\delta = \partial_T$ generates $\mathscr{L}(\mathbb{G}_a)$. Similarly, $\Bbbk[\mathbb{G}_m] = \Bbbk[T, T^{-1}]$ and the left-invariant derivation $\delta = T\partial_T$ generates $\mathscr{L}(\mathbb{G}_m)$.

Proposition 2.7

The Lie algebra of $\mathrm{GL}_n(\Bbbk)$ is $\mathfrak{gl}_n(\Bbbk) = \mathscr{M}_n(\Bbbk)$, and the Lie algebra of $\mathrm{SL}_n(\Bbbk)$ is $\mathfrak{sl}_n(\Bbbk)$.

EXERCISE 2.7 (proof of Proposition 2.7). Recall that

$$\mathrm{GL}_n := \mathrm{GL}_n(\Bbbk) = \{x \in \mathscr{M}_n(\Bbbk) \colon \det(x) \neq 0\},$$

that

$$\Bbbk[\mathrm{GL}_n] = \Bbbk[T_{i,j}, \det^{-1}],$$

and that

$$\Delta(T_{i,j}) = \sum_{k=1}^{n} T_{i,k} \otimes T_{k,j}, \quad \varepsilon(T_{i,j}) = \delta_{i,j}.$$

(1) Verify that a basis of $\mathrm{Der}_\Bbbk(\Bbbk[G], \Bbbk)$ is given by the $e_{i,j}$'s defined by $e_{i,j}(T_{k,\ell}) = \delta_{i,k}\delta_{j,\ell}$.

(2) Check that the assignment $e_{i,j} \mapsto E_{i,j}$, where $E_{i,j}$ is the (i,j)-elementary matrix, is a Lie algebra isomorphism, and conclude for the Lie algebra of GL_n.

Hint: identify $\mathrm{Der}_\Bbbk(\Bbbk[G], \Bbbk)$ with $\mathrm{Der}_\Bbbk(\Bbbk[G])^{\lambda(G)}$ using $D_{e_{i,j}} = (\mathrm{Id} \otimes e_{i,j}) \circ \Delta =: \widetilde{e_{i,j}}$ and compute $\widetilde{e_{a,b}} \circ \widetilde{e_{c,d}}$.

(3) Noticing that

$$\sum_{i,j} \frac{\partial \det}{\partial T_{i,j}}(I_n) T_{i,j} = \sum_i T_{i,i} = \mathrm{Tr}(T),$$

deduce the Lie algebra of $\mathrm{SL}_n(\Bbbk)$.

To compute more examples in practice, the following interpretation of $T_e(G)$ can be useful. Recall that the algebra of global sections of the tangent sheaf $\mathcal{T}_G$ is given by

$$\Gamma(G, \mathcal{T}_G) = \mathrm{Der}_\Bbbk(\Bbbk[G]).$$

This is the set of ***vector fields*** [1] on G. Moreover,

$$\mathrm{Der}_\Bbbk(\Bbbk[G]) \cong \mathrm{Hom}_{\Bbbk-\mathrm{alg.}}(\Bbbk[G], \Bbbk[\varepsilon]/(\varepsilon^2)).$$

Consider the projection

$$\begin{array}{rccc} \pi \colon (TG)(\Bbbk) = \mathrm{Hom}_{\Bbbk-\mathrm{alg.}}(\Bbbk[G], \Bbbk[\varepsilon]/(\varepsilon^2)) & \longrightarrow & G(\Bbbk) = \mathrm{Hom}_{\Bbbk-\mathrm{alg.}}(\Bbbk[G], \Bbbk) \\ \gamma & \longmapsto & s \circ \gamma, \end{array}$$

where $s \colon \Bbbk[\varepsilon]/(\varepsilon^2) \to \Bbbk$, $x \mapsto 0$. Then

$$\pi^{-1}(e) \cong \mathrm{Der}_\Bbbk(\Bbbk[G], \Bbbk(x)) = T_e G,$$

where e is identified with ev_e.

EXAMPLE 2.4. Since $\mathrm{O}_n(\Bbbk) = \{x \in \mathscr{M}_n(\Bbbk) \colon xx^T = I_n\}$, we get that $\mathrm{Lie}(\mathrm{O}_n(\Bbbk)) \cong \mathfrak{so}_n(\Bbbk)$ using the equivalence

$$(I_n + \varepsilon x)(I_n + \varepsilon x)^T = I_n \bmod \varepsilon^2 \iff x + x^T = 0 \bmod \varepsilon^2,$$

for $x \in \mathscr{M}_n(\Bbbk)$.

EXERCISE 2.8. Compute the Lie algebras of the algebraic groups $\mathrm{SO}_n(\Bbbk)$, $\mathrm{Sp}_{2n}(\Bbbk)$, $\mathrm{D}_n(\Bbbk)$, $\mathrm{T}_n(\Bbbk)$, $\mathrm{U}_n(\Bbbk)$.

REMARK 2.3. The Lie algebra of G acts on $\Bbbk[G]$ by ***vector fields***: for $x \in \mathrm{Lie}(G)$ and $f \in \Bbbk[G]$,

$$(x_L f)(g) = \delta_x(\lambda_{g^{-1}} f) \quad \text{for all} \quad g \in G,$$

where x_L is the corresponding element of $\mathscr{L}(G)$ and δ_x that of $T_e(G)$.

1. For $U \subset G$ open, $\Gamma(U, \mathcal{T}_G) = \mathrm{Der}_\Bbbk(\mathcal{O}_G(U))$.

A Lie algebra which is the Lie algebra of some linear algebraic group is called an ***algebraic Lie algebra***. All examples or finite-dimensional Lie algebras we have encountered so far are thus algebraic. As a rule, we write with a gothic letter $\mathfrak{g}$ the Lie algebra of a linear algebraic group G. This justifies in part the notation of Example 1.4 and Exercise 1.2.

EXAMPLE 2.5 (an example of a non algebraic Lie algebra). Let $\mathfrak{g}$ be a Lie algebra with basis $\{x, y, z\}$ and Lie bracket given by

$$[x, y] = y, \quad [x, z] = \alpha z, \quad [y, z] = 0.$$

Then for $\alpha \notin \mathbb{Q}$, the Lie algebra $\mathfrak{g}$ is not algebraic (see [12, §24.8.4]).

EXERCISE 2.9. Let H be a subgroup of G, with defining ideal I, that is, $\Bbbk[H] = \Bbbk[G]/I$. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively. Show that

$$\mathfrak{h} = \{x \in \mathfrak{g} \colon x_L(I) \subset I\}.$$

We shall admit the following useful result. It is based on the fact that, in characteristic zero, the Lie algebras of an intersection of closed subgroups is the intersection of the Lie algebras ([8, Section 12.5]).

This is not true is positive characteristic!

Theorem 2.8 – correspondence between groups and Lie algebras

Let G be a connected linear algebraic group. The correspondence $H \mapsto \mathfrak{h} = \mathrm{Lie}(H)$ is a one-to-one inclusion preserving correspondence between the collection of closed connected subgroups of G and the collection of their Lie algebras, regarded as Lie subalgebras of $\mathfrak{g} = \mathrm{Lie}(G)$.

2.3. Adjoint representation

Let G, G' be a linear algebraic groups, and $\mathfrak{g}, \mathfrak{g}'$ their respective Lie algebras. If $\varphi \colon G \to G'$ is a morphism of linear algebraic groups, then $\mathrm{d}_e\varphi \colon \mathfrak{g} \to \mathfrak{g}'$ is a morphism of Lie algebras. Moreover,

$$\mathrm{Ker}\, \mathrm{d}_e\varphi = \mathrm{Lie}(\mathrm{Ker}\, \varphi).$$

This is not true is positive characteristic!

For example, consider the ***inner automorphism*** $\mathrm{Int}\, x$ for $x \in G$, defined by

$$\mathrm{Int}\, x(y) = xyx^{-1} \quad \text{for all } y \in G.$$

Its differential $\mathrm{d}_e\mathrm{Int}\, x$ is a great importance; we denote it by $\mathrm{Ad}\, x$. It is an ***automorphism*** of the Lie algebra $\mathfrak{g}$, that is, a Lie algebra morphism which is bijective. Indeed,

$$(\mathrm{Ad}\, x)(\mathrm{Ad}\, x^{-1}) = \mathrm{d}_e(\mathrm{Int}\, x) \circ \mathrm{d}_e(\mathrm{Int}\, x^{-1}) = \mathrm{d}_e(\mathrm{Int}\, e) = \mathrm{Id}_{\mathfrak{g}}$$

because $(\mathrm{Ad}\, x)(\mathrm{Ad}\, y) = \mathrm{Ad}\, xy$. So

$$\mathrm{Ad} \colon G \longrightarrow \mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$$

is a morphism (of abstract groups), called the ***adjoint representation***. Here, $\mathrm{Aut}(\mathfrak{g})$ denote the set of all automorphisms of the Lie algebra $\mathfrak{g}$.

EXERCISE 2.10. Show that $\mathrm{Ad}\, x(\delta) = \rho_x \delta \rho_x^{-1}$ for δ in $\mathfrak{g}$ identified with $\mathscr{L}(G)$.

We admit the following facts.

Proposition 2.9

Let G be a linear algebraic group, with Lie algebra $\mathfrak{g}$. Then, we have:

(i) $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is a morphism of algebraic groups; in the case where G is a closed subgroup of $\mathrm{GL}_n(\mathbb{k})$, $\mathrm{Ad}\, x$ is just the conjugation by $x \in G$,

(ii) the differential of Ad is ad, where $(\mathrm{ad}\, x)y = [x, y]$ for $x, y \in G$,

(iii) the center $Z(G)$ equals $\mathrm{Ker\,Ad}$, and

$$\mathrm{Lie}(\mathrm{Ker\,Ad}) = \mathrm{Ker\,ad} = \mathfrak{z}(\mathfrak{g}),$$

where $\mathfrak{z}(\mathfrak{g})$ is the Lie center of $\mathfrak{g}$.

3

Representations and modules

3.1. Definitions and examples

Let $\mathfrak{g}$ be a Lie algebra.

Definition 3.1

A ***representation*** de $\mathfrak{g}$, or $\mathfrak{g}$-***module***, is a pair (V, σ), where V is a $\Bbbk$-vector space and $\sigma \colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra morphism. In other words, σ is a linear map such that:

$$\forall\, (x, y) \in \mathfrak{g} \times \mathfrak{g}, \qquad \sigma(x) \circ \sigma(y) - \sigma(y) \circ \sigma(x) = \sigma([x, y]).$$

If (V, σ) is a representation of $\mathfrak{g}$, we often briefly write $x.v$ in place of $\sigma(x)(v)$ for $x \in \mathfrak{g}$ and $v \in V$ when σ is obvious.

REMARK 3.1. The above definition is equivalent to the datum of a vector space V equipped with a bilinear map

$$\mathfrak{g} \times V \to V, \quad (x, v) \mapsto x.v$$

such that for all $x, y \in \mathfrak{g}$ et $v \in V$:

$$[x, y].v = x.(y.v) - y.(x.v).$$

One defines the notions of ***submodule***, ***quotient module*** and ***direct sum of modules*** in a natural way.

EXAMPLE 3.1 (adjoint representation). The map

$$\begin{array}{rcl} \mathrm{ad} \colon \mathfrak{g} & \longmapsto & \mathfrak{gl}(\mathfrak{g}) \\ x & \longmapsto & \mathrm{ad}\, x, \end{array}$$

defines a representation of $\mathfrak{g}$, called the ***adjoint representation***. A submodule of $(\mathfrak{g}, \mathrm{ad})$ is nothing but an ideal of $\mathfrak{g}$.

> EXERCISE 3.1 (representation adjointe de $\mathfrak{sl}_2(\Bbbk)$). In the notation of Exercise 1.4, compute the matrices of $\mathrm{ad}\, e, \mathrm{ad}\, h, \mathrm{ad}\, f$ in the basis e, h, f.

EXAMPLE 3.2. If (V, σ) is a representation of $\mathfrak{g}$, we define its ***dual representation*** (V^*, σ^*) as follows: for $x \in \mathfrak{g}$ and $\xi \in V^*$,

$$\sigma^*(x)(\xi) = -\xi \circ \sigma(x).$$

For example, the dual representation of $(\mathfrak{g}, \mathrm{ad})$ is

$$\begin{array}{rcl} \mathrm{ad}^* \colon \mathfrak{g} & \longmapsto & \mathfrak{gl}(\mathfrak{g}^*) \\ x & \longmapsto & \left(\begin{array}{rcl} \mathfrak{g}^* & \longrightarrow & \mathfrak{g}^* \\ \xi & \longmapsto & -\xi([x, -]) \end{array} \right). \end{array}$$

A ***morphism of*** $\mathfrak{g}$-***modules*** is a linear map $\theta \colon V \to W$, where (V, σ) and (W, π) are two $\mathfrak{g}$-modules, such as

$$\theta(\sigma(x)(v)) = \pi(x)(\theta(v))$$

for all $x \in \mathfrak{g}$ and $v \in V$, that is,

$$\theta \circ \sigma(x) = \pi(x) \circ \theta(v),$$

for all $x \in \mathfrak{g}$. We denote by $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ the set of such morphisms. If $V = W$, we will simply write $\mathrm{End}_{\mathfrak{g}}(V)$ this space.

We say that the modules V and W of $\mathfrak{g}$ are ***isomorphic*** or ***equivalent*** if there is an isomorphism of $\mathfrak{g}$-modules $\theta\colon V \to W$ between V and W.

A $\mathfrak{g}$-module (V, σ) is called ***simple*** or ***irreducible*** if $V \neq \{0\}$ and if the only submodules of V are V or $\{0\}$. The space V is ***semisimple*** or ***completely reducible*** if V is isomorphic to a direct sum of irreducible submodules.

EXERCISE 3.2. Consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}(V)$, where V is a finite-dimensional vector space. Show that there exists an isomorphism of representations

$$\mathfrak{gl}(V) \cong V^* \otimes V,$$

where $\mathfrak{gl}(V)$ is equipped with the adjoint representation, V is the natural representation, and V^* is the dual representation of V.

Proposition 3.2 – Lemme de Schur

We assume in this proposition that $\Bbbk$ is algebraically closed.

(i) If V, W are two non-isomorphic simple modules of $\mathfrak{g}$, then $\mathrm{Hom}_{\mathfrak{g}}(V, W) = \{0\}$.

(ii) If V is a simple module of $\mathfrak{g}$, then $\mathrm{End}_{\mathfrak{g}}(V)$ is a field. Furthermore, if V is finite-dimensional, then $\mathrm{End}_{\mathfrak{g}}(V) = \Bbbk\,\mathrm{Id}_V$.

PROOF. (i) Since the kernel and the image of a morphism of $\mathfrak{g}$-modules are $\mathfrak{g}$-modules, a morphism of simple $\mathfrak{g}$-modules is either the zero morphism, or an isomorphism.

(ii) The argument of (i) shows that $\mathrm{End}_{\mathfrak{g}}(V)$ is a field. Moreover, this field clearly contains $\Bbbk\,\mathrm{Id}_V$. If now V is of finite dimension, then any $f \in \mathrm{End}_{\mathfrak{g}}(V)$ possesses an eigenvalue λ ($\Bbbk$ is assumed algebraically closed). This implies that $\mathrm{Ker}(f - \lambda\mathrm{Id}_V)$ is a nontrivial submodule of V. Since V is simple, we deduce that $f = \lambda\mathrm{Id}_V$. □

***Issaï Schur**, 1875 – 1941, was a Russian mathematician who worked in Germany for most of his life. He studied at the University of Berlin. He obtained his doctorate in 1901, became lecturer in 1903 and, after a stay at the University of Bonn, professor in 1919.*

A representation (V, σ) is called ***faithful*** if the morphims σ is injective: in that event, one can identify $\mathfrak{g}$ with a Lie subalgebra of $\mathfrak{gl}(V)$. We admit the following theorem [1].

Theorem 3.3 – Ado's theorem

Any finite-dimensional Lie algebra admits a faithful finite-dimensional representation.

***Igor Dmitrievich Ado** (1910-1983) is a Russian mathematician. He obtained his doctorate in Kazan under the supervision of Chebotarev. Ado's theorem, in its most general version as above (on a field of any characteristic) is due to Iwasawa and Harish-Chandra.*

1. The reader is referred for instance to [3, Theorem E4] for a proof.

EXERCISE 3.3 (the natural representation is simple). Let V be a vector space. Show that the natural representation of $\mathfrak{gl}(V)$ on V is simple.

EXERCISE 3.4 (the adjoint representation of $\mathfrak{sl}_2(\Bbbk)$ is simple and faithful if $\mathrm{char}(\Bbbk) \neq 2$). Show that the adjoint representation of $\mathfrak{sl}_2(\Bbbk)$ est is simple and faithful if $\mathrm{char}(\Bbbk) \neq 2$.

3.2. Finite-dimensional representations of $\mathfrak{sl}_2(\Bbbk)$

We assume in this section that $\Bbbk$ is algebraically closed and of zero characteristic.

We aim to study the finite-dimensional representations of $\mathfrak{sl}_2(\Bbbk)$. As we will see in Part 2, they play an essential role in the study of *any* semisimple Lie algebras.

We keep the notation of Exercise 1.4. Let $r \in \mathbb{N}$. Set for $i \in \{1, \dots, r\}$,

$$\mu_i = i(r - i + 1).$$

Consider the representation (V_r, σ_r) of $\mathfrak{sl}_2(\Bbbk)$ where $V_r = \Bbbk^{r+1}$ and σ_r is given by:

$$\sigma_r(h) = \begin{pmatrix} r & 0 & 0 & \cdots & 0 \\ 0 & r-2 & 0 & \cdots & 0 \\ 0 & 0 & r-4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -r \end{pmatrix}, \tag{2}$$

$$\sigma_r(e) = \begin{pmatrix} 0 & \mu_1 & 0 & \cdots & 0 \\ 0 & 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_r \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \sigma_r(f) = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Here we identify $\mathrm{End}(\Bbbk^{r+1})$ with $\mathscr{M}_{r+1}(\Bbbk)$.

Lemma 3.4

The representation σ_r is simple.

EXERCISE 3.5. Prove Lemma 3.4.

Hint: follow the ideas of Exercise 3.4.

Let now (V, σ) be any nonzero finite-dimensional representation of $\mathfrak{sl}_2(\Bbbk)$. Set

$$H = \sigma(h), \quad E = \sigma(e), \quad F = \sigma(f).$$

By induction, we easily obtain for all $i \in \mathbb{N}$, $j \in \mathbb{N}^*$:

$$[H, E^i] = 2iE^i, \tag{3}$$

$$[H, F^i] = -2iF^i, \tag{4}$$

$$[F, E^j] = -j(H - (j-1)\mathrm{Id}_V) \circ E^{j-1}. \tag{5}$$

EXERCISE 3.6. Check the above relations and deduce that E et F are nilpotent.

Hint: observe that $\mathrm{Tr}(F^i) = \mathrm{Tr}(E^i) = 0$ for any $i \in \mathbb{N}^*$.

Let $\lambda = \min\{n \in \mathbb{N} \colon E^{n+1} = 0\}$ and $w \in V$ such that $v = E^\lambda(w) \neq 0$. Set $v_0 := v$ and for $i \in \mathbb{N}^*$, $v_i := F^i(v_0)$. By (5), we get:

$$0 = [F, E^{\lambda+1}](w) \implies H(v_0) = \lambda v_0.$$

Therefore, by (4),

$$H(v_i) = (\lambda - 2i)v_i. \tag{6}$$

It follows that if $v_i \neq 0$, then v_i is an eigenvector of H relatively to the eigenvalue $\lambda - 2i$. Moreover, a rapid induction gives:

$$E(v_i) = i(\lambda - i + 1)v_{i-1}, \tag{7}$$

where $v_{-1} = 0$. Let s be such that $v_s \neq 0$ et $v_i = 0$ for $i > s$, and let W be the subspace of V generated by $v_0, \dots, v_s$. Then W is a submodule of V and $\{v_0, \dots, v_s\}$ is a basis of W by (6). Denote by H_W, E_W, F_W the restrictions of H, E, F to W, respectively. Since $[E_W, F_W] = H_W$, we get

$$0 = \mathrm{Tr}(H_W) = \sum_{i=0}^{s} (\lambda - 2i) = (s+1)(\lambda - s).$$

As a result, $\boxed{s = \lambda}$ and the matrices of H_W, E_W, F_W in the basis $\{v_0, \dots, v_s\}$ are those of (2) with $r = \lambda$. We deduce that $\sigma|_W$ and σ_λ are equivalent.

In conclusion, we have obtained:

Theorem 3.5

Let $r \in \mathbb{N}^*$ and (V, σ) be a simple $\mathfrak{sl}_2(\Bbbk)$-representation of dimension $r + 1$. Then σ is equivalent to σ_r. Moreover, the eigenvalues of $\sigma(h)$ are $-r, -r + 2, \dots, r - 2, r$, and if $v \in V \setminus \{0\}$ verifies $\sigma(e)(v) = 0$ (resp. $\sigma(f)(v) = 0$), then v is an eigenvector of $\sigma(h)$ relative to the eignevalue r (resp. $-r$)).

Next result with be generalized in Part 2:

Theorem 3.6 – Complete reduciblity of finite-dimensional $\mathfrak{sl}_2(\Bbbk)$-representations

Any finite-dimensional $\mathfrak{sl}_2(\Bbbk)$-representation is completely reducible.

PROOF. Let (V, σ) be a finite-dimensional representation of $\mathfrak{sl}_2(\Bbbk)$. If $\dim V = 0$, there is nothing to prove. We therefore assume $\dim V \geqslant 1$. According to the discussion preceding Theorem 3.5, there exists a simple submodule W of V of dimension $\lambda + 1$. Let us show that there exists a submodule U of V such that $V = U \oplus W$. The theorem will then follows from an induction on the dimension.

Consider the dual representation (V^*, π) of V (see Example 3.2). Let $w \in W$ and $\{v_0, \dots, v_\lambda\}$ be the basis of W as in the discussion preceding Theorem 3.5. Set $\xi \in V^*$ such that $\xi(v_0) = 1$. Then

$$\left(\pi(e)^\lambda(\xi)\right)(w) = \xi\left((-E)^\lambda(w)\right) = (-1)^\lambda.$$

Therefore, $\eta = \pi(e)^\lambda(\xi) \neq 0$ and λ is the smallest integer such that $\pi(e)^{\lambda+1} = 0$. According to the discussion preceding Theorem 3.5, the subspace M of V^* generated by $\eta_i = \pi(f)^i(\eta)$, $i = 0, \dots, \lambda$, is a simple submodule of V^* of dimension $\lambda + 1$.

Let U be the orthogonal in V of M, i.e., $U = \{u \in V \colon \eta_i(u) = 0 \text{ for any } i = 0, \dots, \lambda\}$. We have $\dim V = \dim U + \dim W$ and U is a submodule of V. Using (7), we verify that:

$$\eta_i(v_{\lambda-i}) = (-1)^{\lambda-i}(\lambda!)^2, \tag{8}$$

$$\eta_i(v_p) = 0, \text{ if } i + p > \lambda, \tag{9}$$

and (8) and (9) imply that $U \cap W = \{0\}$. It follows that V is the direct sum of the submodules U and V, as desired. □

Corollary 3.7

Let (V, σ) be a finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{k})$.

(i) There exists $r_1, \ldots, r_n \in \mathbb{N}$ such that σ is equivalent to the direct sum of the representations σ_{r_i}, $i = 1, \ldots, n$.

(ii) We have $V = \sigma(e)(V) \oplus \operatorname{Ker} \sigma(f) = \sigma(f)(V) \oplus \operatorname{Ker} \sigma(e)$.

(iii) $\sigma(h)$ is semisimple with $\mathbb{Z}$-eigenvalues.

(iv) If V is nontrivial, then σ is irreducible if and only if the eigenvalues of $\sigma(h)$ are without multiplicity and are either all odd, or all even.

The representations of $\mathfrak{sl}_2$ appear in many other areas of mathematics. For example, they play a crucial role in the Hard Lefschetz Theorem. One of the key point is that the total cohomology of a smooth complex projective variety is a representation of $\mathfrak{sl}_2$, which gives rise to the notion of $\mathfrak{sl}_2$-Hodge structure. We refer for instance to [11] for more details.

EXERCISE 3.7 (tensor product of two representations). Let $\mathfrak{g}$ be a Lie algebra, and (V_1, σ_1), (V_2, σ_2) two representations of $\mathfrak{g}$.

(1) Set for $x \in \mathfrak{g}$ et $(v_1, v_2) \in V_1 \times V_2$,

$$\sigma(x)(v_1 \otimes v_2) = \sigma_1(x)v_1 \otimes v_2 + v_1 \otimes \sigma_2(x)v_2.$$

Verify that this defines a representation $(V_1 \otimes V_2, \sigma)$ of $\mathfrak{g}$, called the ***tensor representation*** of (V_1, σ_1) and (V_2, σ_2). We write $\sigma = \sigma_1 \otimes \sigma_2$.

(2) Assume now $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{k})$. Establish the following isomorphisms of $\mathfrak{sl}_2(\mathbb{k})$-modules:

$$V_3 \otimes V_7 \cong V_4 \oplus V_6 \oplus V_8 \oplus V_{10}.$$

(3) More generally, find a decomposition into a direct sum of irreducible representations of $\mathfrak{sl}_2(\mathbb{k})$ for the tensor product $V_m \otimes V_n$, where $m, n \in \mathbb{N}^*$.

EXERCISE 3.8 (an infinite dimensional $\mathfrak{sl}_2(\mathbb{k})$-module). In this exercise we will see that the complete reducibility does not remain valid for infinite dimensional representations.

Let $\lambda \in \mathbb{k}$, and Z_λ a $\mathbb{k}$-vector space with countable basis $\{v_0, v_1, v_2, \ldots\}$. Define a representation $(Z_\lambda, \sigma_\lambda)$ of $\mathfrak{sl}_2(\mathbb{k})$ setting for $i \in \mathbb{N}$:

(i) $\sigma_\lambda(h)v_i = (\lambda - 2i)v_i$,

(ii) $\sigma_\lambda(f)v_i = (i+1)v_{i+1}$,

(iii) $\sigma_\lambda(e)v_i = (\lambda - i + 1)v_{i-1}$, where by convention $v_{-1} = 0$.

(1) Verify that this indeed defines an $\mathfrak{sl}_2(\mathbb{k})$-module.

(2) Show that any nonzero submodule of Z_λ has at least one ***maximal vector***, that is an element $w \in W \setminus \{0\}$ such that $\sigma_\lambda(e)w = 0$.

(3) Assume in this question that $\lambda + 1 = r \in \mathbb{N}^*$.

(a) Show that Z_λ is not irreducible.

(b) Let $\phi \colon Z_\mu \to Z_\lambda$ be the unique morphism of $\mathfrak{sl}_2(\mathbb{k})$-modules from Z_μ to Z_λ which send v_0 to v_r, where $\mu = \lambda - 2r$ (verify that such morphism does exist). Show that ϕ is injective and deduce that $\operatorname{Im} \phi$ and $Z_\lambda / \operatorname{Im} \phi \cong V_\lambda$ are irreducible.

REMARK 3.2. As a consequence, Z_λ is not completely reducible. To see this, notice that $\operatorname{Im} \phi$ does not admit any complement in Z_λ that is an $\mathfrak{sl}_2(\mathbb{k})$-module: if this were the case, such a module would be finite-dimensional but Z_λ has not non-trivial submodule of finite-dimension (to see this, let $\sigma_\lambda(f)$ acts on any nonzero vector).

(4) Assume in this question that $\lambda + 1 \notin \mathbb{N}^*$. We aim to show that Z_λ is irreducible. Let v be a nonzero vector of Z_λ written as

$$v = a_s v_s + \cdots + a_r v_r \quad \text{with} \quad a_s \neq 0,\ a_r \neq 0,$$

and let W be the submodule of Z_λ generated by v. Using the action of $\sigma_\lambda(e)$ on v repeatidly, show that $v_0 \in W$, and conclude.

REMARK 3.3. For char($\mathbb{k}$) $\neq 0$, the irreducible finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{k})$ can be constructed explicitly as follows. Let $A = \mathbb{k}[X, Y]$ be the algebra of polynomials in two variables. We define a structure of $\mathfrak{sl}_2(\mathbb{k})$-modules on A by letting the basis $\{e, h, f\}$ acting by:

$$e = X\partial Y, \quad h = X\partial X - Y\partial_X, \quad f = Y\partial_X.$$

Then the subspace of homogeneous polynomials of degree m with basis

$$X^m, X^{m-1}Y, \ldots, XY^{m-1}, Y^m$$

is invariant by $\mathfrak{sl}_2(\mathbb{k})$ and irreducible. In other words, the space $\mathbb{k}[X, Y]_m$ of m-degree homogeneous polynomials is isomorphic to V_m.

4

Nilpotent and solvable Lie algebras

We now assume that all Lie algebras are of finite dimension.

Let $\mathfrak{g}$ be a Lie algebra. If H and K are two subspaces of $\mathfrak{g}$, we denote by $[H, K]$ the subspace of $\mathfrak{g}$ generated by the vectors $[x, y]$, where $x \in H$ and $y \in K$. So, for example, the Lie algebra $\mathfrak{g}$ is abelian if and only if $[\mathfrak{g}, \mathfrak{g}] = \{0\}$.

4.1. Nilpotent Lie algebras and Engel's theorem

We define the ***descending central series*** of $\mathfrak{g}$ by induction:

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}^0, \mathfrak{g}], \quad \mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}], \ldots, \mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}], \ldots.$$

Ainsi, $\mathfrak{g}$ est abélienne if and only if $\mathfrak{g}^1 = 0$.

EXERCISE 4.1 (the descending central series is a decreasing sequence of ideals). Check that $\mathfrak{g}^i$, $i \in \mathbb{N}$, are ideals of $\mathfrak{g}$, and that

$$\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \cdots.$$

Definition 4.1 – nilpotent Lie algebra

The Lie algebra $\mathfrak{g}$ is called ***nilpotent*** if $\mathfrak{g}^n = 0$ for some $n \in \mathbb{N}^*$.

Any abelian Lie algebra, for example $\mathfrak{d}_n(\Bbbk)$, is obviously nilpotent.

EXERCISE 4.2 (the adjoint representation of a nilpotent Lie algebra is nilpotent). Show that $\mathfrak{g}$ est nilpotent if and only if there exists $j \in \mathbb{N}^*$ such that $\operatorname{ad} x_1 \circ \operatorname{ad} x_2 \circ \cdots \circ \operatorname{ad} x_j = 0$ for all $x_1, x_2, \ldots, x_j \in \mathfrak{g}$.

EXERCISE 4.3. Let $n \in \mathbb{N}^*$. Show that the Lie algebra $\mathfrak{u}_n^+(\Bbbk)$ is nilpotent.

We will see that the example of Exercise 4.3 is in fact quite general.

EXERCISE 4.4. Show that if $\operatorname{char}(\Bbbk) = 2$, then $\mathfrak{sl}_2(\Bbbk)$ is nilpotent.

Lemma 4.2

Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then any subalgebra of $\mathfrak{g}$ and any quotient of $\mathfrak{g}$ (by an ideal of $\mathfrak{g}$) are nilpotent Lie algebras.

EXERCISE 4.5. Prove the above lemma.

EXERCISE 4.6. Assume that $\mathfrak{g}$ is a nonzero nilpotent Lie algebra. Show that there exists an ideal of $\mathfrak{g}$ of codimension one.

Hint: observe that if $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}^1 \neq \mathfrak{g}$ and consider a subspace I of $\mathfrak{g}$ such that $I \supset \mathfrak{g}^1$ and $\dim I = \dim \mathfrak{g} - 1$.)

Lemma 4.3

Let V be a vector space of dimension $n > 0$.

(i) If $x \in \mathrm{End}(V)$ is nilpotent, then $\mathrm{ad}\, x$ is a nilpotent endomorphism of $\mathrm{End}(\mathrm{End}\, V)$. More precisely, if $x^p = 0$, then $(\mathrm{ad}\, x)^{2p-1} = 0$.

(ii) If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$ consisted of nilpotent endomorphisms, then there exists a nonzero vector v of V such that $x(v) = 0$ for any $x \in \mathfrak{g}$.

PROOF. The assertion (i) is clear because if $y \in \mathfrak{gl}(V)$, then $(\mathrm{ad}\, x)^n(y)$ is a sum of terms of the form $\pm x^i y x^j$ with $i + j = n$.

(ii) We argue by induction on the dimension of $\mathfrak{g}$. If $\mathfrak{g} = \{0\}$, the result is obvious. We assume $\dim \mathfrak{g} > 0$.

The first step is to construct an ideal of codimension one. Let A be a nonzero Lie subalgebra of $\mathfrak{g}$ (such an algebra exists: e.g., $A = \Bbbk x$ with $x \in \mathfrak{g} \setminus \{0\}$). For $x \in A$, the endomorphism of $\mathfrak{g}/A$ induced by $\mathrm{ad}\, x$ is nilpotent by (i). By the induction hypothesis, there exists $y \in \mathfrak{g} \setminus A$ such that $[A, y] \subset A$. Therefore, A is an ideal of $A \oplus \Bbbk y$. In this way, we construct an ideal I of $\mathfrak{g}$ of codimension one: if $A \oplus \Bbbk y = \mathfrak{g}$, $I = A$ does the job, otherwise we repeat the previous argument with the algebra $A' = A \oplus \Bbbk y$.

So let I be an ideal of $\mathfrak{g}$ such that $\mathfrak{g} = I \oplus \Bbbk y$ pour $y \in \mathfrak{g} \setminus I$. Let W be the vector subspace of V consisted of vectors v such that $x(v) = 0$ for any $x \in I$. By the induction hypothesis, $W \neq \{0\}$ since I is a nilpotent Lie algebra by lemma 4.2.

Show that $y(W) \subset W$. Let $w \in W$ and $z \in I$. We have

$$z \circ y(w) = y \circ z(w) + [z, y](w) = 0$$

because $z(w) = 0$ and $[I, \mathfrak{g}] \subset I$ hence $[z, y](w) = 0$. As y is nilpotent, so is its restriction to W, and there exists $v \in W \setminus \{0\}$ such that $y(v) = 0$. The vector v thus satisfies $x(v) = 0$ for any $x \in \mathfrak{g}$, which proves the lemma. □

***Friedrich Engel**, born December 26, 1861 in Lugau near Chemnitz and died September 29, 1941 in Giessen, was a German mathematician.*

Theorem 4.4 – Engel

Let (V, σ) be a representation of $\mathfrak{g}$ of dimension $n > 0$. Suppose that $\sigma(x)$ is a nilpotent endomorphism for any $x \in \mathfrak{g}$. Then there exists a basis $\mathscr{B}$ of V in which the matrix of $\sigma(x)$, for any $x \in \mathfrak{g}$, is of the form

$$\operatorname{Mat}_{\mathscr{B}}(\sigma(x)) = \begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

In other words, there is a ***flag*** [a] of V,

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V,$$

such that $\sigma(\mathfrak{g})V_i \subset V_{i-1}$ for $i \in \{1, \ldots, n\}$.

a. A ***flag*** of a vector space V is a sequence of vector subspaces $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that $\dim V_i = i$ for any $i \in \{0, \ldots, n\}$.

EXERCISE 4.7 (proof of Theorem 4.4). Possibly quotienting by the kernel of σ, we can assume: $\mathfrak{g} \subset \mathfrak{gl}(V)$. Arguing by induction and using the the canonical projection

$$\pi \colon V \longrightarrow V/\Bbbk v = W,$$

where v is as in Lemma 4.3, prove the theorem.

Corollary 4.5

A Lie algebra $\mathfrak{g}$ is nilpotent is and only if for any $x \in \mathfrak{g}$, $\operatorname{ad} x$ is a nilpotent endomorphism of $\mathfrak{g}$.

EXERCISE 4.8. Prove the above corollary using the fact that if I are $\mathfrak{g}/I$ are nilpotent, then so is $\mathfrak{g}$.

4.2. Lie algebras solvable et radical d'une Lie algebra

We define another sequence of ideals of $\mathfrak{g}$, the ***derived series***, by induction:

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], \ldots, \mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}], \ldots.$$

EXERCISE 4.9 (the derived series is a decreasing sequence of ideals). Check that $\mathfrak{g}^{(i)}$, $i \in \mathbb{N}$, are ideals of $\mathfrak{g}$, and that $\mathfrak{g} = \mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots$.

Definition 4.6 – solvable Lie algebra

We say that a Lie algebra $\mathfrak{g}$ is ***solvable*** if $\mathfrak{g}^{(n)} = 0$ for some $n \in \mathbb{N}$.

For example, any nilpotent Lie algebra is solvable. In particular, any abelian Lie algebra is solvable.

EXERCISE 4.10. Let $n \in \mathbb{N}^*$. Show that the Lie algebra $\mathfrak{b}_n^+(\Bbbk)$ is solvable, and compute its derived series.

This example is in fact very general.

EXERCISE 4.11. Let $\mathfrak{g}$ be a Lie algebra, and I, J two ideals of $\mathfrak{g}$.

(1) Show that if $\mathfrak{g}$ is solvable, then so are all subalgebras and homomorphic images of $\mathfrak{g}$,
(2) Show that if I are $\mathfrak{g}/I$ are solvable, then so is $\mathfrak{g}$.
(3) Show that if I, J are solvable, then so is $I + J$.

Question 3 of Exercise 4.11 garanties the existence of a unique maximal solvable ideal. Indeed, let S be a maximal solvable ideal (for the inclusion). If I is another solvable ideal, then $I + S$ is solvable ideal, and so $I + S \subset S$, whence $I + S = S$ (by maximality).

Proposition-definition 4.7 – radical of a Lie algebra

The unique maximal solvable ideal of $\mathfrak{g}$ is called the ***radical*** of $\mathfrak{g}$ and is denoted by $\operatorname{rad}(\mathfrak{g})$. Furthermore, the quotient $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ does not contain any nonzero solvable ideal.

Definition 4.8 – semisimple Lie algebra

A Lie algebra $\mathfrak{g}$ is called ***semisimple*** if $\operatorname{rad}(\mathfrak{g}) = 0$.

We will see next chapter that if a Lie algebra $\mathfrak{g}$ is semisimple (resp. simple) then the representation $(\mathfrak{g}, \operatorname{ad})$ is semisimple (resp. simple).

EXERCISE 4.12. Let $\mathfrak{g}$ be the Lie algebra of upper triangular matrices by blocks:

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a \in \mathscr{M}_p(\mathbb{k}),\, b \in \mathscr{M}_{p,q}(\mathbb{k}),\, c \in \mathscr{M}_q(\mathbb{k}) \} \subset \mathfrak{gl}_n(\mathbb{k}) \right\}.$$

What is $\operatorname{rad}(\mathfrak{g})$? Describe $\mathfrak{g}/\operatorname{rad}(\mathfrak{g})$.

Proposition 4.9

Any simple Lie algebra is semisimple.

PROOF. Suppose that $\mathfrak{g}$ is simple but not semisimple. So $\operatorname{rad}(\mathfrak{g}) \neq 0$. Since $\operatorname{rad}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, we deduce that $\operatorname{rad}(\mathfrak{g}) = \mathfrak{g}$. Therefore $\mathfrak{g}$ is solvable and there exists $n \geqslant 0$ such that $\mathfrak{g}^{(n)} = \{0\}$. In particular, $\mathfrak{g}^{(1)} \neq \mathfrak{g}$. Indeed, $\mathfrak{g}^{(1)} = \mathfrak{g}$ would imply $\mathfrak{g}^{(n)} = \mathfrak{g}$ for any $n \geqslant 1$. But $\mathfrak{g}^{(1)}$ is an ideal of $\mathfrak{g}$, so $\mathfrak{g}^{(1)} = \{0\}$ since $\mathfrak{g}$ is simple, and therefore $[\mathfrak{g}, \mathfrak{g}] = \{0\}$, i.e., $\mathfrak{g}$ is abelian. Since $\mathfrak{g}$ is abelian, any subspace of $\mathfrak{g}$ is an ideal. As $\mathfrak{g}$ is simple, we deduce that $\dim \mathfrak{g} = 1$, which is impossible. □

EXERCISE 4.13. Show that if $\mathfrak{g}$ is nilpotent then its center $\mathfrak{z}(\mathfrak{g})$ is nonzero. Show that this implication is not true for solvable Lie alegbra.

4.3. Representations of solvable Lie algebras

Assume now that $\mathbb{k}$ is algebraically closed and of zero characteristic.

Observe that any representation of dimension one of a Lie algebra is irreducible. In addition, giving a one-dimensional representation of a Lie algebra $\mathfrak{g}$ amounts to giving a linear map $\sigma\colon \mathfrak{g} \to \mathbb{k}$ such that $\sigma([\mathfrak{g}, \mathfrak{g}]) = 0$. Such a linear map is also called a ***character*** of $\mathfrak{g}$.

Theorem 4.10 – Lie

Let $\mathfrak{g}$ be a nonzero solvable Lie algebra, and (V, σ) a finite-dimensional irreducible representation of $\mathfrak{g}$. Then $\dim V = 1$.

Lie's theorem ensures that if $\mathfrak{g}$ is solvable and if (V, σ) is a finite-dimensional representation of $\mathfrak{g}$, then there exists an eigenvector in V common to all $\sigma(x)$, for $x \in \mathfrak{g}$. We have already proved this result for nilpotent Lie algebras: this is Engel's Theorem 4.4.

EXERCISE 4.14. The objective of the exercise is to prove Theorem 4.10. We argue by induction on the dimension of $\mathfrak{g}$.

(1) Prove the theorem for $\dim \mathfrak{g} = 1$.

(2) Assume now that $\dim \mathfrak{g} > 1$, and fix an ideal I of $\mathfrak{g}$ de codimension 1 as in Exercise 4.6 (observe here that the same argument work for "solvable" instead of "nilpotent"). Since V is an I-module, V contains an I-irreducible submodule W of V; by induction, $\dim W = 1$. Then for all $w \in W \setminus \{0\}$ and $y \in I$, we have
$$\sigma(y)w = \lambda(y)w,$$
where $\lambda \colon I \to \Bbbk$ is the one-dimensional representation of I associated with W. Show that for all $x \in \mathfrak{g}$ and $y \in I$,
$$\lambda([x, y]) = 0.$$

(3) Set
$$U = \{u \in V : \sigma(y)u = \lambda(y)u \text{ for any } y \in I\}.$$
Show that U is a submodule of V. Hence $U = V$ since V is irreducible, that is,
$$\sigma(y)v = \lambda(y)v \quad \text{for all} \quad y \in I \text{ et } v \in V.$$

(4) Writing $\mathfrak{g} = I \oplus \Bbbk x$ for $x \in \mathfrak{g} \setminus I$, and letting v an eigenvector of $\sigma(x)$ in V, prove that $V = \Bbbk v$ and conclude.

Corollary 4.11

Let $\mathfrak{g}$ be a solvable Lie algebra, and (V, σ) a representation of $\mathfrak{g}$ of finite dimension $n \in \mathbb{N}^*$. Then there exists a basis $\mathscr{B}$ of V such that for any $x \in \mathfrak{g}$ the matrix of $\sigma(x)$ is of the form
$$\mathrm{Mat}_{\mathscr{B}}(\sigma(x)) = \begin{pmatrix} * & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix}.$$
In other words, via the isomorphism $\mathfrak{gl}(V) \cong \mathfrak{gl}_n(\Bbbk)$ (given by the choice of $\mathscr{B}$), $\sigma(\mathfrak{g})$ is contained in the subalgebra $\mathfrak{b}_n^+(\Bbbk)$.

Corollary 4.12

Let $\mathfrak{g}$ be a solvable Lie algebra of dimension $n \in \mathbb{N}^*$. Then $\mathfrak{g}$ has a sequence of ideals
$$0 = I_0 \subset I_1 \subset \cdots \subset I_n = \mathfrak{g},$$
with $\dim I_i = i$.

EXERCISE 4.15. Show the above corollaries using Theorem 4.10.

EXERCISE 4.16. Let $\mathfrak{g}$ be a Lie algebra and (V, σ) a simple representation of finite dimension of $\mathfrak{g}$. Show that there exists a linear form $\lambda\colon \operatorname{rad}(\mathfrak{g}) \to \Bbbk$, zero on $[\mathfrak{g}, \operatorname{rad}(\mathfrak{g})]$, such that for all $v \in V$ and all $x \in \operatorname{rad}(\mathfrak{g})$, we have

$$\sigma(x)v = \lambda(x)v.$$

Use the ideas of the question 2 of Exercise 4.14.

Exercise 4.16 shows that the morphism $\sigma\colon \mathfrak{g} \to \mathfrak{gl}(V)$ factorizes as

$$\mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \operatorname{rad}(\mathfrak{g})] \to \mathfrak{gl}(V).$$

So the representation V can be obtained from a simple representation of the Lie algebra $\mathfrak{g}/[\mathfrak{g}, \operatorname{rad}(\mathfrak{g})]$ which is ***reductive***, that is, $\operatorname{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$ where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$ (the set of elements that commute with all elements of $\mathfrak{g}$), see Definition 6.3. Check this as an exercise!

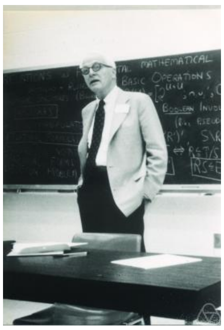
Therefore to understand all simple representations of finite-dimensional Lie algebras, it is enough to understand the simple representations of reductive Lie algebras.

Next, we will see later that any reductive Lie algebra can be written as $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$, with semisimple Lie algebras $[\mathfrak{g}, \mathfrak{g}]$. So to understand all simple representations of finite-dimensional Lie algebras, it is enough to understand the simple representations of semisimple Lie algebras.

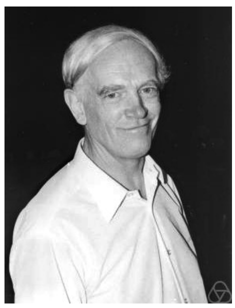
5

Enveloping algebra and PBW theorem

Henri Poincaré, *1854 – 1912, was a French mathematician, theoretical physicist, engineer, and philosopher of science. He is often described as a polymath, and in mathematics as "The Last Universalist" since he excelled in all fields of the discipline as it existed during his lifetime.*



Garrett Birkhoff, *1911 – 1996, was an American mathematician. He is best known for his work in lattice theory.*



Ernst Witt, *1911 – 1991, was a German mathematician, one of the leading algebraists of his time. In 1936, supervised by Emmy Noether at the University of Göttingen, he obtained his PhD on the subject of the Riemann–Roch theorem. He then taught until 1937 at the University of Hamburg. Witt's work mainly focused on algebra and quadratic forms.*

In this chapter, $\Bbbk$ is any commutative field.

5.1. Tensor algebra and symmetric algebra

Let V be a $\Bbbk$-vector space. Let $T^0V = \Bbbk 1$, $T^1V = V, \dots, T^mV = \underbrace{V \otimes \cdots \otimes V}_{m \text{ times}}, \dots$. We define TV as the direct sum of the vector spaces T^mV,

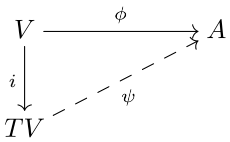
$$TV = \bigoplus_{i=0}^{\infty} T^iV.$$

The ***tensor algebra*** is the associative algebra TV equipped with the associative product defined on homogeneous elements as follows:

$$(v_1 \otimes \cdots \otimes v_k).(w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes v_m, \qquad v_i, w_j \in V,\ k, m \in \mathbb{N}.$$

The element $1 \in T^0V$ is the unit of TV.

The tensor algebra TV satisfies the following universal property: for any $\Bbbk$-linear map $\phi\colon V \to A$, where A is a unital associative $\Bbbk$-algebra, there exists a unique morphism of $\Bbbk$-algebras $\psi\colon TV \to A$ such that $\psi(1) = 1$ and such that the following diagram commutes:



i.e., $\psi \circ i = \phi$, where $i\colon V \hookrightarrow TV$ is inclusion.

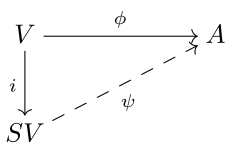
Let I be the two-sided ideal of TV generated by the elements $x \otimes y - y \otimes x$, where $x, y \in V$. The ***symmetric algebra*** of V is the quotient algebra

$$SV = TV/I.$$

Since the generators of I belong to T^2V, we have $I = (I \cap T^2V) \oplus (I \cap T^3V) \oplus \cdots$, therefore the canonical surjection $TV \to SV = TV/I$ induces injections $T^0V = \Bbbk \hookrightarrow SV$ and $T^1V = V \hookrightarrow SV$. Denote by $x_1 \dots x_k$ the image in SV of a homogeneous element $x_1 \otimes \cdots \otimes x_k$ of TV by the canonical surjection $TV \twoheadrightarrow SV$. The symmetric algebra is naturally graded by the degree of elements:

$$SV = \bigoplus_{i=0}^{\infty} S^mV, \quad S^mV = T^mV/(I \cap T^mV),$$

and $(S^kV).(S^mV) \subset S^{k+m}V$ for all $k, m \in \mathbb{N}$. By construction, SV is a commutative algebra. Furthermore, it satisfies the following universal property: for any $\Bbbk$-linear map $\phi\colon V \to A$, where A is a unital commutative $\Bbbk$-algebra, there exists a unique morphism of $\Bbbk$-algebras $\psi\colon SV \to A$ such that $\psi(1) = 1$ and such that the following diagram commutes:



i.e., $\psi \circ i = \phi$, where $i\colon V \hookrightarrow SV$ is the inclusion.

REMARK 5.1. If V is of finite dimension, then SV is canonically isomorphic to $\mathbb{C}[V^*]$, using the assignment

$$V \ni v \longmapsto (\xi \in V^* \mapsto \xi(v)).$$

A ***$\Bbbk$-filtered algebra*** is a $\Bbbk$-algebra A equipped with an increasing filtration $(A_i)_{i\in\mathbb{N}}$ (i.e., $A = \bigcup_{i\in\mathbb{N}} A_i$ and $A_i \subset A_{i+1}$ for any $i \in \mathbb{N}$) such that $1 \in A_0$ and $A_i.A_j \subset A_{i+j}$ for any $i, j \in \mathbb{N}$. The ***graded associated space*** with A is

$$\operatorname{gr} A = \bigoplus_{i\in\mathbb{N}} A_i/A_{i-1},$$

where by convention $A_{-1} = \{0\}$. It is naturally equipped with a structure of an associative $\Bbbk$-algebra structure associative; we define a product,

$$A_i/A_{i-1} \times A_j/A_{j-1} \longrightarrow A_{i+j}/A_{i+j-1}, \qquad i, j \in \mathbb{N},$$

setting

$$(a_i + A_{i-1})(a_j + A_{j-1}) = a_i.a_j + A_{i+j-1}, \qquad a_i \in A_i,\ a_j \in A_j.$$

The unit is the element $1 + A_{-1} \in A_0/A_{-1}$.

REMARK 5.2. The graded algebra $\operatorname{gr} A$ is commutative if and only if for all $i, j \in \mathbb{N}$, $a_i \in A_i$, $a_j \in A_j$, we have $a_i a_j - a_j a_i \in A_{i+j-1}$.

Let A, B be two filtered $\mathbb{k}$-algebras such that $A_0 \cong \mathbb{k}$, $B_0 \cong \mathbb{k}$, and $f\colon A \to B$ is a ***morphism of $\mathbb{k}$-filtered algebras***, that is, f is a morphism of algebras and for all $i \in \mathbb{N}$, we have $f(A_i) \subset B_i$. This morphism induces a morphism

$$\operatorname{gr} f\colon \operatorname{gr} A \to \operatorname{gr} B,$$

by setting for $i \in \mathbb{N}$ and $\bar{a} \in A_i/A_{i-1}$,

$$(\operatorname{gr} f)\bar{a} = f(a) + B_{i-1},$$

where a is any representative of $\bar{a}$ in A_i. The map is well defined since $f(A_{i-1}) \subset B_{i-1}$.

5.2. Universal enveloping algebra

Let $\mathfrak{g}$ be a $\mathbb{k}$-Lie algebra, and J the two-sided ideal of $T(\mathfrak{g})$ generated by the elements

$$x \otimes y - y \otimes x - [x, y], \quad x, y \in \mathfrak{g}.$$

Definition 5.1

The unital associative $\mathbb{k}$-algebra

$$U(\mathfrak{g}) = T(L)/J$$

is called the ***universal enveloping algebra*** of $\mathfrak{g}$.

EXAMPLE 5.1. If $\mathfrak{g}$ is abelian, then $U(\mathfrak{g}) = S(\mathfrak{g})$.

Let $i\colon \mathfrak{g} \to U(\mathfrak{g})$ be the composition of the following linear maps

$$\mathfrak{g} = T^1\mathfrak{g} \longrightarrow T(\mathfrak{g}) \xrightarrow{\ \pi\ } U(\mathfrak{g}), \qquad i = \pi \circ (T^1\mathfrak{g} \to T(\mathfrak{g}))$$

A priori, i is not injective! We will see that this is the case, but it is a subtle result.

REMARK 5.3. Since $J \subset \bigoplus_{i>0} T^i L$, π induces an injection $T^0 L = \mathbb{k} \hookrightarrow U(\mathfrak{g})$, so $U(\mathfrak{g})$ contains at least the scalars.

The term *universal* comes from next proposition.

Proposition 5.2 – universal property

Let A be a unital associative algebra equipped with the Lie bracket: $[a, b] = ab - ba$, for $a, b \in A$. For any Lie algebra morphism $\theta\colon \mathfrak{g} \to A$, there exists a unique morphism of associative algebras $\phi\colon U(\mathfrak{g}) \to A$ such that $\phi \circ i = \theta$ and $\phi(1) = 1$, i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \theta\ } & A \\ {\scriptstyle i}\downarrow & \nearrow_{\phi} & \\ U(\mathfrak{g}) & & \end{array}$$

Moreover, $U(\mathfrak{g})$ is, up to isomorphism, the unique unital associative algebra verifying this property.

EXERCISE 5.1. Prove the proposition.

In particular, if (V, θ) is a representation of $\mathfrak{g}$, i.e., $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra morphism, then θ induces a morphism of associative algebras $\psi\colon U(\mathfrak{g}) \to \mathfrak{gl}(V)$, i.e., (V, ψ) is a $U(\mathfrak{g})$-module. So, a $\mathfrak{g}$-representation is the same as a $U(\mathfrak{g})$-module.

We denote by $x_1 \dots x_k$ the image in $U(\mathfrak{g})$ of a homogeneous element $x_1 \otimes \cdots \otimes x_k$ of $T(\mathfrak{g})$ by the canonical surjection $T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$.

5.3. Poincaré–Birkhoff–Witt Theorem

We define a filtration on $T(\mathfrak{g})$ by setting $T_m(\mathfrak{g}) = \bigoplus_{i=0}^m T^i(\mathfrak{g})$. We then set

$$U_m(\mathfrak{g}) = \pi(T_m(\mathfrak{g})),$$

$U_{-1}(\mathfrak{g}) = \{0\}$, where $\pi \colon T(\mathfrak{g}) \to U(\mathfrak{g})$ is the canonical projection. We clearly have $U_m(\mathfrak{g}).U_p(\mathfrak{g}) \subset U_{m+p}(\mathfrak{g})$ and $(U_m)_{m \in N}$ is an increasing filtration, which makes $U(\mathfrak{g})$ a filtered algebra.

Let $\mathrm{gr}^m U(\mathfrak{g}) = U_m(\mathfrak{g})/U_{m-1}(\mathfrak{g})$, and set

$$\mathrm{gr}\, U(\mathfrak{g}) = \bigoplus_{i=0}^{\infty} \mathrm{gr}^m U(\mathfrak{g}).$$

The multiplication in $U(\mathfrak{g})$ induces a bilinear map $\mathrm{gr}^m U(\mathfrak{g}) \times \mathrm{gr}^p U(\mathfrak{g}) \to \mathrm{gr}^{m+p} U(\mathfrak{g})$. The latter extends into a bilinear map $\mathrm{gr}\, U(\mathfrak{g}) \times \mathrm{gr}\, U(\mathfrak{g}) \to \mathrm{gr}\, U(\mathfrak{g})$ which gives to $\mathrm{gr}\, U(\mathfrak{g})$ a structure of a graded unital associative algebra.

Consider the composition

$$\phi_m \colon T^m(\mathfrak{g}) \xrightarrow{\pi} U_m(\mathfrak{g}) \longrightarrow \mathrm{gr}^m U(\mathfrak{g}).$$

It is surjective because $\pi(T_m(\mathfrak{g}) \setminus T_{m-1}(\mathfrak{g})) = U_m(\mathfrak{g}) \setminus U_{m-1}(\mathfrak{g})$. Consequently, ϕ_m induces a surjective linear map,

$$\phi \colon T(\mathfrak{g}) \to \mathrm{gr}\, U(\mathfrak{g}),$$

which sends 1 to 1.

EXERCISE 5.2. Recall that I is the kernel of the canonical surjection $T(\mathfrak{g}) \twoheadrightarrow S(\mathfrak{g})$. Show that ϕ is an algebra morphism and that $\phi(I) = \{0\}$.

By Exercise 5.2, the map ϕ factorizes into a surjective algebra morphism

$$\omega \colon S(\mathfrak{g}) \longrightarrow\!\!\!\!\rightarrow \mathrm{gr}\, U(\mathfrak{g}).$$

Theorem 5.3 – Poincaré–Birkhoff–Witt (PBW)

The morphism $\omega \colon S(\mathfrak{g}) \to \mathrm{gr}\, U(\mathfrak{g})$ is an isomorphism of graded algebras. In particular, $\mathrm{gr}\, U(\mathfrak{g})$ is a commutative algebra.

Corollary 5.4

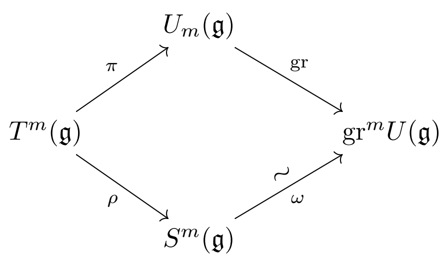
Let W be a subspace of $T^m(\mathfrak{g})$. Suppose that the canonical map $\rho \colon T^m(\mathfrak{g}) \to S^m(\mathfrak{g})$ sends W isomorphically over $S^m(\mathfrak{g})$, i.e.,

$$W \cong \rho(W) = S^m(\mathfrak{g}).$$

Then $\pi(W)$ is a complement to $U_{m-1}(\mathfrak{g})$ in $U_m(\mathfrak{g})$, i.e.,

$$U_m(\mathfrak{g}) = \pi(W) \oplus U_{m-1}(\mathfrak{g}).$$

PROOF. Consider the diagram



By Exercise 5.2, this is a commutative diagram. Since ω is an isomorphism by the PBW Theorem 5.3, the bottom map sends $W \subset T^m(\mathfrak{g})$ isomorphically onto $\mathrm{gr}^m U(\mathfrak{g})$. Reverting to the top map, we get the statement. □

Applying Corollary 5.4 to $W = T^1(\mathfrak{g}) = \mathfrak{g}$ we get the next fundamental corollary of PBW Theorem 5.3.

Corollary 5.5

The canonical map $i\colon \mathfrak{g} \to U(\mathfrak{g})$ is injective. Therefore, we can identify $\mathfrak{g}$ with a subalgebra of $U(\mathfrak{g})$.

Next corollary is as much fundamental (it is sometimes also referred to as PBW Theorem).

Corollary 5.6

Let $(x_i\colon i \in I)$ be an ordered basis of $\mathfrak{g}$, then the unit 1 together with the monomials

$$x_{i_1}^{r_1} \dots x_{i_n}^{r_n},$$

for $n > 0$, $r_i > 0$ and $i_1, \dots, i_n \in I$ such that $i_1 < i_2 < \cdots < i_n$, form a basis of $U(\mathfrak{g})$.

Such a basis of $U(\mathfrak{g})$ is called a ***Poincaré–Birkhoff–Witt basis*** or ***PBW basis***.

PROOF. Let W be the subspace of $T^m(\mathfrak{g})$ spanned by all $x_{i_1} \otimes \ldots \otimes x_{i_m}$ for $i_1, \dots, i_m \in I$ such that $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_m$. Obviously, W maps isomorphically onto $S^m(\mathfrak{g})$, so Corollary 5.4 shows that $\pi(W)$ is a complement to $U_{m-1}(\mathfrak{g})$ in $U_m(\mathfrak{g})$, and we construct the desired basis by induction. □

Corollary 5.7

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, and let us complete a basis $\{h_i\colon i \in J\}$ of $\mathfrak{h}$ into an ordered basis

$$(h_i\colon i \in J) \cup (x_i\colon i \in I)$$

of $\mathfrak{g}$. Then the algebra morphism $U(\mathfrak{h}) \to U(\mathfrak{g})$ induced by the injection $\mathfrak{h} \hookrightarrow \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ is injective, and $U(\mathfrak{g})$ is a free module over $U(\mathfrak{h})$, with basis consisted of the unit 1 and the sets

$$x_{i_1}^{r_1} \dots x_{i_n}^{r_n},$$

$n > 0$, $r_i > 0$, and $i_1, \dots, i_n \in I$ such that $i_1 < \cdots < i_n$.

EXERCISE 5.3. Prove the above corollary using Corollary 5.6.

PROOF OF THEOREM 5.3. Set, to simplify,

$$T = U(\mathfrak{g}), \quad S = S(\mathfrak{g}), \quad U = U(\mathfrak{g}) \quad T^m = T^m(\mathfrak{g}), \quad T_m(\mathfrak{g}) = T_m, \quad \text{etc.}$$

Let us fix an ordered basis $(x_i\colon i \in I)$ of $\mathfrak{g}$ so that S identifies with the algebra of polynomials into the variables z_i, $i \in I$.

We set

$$x_{\underline{i}} = x_{i_1} \otimes \cdots \otimes x_{i_m} \in T^m,$$

and

$$z_{\underline{i}} = z_{i_1} \dots z_{i_m} \in S^m$$

the image of $x_{\underline{i}}$ in S by the canonical projection $T^m \to S^m$, if $\underline{i} = (i_1, \dots, i_m)$ is a sequence of length m. We will say that a sequence $\underline{i}$ is *increasing* if $i_1 \leqslant \cdots \leqslant i_m$. By convention, $z_\varnothing = 1$ and $\varnothing$ is increasing. The set $\{z_{\underline{i}}\colon \underline{i} \text{ increasing}\}$ forms a basis of S. We will write

$$j \leqslant \underline{i} \quad \text{if} \quad j \leqslant i_k \quad \text{for any} \quad i_k \in \underline{i}.$$

The graduation $S = \bigoplus_{i=0}^{\infty} S^i$ induces a filtration $S_m = \bigoplus_{i=0}^{m} S^i$ of S.

The idea of the proof is to show that there exists a representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(S)$ such as

(a) $\rho(x_j) z_{\underline{i}} = z_j z_{\underline{i}}$ for $j \leqslant \underline{i}$.

(b) $\rho(x_j)z_{\underline{i}} \equiv z_j z_{\underline{i}} \bmod S_m$ if $\underline{i}$ is a sequence of length m.

EXAMPLE 5.2. Choose for the ordered basis of $\mathfrak{sl}_2(\mathbb{k})$ the basis (e, h, f). We set

$$\rho(e).1 = e, \quad \rho(h).1 = h, \quad \rho(f).1 = f.$$

In order to get the relation

$$(\rho(h)\rho(e) - \rho(e)\rho(h)).1 = \rho([h, e]).1 = 2e,$$

we must set

$$\rho(h).e = he + 2e.$$

More generally, the existence of such a representation results from the following assertion:

ASSERTION 5.1. For any $m \in \mathbb{N}$, there exists a unique linear map

$$f_m : \mathfrak{g} \otimes S_m \to S$$

verifying the following properties:

(A_m) $f_m(x_j \otimes z_{\underline{i}}) = z_j z_{\underline{i}}$ for $j \leqslant \underline{i}$, $z_{\underline{i}} \in S_m$,

(B_m) $f_m(x_j \otimes z_{\underline{i}}) - z_j z_{\underline{i}} \in S_k$ for $k \leqslant m$ and $z_{\underline{i}} \in S_k$,

(C_m) $f_m(x_j \otimes f_m(x_k \otimes z_{\underline{l}})) = f_m(x_k \otimes f_m(x_j \otimes z_{\underline{l}})) + f_m([x_j, x_k] \otimes z_{\underline{l}})$ for any $z_{\underline{l}} \in S_{m-1}$.

Furthermore, the restriction of f_m to $\mathfrak{g} \otimes S_{m-1}$ coincides with f_{m-1}.

EXERCISE 5.4. Prove the Assertion 5.1 by induction on m.

> This is a little technical!

Assertion 5.1 proves that there exists a representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(S)$ satisfying (a) and (b).

ASSERTION 5.2. If $s \in T_m \cap J$, then the component s_m of degree m of s belongs to I.

PROOF. Write s_m as a linear combination of elements $x_{\underline{i}^{(k)}}$, $1 \leqslant k \leqslant r$, where the $\underline{i}^{(k)}$ are sequences of length m. The representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(S)$ extends by the universal property of U to an algebra morphism, again denoted $\rho\colon U \to \operatorname{End}(S)$, such that $J \subset \operatorname{Ker}\tilde{\rho}$, where $\tilde{\rho} = \rho \circ \pi$:

$$T \xrightarrow{\ \pi\ } U = T/J \xrightarrow{\ \rho\ } \operatorname{End}(S), \qquad \tilde{\rho}\colon T \to \operatorname{End}(S)$$

Since $s \in J$, we have $\tilde{\rho}(s) = 0$. But $\tilde{\rho}(s) = \tilde{\rho}(s).1$ is a polynomial whose highest degree term is a combination linear of the elements $z_{\underline{i}^{(k)}}$, $1 \leqslant k \leqslant r$. This linear combination is therefore zero in S. In other words, $s_m \in I$. □

We are now in a position to prove Theorem 5.3. We have to show that the map $T/I = S \to \operatorname{gr} U$ is injective. Recall that $\pi\colon T \to U$ is the canonical projection. We have to show that if $t \in T^m$ is such that $\pi(t) \in U_{m-1}$, then $t \in I$. But since $\pi(T_{m-1}) = U_{m-1}$, there exists $t' \in T_{m-1}$ such that

$$\pi(t) = \pi(t'),$$

whence

$$t - t' \in T_m \cap J,$$

the kernel of π. By Assertion 5.2, the degree m component of $t - t'$ belongs to I, that is, t belongs to I since $t'_m = 0$. □

EXERCISE 5.5. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and I an ideal of $\mathfrak{g}$. Show that

$$IU(\mathfrak{g}) = \{xu \colon x \in I,\ u \in U(\mathfrak{g})\}$$

is a two-sided ideal of $U(\mathfrak{g})$ and that we have a natural algebra isomorphism

$$U(\mathfrak{g})/IU(\mathfrak{g}) \cong U(\mathfrak{g}/I).$$

> Give two proofs: one using the PBW Theorem, one using the universal property of the enveloping algebra.

5.4. Differential operators on an algebraic group

First, assume that X is any smooth affine variety. Let $\mathcal{O}_X$ the structure sheaf, and $\mathcal{T}_X$ the tangent sheaf. We define the sheaf $\mathscr{D}_X$ as the sheaf of $\Bbbk$-subalgebras of $\mathrm{End}_{\Bbbk}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $\mathcal{T}_X$. Here we identify $\mathcal{O}_X$ with a subsheaf of $\mathrm{End}_{\Bbbk}(\mathcal{O}_X)$ by identifying $f \in \mathcal{O}_X$ with the element $g \mapsto fg$ of $\mathrm{End}_{\Bbbk}(\mathcal{O}_X)$. We call the sheaf $\mathscr{D}_X$ the ***sheaf of differential operators*** on X. For any point of X we can take an affine open neighbourhood U and a local coordinate system $\{x_i, \partial_i \colon 1 \le i \le n\}$. Hence we have

$$\mathscr{D}_U := \mathscr{D}_X(U) = \bigoplus_{\alpha \in \mathbb{Z}^n_{\geq 0}} \mathcal{O}_U \partial_x^\alpha, \quad \partial_x^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}.$$

We define the ***order filtration*** $F_\bullet \mathscr{D}_U$ of $\mathscr{D}_U$ by

$$F_\ell \mathscr{D}_U = \sum_{|\alpha| \leqslant l} \mathcal{O}_U \partial_x^\alpha, \qquad l \in \mathbb{Z}_{\geqslant 0}, \; |\alpha| := \sum_i \alpha_i.$$

More generally, for an arbitrary open subset V of X we define the order filtration $F_\bullet \mathscr{D}_X$ over V by

$$\begin{aligned}&(F_\ell \mathscr{D}_X)(V) \\ &\quad = \{P \in \mathscr{D}_X(V) \colon \mathrm{res}^V_U P \in (F_\ell \mathscr{D}_X)(U) \text{ for any affine open subset } U \text{ of } V\},\end{aligned}$$

where $\mathrm{res}^V_U \colon \mathscr{D}_X(V) \to \mathscr{D}_X(U)$ is the restriction map. For convenience we set $F_p \mathscr{D}_X = 0$ for $p < 0$.

Proposition 5.8

(i) $F_\bullet \mathscr{D}_X$ is an increasing filtration of $\mathscr{D}_X$ such that $\mathscr{D}_X = \bigcup_{\ell \geq 0} F_\ell \mathscr{D}_X$ and each $F_\ell \mathscr{D}_X$ is a locally free module over $\mathcal{O}_X$.

(ii) $F_0 \mathscr{D}_X := \mathcal{O}_X$ and $(F_\ell \mathscr{D}_X)(F_m \mathscr{D}_X) = F_{\ell+m} \mathscr{D}_X$.

(iii) $[F_\ell \mathscr{D}_X, F_m \mathscr{D}_X] \subset F_{\ell+m-1} \mathscr{D}_X$.

REMARK 5.4. One can alternatively define $F_\bullet \mathscr{D}_X$ by the recursive formula:

$$F_\ell \mathscr{D}_X = \{P \in \mathrm{End}_{\Bbbk}(\mathcal{O}_X) \colon [P, f] \in F_{\ell-1} \mathscr{D}_X \text{ for all } f \in \mathcal{O}_X\}, \quad \ell \in \mathbb{Z}_{\geq 0}.$$

Let us consider the sheaf of graded rings

$$\mathrm{gr}\, \mathscr{D}_X = \mathrm{gr}^F \mathscr{D}_X = \bigoplus_{\ell \geq 0} \mathrm{gr}_\ell \mathscr{D}_X,$$

where $\mathrm{gr}_\ell \mathscr{D}_X := F_\ell \mathscr{D}_X / F_{\ell-1} \mathscr{D}_X$, $F_{-1} \mathscr{D}_X = 0$. By Proposition 5.8 (iii), $\mathrm{gr}\, \mathscr{D}_X$ is a sheaf of commutative algebras finitely generated over $\mathcal{O}_X$. Take an affine chart U with a coordinate system $\{x_i, \partial_i\}$ and set

$$\xi_i := (\partial_i \bmod F_0 \mathscr{D}_U = \mathcal{O}_U) \in \mathrm{gr}_1 \mathscr{D}_U.$$

Then we have

$$\begin{aligned}\mathrm{gr}_\ell \mathscr{D}_U &= \bigoplus_{|\alpha| = \ell} \mathcal{O}_U \xi^\alpha, \\ \mathrm{gr}\, \mathscr{D}_U &= \mathcal{O}_U[\xi_1, \ldots, \xi_n].\end{aligned}$$

We can globalize this notion as follows. Let T^*X be the cotangent bundle of X and let $\pi \colon T^*X \to X$ be the projection. We may regard $\xi_1, \ldots, \xi_n$ as the coordinate system of the cotangent space $\bigoplus_{i=1}^n \Bbbk dx_i$ and hence $\mathcal{O}_U[\xi_1, \ldots, \xi_n]$ is canonically identified with the sheaf $\pi_* \mathcal{O}_{T^*X}$ of algebras. Thus we obtain a canonical identification

$$\mathrm{gr}\, \mathscr{D}_X \cong \pi_* \mathcal{O}_{T^*X}. \tag{10}$$

The algebra $\mathscr{D}(X) := \mathscr{D}_X(X)$ is called the ***algebra of differential operators*** on X. We refer to [6, Section 1.1] for more details about this topic.

Let now G be a linear algebraic group, and $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. Recall that, by definition,

$$\mathrm{Lie}(G) = \{\delta \in \mathrm{Der}_{\Bbbk}(\Bbbk[G]) \colon \Delta \circ \delta = (1 \otimes \delta) \circ \Delta\},$$

where $\Delta \colon \Bbbk[G] \to \Bbbk[G] \otimes \Bbbk[G]$ is the coproduct induced by the multiplication in G. Thus, $\delta \in \mathrm{Der}_{\mathbb{C}}(\Bbbk[G])$ is in $\mathrm{Lie}(G)$ if and only if for all $g \in G$, $\lambda_g \delta = \delta \lambda_g$, where $(\lambda_g f)(y) = f(g^{-1}y)$ for $f \in \Bbbk[G]$ and $y \in G$.

The Lie algebra of G is canonically isomorphic, as a Lie algebra, to the Lie algebra $\mathrm{Lie}_r(G)$ of right invariant vector fields δ, that is, the Lie algebra consisted of $\delta \in \mathrm{Der}_{\Bbbk}(\Bbbk[G])$ such that $\rho_g\delta = \delta\rho_g$, where $(\rho_g f)(y) = f(yg)$ for $f \in \Bbbk[G]$ and $y \in G$ (see Remark 2.2).

We know that it is also canonically isomorphic to $T_e(G)$, the tangent space at the identity to G, via the isomorphism (Theorem 2.6),

$$\mathrm{Lie}(G) \longrightarrow T_e(G),$$

sending $\delta \in \mathrm{Lie}(G)$ to $\mathrm{ev}_e \circ \delta$, where ev_e is the evaluation map at the neutral element e, in which we have identified $\mathrm{Der}_{\Bbbk}(\Bbbk[G])$ with $TG = \mathcal{T}_G(G)$.

Thus, we have

$$TG \cong G \times \mathfrak{g} \quad \text{and} \quad T^*G \cong G \times \mathfrak{g}^*.$$

For $x \in \mathfrak{g}$, we write x_L (resp., x_R) the corresponding left (resp., right) invariant vector field on G (see Remark 2.3). Recall that

$$(x_L f)(a) = x(\lambda_{a^{-1}} f)$$

for $f \in \Bbbk[G]$ and $a \in G$.

The embedding $\mathfrak{g} \hookrightarrow \mathrm{Der}_{\Bbbk}(\Bbbk[G])$, $x \mapsto x_L$, induces an isomorphism of left $\Bbbk[G]$-modules

$$\Bbbk[G] \otimes_{\Bbbk} \mathfrak{g} \xrightarrow{\sim} \mathrm{Der}_{\Bbbk}(\Bbbk[G]). \tag{11}$$

Indeed, both sides are free $\Bbbk[G]$-modules of rank the dimension of $\mathfrak{g}$ since G is smooth.

Let now $\mathscr{D}(G)$ be the algebra of differential operators on G. We have a natural embedding

$$\Bbbk[G] \longhookrightarrow \mathscr{D}(G).$$

Moreover, from the embedding $\mathfrak{g} \hookrightarrow TG$, $x \mapsto x_L$, given by the left invariant vector fields, we get an embedding

$$U(\mathfrak{g}) \longhookrightarrow \mathscr{D}(G).$$

This induces a G-equivariant map

$$\iota\colon U(\mathfrak{g}) \otimes \Bbbk[G] \longhookrightarrow \mathscr{D}(G) \tag{12}$$

of $\Bbbk[G]$-modules, where the G-action on the left-hand-side is the G-action on $\Bbbk[G]$ induced by the left translation action of G on itself, that is, the G-action on $U(\mathfrak{g})$ is trivial. On the right-hand side, the G-action is given by: $g.\alpha = \lambda_g \circ \alpha$.

Let $\mathscr{D}_l(G)$ be the algebra of left-invariant differential operators on G, that is, the algebra of elements $\alpha \in \mathscr{D}(G)$ such that for all $g \in G$ and all $f \in \Bbbk[G]$,

$$\lambda_g \circ \alpha = \alpha \circ \lambda_g.$$

Proposition 5.9

The algebra $\mathscr{D}(G)$ is characterized as the algebra such that $\mathscr{D}(G) \cong \Bbbk[G] \otimes U(\mathfrak{g})$ as vector spaces, the natural embeddins $\Bbbk[G] \hookrightarrow \mathscr{D}(G)$ and $U(\mathfrak{g}) \hookrightarrow \mathscr{D}(G)$ are algebra homomorphisms, and $[x, f] = x_L f$ for $x \in \mathfrak{g} \subset U(\mathfrak{g})$, $f \in \Bbbk[G]$. Moreover,

$$U(\mathfrak{g}) \cong \mathscr{D}_l(G) \cong \mathscr{D}(G)^G.$$

where $\mathscr{D}(G)^G$ stands for the algebra of left invariant differential operators on G.

PROOF. For the first statement it is enough to show that ι is an isomorphism. The algebra $\mathscr{D}(G)$ is filtered by the order filtration $F_\bullet\mathscr{D}(G)$. On the other hand, the PBW filtration $U_\bullet(\mathfrak{g})$ on $U(\mathfrak{g})$ induces a filtration $F_\bullet(\Bbbk[G] \otimes U(\mathfrak{g}))$ on $\Bbbk[G] \otimes U(\mathfrak{g})$ by setting

$$F_\ell(\Bbbk[G] \otimes U(\mathfrak{g})) := \Bbbk[G] \otimes U_\ell(\mathfrak{g}), \quad \ell \in \mathbb{N}.$$

The map ι sends $F_l(\Bbbk[G] \otimes U(\mathfrak{g}))$ to $F_\ell\mathscr{D}(G)$, and both filtrations are exhaustive. So it suffices to check that the map on associated graded space is an isomorphism. The associated graded of the right-hand-side is

$$\Bbbk[T^*G] \cong \Bbbk[G \times \mathfrak{g}^*] \cong \Bbbk[G] \otimes \Bbbk[\mathfrak{g}^*],$$

by (10), while by the PBW theorem the associated graded of the left-hand-side is

$$\Bbbk[G] \otimes S(\mathfrak{g}),$$

whence the statement since $\Bbbk[\mathfrak{g}^*] \cong S(\mathfrak{g})$.

Next, since the map ι is G-equivariant,

$$\mathscr{D}(G)^G \cong (\Bbbk[G] \otimes U(\mathfrak{g}))^G \cong \Bbbk[G]^G \otimes U(\mathfrak{g}) \cong U(\mathfrak{g}) \hookrightarrow \mathscr{D}_l(G).$$

To show the other inclusion, observe that $\operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g}) \cong \Bbbk[\mathfrak{g}^*]$ while

$$\operatorname{gr} \mathscr{D}_l(G) \cong (\Bbbk[G] \otimes \Bbbk[\mathfrak{g}^*])^G \cong \Bbbk[G]^G \otimes \Bbbk[\mathfrak{g}^*] \cong \Bbbk[\mathfrak{g}^*],$$

where G acts on on $\Bbbk[G]$ by λ_g, $g \in G$, and trivially on $\Bbbk[\mathfrak{g}^*]$. Hence, $\mathscr{D}_l(G) \cong \mathscr{D}(G)^G \cong U(\mathfrak{g})$ as desired. □

Proposition 5.9 give a more geometrical interpretation of the enveloping algebra of an algebraic Lie algebra $\mathfrak{g} = \operatorname{Lie}(G)$: it can be defined as the algebra of left-invariant differential operators.

Part 2

Representations of semisimple Lie algebras

This part is the heart of the course. The aim is to study the representation theory of semisimple Lie algebras. Some general facts on semisimple Lie algebras are stated in Chapter 6 while Weyl's theorem on the complete reducibility of finite-dimensional representations is proved in Chapter 7. Then we focus on the structure of semisimple Lie algebras (cf. Chapter 8). It turns out that it is governed by combinatorial objects, the abstract root systems (cf. Chapter 9). Using these tools, we initiate the study of highest weight representations in Chapter 10, an important class of representations. Chapter 11 explores more advanced properties of these representations.

6 Semisimple Lie algebras

In this part, the field $\mathbb{k}$ est algebraically closed and of characteristic zero. All Lie algebras are supposed to be finite-dimensional.

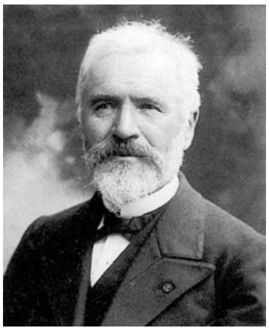
6.1. Cartan's criterion

Let us start with a preliminary exercise.

EXERCISE 6.1. Let V be a $\mathbb{k}$-vector space of finite dimension. Consider the Lie algebra $\mathfrak{gl}(V) = \mathrm{End}(V)$. For $x \in \mathfrak{gl}(V)$, write

$$x = x_s + x_n$$

its ***Jordan decomposition***, that is x_s is semisimple, x_n is nilpotent and $[x_s, x_n] = 0$. Show $\mathrm{ad}\, x = \mathrm{ad}\, x_s + \mathrm{ad}\, x_n$ is the Jordan decomposition of $\mathrm{ad}\, x$ in $\mathrm{End}(\mathrm{End}(V))$.



Marie Ennemond Camille Jordan, *1838 – 1922, was a French mathematician, known both for his foundational work in group theory and for his influential Cours d'analyse.*

Theorem 6.1 – Cartan's criterion

Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}(V)$ where V is a vector space of finite dimension. Suppose that $\mathrm{Tr}(xy) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$. Then $\mathfrak{g}$ is solvable.

Élie Joseph Cartan *1869 – 1951, was an influential French mathematician who did fundamental work in the theory of Lie groups, differential systems (coordinate-free geometric formulation of PDEs), and differential geometry. He also made significant contributions to general relativity and indirectly to quantum mechanics. He is widely regarded as one of the greatest mathematicians of the twentieth century. His son Henri Cartan was an influential mathematician working in algebraic topology.*

EXERCISE 6.2. Let A, B be two subspaces of $\mathfrak{gl}(V)$ such that $A \subset B$. Set

$$M = \{y \in \mathfrak{gl}(V) \colon [y, B] \subset A\}.$$

Let $x \in M$ such that

$$\operatorname{Tr}(xy) = 0 \quad \text{for any} \quad y \in M.$$

Show that x is nilpotent, that is, $x_s = 0$.

Hint: fix a basis of eigenvectors $\{v_1, \dots, v_m\}$ for x_s relative to eigenvalues $\lambda_1, \dots, \lambda_m$, consider the $\mathbb{Q}$-vector space

$$E = \sum_{i=1}^{m} \mathbb{Q}\lambda_i,$$

and show that $E = \{0\}$, or equivalently, that any $\mathbb{Q}$-linear form on E is zero.

PROOF OF THEOREM 6.1. First observe that $\mathfrak{g}$ is solvable if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Furthermore, by Engel's Theorem 4.4, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent if x is a nilpotent endomorphism for any $x \in [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{gl}(V)$.

Apply Exercise 6.2 to $A = [\mathfrak{g}, \mathfrak{g}]$ and $B = \mathfrak{g}$ so that

$$M = \{y \in \mathfrak{gl}(V) \colon [y, \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}]\}.$$

We clearly have: $\mathfrak{g} \subset M$. By Exercise 6.2, it suffices to show that

$$\operatorname{Tr}(xy) = 0 \quad \text{for any} \quad x \in [\mathfrak{g}, \mathfrak{g}] \quad \text{and} \quad y \in M.$$

Since all element $x \in [\mathfrak{g}, \mathfrak{g}]$ is a finite sum of elements of the form $[u, v]$, with $u, v \in \mathfrak{g}$, it suffices to show that for all $u, v \in \mathfrak{g}$ and $y \in M$,

$$\operatorname{Tr}([u, v]y) = 0.$$

But

$$\operatorname{Tr}([u, v]y) = \operatorname{Tr}(u[v, y]) = \operatorname{Tr}([v, y]u)$$

and, by definition of M, we have

$$[v, y] \in [\mathfrak{g}, \mathfrak{g}]$$

and so $\operatorname{Tr}([v, y]x) = 0$ by our hypothesis. Therefore we have proved that all $x \in [\mathfrak{g}, \mathfrak{g}]$ are nilpotent. □

As a consequence of Cartan's Theorem 6.1, we obtain the following criterion which can be seen as an analogue of Corollary 4.5 for solvable Lie algebras.

Corollary 6.2

Let $\mathfrak{g}$ be a Lie algebra such that $\operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$. Then $\mathfrak{g}$ is solvable.

EXERCISE 6.3. Prove the corollary using Theorem 6.1 and Exercise 4.11.

6.2. Killing form

Let $\mathfrak{g}$ be a Lie algebra. For $x, y \in \mathfrak{g}$, we set

$$\kappa_{\mathfrak{g}}(x,y) = \mathrm{Tr}(\mathrm{ad}\, x \,\mathrm{ad}\, y).$$

Then $\kappa_{\mathfrak{g}}$ is a symmetric bilinear form, called the ***Killing form*** of $\mathfrak{g}$. This form is ***invariant***, that is:

(13) $$\forall\, x, y, z \in \mathfrak{g}, \qquad \kappa_{\mathfrak{g}}([x,y],z) = \kappa_{\mathfrak{g}}(x,[y,z]).$$

__Wilhelm Karl Joseph Killing__ 1847 – 1923, is a German mathematician known for his numerous contributions to the theories of algebra de Lie and Lie groups and non-Euclidean geometry.

EXERCISE 6.4 (properties of the Killing form).

(1) Check the relation (13), and show that the kernel of $\kappa_{\mathfrak{g}}$, is an ideal of $\mathfrak{g}$.

(2) Let I be an ideal of $\mathfrak{g}$. Show that the Killing form of I coincides with the restriction of $\kappa_{\mathfrak{g}}$ to I.

(3) Determine the matrix of the Killing form of $\mathfrak{sl}_2(\mathbb{k})$ in the base (e, h, f). Deduce that the Killing form of $\mathfrak{sl}_2(\mathbb{k})$ is nondegenerate.

REMARK 6.1. Using the question 3 of Exercise 6.4 we easily show that

$$\kappa_{\mathfrak{sl}_2(\mathbb{k})}(x,y) = 4\,\mathrm{Tr}(xy) \quad \text{for all} \quad x, y \in \mathfrak{sl}_2(\mathbb{k}).$$

Recall that a nonzero Lie algebra is called ***semisimple*** if $\mathrm{rad}(\mathfrak{g}) = \{0\}$ (see Definition 4.8). This is equivalent to that $\mathfrak{g}$ does not have any nonzero abelian ideal. Indeed, such an ideal is necessarily contained in $\mathrm{rad}(\mathfrak{g})$ and, conversely, the radical of $\mathfrak{g}$, if it is nonzero, contains a nonzero abelian ideal; the last term of its derived series (cf. Exercise 4.9).

In particular, if $\mathfrak{g}$ is semisimple, then its ***center*** $\mathfrak{z}(\mathfrak{g})$ is zero since it is an abelian ideal, where

$$\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \colon [x,y] \text{ for any } y \in \mathfrak{g}\}$$

So the adjoint map is injective.

Definition 6.3 – reductive Lie algebra

A Lie algebra $\mathfrak{g}$ is called ***reductive*** if $\mathrm{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$.

For example, abelian and semisimple Lie algebras are reductive. The Lie algebra $\mathfrak{gl}_n(\mathbb{k})$ is reductive but not semisimple. What is its center?

Theorem 6.4

Let $\mathfrak{g}$ be a nonzero Lie algebra. Then, $\mathfrak{g}$ is semisimple if and only if its Killing form $\kappa_{\mathfrak{g}}$ is nondegenerate.

PROOF. Suppose that $\mathfrak{g}$ is semisimple, i.e., $\mathrm{rad}(\mathfrak{g}) = \{0\}$. Let K be the kernel of $\kappa_{\mathfrak{g}}$. By definition, $\mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y) = 0$ for all $x \in K$ and $y \in \mathfrak{g}$ (in particular for any $y \in [K, K]$). According to Cartan's criterion (or rather Corollary 6.2) we deduce that K is solvable. Since K is an ideal of $\mathfrak{g}$ (cf. Exercise 6.4), we have

$$K \subset \mathrm{rad}(\mathfrak{g}) = \{0\}.$$

Conversely, suppose $K = \{0\}$ and show that $\mathfrak{g}$ does not contain any nonzero abelian ideal. It is enough to show that every abelian ideal is contained in K. Let I be an abelian ideal of $\mathfrak{g}$. Let $x \in I$ and $y \in \mathfrak{g}$. Then the image of $(\mathrm{ad}\, x \circ \mathrm{ad}\, y)^2$ is contained in $[I, [\mathfrak{g}, [I, \mathfrak{g}]]] \subset [I, I] = \{0\}$ since I is abelian. This shows that $\mathrm{ad}\, x\, \mathrm{ad}\, y$ is a nilpotent endomorphism nilpotent. Therefore:

$$0 = \mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y) = \kappa_{\mathfrak{g}}(x, y)$$

for all $x \in I$ and $y \in \mathfrak{g}$, whence $I \subset K = \{0\}$, as desired. □

REMARK 6.2. The second part of the proof remains valid for $\mathrm{char}(\Bbbk) > 0$. On the other hand, the proof shows that the inclusion

$$\mathrm{Ker}\, \kappa_{\mathfrak{g}} \subset \mathrm{rad}(\mathfrak{g})$$

always holds.

Proposition 6.5 – all derivations of a semisimple Lie algebra are inner

If $\mathfrak{g}$ is semisimple, then $\mathrm{ad}\, \mathfrak{g} = \mathrm{Der}\, \mathfrak{g}$.

PROOF. As $\mathfrak{g}$ is semisimple, the adjoint map is injective. In particular, the Killing form of $M = \mathrm{ad}\, \mathfrak{g}$ is nondegenerate. Let $D = \mathrm{Der}\, \mathfrak{g}$. We have $[D, M] \subset M$ since for all $\delta \in D$ and $x \in \mathfrak{g}$,

$$[\delta, \mathrm{ad}\, x] = \mathrm{ad}(\delta x). \tag{14}$$

In other words, M is an ideal of D and it results from Exercise 6.4 that κ_M is the restriction to M of κ_D. In particular, if $I = M^{\perp}$ is the orthogonal of M in D relative to κ_D, then $I \cap M = \{0\}$ since κ_M is nondegenerate.

As I and M are ideals of D, we deduce that $[I, M] = \{0\}$ and therefore that if $\delta \in I$, then $\mathrm{ad}(\delta x) = 0$ for any $x \in \mathfrak{g}$ by (14), whence $\delta x = 0$ for any $x \in \mathfrak{g}$ since ad is injective, i.e., $\delta = 0$. In conclusion, $I = \{0\}$ and $\mathrm{Der}\, L = \mathrm{ad}\, L$. □

EXERCISE 6.5. Let $\mathfrak{g}$ be a Lie algebra with basis $\{x, y, z\}$ and Lie bracket given by:

$$[x, y] = z, \quad [x, z] = [y, z] = 0.$$

Show that

$$\mathfrak{g} \cong \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \Bbbk \right\},$$

that $\dim \mathrm{Der}\, \mathfrak{g} = 6$ and that $\mathrm{Der}\, \mathfrak{g} / \mathrm{ad}\, \mathfrak{g} \cong \mathfrak{gl}_2(\Bbbk)$.

Proposition 6.6

Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there exist simple ideals $I_1, \ldots, I_t$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = I_1 \oplus \cdots \oplus I_t.$$

Moreover, any simple ideal of $\mathfrak{g}$ coincides with some of the I_j.

Since the I_j's are ideals of $\mathfrak{g}$, we have $[I_j, I_k] \subset I_j \cap I_k = \{0\}$, so the above decomposition is adapted to the Lie bracket, in the sense that it can be described from the Lie bracket on each I_j.

EXERCISE 6.6. The aim of the exercice is to show the proposition.

(1) Observe that if I is an ideal of $\mathfrak{g}$, then its orthogonal $I^{\perp}$ with respect to $\kappa_{\mathfrak{g}}$ is also an ideal (this was in fact already used in the proof of Proposition 6.5), and use Cartan's criterion (Corollary 6.2) to show that $I \cap I^{\perp} = \{0\}$.

(2) Show that if I_1 is a nonzero minimal ideal of $\mathfrak{g}$ then I_1 is simple.

(3) Arguing by induction, prove the first part of the proposition.

(4) Prove the ideals I_j's thus constructed are unique up to permutations, which proves the second part.

We deduce from Proposition 6.6 that for semisimple $\mathfrak{g}$:

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$$

since $[I_j, I_j] = I_j$ for each j. Furthermore, if I is an ideal of $\mathfrak{g}$, then I is a sum of I_k's: in fact, we can assume, possibly permuting the indices, that $I \cap I_j \neq 0$ for $i = 1, \ldots, s$. By the simplicity of I_j, we have $I \cap I_j = I_j$. In particular, I is semisimple, too.

REMARK 6.3. Let $\sigma \colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra $\mathfrak{g}$. Then $\sigma(\mathfrak{g})$ est contenue dans $\mathfrak{sl}(V)$. In particular, $\mathfrak{g}$ trivially acts on any $\mathfrak{g}$-module de dimension one.

7

Complete reducibility of finite-dimensional representations

7.1. Casimir element of a representation

We assume in this section that $\mathfrak{g}$ is a semisimple Lie algebra.

Hendrik Brugt Gerhard Casimir *1909– 2000, was a Dutch physicist who made significant contributions to the field of quantum mechanics and quantum electrodynamics. He is best known for his work on the Casimir effect, which describes the attractive force between two uncharged plates in a vacuum due to quantum fluctuations of the electromagnetic field.*

Let $\sigma\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful representation of $\mathfrak{g}$ of finite dimension. We define a symmetric bilinear form by setting for all $x, y \in \mathfrak{g}$:

$$\beta_\sigma(x, y) = \operatorname{Tr}(\sigma(x) \circ \sigma(y)).$$

The form is invariant in the sense of (13) with β_σ instead of $\kappa_{\mathfrak{g}}$. In particular, its kernel is an ideal of $\mathfrak{g}$. Additionally, β_σ is nondegenerate. Indeed, according to Cartan's criterion (Theorem 6.1), the Lie algebra $\sigma(\operatorname{Ker}\beta_\sigma) \cong \operatorname{Ker}\beta_\sigma$ is solvable and therefore $\operatorname{Ker}\beta_\sigma = \{0\}$.

Let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ be dual basis of $\mathfrak{g}$ with respect to β_σ, that is, $i, j \in \{1, \dots, n\}$, $\beta_\sigma(x_i, y_j) = \delta_{i,j}$. Set

$$c_\sigma = \sum_{i=1}^{n} \sigma(x_i) \circ \sigma(y_i) \in \operatorname{End}(V).$$

EXERCISE 7.1.

(1) Let $z \in \mathfrak{g}$. Write for $i \in \{1, \dots, n\}$,

$$[z, x_i] = \sum_{j=1}^{n} a_{i,j} x_j \quad \text{and} \quad [z, y_i] = \sum_{j=1}^{n} b_{i,j} y_j.$$

Using the invariance of β_σ, show that for any $k \in \{1, \dots, n\}$,

$$a_{i,k} = -b_{k,i}.$$

(2) Show that c_σ is an endomorphism of V which commutes with any endomorphism of $\sigma(\mathfrak{g})$, that is,

$$[c_\sigma, \sigma(z)] = 0 \quad \text{for all} \quad z \in \mathfrak{g}.$$

Hint: use the fact that $\operatorname{ad} x$ is a derivation in $\operatorname{End} V$, i.e., $[x, yz] = [x, y]z + y[x, z]$ and the previous question.

(3) What is the trace of c_σ?

(4) We assume in this question that (V, σ) is irreducible. Show using Schur's Lemma 3.2 that c_σ is a scalar, equal to $\dim \mathfrak{g} / \dim V$. Deduce that c_σ is independent of the basis $(x_1, \ldots, x_n)$.

EXAMPLE 7.1. Consider the natural representation $\sigma\colon \mathfrak{sl}_2(\Bbbk) \to \mathfrak{gl}(\Bbbk^2) \cong \mathscr{M}_2(\Bbbk)$. Then

$$\beta_\sigma(x, y) = \mathrm{Tr}(xy) \quad \text{for all} \quad x, y \in \mathfrak{sl}_2(\Bbbk),$$

and the dual basis of (e, h, f) with respect to β_σ is $(f, h/2, e)$. So

$$c_\sigma = ef + \frac{h^2}{2} + fe = \frac{3}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\dim \mathfrak{sl}_2(\Bbbk)}{\dim \Bbbk^2} I_2.$$

The element c_σ is called the ***Casimir element*** of σ. It plays a crucial role in the representation theory of of semisimple Lie algebras.

EXERCISE 7.2. Let β and γ two non-degenerate invariant bilinear form. Show that β and γ are proportional.

Hint: observe that β induces an isomorphism from $\mathfrak{g}$ to $\mathfrak{g}^*$, that γ induces an isomorphism from $\mathfrak{g}^*$ to $\mathfrak{g}$, and that the composition map yields an isomorphism of $\mathfrak{g}$-modules; then use Schur's Lemma 3.2.

Using Exercise 7.2 and Theorem 6.4, we can prove that the classical Lie algebras $\mathfrak{sl}_n$, $n \geqslant 2$, $\mathfrak{sp}_{2n}$, $n \geqslant 2$, and $\mathfrak{so}_n$, $n \geqslant 5$, are semisimple because the bilinear form $(x, y) \mapsto \mathrm{Tr}(xy)$ is invariant and non-degenerate.

7.2. Weyl's theorem and applications

The aim of the section is to prove the following theorem:

Theorem 7.1 – Weyl

Let $\sigma\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a nonzero finite-dimensional representation of a (finite-dimensional) semisimple Lie algebra $\mathfrak{g}$. Then V is completely reducible.

Hermann Weyl*, 1885 – 1955, was a German mathematician, theoretical physicist, logician and philosopher. His research has had major significance for theoretical physics as well as purely mathematical disciplines such as number theory. He was one of the most influential mathematicians of the twentieth century, and an important member of the Institute for Advanced Study during its early years. Weyl contributed to an exceptionally wide range of fields, including works on space, time, matter, philosophy, logic, symmetry and the history of mathematics. He was one of the first to conceive of combining general relativity with the laws of electromagnetism.*

EXERCISE 7.3 (converse of Weyl's Theorem). Show that if any nonzero finite-dimensional representation of a Lie algebra $\mathfrak{g}$ is completely reducible, then $\mathfrak{g}$ est semisimple.

Hint: use the adjoint representation.

Lemma 7.2 – Whitehead

Let $\mathfrak{g}$ be a semisimple Lie algebra, (V, σ) a finite-dimensional representation of $\mathfrak{g}$ and $f\colon \mathfrak{g} \to V$ a linear map. Suppose that for all $x, y \in \mathfrak{g}$,

$$f([x,y]) = \sigma(x)f(y) - \sigma(y)f(x).$$

Then there exists $v \in V$ such that for any $x \in \mathfrak{g}$, $f(x) = \sigma(x)v$.

***John Henry Constantine Whitehead**, 1904 – 1960, was a British mathematician and was one of the founders of homotopy theory. During the Second World War he worked on operations research for submarine warfare. Later, he joined the code-breakers at Bletchley Park, and by 1945 was one of some fifteen mathematicians working in the "Newmanry", a section headed by Max Newman and responsible for breaking a German teleprinter cipher using machine methods. Those methods included the Colossus machines, early digital electronic computers. Whitehead was a Fellow of the Royal Society and President of the London Mathematical Society.*

PROOF. We can assume that $\mathfrak{g}$ is nonzero. The ideal $I = \operatorname{Ker}\sigma$ is semisimple according to the remark which succeeds Proposition 6.6. In particular, $I = [I, I]$ and by the hypothesis, $f(I) = f([I, I]) = \{0\}$. We can therefore assume that (V, σ) is faithful.

Assume first that (V, σ) is simple. Let c_σ be the Casimir element associated with σ. In the notation of Exercise 7.1, we have

$$c_\sigma = \sum_{i=1}^{n} \sigma(x_i) \circ \sigma(y_i) = \frac{n}{\dim V} \operatorname{Id}_V.$$

If the vector v we are looking for does exist, then

$$\sum_{i=1}^{n} \sigma(x_i) \circ f(y_i) = \left(\sum_{i=1}^{n} \sigma(x_i) \circ \sigma(y_i)\right) v.$$

Then it is natural to set:

$$v = c_\sigma^{-1} \left(\sum_{i=1}^{n} \sigma(x_i) \circ f(y_i)\right).$$

Let $z \in \mathfrak{g}$. By Exercise 7.1, we have for $i \in \{1, \ldots, n\}$,

$$[z, x_i] = \sum_{j=1}^{n} a_{i,j} x_j, \qquad [z, y_i] = -\sum_{j=1}^{n} a_{j,i} y_j. \tag{15}$$

We now show that $f(x) = \sigma(x)v$ for any $x \in \mathfrak{g}$, that is, $c_\sigma \circ f(x) = c_\sigma \circ \sigma(x)v$ for any $x \in \mathfrak{g}$, We have,

$$\begin{aligned}
c_\sigma \circ f(z) &= \sum_{i=1}^{n} \sigma(x_i) \circ \sigma(y_i) \circ f(z) \\
&= \sum_{i=1}^{n} \sigma(x_i) \circ f([y_i, z]) + \sum_{i=1}^{n} \sigma(x_i) \circ \sigma(z) \circ f(y_i) \\
&= \underbrace{\sum_{i=1}^{n} \sigma(x_i) \circ f([y_i, z]) + \sum_{i=1}^{n} \sigma([x_i, z]) \circ f(y_i)}_{=0 \text{ by } (15)} + \sum_{i=1}^{n} \sigma(z) \circ \sigma(x_i) \circ f(y_i) = c_\sigma \circ \sigma(z)v,
\end{aligned}$$

whence $\sigma(z)v = f(z)$ for any $z \in \mathfrak{g}$, as desired.

For the general case, we proceed by induction on the dimension of V. From the previous case, we can assume that V contains a proper submodule W. Let $\pi\colon V \to V/W$ be the canonical projection. By the induction hypothesis applied to the quotient module V/W, there exists $v \in V$ such that $(\pi \circ f)(x) = \pi \circ \sigma(x)v$ for any $x \in \mathfrak{g}$. Set for any $x \in \mathfrak{g}$,

$$\theta(x) = f(x) - \sigma(x)v.$$

Then $\theta(x) \in W$ for any $x \in \mathfrak{g}$ and θ is a linear map from $\mathfrak{g}$ to W which satisfies the hypothesis of the proposition. Therefore, by the induction hypothesis, there exists $w \in W$ such that $\theta(x) = \sigma(x)w$ for any $x \in \mathfrak{g}$. In conclusion, $f(x) = \sigma(x)(v + w)$ for any $x \in \mathfrak{g}$, as desired. □

We are now in a position to prove Weyl's Theorem 7.1.

PROOF OF WEYL'S THEOREM 7.1. Let (V, σ) be a representation of $\mathfrak{g}$ of finite dimension, U a nontrivial submodule, $\pi\colon V \to V/U$ the projection canonical and $(V/U, \tau)$ the induced representation. Let

$$M = \mathscr{L}(V/U, V)$$

be the space of linear maps from V to V/U,

If $\phi_0 \in M$ is such that $\pi \circ \phi_0 = \mathrm{Id}_{V/U}$, then obviously $V = \phi_0(V/U) \oplus U$, but, unfortunately, $\phi_0(V/U)$ is not a submodule of V a priori!

Let ϕ_0 arbitrary as above. So we rather seek for $\theta_0 \in M$ such that

$$V = \theta_0(V/U) \oplus U$$

and $\theta_0(V/U)$ is a submodule of V. For this, we search $\psi_0 \in N$, where

$$N = \{\phi \in M \colon \phi(V/U) \subset U\},$$

such that $\theta_0(V/U)$ is a submodule of V, where

$$\theta_0 = \phi_0 - \psi_0$$

so that $V = \theta_0(V/U) \oplus U$.

The map $\lambda\colon \mathfrak{g} \to \mathrm{End}(M)$ defined by:

$$\forall x \in \mathfrak{g},\ \forall \phi \in M, \quad \lambda(x)\phi = \sigma(x) \circ \phi - \phi \circ \tau(x),$$

is a representation of $\mathfrak{g}$ whose N is a submodule. Note that for $\phi \in M$, if $\lambda(x)\phi = 0$ for any $x \in \mathfrak{g}$, then $\phi(V/U)$ is a submodule of V. We thus search $\psi_0 \in N$ such that $\lambda(x)\phi_0 = \lambda(x)\psi_0$ for any $x \in \mathfrak{g}$.

The linear map

$$f\colon \mathfrak{g} \to M, \quad x \mapsto \lambda(x)\phi_0$$

verifies $f(\mathfrak{g}) \subset N$ since for any $w \in V/U$,

$$\begin{aligned} \pi(f(x)w) &= \pi \circ \sigma(x) \circ \phi_0(w) - \pi \circ \phi_0 \circ \tau(x)(w) \\ &= \tau(x) \circ \pi \circ \phi_0(w) - \tau(x)(w) = \tau(x)(w) - \tau(x)(w) = 0. \end{aligned}$$

The condition of Lemma 7.2 is satisfied for the representation (N, μ) induced from λ. As a result, there exists $\psi_0 \in N$ such that

$$f(x) = \mu(x)\psi_0 = \lambda(x)\psi_0 \quad \text{for any} \quad x \in \mathfrak{g}.$$

Set

$$\theta_0 = \phi_0 - \psi_0.$$

Then $\lambda(x)\theta_0 = 0$ for any $x \in \mathfrak{g}$, whence $\pi \circ \theta_0 = \mathrm{Id}_{V/U}$ since for all $w \in V/U$,

$$\pi \circ \theta_0(w) = \pi \circ \phi_0(w) - \pi \circ \psi_0(w) = w - 0 = w,$$

and $\theta_0 \circ \tau(x) = \sigma(x) \circ \theta_0$ for any $x \in \mathfrak{g}$ so $\theta_0(V/U)$ is a $\mathfrak{g}$-submodule.

In conclusion, V is the direct sum of the submodules U and $\theta_0(V/U)$, and we can argue by induction on the dimension. □

Proposition 7.3 – semisimple Lie algebras of matrices contains semisimple and nilpotent parts of elements

Let V be a finite-dimensional vector space, and $\mathfrak{g}$ be a semisimple Lie subalgebra of $\mathfrak{gl}(V)$. Then $\mathfrak{g}$ contains the semisimple and nilpotent parts of all its elements.

EXERCISE 7.4. The objective of the exercise is to prove the proposition. Let $x \in \mathfrak{g}$. We have to show that x_s and x_n are in $\mathfrak{g}$.

(1) Verify that x_s and x_n belongs to the ***normalizer*** of $\mathfrak{g}$ in $\mathfrak{gl}(V)$:
$$\mathfrak{n}_{\mathfrak{gl}(V)}(\mathfrak{g}) = \{y \in \mathfrak{gl}(V) \colon [y, \mathfrak{g}] \subset \mathfrak{g}\}.$$

> If $\mathfrak{n}_{\mathfrak{gl}(V)}(\mathfrak{g})$ were equal to $\mathfrak{g}$, we would conclude immediately. But it is not the case in general! For example, if $\mathfrak{g} = \mathfrak{sl}(V)$, then $\mathfrak{n}_{\mathfrak{gl}(V)}(\mathfrak{g}) = \mathfrak{gl}(V)$!

(2) Let $\mathcal{V}$ be the set of all $\mathfrak{g}$-submodules of V. For $W \in \mathcal{V}$, set
$$\mathfrak{g}_W = \{x \in \mathfrak{gl}(V) \colon x(W) \subset W \text{ and } \operatorname{Tr}(x_W) = 0\}.$$
For example, $\mathfrak{g}_V = \mathfrak{sl}(V)$. Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, we have $\mathfrak{g} \subset \mathfrak{g}_W$ for any $W \in \mathcal{V}$. Set
$$N = \mathfrak{n}_{\mathfrak{gl}(V)}(\mathfrak{g}) \cap \left(\bigcap_{W \in \mathcal{V}} \mathfrak{g}_W \right).$$
Verify that N is a subalgebra of $\mathfrak{n}_{\mathfrak{gl}(V)}(\mathfrak{g})$ containing $\mathfrak{g}$ as ideal (the scalars are not in N however!), and show that x_s and x_n are in N.

(3) Show that $\mathfrak{g} = N$ and conclude.

> Hint: use Weyl's Theorem 7.1 to show that there exists a submodule M such that $N = \mathfrak{g} \oplus M$, notice that the action of $\mathfrak{g}$ in M is trivial, and then show that $M = \{0\}$ using Schur's Lemma 3.2 applied to any irreducible $W \in \mathcal{V}$.

Let $\mathfrak{g}$ be a semisimple Lie algebra and $x \in \mathfrak{g}$. Since the adjoint representation of $\mathfrak{g}$ is faithful, the center of $\mathfrak{g}$ is trivial, and the previous proposition says that we have
$$\operatorname{ad} x = \operatorname{ad} x_s + \operatorname{ad} x_n,$$
with $x_s, x_n \in \mathfrak{g}$, unique, such that $\operatorname{ad} x_s = (\operatorname{ad} x)_s$ and $\operatorname{ad} x_n = (\operatorname{ad} x)_n$. The elements x_s and x_n are called, respectively, the ***semisimple part*** and the ***nilpotent part*** of x.

Thus, we say that an element x is ***semisimple*** if $\operatorname{ad} x$ is semisimple, i.e., $x = x_s$, and that an element x is ***nilpotent*** if $\operatorname{ad} x$ is nilpotent, i.e., $x = x_n$.

> The terminology makes sense only if $\mathfrak{g}$ is semisimple!

7.3. Rational representation of a semisimple algebraic group

Proposition 2.9 (iii) shows that a connected algebraic group G is commutative if and only if its Lie algebra $\mathfrak{g}$ is. Recall that a Lie algebra $\mathfrak{g}$ is semisimple if it has no nonzero commutative ideals.

Definition 7.4 – semisimple algebraic group

A connected algebraic group G is called ***semisimple*** of it has no closed connected commutative normal subgroup except $\{e\}$.

Nicely, we have the following result that we do not prove here.

Proposition 7.5

A connected algebraic group G is semisimple if and only if its Lie algebra is semisimple.

In particular, because $\operatorname{Lie} Z(G) = \mathfrak{z}(\mathfrak{g})$, the center of $\mathfrak{g}$, the center of G is finite if G is semisimple.

A ***rational representation*** of G is a morphism of algebraic groups

$$\varphi\colon G \to \mathrm{GL}(V),$$

where V is a finite-dimensional [1] vector space.

Using Weyl's Theorem 7.1 one can prove the following theorem.

Theorem 7.6 – complete reducibility for rational representations of semisimple algebraic group

Let G be a semisimple algebraic group. Then any rational representation is completely reducible.

PROOF. A rational representation $\varphi\colon G \to \mathrm{GL}(V)$ induces a representation

$$\rho = \mathrm{d}_e\varphi\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V).$$

We can assume without loss of generality that φ is faithful: if $(V, \varphi/\operatorname{Ker}\varphi)$ is completely reducible then so is (V, φ). Thus we identify G with a subgroup of $\mathrm{GL}(V)$.

Let W be an irreducible $\mathfrak{g}$-submodule of V. The set

$$G_W := \{a \in G\colon \varphi(a)(W) \subset W\}$$

is a subgroup whose Lie algebra is

$$\mathfrak{g}_W := \{x \in \mathfrak{g}\colon \rho(x)(W) \subset W\}$$

(we have an inclusion $\operatorname{Lie}(G_W) \subset \mathfrak{g}_W$ and they share the same dimension), which is equal to $\mathfrak{g}$ since W is a $\mathfrak{g}$-submodule. By the correspondence between closed connected subgroups of G and Lie subalgebras of $\mathfrak{g}$ (Theorem 2.8) we deduce that any submodule for $\mathfrak{g}$ is also a submodule for G. Therefore one can conclude thanks to Weyl's Theorem 7.1. □

1. Sometimes, *rational* refers to locally finite representations.

8

Root decomposition of semisimple Lie algebras

In this chapter $\mathfrak{g}$ is a semisimple Lie algebra.

8.1. Toral subalgebras

If $\mathfrak{g}$ consisted only of nilpotent elements (i.e., ad-nilpotent elements), then $\mathfrak{g}$ would be nilpotent according to Engel's Theorem 4.4. But this is not the case. So there exists $x \in \mathfrak{g}$ such that x_s is nonzero. Since $\mathfrak{g}$ contains the semisimple and nilpotent parts of all its elements, $x_s \in \mathfrak{g}$. We deduce that $\mathfrak{g}$ contains a nonzero subalgebra consisted of only semisimple elements (for example $\Bbbk x_s$).

Definition 8.1 – torale subalgebra

A subalgebra of $\mathfrak{g}$ which only consists of semisimple elements is called ***torale***.

Next lemma is an analogue of Engel's Theorem.

Lemma 8.2

Any toral subalgebra is abelian.

PROOF. Let T be a toral subalgebra. We have to show that $\operatorname{ad}_T x = 0$ for any $x \in T$, where $\operatorname{ad}_T \colon T \to \operatorname{End}(T)$, $x \mapsto (\operatorname{ad} x)|_T$. Since $\operatorname{ad} x$ is diagonalizable, it suffices to show that $\operatorname{ad}_T x$ has no nonzero eigenvalues. Suppose the contrary, i.e., $[x, y] = \lambda y$, with $y \in T \setminus \{0\}$ and $\lambda \neq 0$. Write $x = \sum_{i=1}^{n} x_i v_i$, $x_i \in \Bbbk$, in a basis $(v_1, \ldots, v_n)$ of eigenvectors for $\operatorname{ad}_T y$ ($\operatorname{ad}_T y$ is semisimple), associated with the eigenvalues $\mu_1, \ldots, \mu_n$. We have

$$-\lambda y = \operatorname{ad}_T y(x) = \sum_{i=1}^{n} x_i \operatorname{ad}_T y(v_i) = \sum_{i=1}^{n} x_i \mu_i v_i,$$

whence a contraction since y is an eigenvector of $\operatorname{ad}_T y$ associated with the eigenvalue 0. □

From now on, fix a maximal (for the inclusion) toral subalgebra $\mathfrak{h}$. As $\mathfrak{h}$ is abelian by Lemma 8.2, $\operatorname{ad} \mathfrak{h}$ consisted of pairwise commuting semisimple elements. So all the elements of $\operatorname{ad} \mathfrak{h}$ are simultaneously semisimple. Therefore, we have:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \colon [h, x] = \alpha(h)x \text{ for any } h \in \mathfrak{h}\}.$$

We notice that $\mathfrak{g}_0 = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$, the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. It contains $\mathfrak{h}$ according to Lemma 8.2. Let Φ denote the set of $\alpha \in \mathfrak{h}^* \setminus \{0\}$ such that $\mathfrak{g}_\alpha \neq \{0\}$. The elements of Φ are called the ***roots*** of $\mathfrak{g}$. They are finite in number, and we obtain the ***decomposition in root subspaces***:

$$\mathfrak{g} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha. \tag{16}$$

We will see that $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$.

EXERCISE 8.1. Assume $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{k})$.

(1) Check that one can choose for $\mathfrak{h}$ the set of traceless diagonal matrices.
(2) Write the decomposition (16) for $\mathfrak{sl}_n(\mathbb{k})$ with this choice of $\mathfrak{h}$. Check that $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Proposition 8.3

For all $\alpha, \beta \in \mathfrak{h}^*$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. If $x \in \mathfrak{g}_\alpha$, $\alpha \neq 0$, then $\operatorname{ad} x$ is nilpotent. If $\alpha, \beta \in \mathfrak{h}^*$ and $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ relative to $\kappa_{\mathfrak{g}}$.

EXERCISE 8.2.

(1) Prove the proposition using the invariance of the Killing form.
(2) Deduce from the proposition that the restriction of $\kappa_{\mathfrak{g}}$ to $\mathfrak{g}_0 = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$ is nondegenerate.

Proposition 8.4

Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$. In particular, the restriction to $\mathfrak{h}$ of $\kappa_{\mathfrak{g}}$ is nondegenerate.

EXERCISE 8.3. The objective of the exercise it to prove Proposition 8.4. Set $\mathfrak{c} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$.

(1) Show that $\mathfrak{c}$ contains the semisimple and nilpotent parts of its elements.
(2) Show that semisimple elements of $\mathfrak{c}$ belong to $\mathfrak{h}$.
(3) Show that the restriction of $\kappa_{\mathfrak{g}}$ to $\mathfrak{h}$ is nondegenerate.
(4) Show that $\mathfrak{c}$ is nilpotent.
(5) Show that $\mathfrak{h} \cap [\mathfrak{c}, \mathfrak{c}] = \{0\}$.
(6) Prove that $\mathfrak{c}$ is abelian.
(7) Conclude that $\mathfrak{c} = \mathfrak{h}$.

By Proposition 8.4, the decomposition (16) becomes

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \tag{17}$$

and $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, where $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} : [x, \mathfrak{h}] \subset \mathfrak{h}\}$.

REMARK 8.1. For a general Lie algebra $\mathfrak{g}$, a ***Cartan subalgebra*** $\mathfrak{h}$ of $\mathfrak{g}$ is a nilpotent subalgebra equals to its normalizer, that is $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, By the above remark, any maximal toral subalgebra of a semisimple Lie algebra is a Cartan subalgebra. Conversely, it can be shown that any Cartan subalgebra of a semisimple Lie algebra is a maximal toral subalgebra (see [12, 19.8.7]).

Since the restriction of $\kappa_{\mathfrak{g}}$ to $\mathfrak{h}$ is nondegenerate, we can identify $\mathfrak{h}$ to $\mathfrak{h}^*$ via $\kappa_{\mathfrak{g}}$. Thus, every $\phi \in \mathfrak{h}^*$ corresponds a unique $t_\phi \in \mathfrak{h}$ such that

$$\phi(h) = \kappa_{\mathfrak{g}}(t_\phi, h)$$

for any $h \in \mathfrak{h}$. In particular, Φ corresponds to a subset $\{t_\alpha : \alpha \in \Phi\}$ of $\mathfrak{h}$.

Proposition 8.5 – orthogonality

(i) Φ generates $\mathfrak{h}^*$.

(ii) If $\alpha \in \Phi$, then $-\alpha \in \Phi$.

(iii) Let $\alpha \in \Phi$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, then $[x, y] = \kappa_\mathfrak{g}(x, y)t_\alpha$.

(iv) If $\alpha \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one-dimensional, with basis t_α.

(v) $\alpha(t_\alpha) = \kappa_\mathfrak{g}(t_\alpha, t_\alpha) \neq 0$ for $\alpha \in \Phi$.

(vi) If $\alpha \in \Phi$ and if x_α is a nonzero element of $\mathfrak{g}_\alpha$, then there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such as $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ generate a three-dimensional simple subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\Bbbk)$ via

$$x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(vii) $h_\alpha = \dfrac{2t_\alpha}{\kappa_\mathfrak{g}(t_\alpha, t_\alpha)}$, $h_\alpha = -h_{-\alpha}$.

EXERCISE 8.4. Prove the above proposition.

Hint for (v): observe that if $\alpha(t_\alpha) = 0$, then the Lie algebra generated by x, y, t_α with x, y as in (iii) with $\kappa_\mathfrak{g}(x, y) = 1$ would be nilpotent and use Engel's theorem to show that $\mathrm{ad}_\mathfrak{g}(t_\alpha)$ is nilpotent.

For any $\alpha \in \Phi$, denote $S_\alpha \cong \mathfrak{sl}_2(\Bbbk)$ the subalgebra of $\mathfrak{g}$ generated by $x_\alpha, h_\alpha, y_\alpha$. This subalgebra acts in $\mathfrak{g}$ so that $\mathfrak{g}$ is an S_α-module. By the study of $\mathfrak{sl}_2$-representations of Section 3.2, we obtain the following proposition:

Proposition 8.6

(i) If $\alpha \in \Phi$ then $\dim \mathfrak{g}_\alpha = 1$. In particular, $S_\alpha = \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + \mathfrak{h}_\alpha$, where $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, and for any $x_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$, there exists a unique $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[x_\alpha, y_\alpha] = h_\alpha$.

(ii) If $\alpha \in \Phi$, the only multiples of α which are roots are α and $-\alpha$.

(iii) If $\alpha, \beta \in \Phi$, then $\beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Phi$.

(iv) If $\alpha, \beta, \alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

(v) Let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Let p, q be the maximal integers such that, respectively, $\beta - p\alpha$ and $\beta + q\alpha$ are roots. Then $\beta + i\alpha$ is a root for any $i \in \{-p, \dots, q\}$ and $\beta(h_\alpha) = p - q$.

This sequence of roots is called the ***α-chain of roots passing through*** β.

(vi) $\mathfrak{g}$ is generated as a Lie algebra by the weight spaces $\mathfrak{g}_\alpha$.

Since the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ is nondegenerate, we can define a bilinear form on $\mathfrak{h}$ by:

$$(\gamma|\delta) = \kappa_\mathfrak{g}(t_\gamma, t_\delta), \qquad \gamma, \delta \in \mathfrak{h}^*.$$

Recall that Φ generates $\mathfrak{h}^*$ as space vector. Let $\alpha_1, \dots, \alpha_\ell$ be a basis of $\mathfrak{h}^*$ consisted of elements of Φ.

EXERCISE 8.5 (a rationality condition).

(1) Let $\beta \in \Phi$ written as $\beta = \sum_{i=1}^\ell c_i\alpha_i$, with $c_i \in \Bbbk$. Show that all c_i are rational numbers.

The $\mathbb{Q}$-vector space $E_\mathbb{Q}$ of $\mathfrak{h}^*$ generated by all roots is so of dimension $\ell = \dim_\Bbbk \mathfrak{h}^*$.

(2) Show that for all $\gamma, \delta \in E_\mathbb{Q}$, we have: $(\gamma|\delta) \in \mathbb{Q}$.

Hint: observe that $(\gamma|\delta) = \mathrm{Tr}(\mathrm{ad}(t_\gamma)\mathrm{ad}(t_\delta)) = \sum_{\alpha\in\Phi} \alpha(\gamma)\alpha(\delta)$, and first show that $(\beta|\beta) \in \mathbb{Q}$ for any $\beta \in \Phi$.

So $(-|-)$ defines a bilinear form $E_{\mathbb{Q}} \times E_{\mathbb{Q}}$ dans $\mathbb{Q}$.

(3) Show that the symmetric bilinear form $(-|-)$ is definite positive.

According to Exercise 8.5, the bilinear form $(-|-)$ induces a scalar product on the $\mathbb{R}$-vector space $E = \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$. In other words, $E, (-|-)$ is a Euclidean space.

In the language of the Chapter 9, the following theorem ensures that the set Φ is a ***root system*** of the Euclidean space E.

Theorem 8.7

(i) Φ generates E and 0 does not belong to Φ.

(ii) If $\alpha \in \Phi$, then $-\alpha \in \Phi$ and there is no other multiple of α in Φ.

(iii) If $\alpha, \beta \in \Phi$, then $\beta - \dfrac{2(\beta|\alpha)}{(\alpha|\alpha)}\alpha \in \Phi$,

(iv) If $\alpha, \beta \in \Phi$, then $\dfrac{2(\beta|\alpha)}{(\alpha|\alpha)} \in \mathbb{Z}$.

EXERCISE 8.6. Assume that $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{k})$. Let $\mathfrak{h}$ be the Lie subalgebra of diagonal matrices in $\mathfrak{g}$. Show that $\mathfrak{h}$ is a maximal toral Lie subalgebra et determine the corresponding set of roots.

8.2. Automorphisms of a Lie algebra

Assume in this section that $\mathrm{char}(\mathbb{k}) = 0$, $\mathfrak{g}$ is an arbitrary Lie algebra.

An ***automorphism*** of a Lie algebra $\mathfrak{g}$ is an endomorphism of the Lie algebra $\mathfrak{g}$ which is bijective. Denote by $\mathrm{Aut}(\mathfrak{g})$ the set of automorphisms of $\mathfrak{g}$.

EXAMPLE 8.1. Assume that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$, with V a finite-dimensional vector space. If $a \in \mathrm{GL}(V)$ is such that $a\mathfrak{g}a^{-1} \subset \mathfrak{g}$, then the linear map $x \mapsto axa^{-1}$ is an automorphism of $\mathfrak{g}$.

Important examples of automorphisms come from nilpotent derivations.

EXERCISE 8.7 (nilpotent derivations induce Lie algebra automorphisms). Let δ be a nilpotent derivation of $\mathfrak{g}$, i.e., $\delta\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation and $\delta^n = 0$ for n big enough.

(1) Show that the *Leibniz rule* holds: for any $r \in \mathbb{N}$,

$$\delta^r([x, y]) = \sum_{i=0}^{r} \binom{r}{i} [\delta^i x, \delta^{r-i} y].$$

(2) Deduce that $\exp \delta$ belongs to $\mathrm{Aut}(\mathfrak{g})$.

In particular, if $x \in \mathfrak{g}$ is such that $\mathrm{ad}\, x$ is a nilpotent endomorphism, then

$$\exp \mathrm{ad}\, x \in \mathrm{Aut}(\mathfrak{g}).$$

Such automorphisms are called ***elementary***. Denote by $\mathrm{Aut}_e(\mathfrak{g})$ the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by elementary automorphisms of $\mathfrak{g}$. It is a normal subgroup of $\mathrm{Aut}(\mathfrak{g})$. Namely, if $\theta \in \mathrm{Aut}(\mathfrak{g})$ and if $\mathrm{ad}\, x$ is a nilpotent endomorphism, then $\theta \circ (\mathrm{ad}\, x) \circ \theta^{-1} = \mathrm{ad}\, \theta(x)$ and so

$$\theta \circ \exp(\mathrm{ad}\, x) \circ \theta^{-1} = \exp(\mathrm{ad}\, \theta(x)).$$

Definition 8.8 – conjugated subalgebras

We say that two Lie subalgebras $\mathfrak{m}_1$ and $\mathfrak{m}_2$ of $\mathfrak{g}$ are ***conjugated*** if there exists $\theta \in \mathrm{Aut}_e(\mathfrak{g})$ such that $\mathfrak{m}_2 = \theta(\mathfrak{m}_1)$.

EXERCISE 8.8 (elementary automorphisms in $\mathfrak{sl}_2(\Bbbk)$). Assume that $\mathfrak{g} = \mathfrak{sl}_2(\Bbbk)$. Let

$$\sigma = \exp(\operatorname{ad} e) \circ \exp(\operatorname{ad}(-f)) \circ \exp(\operatorname{ad} e)$$

so that $\sigma \in \operatorname{Aut}_e(\mathfrak{g})$.

(1) Show that

$$\sigma(e) = -f, \quad \sigma(f) = -e, \quad \sigma(h) = -h.$$

In particular, σ is of order 2.

(2) Consider the element

$$s = \exp(e)\exp(-f)\exp(e) \in \mathrm{GL}_2(\Bbbk).$$

Check that s is an element of $\mathrm{SL}_2(\Bbbk)$. In particular, the map $z \mapsto szs^{-1}$ is an automorphism of $\mathfrak{g}$. Calculate the matrix s, and deduce that the action by conjugation of s has the same effect as σ on the basis (e, h, f), and therefore on $\mathfrak{g}$.

The phenomenon observed in Exercise 8.8 is not a simple coincidence: if $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$ and if $x \in \mathfrak{g}$ is nilpotent, then we have for all $x, y \in \mathfrak{g}$,

$$(\exp x)y(\exp x)^{-1} = \exp(\operatorname{ad} x)y. \tag{18}$$

To see this, notice that

$$\operatorname{ad} x = \lambda_x + \rho_{-x},$$

where λ_x and ρ_x are the left and right multiplications by $x \in \mathfrak{g}$, respectively, in $\operatorname{End}(V)$. These two endomorphisms commute and are nilpotent. According to the properties of the exponential, we therefore have

$$\exp(\operatorname{ad} x) = \exp(\lambda_x + \rho_{-x}) = \exp(\lambda_x)\exp(\rho_{-x}) = \lambda_{\exp x}\rho_{\exp(-x)},$$

which implies the relation (18).

Recall that if $\mathfrak{g}$ is algebraic, that is, $\mathfrak{g} = \operatorname{Lie}(G)$ for some linear algebraic group G, then we can consider the adjoint representation of G (Section 2.3):

$$\operatorname{Ad}\colon G \longrightarrow \operatorname{Aut}(\mathfrak{g}).$$

Proposition 8.9

Assume that G is a semisimple algebraic group, with Lie algebra $\mathfrak{g}$. Then

$$\operatorname{Ad}(G) \cong (\operatorname{Aut}\mathfrak{g})^\circ = \operatorname{Aut}_e(\mathfrak{g}).$$

In particular, any semisimple Lie algebra is algebraic.

EXERCISE 8.9. The objective is to prove the proposition.

(1) Prove that $\operatorname{Lie}(\operatorname{Aut}\mathfrak{g}) = \operatorname{Der}\mathfrak{g}$ (without any assumption on $\mathfrak{g}$).

(2) Assume that G is semisimple. Justify all the following equalities:

$$\dim G = \dim \mathfrak{g} = \dim \operatorname{ad}\mathfrak{g} = \dim \operatorname{Der}\mathfrak{g} = \dim \operatorname{Aut}\mathfrak{g} = \dim \operatorname{Aut}_e(\mathfrak{g}).$$

(3) Conclude.

Definition 8.10 – adjoint group

For an arbitrary Lie algebra $\mathfrak{g}$, the ***ajoint group*** G_{ad} of $\mathfrak{g}$ is the smallest algebraic group of $\mathrm{GL}(\mathfrak{g})$ whose Lie algebra contains $\operatorname{ad}\mathfrak{g}$.

For an arbitrary Lie algebra $\mathfrak{g}$, we have the following inclusions:

$$\operatorname{Aut}_e(\mathfrak{g}) \subset G_{ad} \subset \operatorname{Aut}(\mathfrak{g}).$$

By Proposition 8.9, if $\mathfrak{g}$ is semisimple, then

$$G_{ad} \cong G/\operatorname{Ker}\operatorname{Ad} \cong (\operatorname{Aut}\mathfrak{g})^\circ = \operatorname{Aut}_e(\mathfrak{g}),$$

for any algebraic group such that $\mathrm{Lie}(G) = \mathfrak{g}$.

In general, for semisimple $\mathfrak{g}$ there are several linear algebraic groups such that $\mathrm{Lie}(G) = \mathfrak{g}$. For example, $\mathrm{PSL}_n(\Bbbk)$ and $\mathrm{SL}_n(\Bbbk)$ share the same Lie algebra $\mathfrak{sl}_n(\Bbbk)$.

8.3. Remark about the unicity

The construction of the root system depends on the initial choice of the maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We will not deal with this in detail, but we can show that if $\mathfrak{h}'$ is another one, then there exists a Lie algebra automorphism φ of $\mathfrak{g}$ such that $\varphi(\mathfrak{h}) = \mathfrak{h}'$. Then φ induces an isomorphism $\mathfrak{h}^* \xrightarrow{\sim} (\mathfrak{h}')^*$ which sends the root system Φ constructed from $\mathfrak{h}$ to the root system Φ' constructed from $\mathfrak{h}'$. If E is constructed as above, and E' is constructed in the same way but using $\mathfrak{h}'$ instead of $\mathfrak{h}$, then φ induces an isomorphism of the Euclidean spaces $E \xrightarrow{\sim} E'$. So the root systems Φ in E and Φ' in E' are isomorphic, in the sense of Chapter 9.

9

Root systems and isomorphism theorems

In this chapter we introduce the notion of a root system (abstract) in a Euclidean space and we state classification results. We will then establish a correspondence between irreducible root systems and Lie simple algebras which will allow them to be classified.

We will omit most of the demonstrations for the part combinatorics of root systems: see [9, 12, 1] for more details.

Throughout this chapter, E is a Euclidean space. In particular, E is a finite-dimensional $\mathbb{R}$-vector space. We denote by $(-|-)$ the scalar product.

9.1. Axioms and examples

Recall that a reflection of E is an orthogonal symmetry with respect to a hyperplane of E. For $\alpha \in E \setminus \{0\}$, we denote by s_α the reflection with respect to $\alpha^\perp = \{x \in E\colon (x|\alpha) = 0\}$. Since $s_\alpha = \mathrm{id}_E - 2p_\alpha$, where p_α is the orthogonal projection of E on $\mathbb{R}\alpha$, we have:

$$\forall x \in E, \qquad s_\alpha(x) = x - \frac{2(x|\alpha)}{(\alpha|\alpha)}\alpha.$$

Since the ratio $\dfrac{2(\beta|\alpha)}{(\alpha|\alpha)}$ will appear very frequently, we set for $\alpha, \beta \in E$,

$$\langle \beta, \alpha \rangle = \frac{2(\beta|\alpha)}{(\alpha|\alpha)}.$$

⚠ The application $(\beta, \alpha) \mapsto \langle \beta, \alpha \rangle$ is only linear in the variable β!

EXERCISE 9.1. Let Φ be a finite set of E which generates E. Suppose that all reflections s_α, $\alpha \in \Phi$, preserve Φ. Let σ be an element of $\mathrm{GL}(E)$ such that σ preserves Φ, σ fixes (point by point) a hyperplane of E and $\sigma(\alpha) = -\alpha$ for a certain $\alpha \in \Phi \setminus \{0\}$. Show: $\sigma = s_\alpha$.

Definition 9.1 – système de racines

A subset Φ of E is called a ***root system*** dans E si les conditions suivantes sont satisfaites:

- **(R1)** Φ is finite, generates E and does not contain 0,
- **(R2)** if $\alpha \in \Phi$, then the only multiples of α in Φ are $\pm\alpha$,
- **(R3)** if $\alpha \in \Phi$, the reflection s_α preserves Φ,
- **(R4)** if $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

EXAMPLE 9.1. Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a maximal toral subalgebra of $\mathfrak{g}$. Then the set Φ of roots associated with $(\mathfrak{g}, \mathfrak{h})$ is a system of roots by Theorem 8.7.

Let Φ be a root system. We denote by $\mathcal{W}(\Phi)$, or simply $\mathcal{W}$ when there is no ambiguity, the subgroup of $\mathrm{GL}(E)$ generated by the reflections s_α, $\alpha \in \Phi$. By **(R3)** , $\mathcal{W}$ permutes the elements of Φ which is, by **(R1)**, is a finite generating set of E. Therefore, $\mathcal{W}$ identifies with a subgroup of the symmetric group of Φ. In particular, $\mathcal{W}$ is finite.

Definition 9.2 – Weyl group

The group $\mathcal{W} = \mathcal{W}(\Phi)$ is called the ***Weyl group*** de Φ.

EXERCISE 9.2. Let $\sigma \in \mathrm{GL}(E)$ such that σ preserves Φ. Show that $\sigma \circ s_\alpha \circ \sigma^{-1} = s_{\sigma(\alpha)}$ for any $\alpha \in \Phi$, and that $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ for all $\alpha, \beta \in \Phi$.

We say that two systems of roots Φ and Φ' of respective Euclidean spaces E and E' are ***isomorphic*** if there exists an isomorphism of vector spaces $f \colon E \to E'$ such that $f(\Phi) = \Phi'$ and such that for all $\alpha, \beta \in \Phi$, $\langle f(\beta), f(\alpha) \rangle = \langle \beta, \alpha \rangle$. We then have for all $\alpha, \beta \in \Phi$, $s_{f(\alpha)}(f(\beta)) = f(s_\alpha(\beta))$. Such isomorphism of root systems induces an isomorphism, $\mathcal{W}(\Phi) \to \mathcal{W}(\Phi')$, $s \mapsto f \circ s \circ f^{-1}$ between Weyl groups.

According to Exercise 9.2, the automorphisms of Φ are the automorphisms of E which preserve Φ. In particular, we can view the Weyl group $\mathcal{W} = \mathcal{W}(\Phi)$ as a subgroup of $\mathrm{Aut}(\Phi)$, where $\mathrm{Aut}(\Phi)$ is the set of automorphisms of Φ.

For $\alpha \in \Phi$, we set

$$\alpha^\vee = \frac{2\alpha}{(\alpha|\alpha)}$$

so that $s_\alpha = \mathrm{id}_E - \alpha\alpha^\vee$, where $\alpha^\vee$ identifies with the linear form of E which to $x \in E$ associates $\dfrac{2(x|\alpha)}{(\alpha|\alpha)}$. We call the set

$$\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$$

the ***dual*** of Φ. It is a root system of $E \cong E^*$, whose Weyl group is canonically isomorphic to that of Φ.

REMARK 9.1. In the case of semisimple Lie algebras, α corresponds to t_α and $\alpha^\vee$ to h_α via the identification of $\mathfrak{h}^*$ with $\mathfrak{h}$ using the Killing form.

Let us see some examples. The ***rank*** of the root system Φ is the dimension ℓ of E. When $\ell \leqslant 2$, we can easily draw examples in a figure.

$*$ For $\ell = 1$, by the axiom **(R2)**, there is only one possibility (up to isomorphism). We denote A_1 this class.

$*$ For $\ell = 2$, there are more possibilities. See Figure 1 for examples (we will see later that there are the only ones up to isomorphisms).

EXERCISE 9.3 (rank two root systems). Check that the systems in Figure 1 give root systems in the Euclidean plane $\mathbb{R}^2$, and show that the Weyl group of $A_1 \times A_1$, A_2, B_2 and G_2 are respectively the dihedral group of order 4, 6, 8, 12. Finally, represent the system $\Phi^\vee$ for each of the examples.

The axiom **(R4)** strongly rigidifies the structure of root systems. Recall that for $\alpha, \beta \in E$, the cosinus of the angle θ between α and β is given by:

$$\|\alpha\| \cdot \|\beta\| \cos\theta = (\alpha|\beta).$$

Therefore, we have:

$$\langle \beta, \alpha \rangle = \frac{2(\beta|\alpha)}{(\alpha|\alpha)} = 2\frac{\|\beta\|}{\|\alpha\|}\cos\theta,$$

hence

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4\cos^2\theta.$$

The number $4\cos^2\theta$ is then a positive integer. But $0 \leqslant \cos^2\theta \leqslant 1$ so $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ must be integers of the same sign whose product is between 0 and 4.

The only possibilities when $\alpha \neq \pm\beta$ and $\|\beta\| \geqslant \|\alpha\|$ are described in Figure 2.

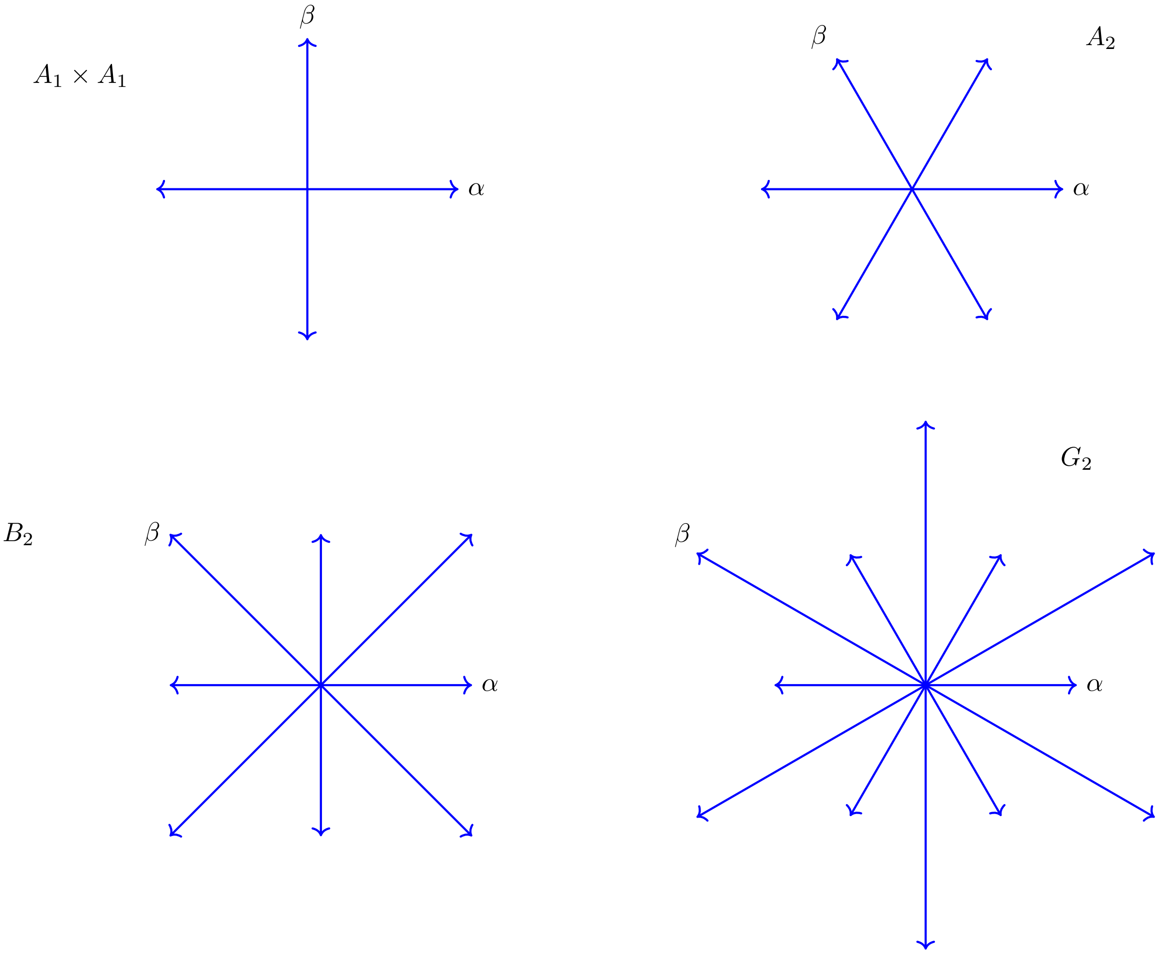


FIGURE 1. Examples of rank two root systems

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\Vert\beta\Vert^2/\Vert\alpha\Vert^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

FIGURE 2. Possibles configurations for $\alpha, \beta \in \Phi$, when $\alpha \neq \pm\beta$ and $\Vert\beta\Vert \geqslant \Vert\alpha\Vert$.

Examples of root systems in rank one and two show that all these configurations can occur.

EXERCISE 9.4 (A_2 corresponds to $\mathfrak{sl}_3(\Bbbk)$). Verify that the root system of the semisimple Lie algebra $\mathfrak{sl}_3(\Bbbk)$, relative to the maximum toral subalgebra of diagonal matrices, is the root system A_2.

Likewise recognize the root system of $\mathfrak{so}_5(\Bbbk)$ and $\mathfrak{sp}_4(\Bbbk)$, relative to the maximum toral subalgebra of diagonal matrices.

EXERCISE 9.5. Let α and β be two non-proportional roots. Show that if $(\alpha|\beta) > 0$, then $\alpha - \beta$ is a root and if $(\alpha|\beta) < 0$, then $\alpha + \beta$ is a root.

Let α and β be two non-proportional roots as in Exercise 9.5 and consider all roots of the form $\beta + i\alpha$, the ***α-root string passing through*** β. We note p (resp. q) the largest integer such that $\beta - p\alpha \in \Phi$ (resp. $\beta + q\alpha \in \Phi$). Exercise 9.5 shows that for any $i \in \{-p, -p+1\ldots, q-1, q\}$, $\beta + i\alpha \in \Phi$. Furthermore, it can be shown that

$$p - q = \langle \beta, \alpha \rangle,$$

and so (see Figure 2) a string of roots is of length at most 4 (the system G_2 contains a root string of length exactly 4).

9.2. Basis of a root system and Weyl chambers

As before, let Φ be a root system of the Euclidean space E.

Definition 9.3 – Basis of a root system

A subset Δ of Φ is called a ***basis*** of Φ if:

(B1) Δ is a basis of E,

(B2) every root $\beta \in \Phi$ is written $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$, where the k_α are integers either all positive, or all negative.

The elements of Δ are called ***simple roots***.

By , we have $|\Delta| = \ell$. Moreover, the expression of β in is unique. We define the ***height*** of a root β by

$$\mathrm{ht}(\beta) = \sum_{\alpha \in \Delta} k_\alpha \in \mathbb{Z}.$$

If all k_α are positive, we say that β is a ***positive root***; if all k_α are negative, we say that β is a ***negative root***. We note Φ^+ the set of positive roots, and Φ^- the set of negative roots. We have $\Phi^- = -\Phi^+$ and,

$$\Phi = \Phi^+ \sqcup \Phi^-.$$

We define a partial order on Φ by setting $\beta \prec \alpha$ if $\alpha - \beta$ is a sum of simple roots or if $\beta = \alpha$. We note $\beta \succ 0$ if $\beta \in \Phi^+$. (this is compatible with the definition of the partial order).

A natural question is: are there basis for Φ? The answer is positive and we construct a basis of Φ as follows.

For any $\gamma \in E$, we set

$$\Phi^+(\gamma) = \{\alpha \in \Phi \colon (\gamma|\alpha) > 0\}.$$

If $\gamma \in E \setminus \bigcup\limits_{\alpha \in \Phi} \alpha^\perp$, we say that γ is ***regular***. Now suppose that γ is regular. We have

$$\Phi = \Phi^+(\gamma) \sqcup -\Phi^+(\gamma).$$

We say that an element $\alpha \in \Phi^+(\gamma)$ is ***decomposable*** if $\alpha = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in \Phi^+(\gamma)$; it is said ***indecomposable*** otherwise. We then show that the set $\Delta(\gamma)$ formed of all the indecomposable roots of $\Phi^+(\gamma)$ is a base Φ. Moreover, all the bases of Φ all obtained in this way.

We thus obtain the following crucial theorem.

Theorem 9.4 – existence of basis for a root system

The root system Φ admits a basis.

The hyperplanes $\alpha^\perp$, $\alpha \in \Phi$, subdivide the space E into a finite number of regions. The connected components of $E \setminus \bigcup\limits_{\alpha \in \Phi} \alpha^\perp$ are called the ***Weyl chambers*** of E. For a regular γ, we denote by $\mathscr{C}(\gamma)$ the unique Weyl chamber to which γ belongs. Saying that $\mathscr{C}(\gamma) = \mathscr{C}(\gamma')$, for regular γ, γ', means exactly that

$$\Phi^+(\gamma) = \Phi^+(\gamma') \quad \text{and} \quad \Delta(\gamma) = \Delta(\gamma').$$

Thus, there is a bijective correspondence between Weyl chambers and basis of Φ.

We write $\mathscr{C}(\Delta) = \mathscr{C}(\gamma)$ if $\Delta = \Delta(\gamma)$. The Weyl chamber $\mathscr{C}(\Delta)$ is called the ***fundamental Weyl chamber*** associated with the basis Δ.

EXERCISE 9.6 (basis and Weyl chambers in rank two). Verify that the sets $\{\alpha, \beta\}$ of root systems in rank two represented in Figure 1 are basis of the corresponding root system. Represent the Weyl chambers and specify the Weyl fundamental chamber relative to $\{\alpha, \beta\}$.

We denote $\rho = \frac{1}{2}\sum_{\beta \succ 0} \beta$ the half-sum of the positive roots. This element will be of great importance in the following chapters.

EXERCISE 9.7 (half-sum of the positive roots). Let Δ be a basis of Φ.

(1) Let $\alpha \in \Delta$. Show that s_α permutes the set of positive roots other than α.

> Hint: for $\beta \in \Phi^+ \setminus \{\alpha\}$, write
> $$\beta = \sum_{\gamma \in \Phi^+} k_\gamma \gamma,$$
> with $k_\gamma \in \mathbb{N}$, note that $k_\gamma \neq 0$ for some $\gamma \neq \alpha$, and compute the coefficient in γ of $s_\alpha(\beta)$.

(2) Deduce from the previous question that $s_\alpha(\rho) = \rho - \alpha$ for any $\alpha \in \Delta$.

By Exercise 9.7, 2, we have
$$(\rho|\alpha) = (s_\alpha(\rho) + \alpha|\alpha) = (\rho + s_\alpha(\alpha)|s_\alpha(\alpha)) = -(\rho|\alpha) + \|\alpha\|^2.$$
So $(\rho|\alpha) > 0$ for any $\alpha \in \Delta$, and thus ρ belongs to the fundamental Weyl chamber.

According to the following theorem, the Weyl group $\mathcal{W}(\Phi)$ acts simply and transitively on the set of bases of Φ. In particular, the order of the Weyl group is equal to the number of Weyl chambers. The parts of the theorem provide the steps of the proof (which we omit here).

Theorem 9.5 – action of the Weyl group on the bases of Φ

Let Δ be a basis of Φ, and $\mathcal{W} = \mathcal{W}(\Phi)$.

(i) If $\gamma \in E$ is regular, then there exists $\sigma \in \mathcal{W}$ such that $(\sigma(\gamma)|\alpha) > 0$ for any $\alpha \in \Delta$. In particular, $\mathcal{W}$ acts transitively on the Weyl chambers.

(ii) If Δ' is a basis of Φ, then there exists $\sigma \in \mathcal{W}$ such that $\sigma(\Delta') = \Delta$. In particular, $\mathcal{W}$ acts transitively on the bases of Φ.

(iii) If α is a root, then there exists $\sigma \in \mathcal{W}$ such that $\sigma(\alpha) \in \Delta$.

(iv) $\mathcal{W}$ is generated by the reflections s_α, for $\alpha \in \Delta$.

(v) If $\sigma(\Delta) = \Delta$, for $\sigma \in \mathcal{W}$, then $\sigma = \mathrm{id}_E$. In particular, $\mathcal{W}$ acts simply and transitively on the bases of Φ.

A root system Φ is said to be ***irreducible*** if it cannot be written as a union of two proper subsets Φ_1 and Φ_2 such that every root in Φ_1 is orthogonal to every root in Φ_2.

EXAMPLE 9.2. It is easily verified that the root systems of type A_2, B_2, and G_2 shown in Figure 1 are irreducible, while $A_1 \times A_1$ is not.

It turns out that Φ is irreducible if and only if for any base Δ of Φ, the set Δ cannot be written as a union of two proper subsets Δ_1 and Δ_2 such that every root in Δ_1 is orthogonal to every root in Δ_2.

9.3. Classification

Fix an ordered basis $(\alpha_1, \dots, \alpha_\ell)$ of simple roots. The matrix $(\langle \alpha_i, \alpha_j \rangle)_{1 \leqslant i,j \leqslant \ell}$ is called the ***Cartan matrix*** of Φ.

EXAMPLE 9.3. For the rank 2 root systems in Figure 1, the Cartan matrices are:
$$A_1 \times A_1 : \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad A_2 : \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B_2 : \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

The Cartan matrix does not depend on the order of the elements in the base $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$, up to isomorphism, thanks to Theorem 9.5. Moreover, since $\{\alpha_1, \dots, \alpha_\ell\}$ is a basis of E, the Cartan matrix is invertible. It turns out that it completely determines the root system Φ:

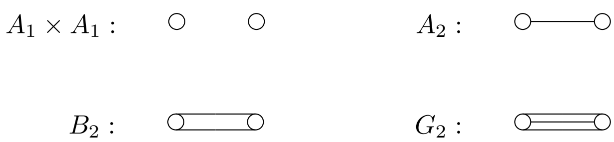
Proposition 9.6

Let $\Phi' \subset E'$ be another root system in a Euclidean space E', with basis $\Delta' = \{\alpha'_1, \dots, \alpha'_\ell\}$. Assume that $(\langle \alpha_i, \alpha_j \rangle)_{1 \leqslant i,j \leqslant \ell} = \left(\langle \alpha'_i, \alpha'_j \rangle\right)_{1 \leqslant i,j \leqslant \ell}$. Then the bijection $\alpha_i \mapsto \alpha'_i$ extends uniquely to an isomorphism $f\colon E \to E'$ that sends Φ to Φ' and satisfies $\langle f(\alpha), f(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Phi$. Consequently, the Cartan matrix of Φ completely determines Φ, up to isomorphism.

Thus, theoretically, one can reconstruct the root system Φ from its Cartan matrix.

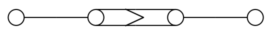
EXERCISE 9.8. Describe an algorithm to determine all the positive roots of Φ from the knowledge of the Cartan matrix.

If α, β are two distinct positive roots, it is known that $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0, 1, 2$, or 3. We define the ***Coxeter graph*** as follows: it is a graph with ℓ vertices, the i-th is joined to the j-th ($i \neq j$) by an edge if $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$. For example, the Coxeter graphs for $A_1 \times A_1$, A_2, B_2, and G_2 are, respectively:



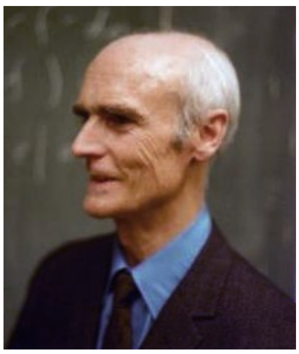
When a double or triple edge appears in the Coxeter graph, one can add an arrow pointing to the shortest root. This information allows to construct the Cartan matrix, and thus Φ, from this new graph. This graph is called the ***Dynkin diagram*** of Φ. For example, for B_2 and G_2, we obtain (for $A_1 \times A_1$ and A_2, the Dynkin diagram and the Coxeter graph coincide):

Here is another example (which will be the root system of type F_4):



The Dynkin diagram of B_2 corresponds here to the choice of a basis $\{\alpha, \beta\}$ where α is the longer root.

***Harold Scott MacDonald Coxeter**, born February 9, 1907 in London and died March 31, 2003 in Toronto (Canada), was a British mathematician. He is considered one of the leading geometers of the 20th century. One of his original ideas was to define a conic as a self-dual curve. He became well-known for his work on regular polytopes and geometry in higher dimensions.*



***Eugene B. Dynkin**, 1924 – 2014, was a Soviet and American mathematician. He made contributions to the fields of probability and algebra, especially semisimple Lie groups, Lie algebras, and Markov processes. The Dynkin diagram, the Dynkin system, and Dynkin's lemma are named after him.*

EXERCISE 9.9. Show that the Cartan matrix associated to the above graph is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

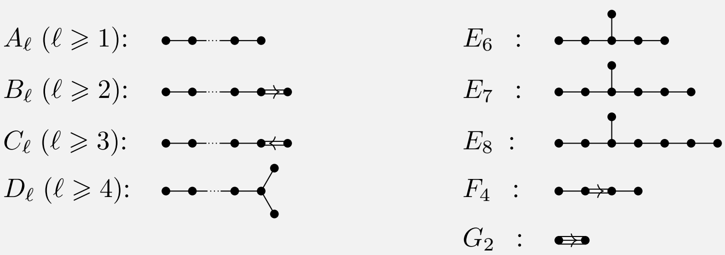
From the definition of an irreducible root system, it is easily deduced that the system Φ decomposes as a union of irreducible components:

$$\Phi = \Phi_1 \sqcup \ldots \sqcup \Phi_t,$$

where the Φ_i are root systems of Euclidean spaces E_i such that $E = E_1 \oplus \cdots \oplus E_t$ is an orthogonal direct sum. It is clear that Φ is irreducible if and only if its Coxeter graph is connected. Therefore, to classify all irreducible root systems, it is enough to classify all connected Dynkin diagrams.

Theorem 9.7 – classification of connected Dynkin diagrams

If Φ is an irreducible root system of rank ℓ, then its Dynkin diagram is one of the following (with ℓ vertices in each case):



All the Dynkin diagrams from Theorem 9.7 correspond to irreducible root systems. For this, we need to explicitly construct root systems of each type. For the families $A_\ell, B_\ell, C_\ell, D_\ell$, one can use the root systems associated with the semisimple classical Lie algebras $\mathfrak{sl}_{\ell+1}(\Bbbk)$, $\ell \geqslant 1$, $\mathfrak{so}_{2\ell+1}(\Bbbk)$, $\ell \geqslant 2$, $\mathfrak{sp}_{2\ell}(\Bbbk)$, $\ell \geqslant 3$, $\mathfrak{so}_{2\ell}(\Bbbk)$, $\ell \geqslant 4$. However, a direct construction can also be provided.

For each type of Dynkin diagram, we can describe a root system Φ in a Euclidean space E, whose Dynkin diagram is of the corresponding type. We refer to [12, 18.14] for such constructions.

9.4. Isomorphism Theorem

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ a maximal toral subalgebra of $\mathfrak{g}$, and Φ the associated root system, as in Chapter 6. Theorem 8.7 ensures that Φ is indeed a root system of the Euclidean space $E = \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$ in the sense of Definition 9.1, where $E_{\mathbb{Q}}$ is the $\mathbb{Q}$-vector space spanned by Φ.

We note that if $\mathfrak{g}$ is simple, then Φ is irreducible (this can be easily verified). Moreover, since any semisimple Lie algebra is an orthogonal direct sum of simple ideals with respect to the Killing form (see Proposition 6.6), the decomposition of the root system Φ into irreducible components $\Phi_1 \cup \ldots \cup \Phi_t$ corresponds to the decomposition of $\mathfrak{g} = I_1 \oplus \cdots \oplus I_t$ into a direct sum of simple ideals, that is, if $\mathfrak{h}_i = \mathfrak{h} \cap I_i$, then $\mathfrak{h}_i$ is a maximal toral subalgebra of I_i, and its root system is Φ_i, which is irreducible.

Thus, the problem of characterizing a semisimple Lie algebra by its root system associated with the choice of a maximal toral subalgebra reduces to the problem of characterizing a simple Lie algebra by its (irreducible) root system associated with the choice of a maximal toral subalgebra. Several very natural questions arise.

Q1 If $\mathfrak{g}'$ is a simple Lie algebra with a maximal toral subalgebra $\mathfrak{h}'$ and associated root system Φ', does an isomorphism of root systems between Φ and Φ' induce an isomorphism of Lie algebras between $\mathfrak{g}$ and $\mathfrak{g}'$?

Theorem 9.9 below answers this question affirmatively.

Q2 A priori, the root system Φ depends on the choice of the maximal toral subalgebra $\mathfrak{h}$; what happens if we choose a different one?

It can be proved that maximal toral subalgebras are all conjugate under an element of the group $\mathrm{Aut}_e(\mathfrak{g})$, and so Φ is independent of this choice. We do not address this problem in the course.

Q3 Do the irreducible root systems appearing in Theorem 9.7 all correspond to a simple Lie algebra?

We already know that the root systems of types A_ℓ, B_ℓ, C_ℓ, and D_ℓ correspond to the simple classical Lie algebras $\mathfrak{sl}_{\ell+1}(\Bbbk)$, $\ell \geqslant 1$, $\mathfrak{so}_{2\ell+1}(\Bbbk)$, $\ell \geqslant 2$, $\mathfrak{sp}_{2\ell}(\Bbbk)$, $\ell \geqslant 3$, $\mathfrak{so}_{2\ell}(\Bbbk)$, $\ell \geqslant 4$ (pairwise non isomorphic). All others also correspond to simple Lie algebras, called ***exceptional***. Here again, we do not address this problem in the course.

As we saw in Proposition 8.6, (vi), the root spaces $\mathfrak{g}_\alpha$, $\alpha \in \Phi$, span $\mathfrak{g}$ as a Lie algebra. Using the concept of basis for a root system, we can refine and improve this result:

Proposition 9.8

Let Δ be a basis of Φ. Then $\mathfrak{g}$ is generated as a Lie algebra by the root spaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$, $\alpha \in \Delta$.

Let $x_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$, $y_\alpha \in \mathfrak{g}_{-\alpha} \setminus \{0\}$, and $h_\alpha = [x_\alpha, y_\alpha]$. The set $x_\alpha, y_\alpha, h_{\alpha\alpha\in\Delta}$ is called a ***standard generating set*** for $\mathfrak{g}$.

EXERCISE 9.10. Prove this proposition.

We now assume that $\mathfrak{g}$ is simple. Let $\mathfrak{g}'$ be another simple Lie algebra, $\mathfrak{h}'$ a maximal toral subalgebra of $\mathfrak{g}'$, and Φ' the corresponding root system. We wish to show that an isomorphism between the root systems Φ and Φ' induces an isomorphism of Lie algebras between $\mathfrak{g}$ and $\mathfrak{g}'$ that sends $\mathfrak{h}$ to H'. This will answer **Q1**.

Let $\psi\colon \Phi \to \Phi'$, $\alpha \mapsto \alpha'$, be an isomorphism of root systems. It induces an isomorphism $\psi\colon E \to E'$ between the corresponding Euclidean spaces. A priori, ψ is not necessarily an isometry! However, the axioms of a root system remain unchanged if the inner product is multiplied by a strictly positive real number. Therefore, we can assume that ψ is an isometry. Furthermore, since Φ and Φ' span $\mathfrak{h}^*$ and $(\mathfrak{h}')^*$ as vector spaces, ψ extends uniquely to a vector space isomorphism $\psi\colon \mathfrak{h}^* \to (\mathfrak{h}')^*$. Via the Killing form, ψ induces an isomorphism $\pi\colon \mathfrak{h} \to \mathfrak{h}'$; specifically, $\pi(t_\alpha) = t_{\alpha'}$ for any $\alpha \in \Phi$. Since the isomorphism arises from a linear isometry, we deduce that $\pi(h_\alpha) = h_{\alpha'}$ for any $\alpha \in \Phi$, as $h_\alpha = 2t_\alpha/(\alpha|\alpha)$.

Since $\mathfrak{h}$ and $\mathfrak{h}'$ are abelian, π can be viewed as a Lie algebra morphism. We wish to extend it to an isomorphism $\mathfrak{g} \to \mathfrak{g}'$. If such an isomorphism exists, we expect that $x_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ maps to a nonzero element of $\mathfrak{g}'_{\alpha'}$. This certainly cannot be done arbitrarily. In fact, if α, β are roots such that $\alpha + \beta \in \Phi$, and if $x_\alpha \in \mathfrak{g}_\alpha \setminus 0$, $x_\beta \in \mathfrak{g}_\beta \setminus \{0\}$, and $x_{\alpha+\beta} \in \mathfrak{g}_{\alpha+\beta} \setminus \{0\}$ satisfy $[x_\alpha, x_\beta] = x_{\alpha+\beta}$, then we require $[x'_\alpha, x'_\beta] = x'_{\alpha+\beta}$, where $x'\alpha$ is the image of x_α under the isomorphism. The choices of x'_α and x'_β thus determine $x'_{\alpha+\beta}$.

The above discussion suggests that it suffices to determine the images of the vectors x_α for α in a basis Δ of Φ, and that these can be chosen arbitrarily.

We admit the following theorem.

Theorem 9.9

Let $\psi\colon \Phi \to \Phi'$, $\alpha \mapsto \alpha'$, be an isomorphism of root systems as above, which induces an isomorphism $\pi\colon \mathfrak{h} \to \mathfrak{h}'$. Fix a basis Δ of Φ so that $\Delta' = \{\alpha' \colon \alpha \in \Delta\}$ is a basis of Φ'. For each $\alpha \in \Delta$, $\alpha' \in \Delta'$, choose arbitrarily $x_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ and $x'_\alpha \in \mathfrak{g}_{\alpha'} \setminus \{0\}$. In other words, fix an isomorphism of Lie algebras

$$\pi_\alpha \colon \mathfrak{g}_\alpha \to \mathfrak{g}_{\alpha'}, \quad x_\alpha \mapsto x'_\alpha.$$

Then there exists a unique isomorphism $\pi\colon \mathfrak{g} \to \mathfrak{g}'$ that extends $\pi\colon \mathfrak{h} \to \mathfrak{h}'$ and all the isomorphisms $\pi_\alpha \colon \mathfrak{g}_\alpha \to \mathfrak{g}_{\alpha'}$, $\alpha \in \Delta$.

What follows is inspired by the example in Exercise 8.8. Apply Theorem 9.9 to the following situation: $\mathfrak{g}$ is semisimple, $\mathfrak{h}$ is a maximal toral subalgebra of $\mathfrak{g}$, Φ is the corresponding root system, and $\psi\colon \Phi \longrightarrow \Phi$ is the automorphism of root systems that sends α to $-\alpha$.

The automorphism ψ induces a Lie algebra morphism $\sigma\colon \mathfrak{h} \to \mathfrak{h}$, $h \mapsto -h$. In particular,

$$\sigma(h_\alpha) = -h_\alpha = h_{-\alpha}.$$

In order to apply Theorem 9.9, we send x_α to $-y_\alpha$ for any $\alpha \in \Delta$, where Δ is a basis of Φ [1]. According to Theorem 9.9, σ uniquely extends to an automorphism of $\mathfrak{g}$ that sends x_α to $-y_\alpha$, $\alpha \in \Delta$. Such an automorphism is necessarily of order two.

In summary, we have obtained:

Proposition 9.10

Let $\mathfrak{g}$ be as in Theorem 9.9. Fix for each $\alpha \in \Delta$, $x_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ and $y_\alpha \in \mathfrak{g}_{-\alpha} \setminus \{0\}$ such that $[x_\alpha, y_\alpha] = h_\alpha$. Then there exists a two-order automorphism σ of $\mathfrak{g}$, of order 2, such that

$$\sigma(x_\alpha) = -y_\alpha, \quad \sigma(y_\alpha) = -x_\alpha, \quad \sigma(h) = -h$$

for all $\alpha \in \Delta$ and $h \in \mathfrak{h}$.

The Weyl group $\mathcal{W}$ describes almost all automorphisms of Φ [2]. Theorem 9.9 ensures the existence of automorphisms of $\mathfrak{g}$ coming from the action of $\mathcal{W}$ on $\mathfrak{h}$. If $\sigma \in \mathcal{W}$, then the extension of σ to an automorphism of $\mathfrak{g}$ sends $\mathfrak{g}_\beta$ to $L_{\sigma\beta}$, for $\beta \in \Delta$.

A direct construction of such an automorphism can also be given inspired by Exercise 8.8. It suffices to construct it for the reflections s_α, $\alpha \in \Delta$. Since x_α is nilpotent, the automorphism

$$\tau_\alpha = \exp(\operatorname{ad} x_\alpha)\exp(\operatorname{ad}(-y_\alpha))\exp(\operatorname{ad} x_\alpha)$$

is well-defined. What is the action of τ_α on $\mathfrak{h}$? We write $\mathfrak{h} = \operatorname{Ker}\alpha \oplus \mathbb{C}h_\alpha$. Clearly, $\tau_\alpha(h) = h$ for all $h \in \operatorname{Ker}\alpha$, and $\tau_\alpha(h_\alpha) = -h_\alpha$. Therefore, s_α and τ_α coincide on $\mathfrak{h}$. Moreover, τ_α also sends $\mathfrak{g}_\beta$ to $L_{\sigma\beta}$.

This way of representing a reflection in $\mathcal{W}$ (and thus any element of $\mathcal{W}$) by an element of $\operatorname{Aut}_e(\mathfrak{g})$ does not always allow identifying $\mathcal{W}$ as a subgroup of $\operatorname{Aut}_e(\mathfrak{g})$ (see Exercise 9.11).

EXERCISE 9.11 (the Weyl group is not always a subgroup of $\operatorname{Aut}_e(\mathfrak{g})$). Suppose that $\mathfrak{g} = \mathfrak{sl}_3(\Bbbk)$ (type A_2). Show that the subgroup of $\operatorname{Aut}_e(\mathfrak{g})$ generated by all the reflections τ_α, $\alpha \in \Delta$, is strictly larger than the Weyl group $\mathcal{W}$ (which is equal to $\mathfrak{S}_3$ here).

1. Note that the unique $z \in \mathfrak{g}_\alpha$ such that $[-y_\alpha, z] = h_{-\alpha}$ is $-x_\alpha$.
2. It can be shown that $\operatorname{Aut}(\Phi) = \mathcal{W} \rtimes \Gamma$, where $\Gamma = \{\sigma \in \operatorname{Aut}(\Phi)\colon \sigma(\Delta) = \Delta\}$ is the ***group of automorphisms of the Dynkin diagram***. We have $\Gamma = \mathbb{Z}/2\mathbb{Z}$ if Φ is of type A_ℓ, D_ℓ, $\ell > 4$, or E_6, $\Gamma = \mathfrak{S}_3$ if Φ is of type D_4, and $\Gamma = 1$ for all other types.

10

Highest weight representations

We assume that $\mathfrak{g}$ is a semisimple Lie algebra over $\Bbbk$, where $\Bbbk$ is an algebraically closed field of characteristic zero.

10.1. Borel subalgebras and triangular decomposition

We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and we denote by Φ the root system of $(\mathfrak{g}, \mathfrak{h})$. Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ be a basis of Φ, and $\Phi^+ = \{\beta_1, \ldots, \beta_m\}$ the corresponding set of positive roots. Set

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$$

EXERCISE 10.1. Show that the subspaces $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent Lie subalgebras of $\mathfrak{g}$ and that $\mathfrak{b}^+$ is a solvable Lie subalgebra of $\mathfrak{g}$.

The Lie subalgebra $\mathfrak{b}^+$ is called a ***Borel subalgebra*** of $\mathfrak{h}$. Of course, this subalgebra depends on the choice of $\mathfrak{h}$ and of Δ. Using the root space decomposition of Chapter 8 and the decomposition

$$\Phi = \Phi^+ \sqcup (-\Phi^+)$$

we see that

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

This decomposition is called a ***triangular decomposition*** of $\mathfrak{g}$.

EXERCISE 10.2. Show that the linear map

$$U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+) \to U(\mathfrak{g})$$

which sends a tensor $u \otimes x \otimes v$ to uxv is an isomorphism of vector spaces.

The isomorphism of Exercise 10.2 is not an isomorphism of algebras!

10.2. Weights and maximal Vectors

Let V be a finite-dimensional representation of $\mathfrak{g}$. It follows from Proposition 7.3 that $\mathfrak{h}$ acts diagonally in V. Therefore, we can write

$$V = \sum_{\lambda} V_\lambda,$$

where

$$V_\lambda = \{v \in V : h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

When V is not necessarily finite-dimensional, the spaces V_λ are still well-defined. If $V_\lambda \neq \{0\}$, we say that V_λ is a ***weight space*** and that λ is a ***weight*** of V (more precisely, a ***weight*** of $\mathfrak{h}$ in V).

EXAMPLE 10.1.
1) If $V = \mathfrak{g}$ is the adjoint representation, then the weights of $\mathfrak{h}$ in $\mathfrak{g}$ are the roots of Φ and 0 (the weight space associated with 0 is $\mathfrak{h}$; its dimension is ℓ).
2) If $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{k})$ and $\mathfrak{h} = \mathbb{k}h$, a linear form λ on $\mathfrak{h}$ is fully determined by $\lambda(h)$, and we have already seen the importance of $\lambda(h)$ in the theory of representations of $\mathfrak{sl}_2(\mathbb{k})$.

When V is infinite-dimensional, it is not guaranteed that V is a direct sum of its weight spaces. However, just as in finite dimensions, one can show that if $\lambda_1, \dots, \lambda_r$ are pairwise distinct weights, then the sum

$$\sum_{i=1}^{r} V_{\lambda_i}$$

is direct. We define

$$V' = \bigoplus_{\lambda, V_\lambda \neq 0} V_\lambda.$$

EXERCISE 10.3. Show that V' is a submodule of V.

Hint: observe that $\mathfrak{g}_\alpha(V_\lambda) \subset V_{\alpha+\lambda}$ for any $\alpha \in \Phi$.

Definition 10.1

Let $v \in V$. We say that v is a ***maximal vector*** or a ***primitive vector*** if there exists a weight λ of V such that $v \in V_\lambda \setminus \{0\}$ and $\mathfrak{g}_\alpha \cdot v = 0$ for all $\alpha \in \Phi^+$ (or equivalently, for all $\alpha \in \Delta$).

For example, if $\mathfrak{g}$ is simple, then every nonzero element of $\mathfrak{g}_\theta$ is maximal for the adjoint representation, where θ is the highest positive root of Φ.

If V is finite-dimensional, a maximal vector always exists. Indeed, by applying Lie's theorem to the Borel subalgebra $\mathfrak{b}^+$ we obtain a common eigenvector for all elements of $\mathfrak{b}^+$, which is annihilated by all elements of $\bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$, since $\bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ is the derived algebra of $\mathfrak{b}^+$. This vector is therefore a maximal vector in the previous sense.

If V is infinite-dimensional, the existence of a maximal vector is not guaranteed.

Fix $x_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ for $\alpha \in \Phi^+$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ so that $(x_\alpha, h_\alpha, y_\alpha)$ forms an $\mathfrak{sl}_2$-triple. Define a partial order on $\mathfrak{h}^*$ by setting

$$\lambda \succ \mu \quad \text{if} \quad \lambda - \mu \in \sum_{j=1}^{m} \mathbb{N}\beta_j.$$

Since V is a representation of $\mathfrak{g}$, by the universal property of the enveloping algebra, there exists a Lie algebra homomorphism $U(\mathfrak{g}) \to \operatorname{End}(V)$ that makes V a $U(\mathfrak{g})$-module.

Theorem 10.2

Suppose $V = U(\mathfrak{g})v^+$, where v^+ is a maximal vector of weight λ. Then:

(i) V is generated by the vectors $y_{\beta_1}^{i_1} \dots y_{\beta_m}^{i_m} v^+$, $i_j \in \mathbb{N}$; in particular, V is a direct sum of its weight spaces,

(ii) the weights of V are of the form $\mu = \lambda - \sum_{i=1}^{\ell} k_i \alpha_i$, where $k_i \in \mathbb{N}$; in other words, all weights μ of V satisfy $\mu \prec \lambda$.

(iii) For any $\mu \in \mathfrak{h}^*$, V_μ is finite-dimensional and $\dim V_\lambda = 1$,

(iv) V has a unique maximal proper submodule.

EXERCISE 10.4. Prove this theorem using the PBW theorem (Corollaries 5.5 and 5.7).

Definition 10.3 – standard cyclic representation/highest weight representation

If $V = U(\mathfrak{g})v^+$, with v^+ a maximal vector of weight λ, then the weight λ is called the ***highest weight*** of V, and we say that V is a ***cyclic representation*** of highest weight λ, or simply that V is a ***highest weight representation***.

Corollary 10.4

If V is a cyclic highest weight representation, its highest weight is unique. Moreover, if $V = U(\mathfrak{g})v^+$, where v^+ is a maximal vector of weight λ, is irreducible, then v^+ is the unique maximal vector of V, up to nonzero multiples.

⚠ Saying that w is a maximal vector does not mean that $V = U(\mathfrak{g})w$ (it is the case if V is irreducible of course).

EXERCISE 10.5. Prove this corollary using (ii) and (iii) of Theorem 10.2.

We now wish to show that for any $\lambda \in \mathfrak{h}^*$, there exists a unique irreducible cyclic standard representation of highest weight λ (up to isomorphism). The following theorem addresses the uniqueness problem:

Theorem 10.5

Let $\lambda \in \mathfrak{h}^*$. Let V and W be two irreducible cyclic representations of highest weight λ, i.e., $V = U(\mathfrak{g})v^+$ and $W = U(\mathfrak{g})w^+$, where v^+ and w^+ are maximal vectors in V and W, respectively, of highest weight λ. Then V and W are isomorphic.

EXERCISE 10.6. Prove the above theorem.

Hint: consider the module $X = V \times W$, observe that $x^+ = (v^+, w^+)$ is a maximal vector of highest weight λ for X, consider the projections $p\colon Y \to V$, $p'\colon Y \to W$, where $Y = U(\mathfrak{g})x^+$, and observe that Y has a unique irreducible quotient by Theorem 10.2, (iv).

We now present two constructions that allow us to solve the problem of the existence of a standard cyclic representation of highest weight λ.

1) By induction. Let $\lambda \in \mathfrak{h}^*$ and $\Bbbk_\lambda \cong \Bbbk$ be a one-dimensional vector space with basis 1. We define an action of $\mathfrak{b}^+$ on $\Bbbk_\lambda$ by setting

$$(h + \sum_{\alpha \succ 0} x_\alpha).1 = \lambda(h), \quad h \in \mathfrak{h}.$$

It is easy to verify that this makes $\Bbbk_\lambda$ a representation of $\mathfrak{b}^+$, and thus $\Bbbk_\lambda$ is a $U(\mathfrak{b}^+)$-module.

On the other hand, $\mathfrak{b}$ acts in $U(\mathfrak{g})$ by right multiplication. We can then define [1]:

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \Bbbk_\lambda.$$

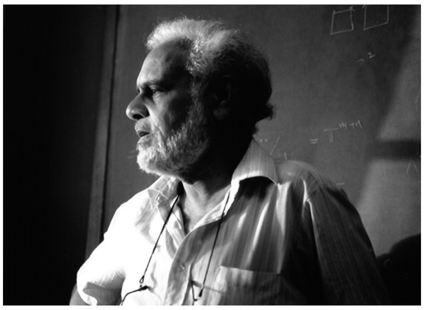
EXERCISE 10.7. Let $v^+ = 1 \otimes 1$.

(1) Verify that v^+ generates $M(\lambda)$, that v^+ is nonzero, and that v^+ is of weight λ.

(2) Show that $M(\lambda)$ is isomorphic, as a $U(\mathfrak{n}^-)$-module, to $U(\mathfrak{n}^-).v^+$.

By Exercise 10.7, $M(\lambda)$ is a cyclic module of highest weight λ. T he module $M(\lambda)$ is called a ***Verma module***. It is sometimes denoted $\mathrm{Ind}_{\mathfrak{b}^+}^{\mathfrak{g}}(\Bbbk_\lambda)$.

__Daya-Nand Verma__, born on June 25, 1933, in Varanasi and passed away on June 10, 2012, in Mumbai, was an Indian mathematician at the Tata Institute of Fundamental Research during the period 1968–1993. The construction of Verma modules appears in his thesis, which he completed under the supervision of Nathan Jacobson at Yale University.



2) By generators and relations Let $I(\lambda)$ be the left ideal of $U(\mathfrak{g})$ generated by the elements x_α for $\alpha \in \Phi^+$, and $h_\alpha - \lambda(h_\alpha).1$ for $\alpha \in \Phi$, that is,

$$I(\lambda) = \sum_{\alpha \in \Phi^+} U(\mathfrak{g})x_\alpha + \sum_{\alpha \in \Phi} U(\mathfrak{g})(h_\alpha - \lambda(h_\alpha).1).$$

The generators of $I(\lambda)$ annihilate the maximal vector v^+ of $M(\lambda)$. Thus, $I(\lambda)$ also annihilates v^+. We deduce that there exists a morphism of left $U(\mathfrak{g})$-modules

$$U(\mathfrak{g})/I(\lambda) \to M(\lambda)$$

which sends $1 + I(\lambda)$ to v^+.

EXERCISE 10.8. Using the PBW Theorem 5.3, show that the morphism $U(\mathfrak{g})/I(\mathfrak{g}) \to M(\lambda)$ is an isomorphism.

Theorem 10.6

Let $\lambda \in \mathfrak{h}^*$. There exists an irreducible cyclic representation of highest weight λ.

PROOF. Let $M(\lambda)$ be as above. Then $M(\lambda)$ is a cyclic module by Exercise 10.7. Therefore, it has a unique proper submodule $N(\lambda)$ by Theorem 10.2, (iv) . The quotient

$$V(\lambda) = M(\lambda)/N(\lambda)$$

is then irreducible and is a cyclic module of highest weight λ. □

By combining Theorems 10.5 and 10.6, we obtain that there exists a unique irreducible cyclic representation (up to isomorphism) of highest weight λ. We denote it by $V(\lambda)$.

Two natural questions arise, which we will address in the following sections:

1) For which $\lambda \in \mathfrak{h}^*$ is the representation $V(\lambda)$ finite-dimensional?

2) For such a λ, what are the weights μ of $V(\lambda)$, and what are their ***multiplicities***, i.e., $\dim V(\lambda)_\mu$?

1. By the definition of the tensor product $U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \Bbbk_\lambda$, we have for all $u \in U(\mathfrak{g})$, $t \in \Bbbk_\lambda$, and $b \in B$, $ub \otimes t = u \otimes b.t$

11

Finite-dimensional representations and multiplicity

Suppose that V is an irreducible finite-dimensional representation of $\mathfrak{g}$. Since V is finite-dimensional, it has a maximal vector v^+. Since V is irreducible, we have $V = U(\mathfrak{g}).v^+$ and thus $V \cong V(\lambda)$, where $\lambda \in \mathfrak{h}^*$, by Theorem 10.2.

For $i \in \{1, \dots, \ell\}$, let S_i be the subspace spanned by $x_i = x_{\alpha_i}$, $h_i = h_{\alpha_i}$, and $y_i = x_{-\alpha_i}$ so that $S_i \cong \mathfrak{sl}_2(\mathbb{k})$ as a Lie algebra. Let $i \in \{1, \dots, \ell\}$. The $\mathfrak{g}$-module $V(\lambda)$ is an S_i-module, and every maximal vector of $V(\lambda)$ for $\mathfrak{g}$ is also maximal for S_i. It follows from the theory of $\mathfrak{sl}_2(\mathbb{k})$-modules that $\lambda(h_i) \in \mathbb{N}$. More generally, if μ is a weight of $V(\lambda)$, then $\mu(h_i) \in \mathbb{Z}$.

11.1. Integral weights and dominant weights

Let Λ be the set of $\lambda \in \mathfrak{h}^*$ such that

$$\lambda(h_i) = \langle \lambda, h_i \rangle \in \mathbb{Z}$$

for any $i \in \{1, \dots, \ell\}$. This is a lattice of E with a basis $\varpi_1, \dots, \varpi_\ell$, where $(\varpi_1, \dots, \varpi_\ell)$ is the dual basis of $(h_1, \dots, h_\ell)$, i.e.,

$$\varpi_i(h_j) = \delta_{i,j}, \qquad i, j \in \{1, \dots, \ell\}.$$

Recall that E is the $\mathbb{R}$-vector space spanned by the roots of Φ. It is the ambient Euclidean space of the root system Φ (see the discussion preceding Theorem 8.7). The lattice Λ is called the ***weight lattice***. It contains the ***root lattice***

$$\Lambda_r := \sum_{i=1}^{\ell} \mathbb{Z}\alpha_i.$$

The elements $\varpi_1, \dots, \varpi_\ell$ are called the ***fundamental weights*** associated to $\alpha_1, \dots, \alpha_\ell$, respectively. An element $\lambda \in \Lambda$ is called an ***integral weight***. We will simply refer to ***weights*** for the elements of $\mathfrak{h}^*$. If $\lambda(h_i) \geqslant 0$ for all $i \in \{1, \dots, \ell\}$, we say that λ is a ***dominant weight***. Thus, when $\lambda(h_i) \in \mathbb{N}$ for any $i \in \{1, \dots, \ell\}$, we say that λ is a ***dominant integral weight***. We denote by Λ^+ the set of dominant integral weights. Thus Λ^+ is the set of integral weights belonging to the closure of the dominant Weyl chamber $\mathscr{C}(\Delta)$.

EXERCISE 11.1. Represent the fundamental weight for the basis α of a root system of type A_1. Represent the fundamental weights for a root systems of type A_2, B_2, G_2 associated with the basis $\{\alpha, \beta\}$ of Figure 1.

EXERCISE 11.2. Recall that ρ denotes the half-sum of the positive roots. Show that:

$$\rho = \sum_{i=1}^{\ell} \varpi_1.$$

Hint: notice that it is enough to show that $\langle \rho, \alpha_i \rangle = 1$ for any $i \in \{1, \dots, \ell\}$, and calculate $(s_{\alpha_i}(\rho)|\alpha_i)$ to do so, using the question 2 of Exercise 9.7 and that the inner product $(\ |\ )$ is $\mathcal{W}$-invariant.

REMARK 11.1. By definition, the transpose of the Cartan matrix $(\langle\alpha_i, \alpha_j\rangle)_{1\leq i,j\leq \ell}$ is the transition matrix of the basis $\{\alpha_1, \dots, \alpha_\ell\}$ written in the basis $\{\varpi_1, \dots, \varpi_\ell\}$. Thus, one can compute the fundamental weights in the Δ-basis by inverting the transpose of the Cartan matrix. The lattice Λ_r has finite index in Λ, and the cardinality of the quotient Λ/Λ_r (called the ***fundamental group*** of Φ) is given by the determinant of the Cartan matrix if Φ is irreducible. The fundamental group of each simple root system is described in Figure 3.

Type	A_ℓ	B_ℓ	C_ℓ	D_ℓ (ℓ even)	D_ℓ (ℓ odd)	E_6	E_7	E_8	F_4	G_2
Λ/Λ_r	$\mathbb{Z}/(\ell+1)\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	0	0

FIGURE 3. Fundamental group of the irreducible root systems

EXERCISE 11.3.

(1) Show the following relations in $U(\mathfrak{g})$ for $k \geqslant 0$ and $i, j \in \{1, \dots, \ell\}$:
   (a) $[x_j, y_i^{k+1}] = 0$ if $i \neq j$,
   (b) $[h_j, y_i^{k+1}] = -(k+1)\alpha_i(h_j)y_i^{k+1}$,
   (c) $[x_i, y_i^{k+1}] = -(k+1)y_i^k(k.1 - h_i)$,

(2) Let $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta$. Suppose that $n := \langle\lambda, \alpha\rangle \in \mathbb{N}$. Show that if v^+ is a highest weight vector of $V(\lambda)$ (of weight λ), then
$$y_\alpha^{n+1}.v^+ = 0.$$

From the discussion at the beginning of this section, we obtain the "only if" part of the following theorem:

Theorem 11.1

The simple module $V(\lambda)$ is finite-dimensional if and only if $\lambda \in \Lambda^+$. This is the case if and only if
$$\dim V(\lambda)_\mu = \dim V(\lambda)_{w\mu} \quad \text{for all} \quad \mu \in \mathfrak{h}^*,\ w \in \mathcal{W},$$
where $\mathcal{W} = \mathcal{W}(\Phi)$ is the Weyl group of the root system Φ.

REMARK 11.2. We know this theorem for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{k})$ by Exercise 3.8.

PROOF. For the first part of the theorem, we need to show that if $\lambda \in \Lambda^+$, then $V(\lambda)$ is finite-dimensional (the other direction has been already seen). The idea is as follows: for any $\mu \in \mathfrak{h}^*$, $V(\lambda)_\mu$ is finite-dimensional by Theorem 10.2, (iii). We will therefore show that $V(\lambda)$ has a finite number of weights. This will be enough to conclude, since $V(\lambda)$ is a direct sum of its weight spaces by Theorem 10.2, (i).

To do so, we again consider the structure of $V(\lambda)$ as an S_i-module for each $i \in \{1, \dots, \ell\}$. Here are the steps of the proof.

1) Let $i \in \{1, \dots, \ell\}$. Since $n = \langle\lambda, \alpha_i\rangle \in \mathbb{N}$ by assumption, the sub-S_i-module generated by v^+ is finite-dimensional (this follows from Exercise 11.3), where v^+ is the highest weight vector of $V(\lambda)$.

2) For each $i \in \{1, \dots, \ell\}$, we show that $V(\lambda)$ is the sum of all its sub-S_i-modules of finite dimension.

   Let us denote this sum by M_i. By Step 1), M_i is non-zero. Let N be a finite-dimensional sub-S_i-module of $V(\lambda)$. Then $N \subset M_i$ and $L \otimes N$ is an S_i-module of finite dimension (here S_i operates in $\mathfrak{g}$ by the adjoint action). The map $L \otimes N \to V(\lambda)$ which to $x \otimes v$ associates $x.v$ is a morphism of S_i-modules. Its image is therefore contained in M_i. Thus M_i is a non-zero sub-$\mathfrak{g}$-module of $V(\lambda)$ since M_i is the sum of such N. As $V(\lambda)$ is irreducible, we deduce that $V(\lambda) = M_i$.

3) By Step 2), every vector $v \in V(\lambda)$ belongs to a sub-S_i-module of finite dimension. Therefore, x_i and y_i are ***locally nilpotent*** operators in $V(\lambda)$, that is, for any $v \in V(\lambda)$ there exists a subspace finite-dimensional vector V' of $V(\lambda)$ in which x_i and y_i operate nilpotently. Denote by
$$\sigma\colon U(\mathfrak{g}) \to \operatorname{End} V(\lambda)$$
the morphism induced by the representation $V(\lambda)$. The operators
$$\exp\sigma(x_i) \circ \exp\sigma(-y_i) \circ \exp\sigma(x_i)$$

are therefore well-defined locally, since they operate in all sub-S_i-modules of finite dimension. We thus define an automorphism r_i of $V(\lambda)$.

4) If μ is a weight of $V(\lambda)$, then $r_i(V(\lambda)_\mu) = V(\lambda)_{s_i\mu}$ for any i. Indeed, the weight subspace $V(\lambda)_\mu$ belongs to a sub-S_i-module N of finite dimension by 2) and because $V(\lambda)_\mu$ is of finite dimension. The assertion then results from the theory $\mathfrak{sl}_2$-modules (see Exercise 8.8). Notice as a by-product that $\mu(h_i) \in \mathbb{Z}$ according to the theory $\mathfrak{sl}_2$-modules.
5) Since $\mathcal{W}$ is generated by the simple reflections s_i, Step 4) implies that all weight spaces $V(\lambda)_{w\mu}$, $w \in \mathcal{W}$, have the same dimension. Since λ is a dominant integer weight by hypothesis, all weights of $V(\lambda)$ are therefore $\mathcal{W}$-conjugates of integer dominant weights $\mu \prec \lambda$. But there are only a finite number of such weights. It follows that $V(\lambda)$ has a finite number of weights, and therefore is of finite dimension.

Step 5) and the discussion at the beginning of the proof proves the second assertion. □

Suppose that $V(\lambda)$ has finite dimension, and let $\alpha \in \Phi$. By considering $V(\lambda)$ as an S_α-module, we see that if μ and $\mu + k\alpha$, with $k \in \mathbb{Z}$, are weights of $V(\lambda)$, then all the intermediate weights $\mu + i\alpha$ are also weights of $V(\lambda)$: we thus obtain the ***α-weight chain passing through*** μ. This generalizes the notion of the α-root chain passing through a given root.

This observation, combined with the previous theorem, implies the following result:

Proposition 11.2

If $\lambda \in \Lambda^+$, then a necessary and sufficient condition for $\mu \in \Lambda$ to be a weight of $V(\lambda)$ is that μ and all of its $\mathcal{W}$-conjugates are $\prec \lambda$.

EXERCISE 11.4. Represent the weights of $V(\lambda)$ for $\mathfrak{g}$ of type A_2 and $\lambda = 4\varpi_1 + 3\varpi_2$, and for $\mathfrak{g}$ of type B_2 and $\lambda = \varpi_1 + \varpi_2$.

Let us make a small digression on the Weyl group in view of Exercise 11.5 below.

Let $w \in W$. Set $\ell(w) = n$ if $w = s_1 \dots s_n$ where the s_i are simple reflections, i.e., $s_i = s_\alpha$ for $\alpha \in \Delta$, and n is minimal for this property. Such an expression of w is called a ***reduced*** expression. The integer $\ell(w)$ is called the ***length*** of w. Here are some properties of the length of elements in W:

(1) The number of $\alpha \in \Phi^+$ such that $w\alpha < 0$ is $\ell(w)$. In particular, if $\alpha \in \Delta$, i.e., $\ell(s_\alpha) = 1$, then we have $s_\alpha\beta > 0$ for any $\beta \neq \alpha \in \Phi^+$. Moreover, w is fully determined by the set of $\alpha \in \Phi^+$ such that $w\alpha < 0$.
(2) If $w \in W$, then $\ell(w) = \ell(w^{-1})$. Thus, $\ell(w) = |\Phi^+ \cap w(\Phi^-)|$.
(3) There exists a unique element $w_0 \in W$ of maximal length $|\Phi^+|$; it sends Φ^+ to $-\Phi^+$. Moreover, $\ell(w_0 w) = \ell(w_0) - \ell(w)$ for any $w \in W$.
(4) If $\alpha > 0$ and $w \in W$ satisfy $\ell(ws_\alpha) > \ell(w)$, then $w\alpha > 0$, while $\ell(ws_\alpha) < \ell(w)$ implies $ws_\alpha < 0$. It follows that $\ell(s_\alpha w) > \ell(w) \iff w^{-1}\alpha > 0$.

EXERCISE 11.5. Let $\lambda \in \Lambda_+$. Show that the dual representation $V(\lambda)^*$, given by the action

$$(x.f)(v) = -f(x.v)$$

for $x \in \mathfrak{g}$, $f \in V(\lambda)^*$ and $v \in V(\lambda)$, is isomorphic to $V(-w_0\lambda)$, where $w_0 \in W$ is the element of maximal length.

Hint: observe that $V(\lambda)^*$ is simple, that its weights relative to $\mathfrak{h}$ are the opposites of those of $V(\lambda)$, and that $w_0\lambda$ is the smallest weight for the order $\prec$ in $V(\lambda)$.

11.2. The Casimir element revisited

Recall that the Casimir element was defined in Section 7.1. For the adjoint representation, its trace is by definition the Killing form. Choose a basis of $\mathfrak{g}$ adapted to the triangular decomposition,

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Let $\{k_1, \dots, k_\ell\}$ be the dual basis of the basis $\{h_1, \dots, h_\ell\}$ with respect to the restriction of $\kappa_{\mathfrak{g}}$ to $\mathfrak{h}$. Next, let $x_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$, and let z_α be the unique element of $\mathfrak{g}_{-\alpha}$ such that $\kappa_{\mathfrak{g}}(x_\alpha, z_\alpha) = 1$.

Notice that $z_\alpha \neq y_\alpha$, where $y_\alpha \in \mathfrak{g}_{-\alpha}$ is the unique element of $\mathfrak{g}_{-\alpha}$ such that $[x_\alpha, y_\alpha] = h_\alpha$. We have (see Proposition 8.5)

$$[x_\alpha, z_\alpha] = t_\alpha = \frac{(\alpha|\alpha)}{2} h_\alpha.$$

By definition, the Casimir element associated with the adjoint representation is

$$c_{\mathrm{ad}} = \sum_{i=1}^{\ell} \operatorname{ad} h_i \operatorname{ad} k_i + \sum_{\alpha \in \Phi} \operatorname{ad} x_\alpha \operatorname{ad} z_\alpha.$$

This construction suggests to consider the following element of the enveloping algebra $U(\mathfrak{g})$:

$$c_{\mathfrak{g}} = \sum_{i=1}^{\ell} h_i k_i + \sum_{\alpha \in \Phi} x_\alpha z_\alpha \in U(\mathfrak{g}).$$

Extend the adjoint representation ad to a unique morphism, still denoted ad,

$$\operatorname{ad} \colon U(\mathfrak{g}) \to \mathfrak{gl}(\mathfrak{g}).$$

Then $\operatorname{ad} c_{\mathfrak{g}}$ is exactly c_{ad}. For this reason, the element $c_{\mathfrak{g}}$ is called the ***universal Casimir element***. The arguments seen in Exercise 7.1 show that if (V, σ) is a representation of $\mathfrak{g}$, then $\sigma(c_{\mathfrak{g}})$ commutes with $\sigma(\mathfrak{g})$, and thus acts by scalar on V if V is irreducible (Schur's Lemma3.2).

EXERCISE 11.6. Compute the Casimir element $c_{\mathfrak{g}}$ for $L = \mathfrak{sl}_2(\mathbb{k})$ relative to the standard basis $\{e, h, f\}$. Verify that $c_{\mathfrak{g}}$ belongs to the ***center*** of $U(\mathfrak{g})$, i.e.,

$$u c_{\mathfrak{g}} = c_{\mathfrak{g}} u, \quad \text{for any} \quad u \in U(\mathfrak{g}).$$

Assume now that $\mathfrak{g}$ is simple. Let β_σ be the bilinear form associated with σ. (see Section 7.1). Recall that this form is non-degenerate and invariant. It is therefore proportional to the Killing form (see Exercise 7.2):

$$\beta_\sigma = a \kappa_{\mathfrak{g}}, \quad \text{where} \quad a \in \mathbb{k}^*.$$

In particular, the dual basis relative to $\kappa_{\mathfrak{g}}$ is obtained by multiplying the one relative to β_σ by a. Then we have

$$\sigma(c_{\mathfrak{g}}) = a c_\sigma.$$

In particular, $\sigma(c_{\mathfrak{g}}) \neq 0$.

If $\mathfrak{g}$ is semisimple, then $\sigma(c_{\mathfrak{g}})$ is not necessarily proportional to c_σ, but if $I_1, \dots, I_t$ are the simple ideals of $\mathfrak{g}$, then $\sigma(c_{I_j})$ is proportional to the restriction to I_j of c_σ, for $j = 1, \dots, t$. In particular, we recover that $\sigma(c_{\mathfrak{g}})$ commutes with $\sigma(\mathfrak{g})$. Furthermore, $\sigma(c_{\mathfrak{g}})$ acts by a non-zero scalar in V if V is irreducible.

11.3. Freudenthal's Formula

Let $\lambda \in \Lambda^+$ so that $V = V(\lambda)$ is finite-dimensional. For $\mu \in \mathfrak{h}^*$, we define the ***multiplicity*** of μ in V as the integer

$$m_\lambda(\mu) = \dim V_\mu.$$

We have $m_\lambda(\mu) = 0$ if μ is not a weight of $V(\lambda)$. We will simply denote $m(\mu)$ for $m_\lambda(\mu)$ when λ is fixed.

Let $\sigma \colon \mathfrak{g} \to \mathfrak{gl}(V)$ be the underlying representation of V, and let μ be a weight of V. We aim to compute $m(\mu)$. The element $\sigma(c_{\mathfrak{g}})$, where $c_{\mathfrak{g}}$ is the universal Casimir element, acts by scalar on V because V is irreducible; let c be this scalar. The idea of the Freudenthal formula below (Theorem 11.5) is to observe that we have:

$$\operatorname{Tr}_{V_\mu} \sigma(c_{\mathfrak{g}}) = c\, m(\mu).$$

We therefore aim to express $\operatorname{Tr}_{V_\mu} \sigma(c_{\mathfrak{g}})$ and c in terms of the root system Φ and the weight μ.

EXERCISE 11.7. Show that the Casimir element $c_{\mathfrak{g}}$ acts on $V(\lambda)$ by the scalar

$$c = (\lambda|\lambda + 2\rho) = (\lambda + \rho|\lambda + \rho) - (\rho|\rho),$$

where ρ is the half-sum of the positive roots:

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

Lemma 11.3

Let $i \in \{1, \dots, \ell\}$ and h_i, k_i as in the previous section. We have

$$\mathrm{Tr}_{V_\mu} \sum_{i=1}^{\ell} \sigma(h_i)\sigma(k_i) = m(\mu)(\mu|\mu).$$

EXERCISE 11.8. Prove this lemma.

Lemma 11.4

Let $\alpha \in \Phi$, and $x_\alpha, z_\alpha, t_\alpha$ as in the previous section. We have

$$\mathrm{Tr}_{V_\mu} \sigma(x_\alpha)\sigma(z_\alpha) = \sum_{i=0}^{\infty} m(\mu + i\alpha)(\mu + i\alpha|\alpha).$$

EXERCISE 11.9. Prove this lemma.

Hint: the idea is, as usual, to view V as an S_α-module; the calculations are quite technical.

Theorem 11.5 – Freudenthal's Formula

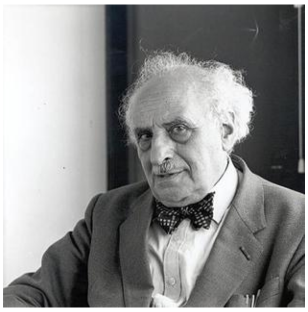
Let $V(\lambda)$ be a simple finite-dimensional module with highest weight $\lambda \in \Lambda^+$. If $\mu \in \Lambda$, then the multiplicity $m_\lambda(\mu)$ of μ in $V(\lambda)$ is given recursively as follows:

$$((\lambda + \rho|\lambda + \rho) - (\mu + \rho|\mu + \rho))\, m_\lambda(\mu) = 2 \sum_{\alpha \in \Phi^+} \sum_{i=1}^{\infty} m_\lambda(\mu + i\alpha)(\mu + i\alpha|\alpha).$$

EXERCISE 11.10. Prove the theorem.

Later we will see Weyl's formula (Theorem 13.8) which also gives the ***multiplicities*** of $V(\lambda)$. Weyl's formula is remarkable from a theoretical point of view. Moreover, it has some analogues for infinite-dimensional Lie algebras.

__Hans Freudenthal__, born on September 17, 1905, and passed away on October 13, 1990, was a German-Jewish mathematician, naturalized Dutch, specializing in algebraic topology but whose contributions extended far beyond this field.



The Freudenthal formula above is very useful in practice because it provides an effective method for calculating the multiplicities. It is used to implement certain computer programs. The computation softwares `LIE`, or `GAP4` allows such calculations: https://www.science.unitn.it/~degraaf/sla.html

For example, with $\mathfrak{g}$ of type A_2, and $\lambda = \varpi_1 + 3\varpi_2$, we obtain the following weights:

μ	$m_\lambda(\mu)$	μ	$m_\lambda(\mu)$
$\varpi_1 + 3\varpi_2$	1	$2\varpi_1 - 2\varpi_2$	2
$-\varpi_1 + 4\varpi_2$	1	$3\varpi_1 - 4\varpi_2$	1
$2\varpi_1 + \varpi_2$	1	$-\varpi_2$	2
$2\varpi_2$	2	$-3\varpi_1 + 2\varpi_2$	1
$3\varpi_1 - \varpi_2$	1	$\varpi_1 - 3\varpi_2$	1
ϖ_1	2	$-2\varpi_1$	2
$-2\varpi_1 + 3\varpi_2$	1	$-\varpi_1 - 2\varpi_2$	1
$4\varpi_1 - 3\varpi_2$	1	$-4\varpi_1 + \varpi_2$	1
$-\varpi_1 + \varpi_2$	2	$-3\varpi_1 - \varpi_2$	1

TABLE 1. Weights for $V(\lambda)$, with $\lambda = \varpi_1 + 3\varpi_2$, and $\mathfrak{g} \cong \mathfrak{sl}_3(\mathbb{k})$ of type A_2

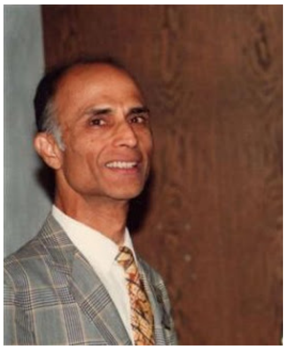
Part 3

Harish-Chandra projection, nilpotent cone and applications

In this part, we state several remarkable about the algebra of invariant polynomial functions (under the action of the adjoint group) on a semisimple Lie algebra (cf. Chapter 12). These results allow, on the one hand, to prove the Weyl formula for the characters of finite-dimensional representations (cf. Chapter 13) and, on the other hand, to study some properties of the nilpotent cone of a semisimple Lie algebra, i.e., the set of its nilpotent elements. We will then establish a link between the highest weight representations of a semisimple Lie algebra and the nilpotent cone (cf. Chapter 14).

Throughout this part, we assume that $\mathfrak{g}$ is a semisimple Lie algebra over $\Bbbk$, where $\Bbbk$ is an algebraically closed field of characteristic zero, with adjoint group $G = G_{ad}$ (see Definition 8.10). We keep all the notation from the previous chapters. In particular, a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is fixed, Φ denotes the root system of $(\mathfrak{g}, \mathfrak{h})$, $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ is a basis of Φ, and $\Phi^+ = \{\beta_1, \ldots, \beta_m\}$ is the associated set of positive roots.

Harish Chandra*, born Harish Chandra Mehrotra on October 11, 1923 in Kanpur (Uttar Pradesh), India, and died on October 16, 1983 in Princeton, USA, was an Indian mathematician who made fundamental contributions to the theory of representations, especially in harmonic analysis of semisimple Lie groups.*



12

Invariant polynomials

Let V be a finite-dimensional vector space of dimension $n \in \mathbb{N}^*$. Let $\Bbbk[V]$ denote the algebra of polynomial functions on V. Recall that there is a canonical isomorphism $S(V^*) \cong \Bbbk[V]$. If we fix a basis $(e_1, \ldots, e_n)$ of V, then $\Bbbk[V] \cong \Bbbk[e_1^*, \ldots, e_n^*]$, where $(e_1^*, \ldots, e_n^*)$ is the dual basis of $(e_1, \ldots, e_n)$. In particular, we have

$$\Bbbk[\mathfrak{h}] \cong \Bbbk[\varpi_1, \ldots, \varpi_\ell],$$

since $(\varpi_1, \ldots, \varpi_\ell)$ is the dual basis of the basis $(h_1, \ldots, h_\ell)$ of the Cartan subalgebra $\mathfrak{h}$.

12.1. Chevalley projection

Recall that the Weyl group $\mathcal{W} = \mathcal{W}(\Phi)$ acts on $\mathfrak{h}$ and $\mathfrak{h}^*$, and thus on $\Bbbk[\mathfrak{h}] = S(\mathfrak{h}^*)$. We denote by $\Bbbk[\mathfrak{h}]^{\mathcal{W}}$, or $S(\mathfrak{h}^*)^{\mathcal{W}}$, the algebra of ***invariant polynomial functions*** under the action of the Weyl group.

EXERCISE 12.1. Describe $\Bbbk[\mathfrak{h}]^{\mathcal{W}}$ for $\mathfrak{g} = \mathfrak{sl}_2(\Bbbk)$.

Recall that that the adjoint group $G = G_{ad}$ equals $\mathrm{Aut}_e(\mathfrak{g})$, the group of elementary automorphisms. The group G acts on $\Bbbk[\mathfrak{g}]$ as follows: for $\sigma \in G$, $f \in \Bbbk[\mathfrak{g}]$, and $x \in \mathfrak{g}$, we define

$$(\sigma.f)(x) = f(\sigma^{-1}x).$$

We denote by $\Bbbk[\mathfrak{g}]^G$, or $S(\mathfrak{g}^*)^G$, the set of fixed points of $\Bbbk[\mathfrak{g}]$ for this action. It is a subalgebra of $\Bbbk[\mathfrak{g}]$ called the ***algebra of G-invariant polynomials***.

REMARK 12.1. Because $G = G_{ad}$ is connected (remember that $G_{ad} = (\mathrm{Aut}\,\mathfrak{g})^\circ$), we have

$$S(\mathfrak{g}^*)^G = S(\mathfrak{g}^*)^{\mathfrak{g}} = \{x \in S(\mathfrak{g}^*) \colon (\mathrm{ad}^* x)s = s \text{ for all } s \in S(\mathfrak{g}^*)\}.$$

Here the action of $\mathrm{ad}^* x$ on $S(\mathfrak{g}^*)$ is defined on homogeneous elements by:

$$(\mathrm{ad}^* x)\xi_1 \ldots \xi_d = \sum_{i=1}^{d} \xi_1 \ldots \xi_{i-1} \left((\mathrm{ad}^* x)\xi_i\right) \xi_{i+1} \ldots \xi_d.$$

The goal of the following exercise is to produce a large number of G-invariant polynomials using representation theory.

EXERCISE 12.2. Let $\phi \colon \mathfrak{g} \to \mathfrak{gl}(V)$ be an irreducible representation of finite dimension. Let $\lambda \in \Lambda^+$ be its highest weight. Let $z \in \mathfrak{n}^+$ and set

$$\sigma = \exp(\mathrm{ad}\, z).$$

(1) Denote by ϕ^σ the representation of $\mathfrak{g}$ defined by $\phi^\sigma(x) = \phi(\sigma x)$ for any $x \in \mathfrak{g}$. Verify that ϕ^σ thus defined is indeed a representation of $\mathfrak{g}$ and that ϕ^σ is irreducible.

(2) What is the highest weight of ϕ^σ? Deduce that ϕ and ϕ^σ are equivalent.

(3) Let $k \in \mathbb{N}^*$. Show that the map $x \mapsto \mathrm{Tr}(\phi(x)^k)$ is a polynomial function invariant under σ.

(4) Conclude that $x \mapsto \mathrm{Tr}(\phi(x)^k)$ is a G-invariant polynomial function, i.e., an element of $\Bbbk[\mathfrak{g}]^G$.

We will now compare $\Bbbk[\mathfrak{g}]^G$ and $\Bbbk[\mathfrak{h}]^{\mathcal{W}}$. If $f \in \Bbbk[\mathfrak{g}]$, then its restriction to $\mathfrak{h}$ is an element of $\Bbbk[\mathfrak{h}]$ (this needs to be verified). Suppose further that $f \in \Bbbk[\mathfrak{g}]^G$. Then f is in particular invariant under the action of all automorphisms τ_α, $\alpha \in \Phi$ (defined after Proposition 9.10). But the restriction of τ_α to $\mathfrak{h}$ acts as the reflection s_α. Since these generate $\mathcal{W}$, we deduce that $f|_{\mathfrak{h}}$ belongs to $\Bbbk[\mathfrak{h}]^{\mathcal{W}}$. Thus, we obtain an algebra homomorphism, called the ***Chevalley restriction***,

$$\theta\colon \Bbbk[\mathfrak{g}]^G \to \Bbbk[\mathfrak{h}]^{\mathcal{W}}.$$

Theorem 12.1 – Chevalley

The map θ is an algebra isomorphism.

For the moment, we will only need of the surjectivity of θ. However, the injectivity can also be established using quite elementary arguments from algebraic geometry, as we will see later.

EXERCISE 12.3 (Proof of the surjectivity). The goal of this exercise is to establish the surjectivity of the map θ.

(1) Show that the elements λ^k, where $k \in \mathbb{N}$ and $\lambda \in \Lambda$, generate the algebra $\Bbbk[\mathfrak{h}]$.

(2) For $k \in \mathbb{N}$ and $\lambda \in \Lambda^+$, define

$$\operatorname{Sym} \lambda^k = \sum_{w \in \mathcal{W}} w(\lambda^k).$$

Show that the elements $\operatorname{Sym} \lambda^k$, where $k \in \mathbb{N}$ and $\lambda \in \Lambda^+$, generate $\Bbbk[\mathfrak{h}]^{\mathcal{W}}$.

(3) Let $\lambda \in \Lambda^+$ be minimal with respect to the partial order $\prec$ (possibly 0). Show that $\operatorname{Sym} \lambda^k$ belongs to the image of θ.

> Hint: consider the map $f\colon x \mapsto \operatorname{Tr}(\phi(x)^k)$ where $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is an irreducible representation (of finite dimension) with highest weight λ.

(4) By induction on the partial order $\prec$ on Λ^+, show that θ is surjective.

> Hint: again, consider the map $f\colon x \mapsto \operatorname{Tr}(\phi(x)^k)$ where $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is an irreducible representation (of finite dimension) with highest weight $\lambda \in \Lambda^+$.

The proof of the surjectivity given in the previous exercise shows that the algebra A generated by the *trace functions*, i.e., the functions like those obtained in Exercise 12.2, maps surjectively to $\Bbbk[\mathfrak{g}]^{\mathcal{W}}$. This is used to establish the injectivity of the restriction of θ to A, as shown in the next exercise. This is a weaker result than the injectivity of θ, but it can be obtained in an elementary way.

EXERCISE 12.4.

(1) Show that a trace function $x \mapsto \operatorname{Tr}(\phi(x)^k)$, where ϕ is an irreducible representation of finite dimension of $\mathfrak{g}$, is completely determined by its value on the set of semisimple elements.

(2) Using the fact that the Cartan subalgebras are G-conjugate (that we admit here), show that the restriction of θ to A is injective.

EXERCISE 12.5. Describe the algebra $\Bbbk[\mathfrak{h}]^{\mathcal{W}}$ for $L = \mathfrak{sl}_n(\Bbbk)$. Deduce that $\Bbbk[\mathfrak{g}]^G$ is an algebra of polynomials with $n-1$ generators.

The observation from Exercise 12.5 is general: if $\mathfrak{g}$ is a simple Lie algebra of rank ℓ, then $\Bbbk[\mathfrak{g}]^G \cong \Bbbk[\mathfrak{h}]^{\mathcal{W}}$ is an algebra of polynomials with ℓ generators. This is a famous result of Chevalley–Kostant (which we admit here). Let $d_1, \dots, d_\ell$ be the degrees of homogeneous generators of $\Bbbk[\mathfrak{g}]^G$, so that $d_1 \leqslant \cdots \leqslant d_\ell$. We have

$$d_1 \times \cdots \times d_\ell = |\mathcal{W}|.$$

For $i \in \{1, \ldots, \ell\}$, let $m_i = d_i - 1$. The elements $m_1, \ldots, m_\ell$ only depend on the simple Lie algebra $\mathfrak{g}$ and are called the ***exponents*** of $\mathfrak{g}$. We have

$$\sum_{i=1}^{\ell} m_i = |\Phi^+|.$$

EXAMPLE 12.1. Suppose that $\mathfrak{g} = \mathfrak{sl}_n(\Bbbk)$. The Weyl group identifies with the symmetric group $\mathfrak{S}_n$ of order n, and we have (see Exercise 12.5):

$$\Bbbk[\mathfrak{h}]^{\mathcal{W}} \cong (\Bbbk[X_1, \ldots, X_n]/(X_1 + \cdots + X_n))^{\mathfrak{S}_n} \cong \Bbbk[\Sigma_2, \ldots, \Sigma_n],$$

where for $k \in \{1, \ldots, n\}$, Σ_k is the elementary symmetric polynomial of degree k. If $\Bbbk = \mathbb{C}$, we also have

$$\Bbbk[\mathfrak{h}]^{\mathcal{W}} \cong \Bbbk[T_2, \ldots, T_n],$$

where for $k \in \{1, \ldots, n\}$, T_k is the Newton polynomial of degree k:

$$T_k = \sum_{i=1}^{n} X_i^k.$$

The sequence of degrees of the generators is $2, 3, \ldots, n$. Thus the exponents are $1, 2, \ldots, n-1$.

EXAMPLE 12.2. Suppose that $\mathfrak{g}$ is the exceptional Lie algebra of type G_2. Then the sequence of degrees of homogeneous generators of $\Bbbk[\mathfrak{g}]^G$ is $2, 6$. Thus the exponents are $1, 5$. We can choose generators as follows. It is known that there is a faithful irreducible representation $(\sigma, \Bbbk^7)$ of G_2 (7 is the minimal dimension of irreducible representations of G_2). Then the trace functions $x \mapsto \mathrm{Tr}(\sigma(x)^2)$ and $x \mapsto \mathrm{Tr}(\sigma(x)^6)$ generate $\Bbbk[\mathfrak{g}]^G$.

PROOF OF THE INJECTIVITY OF θ. For $x \in \mathfrak{g}$, let

$$p_x(X) = \sum_{i=0}^{n} c_i(x) X^i$$

be the characteristic polynomial of $\operatorname{ad} x$. Each function c_i is a polynomial function on $\mathfrak{g}$. Let m be the smallest integer such that c_m is not identically zero on $\mathfrak{g}$. Since 0 is always an eigenvalue of $\operatorname{ad} x$ for $x \in \mathfrak{g}$, we have $m \geqslant 1$, and since there are non-nilpotent elements, we even have $m > 1$ (in fact, $\operatorname{ad} x$ is nilpotent if and only if $c_i(x) = 0$ for $i = 1, \ldots, n$). Let

$$\mathscr{R} = \{x \in \mathfrak{g} \colon c_m(x) \neq 0\}.$$

We have $x \in \mathscr{R}$ if and only if the eigenvalue 0 of $\operatorname{ad} x$ has the smallest possible multiplicity. This shows that $x \in \mathscr{R}$ if and only if $x_s \in \mathscr{R}$. In particular, there exist semisimple elements in $\mathscr{R}$.

Clearly, the set $\mathscr{R}$ is Zariski-open in $\mathfrak{g}$. Moreover, it is non-empty. So the set $\mathscr{R}$ is Zariski-dense in $\mathfrak{g}$.

Let $x \in \mathfrak{g}$ be a semisimple element. Then it belongs to a maximal toral subalgebra of $\mathfrak{g}$, that is, a Cartan subalgebra of $\mathfrak{g}$. Since Cartan subalgebras are all G-conjugate, x is G-conjugate to an element of $\mathfrak{h}$. But if $h \in \mathfrak{h}$, we know that its centralizer $c_{\mathfrak{g}}(h)$ in $\mathfrak{g}$ has dimension $\geqslant \ell$. We also know (admitted) that $\mathfrak{h}$ contains *regular* elements, that is, elements whose centralizer has dimension exactly ℓ. Since there are semisimple elements in $\mathscr{R}$, these are exactly the regular semisimple elements (so $m = \ell$). But there is no nilpotent element other than 0 that centralizes a regular semisimple element. Given a previous remark, we deduce that if $x \in \mathscr{R}$, then $x = x_s$. Thus we have shown that the set $\mathscr{R}$ coincides with the set of regular semisimple elements of $\mathfrak{g}$.

We are now in a position to show that the morphism θ is injective. Let $f \in \Bbbk[\mathfrak{g}]^G$ such that $\theta(f) = f|_{\mathfrak{h}} = 0$. The Cartan subalgebra $\mathfrak{h}$ and its G-conjugates contain all the regular semisimple elements of $\mathfrak{g}$. Therefore, f vanishes on the set $\mathscr{R}$, which is Zariski-dense in $\mathfrak{g}$, so f is identically zero on $\mathfrak{g}$. □

12.2. Harish-Chandra Projection

Let $Z(\mathfrak{g})$ be the ***center*** of the enveloping algebra $U(\mathfrak{g})$, i.e., the set of $z \in U(\mathfrak{g})$ such that $zu = uz$ for any $u \in U(\mathfrak{g})$.

REMARK 12.2. The adjoint representation of the Lie algebra $\mathfrak{g}$ induces a representation in the tensor algebra $T(\mathfrak{g})$, and this action passes to the quotient $U(\mathfrak{g}) = T(\mathfrak{g})/J$. We denote by $\operatorname{ad} \colon \mathfrak{g} \to \mathfrak{gl}(U(\mathfrak{g}))$ this representation. For $x \in \mathfrak{g}$ and $y = y_1 \ldots y_n \in U(\mathfrak{g})$, we have:

$$(\operatorname{ad} x)u = \sum_{i=1}^{n} y_1 \ldots y_{i-1}[x, y_i] y_{i+1} \ldots y_n.$$

It is easily checked that for $z \in U(\mathfrak{g})$, we have

$$z \in Z(\mathfrak{g}) \iff (\operatorname{ad} x)z = 0 \text{ for any } x \in \mathfrak{g}.$$

An automorphism $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ extends uniquely to an automorphism of $U(\mathfrak{g})$. In particular, the group G acts on $U(\mathfrak{g})$.

Proposition 12.2

The center $Z(\mathfrak{g})$ is the set of G-invariant elements of $U(\mathfrak{g})$.

EXERCISE 12.6. Prove the proposition.

EXERCISE 12.7. Verify that the universal Casimir element $c_\mathfrak{g}$ belongs to the center of $U(\mathfrak{g})$.

Hint: revisit the arguments from Exercise 7.1 without referring to the representation σ.

12.3. Central Characters

Let $M = U(\mathfrak{g})v^+$ be a cyclic module of highest weight λ, generated by a maximal vector v^+. If $z \in Z(\mathfrak{g})$ and $h \in \mathfrak{h}$, we have:

$$h.(z.v^+) = z.(h.v^+) = \lambda(h)(z.v^+).$$

Since $\dim M_\lambda = 1$, we deduce that $z.v^+ = \chi_\lambda(z)v^+$, where $\chi_\lambda(z) \in \Bbbk$. Moreover, for any $u \in U(\mathfrak{g})$, we have $z.(u.v^+) = \chi_\lambda(z)(u.v^+)$ because $zu = uz$. In other words, z acts by the scalar $\chi_\lambda(z)$ in M.

For a fixed $\lambda \in \mathfrak{h}^*$, the map

$$\chi_\lambda\colon Z(\mathfrak{g}) \to \Bbbk, \quad z \mapsto \chi_\lambda(z)$$

is an algebra homomorphism.

Definition 12.3 – Central character

Let $\lambda \in \mathfrak{h}^*$. The algebra homomorphism χ_λ is called the ***central character associated with*** λ. More generally, any algebra homomorphism $Z(\mathfrak{g}) \to \Bbbk$ is called a ***central character***.

We can describe χ_λ more concretely using the PBW Theorem 5.3. Consider the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Since $\mathfrak{n}^+ \cdot v^+ = 0$, $\mathfrak{h} \cdot v^+ \subset \Bbbk v^+$, and an element of $\mathfrak{n}^-$ sends v^+ to a linear combination of weight vectors $\prec \lambda$, we see that $z.v^+$, for $z \in Z(\mathfrak{g})$, depends only on the monomials in $\mathfrak{h}$. In other words,

$$\chi_\lambda(z) = \lambda(\mathrm{pr}(z)), \tag{19}$$

where $\mathrm{pr}\colon U(\mathfrak{g}) \to U(\mathfrak{h})$ is the projection of $U(\mathfrak{g})$ onto $U(\mathfrak{h})$ relative to the decomposition

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+).$$

Here, we have extended the linear form λ to an algebra homomorphism $\lambda\colon U(\mathfrak{h}) \to \Bbbk$. Note that $U(\mathfrak{h}) = S(\mathfrak{h})$ since $\mathfrak{h}$ is commutative.

EXERCISE 12.8. Using relation (19), show that the restriction to $Z(\mathfrak{g})$ of pr is an algebra homomorphism.

The map pr is not an algebra homomorphism in general!

Definition 12.4 – Harish-Chandra projection

The restriction to $Z(\mathfrak{g})$ of pr is called the ***Harish-Chandra morphism***. It is denoted

$$\xi \colon Z(\mathfrak{g}) \to S(\mathfrak{h}), \quad z \mapsto \mathrm{pr}(z).$$

Two very natural questions arise:

(1) is the algebra homomorphism ξ injective?

(2) what is its image?

We answer these two questions in the next section.

12.4. Twisted Action of the Weyl Group

In order to answer the two previous questions, it is important to understand under what conditions two weights λ and μ satisfy $\chi_\lambda = \chi_\mu$.

First, consider the case where $\lambda \in \Lambda$. Suppose there exists $\alpha \in \Delta$ such that $n = \langle \lambda, \alpha \rangle = \lambda(h_\alpha) \in \mathbb{N}$. Then, the Verma module $M(\lambda)$ has a maximal vector of highest weight $\lambda - (n+1)\alpha \prec \lambda$ (see the proof of Exercise 11.3). In this case, we have $\chi_\lambda = \chi_\mu$, where $\mu = \lambda - (n+1)\alpha$.

This can be rewritten as:

$$\begin{aligned} s_\alpha(\lambda + \rho) &= \lambda - \langle \lambda, \alpha \rangle \alpha + \rho - \alpha \\ &= \lambda - (n+1)\alpha + \rho \\ &= \mu + \rho. \end{aligned}$$

Indeed, $s_\alpha(\rho) = \rho - \alpha$ because α is a simple root (see Exercise 9.7).

This motivates the following definition.

Definition 12.5

We define a ***twisted action*** of the Weyl group $\mathcal{W}$ on $\mathfrak{h}^*$, denoted $\circ$, as follows: for all $w \in \mathcal{W}$ and $\lambda \in \mathfrak{h}^*$,

$$w \circ \lambda = w(\lambda + \rho) - \rho.$$

Let $\lambda, \mu \in \mathfrak{h}^*$. We say that λ and μ are ***$\mathcal{W}$-linked***, and we write $\lambda \sim \mu$, if $\lambda = w \circ \mu$ for some $w \in \mathcal{W}$.

The relation $\sim$ is clearly an equivalence relation. According to the definition, $\lambda \sim \mu$ if and only if $\lambda + \rho$ and $\mu + \rho$ are in the same $\mathcal{W}$-orbit (under the usual action of $\mathcal{W}$).

EXERCISE 12.9. Describe the orbit of $-\rho$. What condition on $\lambda \in \mathfrak{h}^*$ ensures that $|\mathcal{W} \circ \lambda| = |\mathcal{W}|$?

EXERCISE 12.10. Let $\lambda \in \Lambda$, and $\mu \in \mathfrak{h}^*$. Show that if λ and μ are $\mathcal{W}$-linked, then $\chi_\lambda = \chi_\mu$.

Hint: notice that it suffices to show that if $\lambda = s_\alpha \circ \mu$, for $\alpha \in \Delta$, then $\chi_\lambda = \chi_\mu$; fix $\alpha \in \Delta$, observe that the case where $n = \langle \lambda, \alpha \rangle \in \mathbb{N}$ has already been treated, verify that the case $n = -1$ is easy, and then reduce to known cases by considering $\mu = s_\alpha \circ \lambda$.

Proposition 12.6

Let $\lambda, \mu \in \mathfrak{h}^*$. If $\lambda \sim \mu$, then $\chi_\lambda = \chi_\mu$.

PROOF. We know that $\chi_\lambda = \chi_{w \circ \lambda}$ for all $w \in \mathcal{W}$ and $\lambda \in \Lambda$ by Exercise 12.10. Now, $\chi_\lambda(z) = \lambda(\xi(z))$ for any $z \in Z(\mathfrak{g})$. This implies that the polynomial functions $\xi(z)$ and $w^{-1} \circ \xi(z)$ coincide on Λ. Since this set is Zariski-dense in $\mathfrak{h}^*$, we conclude that they coincide on $\mathfrak{h}^*$. □

Consider the map

$$\tau_\rho \colon \Bbbk[\mathfrak{h}^*] \longrightarrow \Bbbk[\mathfrak{h}^*], \quad p \mapsto (\lambda \longmapsto p(\lambda - \rho)),$$

and let $\psi\colon Z(\mathfrak{g}) \to S(\mathfrak{h})$ be the composition $\tau_\rho \circ \xi$. The map ψ is called the ***twisted Harish-Chandra projection***. For any $z \in Z(\mathfrak{g})$ and $\lambda \in \mathfrak{h}^*$, we have:

$$\chi_\lambda(z) = (\lambda + \rho)(\psi(z)).$$

Proposition 12.7

The image of the twisted Harish-Chandra projection is contained in the algebra $S(\mathfrak{h})^{\mathcal{W}}$ of $\mathcal{W}$-invariant polynomial functions (for the usual action of $\mathcal{W}$).

EXERCISE 12.11.

(1) Prove this theorem.

(2) Suppose $\mathfrak{g} = \mathfrak{sl}_2(\Bbbk)$. Calculate the image of $\chi_\lambda(c)$ for $\lambda \in \mathfrak{h}^* \cong \Bbbk$, where c is the Casimir element. In this special case, show that $\chi_\lambda = \chi_\mu$, for $\lambda, \mu \in \Bbbk$, if and only if λ and μ are $\mathcal{W}$-linked, and explicitly describe this condition.

For the following, we will have the stronger result:

Theorem 12.8 – Harish-Chandra

(i) The morphism $\psi\colon Z(\mathfrak{g}) \to S(\mathfrak{h})^{\mathcal{W}}$ is an isomorphism.

(ii) For $\lambda, \mu \in \mathfrak{h}^*$, we have $\chi_\lambda = \chi_\mu$ if and only if $\lambda \sim \mu$.

SKETCH OF THE PROOF. We have two steps.

1) The idea is to compare the morphism ψ with the Chevalley morphism $\theta\colon \Bbbk[\mathfrak{g}]^G \to \Bbbk[\mathfrak{h}]^{\mathcal{W}}$. We can identify $\Bbbk[\mathfrak{g}]$ with $\Bbbk[\mathfrak{g}^*] \cong S(\mathfrak{g})$, and $\Bbbk[\mathfrak{h}]$ with $\Bbbk[\mathfrak{h}] \cong S(\mathfrak{h})$, using the Killing form. Chevalley's Theorem 2.2 ensures that the morphism $\theta\colon \Bbbk[\mathfrak{g}]^G \to \Bbbk[\mathfrak{h}]^{\mathcal{W}}$ is an isomorphism. Now, through the previous identifications, the latter "looks" very similar to the morphism ψ...

   Although the following diagram does not commute:

$$\begin{array}{ccccc} S(\mathfrak{g})^G & \xrightarrow{\text{sym}} & Z(\mathfrak{g}) & \xrightarrow{\psi} & S(\mathfrak{h})^{\mathcal{W}} \\ \downarrow\sim & & & & \downarrow\sim \\ \Bbbk[\mathfrak{g}]^G & & \xrightarrow[\theta]{} & & \Bbbk[\mathfrak{h}]^{\mathcal{W}} \end{array}$$

   (see Exercise 12.12 below), the comparison between gr, ψ and θ will suffice to show that ψ is an isomorphism. We can show that

$$\operatorname{gr}\psi\colon \operatorname{gr} Z(\mathfrak{g}) \longrightarrow \operatorname{gr} S(\mathfrak{h})^{\mathcal{W}} = S(\mathfrak{h})^{\mathcal{W}}$$

   is an isomorphism of vector spaces. we have $\operatorname{gr} Z(\mathfrak{g}) \cong S(\mathfrak{g})^G$ and, through the identifications $S(\mathfrak{g}) \cong \Bbbk[\mathfrak{g}]$ and $S(\mathfrak{h}) \cong \mathbb{C}[H]$, we check that $\operatorname{gr}\psi$ coincides with the isomorphism θ. Thus we conclude that ψ is an isomorphism.

2) Recall that $\chi_\lambda(z) = (\lambda+\rho)(\psi(z))$ for $\lambda \in \mathfrak{h}^*$ and $z \in Z(\mathfrak{g})$. Suppose $\chi_\lambda = \chi_\mu$, with $\lambda, \mu \in \mathfrak{h}^*$. Then $\lambda + \rho$ and $\mu + \rho$ coincide on $\psi(Z(\mathfrak{g}))$ which is equal to $S(\mathfrak{h})^{\mathcal{W}}$ by Step 1). The task is to deduce that $\lambda + \rho$ and $\mu + \rho$ are $\mathcal{W}$-conjugated. This follows from the below Assertion 12.1:

□

ASSERTION 12.1. If λ and μ belong to distinct $\mathcal{W}$-orbits, then λ and μ take distinct values on certain elements of $S(\mathfrak{h})^{\mathcal{W}}$.

PROOF. Since $\mathcal{W}$ is finite, we can choose a polynomial function p in $S(\mathfrak{h})$ which is equal to 1 at λ and vanishes at all other $\mathcal{W}$-conjugates of λ as well as at $\mathcal{W}.\mu$ (using Lagrange interpolation). Let us define

$$\tilde{p} = \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} w.p.$$

Then $\tilde{p}$ is an element of $S(\mathfrak{h})^{\mathcal{W}}$ which vanishes at μ but not at λ. □

EXERCISE 12.12. Illustrate with $\mathfrak{g} = \mathfrak{sl}_2(\Bbbk)$ that the following diagram does not commute:

$$\begin{array}{ccccc} S(\mathfrak{g})^G & \xrightarrow{\text{sym}} & Z(\mathfrak{g}) & \xrightarrow{\psi} & S(\mathfrak{h})^{\mathcal{W}} \\ \downarrow{\scriptstyle\sim} & & & & \downarrow{\scriptstyle\sim} \\ \Bbbk[\mathfrak{g}]^G & & \xrightarrow[\theta]{\hspace{4em}} & & \Bbbk[\mathfrak{h}]^{\mathcal{W}} \end{array}$$

where $\mathrm{sym}\colon S(\mathfrak{g})^G \to Z(\mathfrak{g})$ is the ***symmetrization map***,

$$x_1 \dots x_d \longmapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)} \dots x_{\sigma(d)}.$$

13

Weyl and Kostant character formulas

We start with an illustrating exercise. In the chapter we will generalize this example.

EXERCISE 13.1 (character for the finite-representations of $\mathfrak{sl}_2(\Bbbk)$). If V is a finite-representation representation of $\mathfrak{sl}_2(\Bbbk)$, recall that

$$V = \bigoplus_{m \in \mathbb{Z}} V(m), \qquad \text{where} \qquad V(m) := \{v \in V \colon h.v = mv\}.$$

Define the ***character*** of V by

$$\mathrm{ch}(V) = \sum_{m \in \mathbb{Z}} (\dim V(m)) q^m \in \mathbb{Z}[q, q^{-1}].$$

(1) Show that if V, W are two finite-dimensional representations of $\mathfrak{sl}_2(\Bbbk)$, then

$$\mathrm{ch}(V \otimes W) = \mathrm{ch}(V) \cdot \mathrm{ch}(W).$$

(2) Denote as usual by V_m the irreducible representation of $\mathfrak{sl}_2(\Bbbk)$ of highest weight m (that is, V_m is of dimension $m+1$). Show that

$$\mathrm{ch}(V_m) = \sum_{i=0}^{m} q^{m-2i}.$$

(3) Show that the vectors $\mathrm{ch}(V_m)$, for $m \in \mathbb{N}$, are linearly independent in the $\mathbb{Z}$-module $\mathbb{Z}[q, q^{-1}]$.

(4) Deduce that if V, W are two finite-dimensional representations of $\mathfrak{sl}_2(\Bbbk)$, then $V \cong W$ if and only if

$$\mathrm{ch}(V) = \mathrm{ch}(W).$$

(5) For $m, n \in \mathbb{N}$, with $m \leqslant n$, show that

$$V_m \otimes V_n \cong \bigoplus_{i=0}^{m} V_{m+n-2i}.$$

(Compare with Exercise 3.7.)

13.1. Formal characters

Let $\mathbb{Z}[\Lambda]$ be the $\mathbb{Z}$-free module of basis $\{e(\lambda)\}_{\lambda \in \Lambda}$ in bijective correspondence with the elements $\lambda \in \Lambda$. Thus, every element of $\mathbb{Z}[\Lambda]$ is a finite linear combination with coefficients in $\mathbb{Z}$ of elements $e(\lambda)$, $\lambda \in \Lambda$, and the writing is unique. We define a ring structure on $\mathbb{Z}[\Lambda]$ by setting

$$e(\lambda)\, e(\mu) = e(\lambda + \mu), \qquad 1 = e(0).$$

Definition 13.1

Let $\lambda \in \Lambda^+$. The ***formal character*** of $V(\lambda)$, which we denote by $\operatorname{ch} V(\lambda)$, is the element of $\mathbb{Z}[\Lambda]$ defined by:

$$\operatorname{ch} V(\lambda) = \sum_{\mu\in\Lambda} m_\lambda(\mu)e(\mu),$$

with the usual convention that $m_\lambda(\mu) = 0$ if μ is not a weight of $V(\lambda)$.

EXAMPLE 13.1. If $\mathfrak{g} = \mathfrak{sl}_2(\Bbbk)$, and if $\lambda \in \Lambda^+ \cong \mathbb{N}$, show that $\operatorname{ch} V(\lambda) = e(\lambda) + e(\lambda - \alpha) + \cdots + e(\lambda - m\alpha)$, where $m = \langle \lambda, \alpha^\vee \rangle$.

Thanks to Weyl's Theorem, every $\mathfrak{g}$-module V of finite dimension is a sum direct of simple submodules, $V = \bigoplus_{i=1}^n V_i$, and each V_i is isomorphic to a simple module $V(\lambda_i)$ for some $\lambda_i \in \Lambda^+$. The element $\operatorname{ch} V$ defined by

$$\operatorname{ch} V = \sum_{i=1}^n V(\lambda_i)$$

is called the ***formal character*** of V.

EXERCISE 13.2. Let V, W be two $\mathfrak{g}$-modules of finite dimension. Show:

$$\operatorname{ch}(V \otimes W) = \operatorname{ch}(V) \cdot \operatorname{ch}(W).$$

We would like to extend the notion of formal character to modules possibly of infinite dimension for which the multiplicities $\dim V_\mu$, $\mu \in \mathfrak{h}^*$ are well defined (for example, cyclic modules). To do this, we will use a slightly different formalism.

We can see $\mathbb{Z}[\Lambda]$ as the set of functions defined on Λ with values in $\mathbb{Z}$, and zero outside a finite set. The product becomes the convolution product:

$$(f * g)(\lambda) = \sum_{\mu+\nu=\lambda} f(\mu)g(\nu), \qquad f, g \in \mathbb{Z}[\Lambda],$$

and $e(\lambda)$ identifies with the function $e^\lambda \colon \Lambda \to \mathbb{Z}$ which is 1 on λ and 0 for any $\mu \in \Lambda$, $\mu \neq \lambda$. We will again denote by $\operatorname{ch} V$ the formal character associated with a $\mathfrak{g}$-module of finite dimension, seen as a function on Λ. In particular,

$$\operatorname{ch} V(\lambda) = \sum_{\mu\in\Lambda} m_\lambda(\mu)e^\mu,$$

so that

$$(\operatorname{ch} V(\lambda))(\mu) = m_\lambda(\mu), \qquad \mu \in \Lambda.$$

Let $\mathfrak{X}$ be the space of functions defined on $\mathfrak{h}^*$ with values in $\Bbbk$ whose ***support***,

$$\operatorname{supp}(f) := \{\lambda \in \mathfrak{h}^* \colon f(\lambda) \neq 0\},$$

is contained in a finite union of sets of the form $\lambda - \Gamma$, where $\lambda \in \mathfrak{h}^*$ and

$$\Gamma = \sum_{\alpha\in\Phi^+} \mathbb{N}\alpha.$$

We notice that $\mathfrak{X}$ is stable by convolution. This makes it a commutative associative $\Bbbk$-algebra, whose identity is e^0. We can extend the definition of e^λ, $\lambda \in \Lambda$, to $\lambda \in \mathfrak{h}^*$ by denoting e^λ the function from $\mathfrak{h}^*$ to $\Bbbk$ which is 1 on λ and 0 for any $\mu \in \mathfrak{h}^*$, $\mu \neq \lambda$. Thus, any element of $\mathfrak{X}$ is written as a formal linear combination of elements e^λ.

Let $\lambda \in \mathfrak{h}^*$. We define the ***formal character*** of the cyclic module $M(\lambda)$ by:

$$\operatorname{ch} M(\lambda) = \sum_{\mu\in\mathfrak{h}^*} (\dim M(\lambda)_\mu)e^\mu.$$

It is an element of $\mathfrak{X}$ according to Theorem 10.2.

The Weyl group $\mathcal{W}$ acts on $\mathfrak{X}$ by:

$$(wf)(\lambda) = f(w^{-1}\lambda), \qquad w \in \mathcal{W},\ f \in \mathfrak{X},\ \lambda \in \mathfrak{h}^*.$$

By Theorem 11.1, the element $\operatorname{ch} V(\lambda)$ is $\mathcal{W}$-invariant for $\lambda \in \Lambda^+$.

Here are some special elements of $\mathfrak{X}$ that will be very useful to us.

Definition 13.2 – Kostant function

For $\lambda \in \mathfrak{h}^*$, we denote by $p(\lambda)$ the number of sets of nonnegative integers $\{k_\alpha \in \mathbb{N} \colon \alpha \in \Phi^+\}$ for which $-\lambda = \sum_{\alpha \in \Phi^+} k_\alpha \alpha$. It is an element of $\mathfrak{X}$ that we call the ***Kostant function***.

We notice that $p(\lambda) \neq 0$ if and only if λ belongs to the lattice $\Lambda_r = \sum_{i=1}^{\ell} \mathbb{Z}\alpha_i$. We have $\operatorname{ch} M(0) = p$ by Theorem 10.2.

Bertram Kostant, *Bertram Kostant (May 24, 1928 – February 2, 2017) was an American mathematician who worked in representation theory, differential geometry, and mathematical physics. Kostant grew up in New York City. He earned his Ph.D. from the University of Chicago in 1954, under the direction of Irving Segal, where he wrote a dissertation on representations of Lie groups.*

Definition 13.3 – Weyl function

Set
$$q = \prod_{\alpha \succ 0} (e^{\alpha/2} - e^{-\alpha/2}),$$
the product being the convolution product. It is an element of $\mathfrak{X}$ that we call the ***Weyl function***.

EXERCISE 13.3. Let $\alpha \in \Phi^+$. We define a new element of $\mathfrak{X}$ by setting:
$$f_\alpha(-k\alpha) = 1, \text{ for } k \in \mathbb{N}, \qquad f_\alpha(\lambda) = 0, \text{ for } \lambda \in \mathfrak{h}^* \setminus (-\mathbb{N}\alpha).$$
We have
$$f_\alpha = e^0 + e^{-\alpha} + e^{-2\alpha} + \cdots .$$
Prove the following assertions:

(a) $p = \prod_{\alpha \succ 0} f_\alpha$,
(b) $(e^0 - e^{-\alpha}) * f_\alpha = e^0$,
(c) $q = \prod_{\alpha \succ 0} (e^0 - e^{-\alpha}) * e^\rho$,
(d) $wq = (-1)^{\ell(w)} q$, for any $w \in \mathcal{W}$,
(e) $q * p * e^{-\rho} = e^0$,
(f) $\operatorname{ch} M(\lambda)(\mu) = p(\mu - \lambda) = (p * e^\lambda)(\mu)$, for $\mu \in \mathfrak{h}^*$,
(g) $q * \operatorname{ch} M(\lambda) = e^{\lambda+\rho}$.

REMARK 13.1. The sign $(-1)^{\ell(w)}$ that appears in Assertion (d) can be viewed as a signature for $w \in \mathcal{W}$ (it is often referred to as such). Indeed, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{k})$, the Weyl group of Φ is isomorphic to the symmetric group of order n, and $(-1)^{\ell(w)}$ corresponds to the usual sign of w.

13.2. Jordan–Hölder series and Verma modules

We will now intend to express $\operatorname{ch} V(\lambda)$, for $\lambda \in \mathfrak{h}^*$, as a linear combination with coefficients in $\mathbb{Z}$ of the $\operatorname{ch} M(\mu)$ using the Harish-Chandra Theorem 12.8.

For an infinite-dimensional $\mathfrak{g}$-module Weyl's theorem no longer applies. We cannot therefore a priori write its formal character (if it is well defined) like a sum of simple module characters. For certain classes of $\mathfrak{g}$-modules, we will achieve get around the problem thanks to the existence of a *Jordan–Hölder* sequence (see Definition 13.4).

Let $\mathfrak{M}_\lambda$ be the collection of $\mathfrak{g}$-modules M having the following properties, for fixed $\lambda \in \mathfrak{h}^*$:

(1) M is the direct sum of its weight spaces (relative to $\mathfrak{h}$),

(2) the center $Z(\mathfrak{g})$ operates in M by the scalar χ_λ, that is, for all $z \in Z(\mathfrak{g})$ and $m \in M$,

$$z.m = \chi_\lambda(z)m,$$

(3) the formal character of M belongs to $\mathfrak{X}$, where this is defined by:

$$\operatorname{ch} M = \sum_{\mu \in \mathfrak{h}^*} (\dim M_\mu) e^\mu.$$

(In particular, M_μ is of finite dimension for any $\mu \in \mathfrak{h}^*$.)

Of course, all cyclic modules of weight λ are in $\mathfrak{M}_\lambda$, as well as their submodules and their images by a morphism of $\mathfrak{g}$-modules; the direct sum of modules of $\mathfrak{M}_\lambda$ is also in $\mathfrak{M}_\lambda$.
According to the Harish-Chandra theorem, we have $\mathfrak{M}_\lambda = \mathfrak{M}_\mu$ if only if $\lambda \sim \mu$.

EXERCISE 13.4. Let $\lambda \in \mathfrak{h}^*$.

(1) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of elements of $\mathfrak{M}_\lambda$. Show that

$$\operatorname{ch} M = \operatorname{ch} M' + \operatorname{ch} M''.$$

(2) Let $M \in \mathfrak{M}_\lambda$ and N be a $\mathfrak{g}$-module of finite dimension. Show that $M \otimes N$ belongs to $\mathfrak{M}_\lambda$ and that

$$\operatorname{ch}(M \otimes N) = \operatorname{ch} M * \operatorname{ch} N.$$

Definition 13.4 – Jordan–Hölder series

Let M be a $\mathfrak{g}$-module. We say that M admits a ***Jordan–Hölder series*** if there exists a strictly decreasing sequence of $\mathfrak{g}$-modules,

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_r = \{0\},$$

not admitting any other strictly decreasing refinement than itself and such that for any $i \in \{1, \ldots, r\}$, M_i is a proper module of M_{i-1} with simple quotient M_{i-1}/M_i.

Otto Ludwig Hölder *(1859 – 1937) was a German mathematician born in Stuttgart. He first studied at the Polytechnikum (which today is the University of Stuttgart) and then in 1877 went to Berlin where he was a student of Leopold Kronecker, Karl Weierstrass, and Ernst Kummer. In 1877, he entered the University of Berlin and took his doctorate from the University of Tübingen in 1882, with title* Beiträge zur Potentialtheorie *(Contributions to potential theory). Following this, he went to the University of Leipzig but was unable to habilitate there, instead earning a second doctorate and habilitation at the University of Göttingen, both in 1884.*

EXERCISE 13.5. Let $M \in \mathfrak{M}_\lambda$ be non-zero. Show that M has a maximal vector.

Hint: notice using property (3) that for any weight μ of M and all $\alpha \in \Phi^+$, $\mu + k\alpha$ is no longer a weight of M for k large enough.

Proposition 13.5 – any Verma module admits a Jordan–Hölder series

Any Verma module admits a Jordan–Hölder series.

PROOF. Let $\lambda \in \mathfrak{h}^*$ and show that the Verma module $M(\lambda)$ admits a Jordan–Hölder series. If $M(\lambda) = V(\lambda)$, there is nothing to show. Otherwise, then $M(\lambda)$ admits a nonzero proper submodule N which belongs to $\mathfrak{M}_\lambda$.
Since $\dim M(\lambda)_\lambda = 1$, the weight λ does not appear as weight of N. By Exercise 13.5, N admits a maximal vector v of weight $\mu \precneqq \lambda$. In particular, $\chi_\lambda = \chi_\mu$ and therefore $\lambda \sim \mu$ according to the Harish-Chandra Theorem 12.8.

Moreover, N contains a submodule W, image of $M(\mu)$ (this can be seen easily by noticing that $M(\mu)$ is a cyclic generated by a vector of higher weight μ).

We now consider the $\mathfrak{g}$-modules $M(\lambda)/W$ and W. Both are cyclic, belong to $\mathfrak{M}_\lambda$, and have either strictly less weight linked to λ than $M(\lambda)$, or have the same weights as $M(\lambda)$ but with certain multiplicities strictly smaller than in $M(\lambda)$. By induction, we prove the result by applying the preceding arguments to $M(\lambda)/W$ and W. The process stops after a finite number of steps. To be convinced of this, set

$$V = \sum_{w \in \mathcal{W}} M(\lambda)_{w \circ \lambda}.$$

It is a finite-dimensional vector space. Since $\mu \sim \lambda$, we have $N \cap V \neq \{0\}$ and $W \cap V \neq \{0\}$. Furthermore, if $W' \subset W$ is a proper submodule of W', then by the same arguments as previously, $W' \cap V \neq \{0\}$ and $\dim(W' \cap V) < \dim(W \cap V)$ (because the weight μ is no longer a weight of W'). It follows that any chain of proper submodules in $M(\lambda)$ ends after a finite number of steps. □

It follows from the above proof that each successive quotient of the sequence is in $\mathfrak{M}_\lambda$. Therefore it has a maximum vector by Exercise 13.5. Each of these quotients is so cyclic since it is irreducible. Consequently, each successive quotient of the sequence is isomorphic to $V(\mu)$ for a certain $\mu \preccurlyeq \lambda$, $\mu \sim \lambda$. Additionally, $V(\lambda)$ only appears once in this sequence of quotients since $\dim M(\lambda)_\lambda = 1$.

Proposition 13.5 allows to write

$$\operatorname{ch} M(\lambda) = \operatorname{ch} V(\lambda) + \sum_{\substack{\mu \precnsim \lambda \\ \mu \sim \lambda}} d_\lambda(\mu) \operatorname{ch} V(\mu), \quad d_\lambda(\mu) \in \mathbb{N}.$$

Thus we can order, $\mu_1, \dots, \mu_t$, the sequence of elements $\mu \preccurlyeq \lambda$ such that $\mu \sim \lambda$ so that $\mu_i \preccurlyeq \mu_j$ implies $i \leqslant j$. In particular, $\lambda = \mu_t$. Then we obtain that the matrix $\big(d_{\mu_j}(\mu_i)\big)_{1 \leqslant i,j \leqslant t}$ is upper triangular with coefficients in $\mathbb{N}$ and with 1's on the diagonal. So this matrix is invertible, and we obtain the following corollary.

Corollary 13.6

Let $\lambda \in \mathfrak{h}^*$. Then $\operatorname{ch} V(\lambda)$ is a linear combination with coefficients in $\mathbb{Z}$ of the $\operatorname{ch} M(\mu)$:

$$\begin{aligned} \operatorname{ch} V(\lambda) &= \sum_{\substack{\mu \preccurlyeq \lambda \\ \mu \sim \lambda}} c_\lambda(\mu) \operatorname{ch} M(\mu), \quad c_\lambda(\mu) \in \mathbb{Z}, \\ &= \sum_{\substack{w \in \mathcal{W} \\ w \circ \lambda \preccurlyeq \lambda}} \tilde{c}_\lambda(w) \operatorname{ch} M(w \circ \lambda), \quad \tilde{c}_\lambda(w) \in \mathbb{Z}, \end{aligned}$$

with $c_\lambda(\lambda) = \tilde{c}_\lambda(1) = 1$.

13.3. Formulas et applications

We now apply Corollary 13.6 to the case where $V(\lambda)$ has finite dimension, that is to say $\lambda \in \Lambda^+$.

EXAMPLE 13.2. Let $\mathfrak{g} = \mathfrak{sl}_2(\Bbbk)$. Then $\Lambda^+ \cong \mathbb{N}$, $|\mathcal{W}| = 2$ and we obtain:

$$\operatorname{ch} V(\lambda) = \operatorname{ch} M(\lambda) - \operatorname{ch} M(-\lambda - 2),$$

since $s_\alpha \circ \lambda = -\lambda - 2$.

In general, calculating the coefficients $c_\lambda(\mu)$ or $\tilde{c}_\lambda(w)$ is a tricky problem!

By Exercise 13.3 (g), we have

$$q * \operatorname{ch} V(\lambda) = \sum_{\substack{\mu \prec \lambda \\ \mu \sim \lambda}} c_\lambda(\mu) e^{\mu + \rho},$$

and by Exercise 13.3 (d), for $w \in \mathcal{W}$,

$$w(q * \operatorname{ch} V(\lambda)) = w(q) * w(\operatorname{ch} V(\lambda)) = (-1)^{\ell(w)} q * \operatorname{ch} V(\lambda) = \sum_{\substack{\mu \prec \lambda \\ \mu \sim \lambda}} c_\lambda(\mu)(-1)^{\ell(w)} e^{\mu + \rho}, \tag{20}$$

since $w(\operatorname{ch} V(\lambda)) = \operatorname{ch} V(\lambda)$ for any $w \in \mathcal{W}$ as we have already observed it. On the other hand,

$$w\left(\sum_{\substack{\mu\prec\lambda\\ \mu\sim\lambda}} c_\lambda(\mu)e^{\mu+\rho}\right) = \sum_{\substack{\mu\prec\lambda\\ \mu\sim\lambda}} c_\lambda(\mu)e^{w^{-1}(\mu+\rho)}. \tag{21}$$

Since $\mathcal{W}$ transitively permutes the elements $\mu+\rho$ (the $\mu+\rho$ are $\mathcal{W}$-conjugated to $\lambda+\rho$) and that $c_\lambda(\lambda) = 1$, the identities (20) and (21) yields

$$c_\lambda(\mu) = (-1)^{\ell(w)} \quad \text{if} \quad w^{-1}(\mu+\rho) = \lambda+\rho.$$

Consequently,

$$q * \operatorname{ch} V(\lambda) = \sum_{w\in\mathcal{W}} (-1)^{\ell(w)} e^{w(\lambda+\rho)}. \tag{22}$$

Finally, Exercise 13.3 (e) gives

$$\begin{aligned}\operatorname{ch} V(\lambda) = q * p * e^{-\rho} * \operatorname{ch} V(\lambda) &= p * e^{-\rho} * \left(\sum_{w\in\mathcal{W}} (-1)^{\ell(w)} e^{w(\lambda+\rho)}\right)\\ &= p * \left(\sum_{w\in\mathcal{W}} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}\right) = \sum_{w\in\mathcal{W}} (-1)^{\ell(w)} p * e^{w(\lambda+\rho)-\rho}.\end{aligned}$$

Combining with Exercise 13.3 (f), we have obtained:

Theorem 13.7 – Kostant

If $\lambda \in \Lambda^+$, then

$$m_\lambda(\mu) = (\operatorname{ch} V(\lambda))(\mu) = \sum_{w\in\mathcal{W}} (-1)^{\ell(w)} p(\mu+\rho-w(\lambda+\rho)).$$

This theorem has the advantage of giving a direct formula for the multiplicities of $V(\lambda)$. However, Freudenthal's recursive formula is often more simple in practice. In fact, summing over the elements of the Weyl group can become extremely tedious when the rank of $\mathfrak{g}$ increases.

Theorem 13.8 – Weyl's character formula

(i) We have the formula

$$q = \sum_{w\in\mathcal{W}} (-1)^{\ell(w)} e^{w(\rho)},$$

or, equivalently,

$$\prod_{\alpha\in\Phi^+} (1-e^{-\alpha}) = \sum_{w\in\mathcal{W}} (-1)^{\ell(w)} e^{w(\rho)-\rho}.$$

(ii) For $\lambda \in \Lambda^+$, we have:

$$\left(\sum_{w\in\mathcal{W}} (-1)^{\ell(w)} e^{w(\rho)}\right) * \operatorname{ch} V(\lambda) = \sum_{w\in\mathcal{W}} (-1)^{\ell(w)} e^{w(\lambda+\rho)}.$$

or, equivalently,

$$\operatorname{ch} V(\lambda) = \frac{\sum_{w\in\mathcal{W}} (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\sum_{w\in\mathcal{W}} (-1)^{\ell(w)} e^{w(\rho)}.}.$$

EXERCISE 13.6. Proof Part (i) of the theorem using (22) applied to $\lambda = 0$, and then Part (ii) using (22) and Part (i).

We now wish to obtain a formula for the dimension of $V(\lambda)$. The idea is that this is obtained by adding the $m_\lambda(\mu)$. In the previous formalism, this amounts to adding all the values of $\operatorname{ch} V(\lambda)$, seen as an element of $\mathfrak{X}$. We introduce the subring $\mathfrak{X}_0$ of $\mathfrak{X}$ generated by the e^μ, $\mu \in \Lambda$ (identified with the ring $\mathbb{Z}[\Lambda]$). The morphism $v\colon \mathfrak{X}_0 \to \mathbb{Z}$ which to $f \in \mathfrak{X}_0$ associates the sum of all its values, i.e.,

$$v(f) = \sum_{\mu \in \Lambda} f(\mu) \in \mathbb{Z}, \qquad f \in \mathfrak{X}_0,$$

is well defined.

⚠ If we apply the morphism v in Weyl's Theorem 13.8, we run into a problem because the numerator and denominator give 0.

EXAMPLE 13.3. Consider again $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{k})$. We have

$$\operatorname{ch} V(\lambda) = e^\lambda + e^{\lambda-2} + \cdots + e^{-\lambda} = \frac{e^{\lambda+1} - e^{-\lambda-1}}{e^1 - e^{-1}}.$$

Recall that $\rho = 1$ via the isomorphism $\mathfrak{h}^* \to \mathbb{k}$, $\mu \mapsto \mu(h_\alpha)$. Thus $\dim V(\lambda) = \lambda + 1$, but if we apply the morphism v to the numerator and to the denominator of the fraction we get $\frac{0}{0}$.

This is essentially the same problem as evaluating $t = 1$ in the rational fraction $\dfrac{t^{\lambda+1} - t^{-\lambda-1}}{t - t^{-1}}$. It is an elementary analysis exercise; we perform Taylor expansion in $t = 1$, we simplify by $t - 1$, then we evaluate in $t = 1$.

Using this idea we will prove the next theorem.

Theorem 13.9 – Weyl's dimension formula

If $\lambda \in \Lambda^+$, then

$$\dim V(\lambda) = \frac{\prod_{\alpha \succ 0} \langle \lambda + \rho, \alpha \rangle}{\prod_{\alpha \succ 0} \langle \rho, \alpha \rangle}.$$

PROOF. Here are the steps of the proof.

(1) For each $\alpha \succ 0$, define an operator $\partial_\alpha \colon \mathfrak{X}_0 \to \mathfrak{X}_0$ extending by linearity the formula $\partial_\alpha e^\mu = \langle \mu, \alpha \rangle e^\mu$, $\mu \in \Lambda$.

We easily verify that ∂_α is a derivation of $\mathfrak{X}_0$, that is, for all $f, g \in \mathfrak{X}_0$,

$$\partial_\alpha (f * g) = \partial_\alpha(f) * g + f * \partial_\alpha(g).$$

The operators ∂_α, $\alpha \succ 0$ pairwise commute. Set

$$\partial = \prod_{\alpha \succ 0} \partial_\alpha.$$

(2) Recall that

$$q = e^{-\rho} * \prod_{\alpha \succ 0} (e^\alpha - 1) = \sum_{w \in \mathcal{W}} (-1)^{\ell(w)} e^{w\rho}$$

(see Theorem 13.8 and Exercise 13.3). We wish to compute $v(\partial(q * \operatorname{ch} V(\lambda)))$. We first use the first expression of q. We notice that $v(e^\alpha - 1) = 0$. Thus, by Leibniz's rule, we observe that the only term that survives is $v(\partial q) v(\operatorname{ch} V(\lambda))$.

In other words,

$$v(\partial(q * \operatorname{ch} V(\lambda))) = v(\partial q) \dim V(\lambda),$$

and so we have to compute $v(\partial q)$ and $v(\partial(q * \operatorname{ch} V(\lambda)))$.

(3) We now calculate $v(\partial q)$ using the second expression of q. By linearity, it suffices to calculate $v(\partial(e^{w\rho}))$. Since $\partial_\alpha e^\rho = \langle \rho, \alpha \rangle e^\rho$, we have

$$v(\partial e^\rho) = \prod_{\alpha \succ 0} \langle \rho, \alpha \rangle.$$

We similarly calculate $v(\partial e^{w\rho})$. Here we obtain

$$\prod_{\alpha \succ 0} \langle w\rho, \alpha \rangle = \prod_{\alpha \succ 0} \langle \rho, w^{-1}\alpha \rangle.$$

(4) The number of positive roots sent to negative roots by w^{-1} equals $\ell(w^{-1}) = \ell(w)$. Since $\langle \rho, -\alpha \rangle = -\langle \rho, \alpha \rangle$ for $\alpha \succ 0$, we can rewrite the previous step, and conclude that:

$$v(\partial q) = \sum_{w \in \mathcal{W}} (-1)^{\ell(w)} v(\partial e^{w\rho}) = \sum_{w \in \mathcal{W}} \prod_{\alpha \succ 0} \langle \rho, \alpha \rangle = |\mathcal{W}| \prod_{\alpha \succ 0} \langle \rho, \alpha \rangle.$$

(5) The same method applied to the numerator of Weyl's formula (ii) for $\operatorname{ch} V(\lambda)$ gives:

$$v\left(\partial \sum_{w \in \mathcal{W}} (-1)^{\ell(w)} e^{w(\lambda+\rho)}\right) = |\mathcal{W}| \prod_{\alpha \succ 0} \langle \lambda + \rho, \alpha \rangle.$$

The theorem deduces by simplifying by $|\mathcal{W}|$.

□

EXERCISE 13.7. Let $\alpha \in \Phi^+$. Write $\alpha^\vee = \sum_{i=1}^{\ell} c_i^{(\alpha)} \alpha_i^\vee$. Verify that

$$\langle \lambda + \rho, \alpha \rangle = \sum_{i=1}^{\ell} c_i^{(\alpha)} (m_i + 1),$$

if $\lambda = \sum_{i=1}^{\ell} m_i \varpi_i$.

EXAMPLE 13.4 (rank one). For the type A_1, we have $\varpi_1 = \dfrac{\alpha}{2} = \rho$ and the formula gives $\dim V(\lambda) = m + 1$ si $\lambda = m\varpi_1$.

EXAMPLE 13.5 (rank two). Suppose that Φ sis of rank two. Write $\lambda = m_1\varpi_1 + m_2\varpi_2$.

For the type A_2, the postive roots are $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$. The numerator of the Weyl's dimension formula gives $1 \times 1 \times 2$, while the denominator equals $(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)$. The computations are similar for B_2 et G_2. To summarize, we obtain (take here α_2 for the short simple root of B_2, and α_1 for the short simple root of G_2):

type	dimension de $V(\lambda)$, avec $\lambda = m_1\varpi_1 + m_2\varpi_2$
A_2	$\frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$
B_2	$\frac{1}{3!}(m_1+1)(m_2+1)(m_1+m_2+2)(2m_1+m_2+3)$
G_2	$\frac{1}{5!}(m_1+1)(m_2+1)(m_1+m_2+2)(m_1+2m_2+3)(m_1+3m_2+4)(2m_1+3m_2+5)$

The representations $V(\varpi_1), \dots, V(\varpi_\ell)$ associated with the fundamental weights are called the ***fundamental representations*** of $\mathfrak{g}$.

EXERCISE 13.8 (fundamental representations of $\mathfrak{sl}_4(\mathbb{k})$). Suppose in the exercise that $\mathfrak{g}$ is of type A_3.

(1) Compute the dimensions of the fundamental representations $V(\varpi_1)$, $V(\varpi_2)$, $V(\varpi_3)$ of $\mathfrak{g}$. Recognise these representations.

(2) Compute the dimension of $V(2\varpi_1)$ and recognise $V(2\varpi_1)$.

EXERCISE 13.9 (the Lie algebra G_2 as a representation of $\mathfrak{sl}_3$). Let $\mathfrak{g}_2$ be the Lie algebra of type G_2, and $\mathfrak{g}_0$ the Lie subalgebra of $\mathfrak{g}$ generated by the long root vectors.

(1) Show that $\mathfrak{g}_0 \cong \mathfrak{sl}_3(\mathbb{k})$.

(2) The Lie algebra $\mathfrak{g}_0$ acts on $\mathfrak{g}_2$ by the adjoint action so that $\mathfrak{g}_2$ is a representation of $\mathfrak{sl}_3(\mathbb{k})$. Show that $\mathfrak{g}_2$, as an $\mathfrak{sl}_3(\mathbb{k})$-representation, decomposes as:

$$\mathfrak{g}_2 \cong \mathfrak{sl}_3(\mathbb{k}) \oplus V \oplus V^*,$$

where V is the standard representation of $\mathfrak{sl}_3(\mathbb{k})$ and V^* its dual representation.

EXERCISE 13.10 (the Lie algebra G_2 as a subalgebra of $\mathfrak{so}_7$ and $\mathfrak{so}_8$.). Let $\mathfrak{g}_2$ be the Lie algebra of exceptional type G_2, and $\mathfrak{d}_4$ the classical Lie algebra of type D_4 with root system Φ. Recall that the group of automorphisms of the Dynkin diagram of Φ is defined by:

$$\Gamma = \{\sigma \in \mathrm{Aut}(\Phi) \colon \sigma(\Delta) = \Delta\}.$$

(1) Show that $\Gamma \cong \mathfrak{S}_3$.

For $\sigma \in \Gamma$, let $\tilde{\sigma}$ the automorphism of the Lie algebra $\mathfrak{d}_4$ induced by σ, and write $\tilde{\Gamma}$ the subgroup of $\mathrm{Aut}(\mathfrak{d}_4)$ generated by $\tilde{\Gamma}$.

(2) Show that the invariant Lie algebra $(\mathfrak{d}_4)^{\tilde{\Gamma}}$ is isomorphic to $\mathfrak{g}_2$.

(3) Let $\sigma \in \Gamma$ of order 2. Show that the invariant Lie algebra $(\mathfrak{d}_4)^{\langle\tilde{\sigma}\rangle}$ is isomorphic to a Lie algebra of type B_3, where $\langle\tilde{\sigma}\rangle$ is the subgroup of $\tilde{\Gamma}$ generated by $\tilde{\sigma}$.

(4) Deduce that the exceptional Lie algebra $\mathfrak{g}_2$ embeds into classical Lie algebras of type B_3 and D_4.

EXERCISE 13.11. Suppose in the exercise that $\mathfrak{g}$ is of type G_2. Compute the dimension of the fundamental representations $V(\varpi_1)$ and $V(\varpi_2)$. Recognise this two representations using Exercise 13.10.

14

Nilpotent cone

This chapter presents a more geometric aspect of the theory of representations of simple Lie algebras. We state some results without full proof. In this chapter, $\mathfrak{g}$ is a semisimple Lie algebra with adjoint group $G = G_{\mathrm{ad}}$.

14.1. Nilpotent elements and invariant polynomials

Recall that an element x of $\mathfrak{g}$ is nilpotent if $\operatorname{ad} x$ is a nilpotent element of $\operatorname{End}(\mathfrak{g})$. If $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$, where V is a finite-dimensional vector space, an element x of $\mathfrak{g}$ is nilpotent if and only if x is nilpotent as an endomorphism of $\mathfrak{g}$.

We denote by $\mathcal{N}$ the set of nilpotent elements of $\mathfrak{g}$. It is a *cone* in the sense that if $x \in \mathcal{N}$, then $tx \in \mathcal{N}$ for any $t \in \mathbb{k}^*$.

Definition 14.1 – Nilpotent Cone

The set $\mathcal{N}$ of nilpotent elements of $\mathfrak{g}$ is called the ***nilpotent cone*** of $\mathfrak{g}$.

Recall that if $\phi \in \operatorname{Aut}(\mathfrak{g})$, then

$$\phi \circ \operatorname{ad}(x) \circ \phi^{-1} = \operatorname{ad}(\phi(x)), \qquad x \in \mathcal{N}.$$

Therefore, if x is nilpotent, then $\phi(x)$ is also nilpotent for any $\phi \in \operatorname{Aut}(\mathfrak{g})$. In particular, the group G of inner automorphisms of $\mathfrak{g}$ acts on $\mathcal{N}$. The set

$$G.x = \{g(x) \colon g \in G\},$$

for $x \in \mathcal{N}$, is called the ***nilpotent orbit*** of x.

REMARK 14.1. If $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{k})$, $n \in \mathbb{N}^*$, then

$$G.x = \{PxP^{-1} \colon P \in \mathrm{SL}_n(\mathbb{k})\}.$$

This follows from equation (18).

Recall that the algebra $S(\mathfrak{g}^*) \cong \mathbb{k}[\mathfrak{g}]$ of polynomial functions on $\mathfrak{g}$ is graded by the degree of the elements. This induces a grading on the algebra $\mathbb{k}[\mathfrak{g}]^G$ of G-invariant polynomial functions on $\mathfrak{g}$ by setting, for $i \in \mathbb{N}$,

$$\mathbb{k}[\mathfrak{g}]^G_i = \mathbb{k}[\mathfrak{g}]_i \cap \mathbb{k}[\mathfrak{g}]^G, \qquad \mathbb{k}[\mathfrak{g}]^G = \bigoplus_{i \in \mathbb{N}} \mathbb{k}[\mathfrak{g}]^G_i.$$

We define

$$\mathbb{k}[\mathfrak{g}]^G_+ = \bigoplus_{i \in \mathbb{N}^*} \mathbb{k}[\mathfrak{g}]^G_i$$

so that $\mathbb{k}[\mathfrak{g}]^G_+$ is the ***augmentation ideal*** of $\mathbb{k}[\mathfrak{g}]^G$. In other words, $\mathbb{k}[\mathfrak{g}]^G_+$ is the ideal formed by G-invariant polynomial functions without a constant term. It follows from the proof of the surjectivity of the Chevalley map θ (see Exercise 12.3) that $\mathbb{k}[\mathfrak{g}]^G_+$ is generated by the trace functions

$$x \mapsto \operatorname{Tr}(\phi(x)^k), \quad k \in \mathbb{N}^*,$$

where $\phi \colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a non-zero finite-dimensional irreducible representation.

Recall that an endomorphism x of a finite-dimensional vector space V is nilpotent if and only if $\operatorname{Tr}(x^k) = 0$ for all $k \in \mathbb{N}^*$. Thus, an element x of $\mathfrak{g}$ is nilpotent if and only if $\operatorname{Tr}((\operatorname{ad} x)^k) = 0$ for all $k \in \mathbb{N}^*$.

On the other hand, as a consequence of Proposition 7.3, if $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a non-zero finite-dimensional representation of $\mathfrak{g}$, then $x \in \mathfrak{g}$ is nilpotent if and only if $\phi(x)$ is nilpotent.

We have proven the following result:

Proposition 14.2

The nilpotent cone $\mathcal{N}$ is the variety of zeros of the augmentation ideal $\Bbbk[\mathfrak{g}]^G_+$, i.e.,

$$\mathcal{N} = \{x \in \mathfrak{g} \colon f(x) = 0 \text{ for any } f \in \Bbbk[\mathfrak{g}]^G_+\}.$$

In particular, $\mathcal{N}$ is an affine algebraic variety.

REMARK 14.2. (1) If $\mathfrak{g} = \mathfrak{sl}_n(\Bbbk)$, we recover that $x \in \mathfrak{g}$ is nilpotent if and only if $\mathrm{Tr}(x^k) = 0$ for all $k = 2, \dots, n$ (see Example 12.1). More generally, since the algebra $\Bbbk[\mathfrak{g}]^G$ is a polynomial algebra (see the discussion following Exercise 12.5) generated by ℓ homogeneous elements $p_1, \dots, p_\ell$, the nilpotent cone is the variety of zeros of the polynomials $p_1, \dots, p_\ell$.

(2) According to a remarkable result by Kostant, $\mathcal{N}$ is an irreducible and reduced, that is, the radical of $\Bbbk[\mathfrak{g}]^G_+$ is equal to $\Bbbk[\mathfrak{g}]^G_+$.

Sometimes, the nilpotent cone of $\mathfrak{g}$ is defined as a subset of $\mathfrak{g}^*$ by:

$$\mathcal{N}^* = \{x \in \mathfrak{g}^* \colon f(x) = 0 \text{ for any } f \in S(\mathfrak{g})^G_+ \cong \Bbbk[\mathfrak{g}^*]^G_+\},$$

where $\Bbbk[\mathfrak{g}^*]^G_+$ is the augmentation ideal of $\Bbbk[\mathfrak{g}^*]^G$. The Killing isomorphism

$$\kappa^\sharp_{\mathfrak{g}} \colon \mathfrak{g} \to \mathfrak{g}^*, \quad x \mapsto \kappa_{\mathfrak{g}}(x, \cdot),$$

allows us to identify $\mathcal{N}$ and $\mathcal{N}^*$.

14.2. Jacobson–Morosov Theorem and consequences

In the case of classical Lie algebras, the nilpotent orbits are conjugation classes of nilpotent matrices. In the case where $\mathfrak{g} = \mathfrak{sl}_n(\Bbbk)$, it is well known that such classes are in finite number, and in bijection with the set

$$\mathscr{P}(n) = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t) \colon \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_t, \sum_{i=1}^t \lambda_i = n\}$$

of partitions of n. We obtain this bijection by choosing a representative of the nilpotent conjugation class a diagonal matrix by Jordan blocks:

$$\begin{pmatrix} J_{\lambda_1} & 0 & \cdots & 0 \\ 0 & J_{\lambda_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{\lambda_t} \end{pmatrix}, \qquad J_{\lambda_k} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We obtain similar results for the other classical Lie algebras although the correspondence between nilpotent orbits and partitions is a bit trickier.

More generally, we will see that there is always a finite number of nilpotent orbits. This is essentially a consequence of the following theorem (which we admit here). See [2, Theorem 3.3.1] for more details; there are more direct proofs (see for example [6, Theorem 10.2.4]).

Theorem 14.3 – Jacobson–Morozov theorem

Let x be an nonzero nilpotent element of $\mathfrak{g}$. Then x belongs to a $\mathfrak{sl}_2$-***triple***, that is, a triple $\{x, h, y\}$ of elements of $\mathfrak{g}$ such that

$$[h, x] = 2x, \qquad [x, y] = h, \qquad [h, y] = -2y.$$

The adjoint group G acts on the set of $\mathfrak{sl}_2$-triples by $g.\{x, h, y\} = \{g.x, g.h, g.y\}$, and we have a bijection between the set of nilpotent orbits of $\mathfrak{g}$ and the set of G-orbits of $\mathfrak{sl}_2$-triples.

Nathan Jacobson *(1910–1999) was an American mathematician born in Warsaw. Renowned as one of the leading algebraists of his generation, he is also known for having written more than a dozen reference manuals.*

Theorem 14.4 – Kostant

There is a finite number of nilpotent orbits in $\mathfrak{g}$.

PROOF. We give the main steps of the proof (which is not exactly that of Kostant).

1) According to the Jacobson-Morosov Theorem, the nilpotent orbits of $\mathfrak{g}$ are in bijection with the set of G-orbits of $\mathfrak{sl}_2$-triples.
2) If $\{x, h, y\}$ is a $\mathfrak{sl}_2$-triplet of $\mathfrak{g}$, then there exists a G-conjugate $\tilde{h}$ of $\mathfrak{h}$ such that $\alpha_i(\tilde{h}) \in \{0, 1, 2\}$ for any $i \in \{1, \dots, \ell\}$. It is easy to establish that there exists $\tilde{h}$ in the dominant Weyl chamber, that is, such that $\alpha_i(\tilde{h}) \geqslant 0$ for any $i \in \{1, \dots, \ell\}$, then the theory of $\mathfrak{sl}_2$-modules ensures that $\alpha_i(\tilde{h}) \in \mathbb{N}$ for any $i \in \{1, \dots, \ell\}$. The difficulty is to obtain the very restrictive property that $\alpha_i(\tilde{h}) \in \{0, 1, 2\}$ for any $i \in \{1, \dots, \ell\}$.
3) According to Step 2), we can "weight" the Dynkin diagram of Φ by assigning to the vertex corresponding to the simple root α_i the value $\alpha_i(\tilde{h}) \in \{0, 1, 2\}$. We refer to [2, §3.5] for more details. We see then that there are at most 3^ℓ nilpotent orbits: at most three choices for each top of the Dynkin diagram. This completes the proof of the theorem.

□

REMARK 14.3. 1) The ***weighted Dynkin diagram*** (see Step 3) of the above proof) is a complete invariant, that is, two nilpotent orbits $\mathbb{O}$ and $\mathbb{O}'$ coincide if and only if their weighted Dynkin diagrams are the same.

2) The bound 3^ℓ of the number of nilpotent orbits obtained during the proof is far from being optimal. As an example, here are the weighted Dynkin diagrams corresponding to the nonzero nilpotent orbits of the exceptional Lie algebra G_2:

There are 5 nilpotent orbits (4 nonzero plus the zero orbit), while $3^2 = 9$.

14.3. Associated variety to an irreducible highest weight representation

In this section we associate with any irreducible representation $V(\lambda)$ of highest weight $\lambda \in \mathfrak{h}^*$ a nilpotent orbit of $\mathfrak{g}$. This orbit is a particularly interesting invariant in the case where $V(\lambda)$ is not of finite dimension, i.e., $\lambda \notin \Lambda^+$, since then we do not have a simple formula for the multiplicities. The dimension of this orbit is a sort of replacement in this framework.

Recall that $U(\mathfrak{g})$ is a filtered algebra and that $\operatorname{gr} U(\mathfrak{g}) = S(\mathfrak{g})$ (PBW Theorem 5.3). Let $J \subset U(\mathfrak{g})$ be a two-sided ideal $U(\mathfrak{g})$. Then

$$\operatorname{gr} J = \bigoplus_{i \in \mathbb{N}} (J \cap U_i(\mathfrak{g}))/(J \cap U_{i-1}(\mathfrak{g})), \quad U_{-1}(\mathfrak{g}) = \{0\},$$

is an ideal of $S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*]$. We define its ***associated variety*** by:

$$\mathcal{V}(J) := \{\lambda \in \mathfrak{g}^* : f(\lambda) = 0 \text{ for any } f \in \operatorname{gr} J\}.$$

Let $\lambda \in \mathfrak{h}^*$. The ***annihilator*** of $V(\lambda)$ in $U(\mathfrak{g})$ is:

$$\operatorname{Ann}_{U(\mathfrak{g})} V(\lambda) = \{x \in U(\mathfrak{g}) : x.v = 0 \text{ for any } v \in V(\lambda)\}.$$

EXERCISE 14.1. Show that $\operatorname{Ann}_{U(\mathfrak{g})} V(\lambda)$ is a two-sided ideal $U(\mathfrak{g})$.

Definition 14.5 – associated variety to an irreducible representation

Let $\lambda \in \mathfrak{h}^*$. The ***associated variety*** to the irreducible highest weight representation $V(\lambda)$ is

$$\mathcal{V}(\mathscr{I}_\lambda) = \{\lambda \in \mathfrak{g}^* : f(\lambda) = 0 \text{ for any } f \in \mathrm{gr}\, \mathscr{I}_\lambda\},$$

where $\mathscr{I}_\lambda = \mathrm{Ann}_{U(\mathfrak{g})} V(\lambda)$.

Proposition 14.6

Identify $\mathfrak{g}$ et $\mathfrak{g}^*$ using the Killing isomorphism $\kappa_{\mathfrak{g}}^\sharp$. Let $\lambda \in \mathfrak{h}^*$. Then the associated variety $\mathcal{V}(\mathscr{I}_\lambda)$ of $V(\lambda)$ is contained in the nilpotent cone $\mathcal{N}^* \cong \mathcal{N}$.

PROOF. Consider the central character $\chi_\lambda : Z(\mathfrak{g}) \longrightarrow \Bbbk$ introduced in Section 12.3 of Chapter 12. Its kernel is a maximal ideal of $Z(\mathfrak{g})$ [1]. The ideal $\mathscr{I}_\lambda$ contains $\mathrm{Ker}\,\chi_\lambda$ by definition of χ_λ. We deduce that $\mathrm{gr}\,\mathscr{I}_\lambda$ contains all ***symbols*** [2] of the elements of $\mathrm{Ker}\,\chi_\lambda$.

Since $\mathrm{gr}\, Z(\mathfrak{g}) = S(\mathfrak{g})^G$, we easily show that these symbols generate $S(\mathfrak{g})^G_+$. Therefore,

$$\mathrm{gr}\,\mathscr{I}_\lambda \supset S(\mathfrak{g})^G_+,$$

hence $\mathcal{V}(\mathscr{I}_\lambda) \subset \mathcal{N}^*$ according to Proposition 14.2 and the isomorphism $\mathcal{N}^* \cong \mathcal{N}$. □

The following theorem is extremely subtle and goes far beyond the scope of the course.

Theorem 14.7 – Joseph and Kashiwara (independently)

Let $\lambda \in \mathfrak{h}^*$. Then the associated variety $\mathcal{V}(\mathscr{I}_\lambda)$ is irreducible. In particular, $\mathcal{V}(\mathscr{I}_\lambda)$ is the Zariski closure of some nilpotent orbit in $\mathfrak{g}$, i.e., there exists a nilpotent element e_λ of $\mathfrak{g}$ such that

$$\mathcal{V}(\mathscr{I}_\lambda) = \overline{\mathbb{O}_\lambda}, \quad \text{where} \quad \mathbb{O}_\lambda = G.e_\lambda.$$

Anthony Joseph*, born July 9, 1942 is a French mathematician, who deals with enveloping algebras. He is a emeritus professor at the Weizman Institute of Science in Israel and was a professor at the Pierre and Marie Curie University in Paris.*

Masaki Kashiwara 柏原正樹 *is a Japanese mathematician born January 30, 1947 in Yuki. He is a student under Mikio Sato at the University of Tokyo. He obtained his doctorate in 1974 with a thesis entitled* On the maximality overdetermined system of linear differential equations. *He made major contributions to algebraic analysis, micro-local analysis, theory of D-modules, Hodge theory, pre-sheaf theory and representation theory.*

The second part of the theorem comes from the fact that the variety $\mathcal{V}(\mathscr{I}_\lambda)$ is closed, G-invariant and irreducible according to the first part of the theorem. Since the nilpotent cone has a finite number of orbits (Theorem 14.4), the assertion follows. The first part uses fine methods of microlocalisation in algebraic geometry.

1. We use here the canonical bijection $\mathrm{Hom}_\Bbbk(A, \Bbbk) \longrightarrow \mathrm{Specm}(A)$, $\chi \mapsto \mathrm{Ker}\,\chi$, where A is a $\Bbbk$-algebra and where $\mathrm{Specm}(A)$ denotes the set of maximal ideals of A. The converse map is given by the projection $A = \Bbbk.1 \oplus M \to \Bbbk = A/M$ if $M \in \mathrm{Specm}(A)$.
2. if $x \in U_i(\mathfrak{g}) \setminus U_{i-1}(\mathfrak{g})$, $i \in \mathbb{N}$, its symbol $\sigma(x)$ is its image in $\mathrm{gr}\, U(\mathfrak{g})$ by projection $U_i \to U_i(\mathfrak{g})/U_{i-1}(\mathfrak{g})$.

The nilpotent orbit $\mathbb{O}_\lambda$ is not a complete invariant of $V(\lambda)$. Indeed, it is possible that two irreducible representations $V(\lambda)$ and $V(\mu)$ have the same associated variety, i.e., $\overline{\mathbb{O}_\lambda} = \overline{\mathbb{O}_\mu}$, without them being isomorphic (see Exercise 14.2).

EXERCISE 14.2. Let $\lambda \in \Lambda^+$.

(1) Denote by θ the highest positive root of Φ. Show that there exists $k \in \mathbb{N}$ such that $y_\theta^k.v = 0$ for any $v \in V(\lambda)$, and deduce that y_θ belongs to the radical of $\mathscr{I}_\lambda$.

(2) Noting that $\mathscr{I}_\lambda$ is $(\operatorname{ad}\mathfrak{g})$-invariant, and so is its radical $\sqrt{\mathscr{I}_\lambda}$, deduce from the previous question that $\sqrt{\mathscr{I}_\lambda}$ contains $\mathfrak{g}$.

(3) Conclude that $\mathcal{V}(\mathscr{I}_\lambda) = \{0\}$.

The converse of Exercise 14.2 is also true, that is, $\mathcal{V}(\mathscr{I}_\lambda) = \{0\}$ if and only if $\lambda \in \Lambda^+$, i.e., $V(\lambda)$ is finite dimensional. This gives a geometric characterization of irreducible finite dimensional representations.

To finish, here is an example of an irreducible highest weight representation of infinite dimension for which we know the associated variety.

EXAMPLE 14.1. The example is due to Levasseur and Smith [10], and based on ***Joseph ideals***.

We assume in this example that $\mathfrak{g} = \mathfrak{so}_7(\mathbb{C})$. One can choose Φ and Δ as follows:

$$\Phi = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j : 1 \leqslant i < j \leqslant 3\}, \quad \Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3\},$$

where $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is the canonical basis of $\mathbb{R}^3$.

Let $X = \mathcal{V}(I)$ be the affine algebraic subset of $\mathbb{A}^5$, where I is the ideal of $\mathbb{C}[x, u_1, u_2, y_1, y_2]$ generated by $x^2 + u_1y_1 + u_2y_2$. Then

$$\mathcal{O}(X) = \mathbb{C}[x, u_1, u_2, y_1, y_2]/I$$

is the algebra of regular functions on X. Denote by $\mathscr{D}(X)$ the algebra of differential operators on X (see Section 5.4 of Chapter 5). The algebra $\mathscr{D}(X)$ is naturally a Lie algebra, with the bracket coming from that of $\operatorname{End}\mathcal{O}(X)$. We have

$$[\partial_t, \partial_s] = 0, \qquad [\partial_t, s] = \delta_{t,s}, \qquad s, t \in \{x, u_1, u_2, y_1, y_2\}.$$

Set

$$\begin{aligned} I &= x\partial/\partial x + \sum_{i=1}^{2}(u_i\partial/\partial u_i + y_i\partial/\partial y_i), \\ \varDelta &= \frac{1}{2}\partial^2/\partial x^2 + 2\sum_{i=1}^{2}\partial^2/\partial u_i\partial y_i. \end{aligned}$$

We define a morphism $\varphi\colon U(\mathfrak{so}_7(\mathbb{C})) \to \mathscr{D}(X)$ as follows (we identify below $\mathfrak{so}_7(\mathbb{C})$ with its image in $\mathscr{D}(X)$):

$$\begin{aligned}
X_{\varepsilon_1-\varepsilon_2} &= \frac{1}{2}u_1\varDelta - \partial/\partial y_1(I + \frac{1}{2}), & X_{-(\varepsilon_1-\varepsilon_2)} &= y_1, \\
X_{\varepsilon_2-\varepsilon_3} &= y_1\partial/\partial y_2 - u_2\partial/\partial u_1, & X_{-(\varepsilon_2-\varepsilon_3)} &= y_2\partial/\partial y_1 - u_1\partial/\partial u_2, \\
X_{\varepsilon_3} &= y_2\partial/\partial x - 2x\partial/\partial u_2, & X_{-\varepsilon_3} &= 2x\partial/\partial y_2 - u_2\partial/\partial x, \\
X_{\varepsilon_1-\varepsilon_3} &= \frac{1}{2}u_2\varDelta - \partial/\partial y_2(I + \frac{1}{2}), & X_{-(\varepsilon_1-\varepsilon_3)} &= y_2, \\
X_{\varepsilon_2} &= y_1\partial/\partial x - 2x\partial/\partial u_1, & X_{-\varepsilon_2} &= 2x\partial/\partial y_1 - u_1\partial/\partial x, \\
X_{\varepsilon_1} &= x\varDelta - \partial/\partial x(I + \frac{1}{2}), & X_{-\varepsilon_1} &= 2x, \\
X_{\varepsilon_2+\varepsilon_3} &= y_1\partial/\partial u_2 - y_2\partial/\partial u_1, & X_{-(\varepsilon_2+\varepsilon_3)} &= u_2\partial/\partial y_1 - u_1\partial/\partial y_2, \\
X_{\varepsilon_1+\varepsilon_3} &= \frac{1}{2}y_2\varDelta - \partial/\partial u_2(I + \frac{1}{2}), & X_{-(\varepsilon_1+\varepsilon_3)} &= u_2, \\
X_{\varepsilon_1+\varepsilon_2} &= \frac{1}{2}y_1\varDelta - \partial/\partial u_1(I + \frac{1}{2}), & X_{-(\varepsilon_1+\varepsilon_2)} &= u_1,
\end{aligned}$$

$$\begin{aligned}
H_{\varepsilon_1-\varepsilon_2} &= (u_1\partial/\partial u_1 - y_1\partial/\partial y_1) - (I + \frac{3}{2}), \\
H_{\varepsilon_2-\varepsilon_3} &= y_1\partial/\partial y_1 - y_2\partial/\partial y_2 + u_2\partial/\partial u_2 - u_1\partial/\partial u_1, \\
H_{\varepsilon_3} &= 2(y_2\partial/\partial y_2 - u_2\partial/\partial u_2).
\end{aligned}$$

We should of course verify that the subalgebra of $\mathscr{D}(X)$ thus defined is indeed isomorphic to the Lie algebra $\mathfrak{so}_7(\mathbb{C})$.

ASSERTION 14.1. As a $\mathfrak{so}_7(\mathbb{C})$-module, $\mathcal{O}(X) \cong V(-\frac{3}{2}\varpi_1)$.

PROOF. First, $\mathcal{O}(X)$ is generated, as a $\mathfrak{so}_7(\mathbb{C})$-module, by the constant function 1 since

$$\mathcal{O}(X) \subset \varphi(U(\mathfrak{so}_7(\mathbb{C}))).$$

The elements of $\mathfrak{n}^+$ (corresponding to the positive roots, i.e., those in the left column in the previous equations) kill 1. Moreover, the action of the Cartan subalgebra (given by the last three equations) shows that 1 is a highest weight vector of height $-\frac{3}{2}\varepsilon_1 = -\frac{3}{2}\varpi_1$.

Notice here that the fundamental weights of $\mathfrak{so}_7(\mathbb{C})$ are given by:

$$\varpi_1 = \varepsilon_1, \qquad \varpi_2 = \varepsilon_1 + \varepsilon_2, \qquad \varpi_3 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3).$$

This shows that $\mathcal{O}(X)$ contains an image homomorphic to $M(-\frac{3}{2}\varpi_1)$.

It would remain to show that $\mathcal{O}(X)$ is simple as an $\mathfrak{so}_7(\mathbb{C})$-module which we admit here. In fact, $\mathcal{O}_X$ is simple even as a $\mathfrak{g}_2$-module, where $\mathfrak{g}_2$ is the Lie simple algebra of exceptional type G_2 which embeds $\mathfrak{so}_7(\mathbb{C})$. □

REMARK 14.4. The central character acts by $-\frac{3}{2}\varpi_1 + \rho = \varepsilon_1 + \frac{3}{2}\varepsilon_2 + \frac{1}{2}\varepsilon_3$.

The kernel $\mathscr{I}_{-\frac{3}{2}\varpi_1}$ of the morphism φ is an ideal of $U(\mathfrak{so}_7(\mathbb{C}))$ such that

$$\mathcal{V}(\mathscr{I}_{-\frac{3}{2}\varpi_1}) = \overline{\mathbb{O}_{min}}, \tag{23}$$

where $\mathbb{O}_{min}$ is the unique nonzero nilpotent orbit of $\mathfrak{so}_7(\mathbb{C})$ of minimal dimension. It is the one associated with the partition $(2^2, 1^3)$ of 7 and it is of dimension 8. In other words, the associated variety of the simple $\mathfrak{so}_7(\mathbb{C})$-module $\mathcal{O}(X)$ is $\overline{\mathbb{O}_{min}}$.

The ideal $\mathscr{I}_{-\frac{3}{2}\varpi_1}$ is called a ***Joseph ideal***. It is defined in a more general framework and plays a very important role in representation theory. We admit here the equality (23) which uses the geometry of nilpotent orbits and an explicit description of the ideal $\mathscr{I}_{-\frac{3}{2}\varpi_1}$.

Part 4

Borel–Weil–Bott Theorem

In this part, we propose to give an overview of an important result, the theorem of Borel–Weil–Bott. It is about constructing geometrically any irreducible representation of a reductive group G. For simplicity we will consider only the case of semisimple groups. We will realize such a representations in term of the flag variety (cf. Section 15.2) G/B, where B is a Borel subgroup. Chapter 15 presents some properties of Borel subgroups and the flag variety. We know that the irreducible representation of a semisimple Lie algebra are parameterized by integral dominant weights. One task is then to explain an analogue results for semisimple groups. This is done in Chapter 16. Borel–Weil Theorem and Borel–Weil–Bott Theorem are stated in Chapter 17. Only Borel–Weil Theorem is proven in the course.

__Armand Borel__, 1923 – 2003, was a Swiss mathematician, born in La Chaux-de-Fonds, and was a permanent professor at the Institute for Advanced Study in Princeton, New Jersey, United States from 1957 to 1993. He worked in algebraic topology, in the theory of Lie groups, and was one of the creators of the contemporary theory of linear algebraic groups.

__André Weil__, 1906 – 1998, was a French mathematician, known for his foundational work in number theory and algebraic geometry. He was one of the most influential mathematicians of the twentieth century. His influence is due both to his original contributions to a remarkably broad spectrum of mathematical theories, and to the mark he left on mathematical practice and style, through some of his own works as well as through the Bourbaki group, of which he was one of the principal founders.

__Raoul Bott__, 1923 – 2005, was a Hungarian-American mathematician known for numerous foundational contributions to geometry in its broad sense. He is best known for his Bott periodicity theorem, the Morse–Bott functions which he used in this context, and the Borel–Bott–Weil theorem.

15

Borel subgroups and the flag variety

15.1. Reductive and semisimple groups

By Definition 4.8, a Lie algebra $\mathfrak{g}$ is semisimple if $\mathrm{rad}(\mathfrak{g}) = \{0\}$. We have defined semisimple algebraic groups in Section 7.3 using connected commutative normal ideals. We have also an analogue to the above definition.

For an arbitrary linear algebraic group G, define the ***derived series*** of G inductively by

$$\mathscr{D}^0 G = G, \quad \mathscr{D}^{i+1} = (\mathscr{D}^i G, \mathscr{D}^i G), \quad \dots,$$

where for A, B closed subgroups of G, (A, B) denotes the group generated by the commutators $xyx^{-1}y^{-1}$ for $x \in A$, $y \in B$. Note that if A, B are closed, connected and normal, then so is (A, B). Say that an algebraic group is ***solvable*** if its derived series terminates in $\{e\}$.

The ***radical*** of G, denoted by $\mathrm{Rad}(G)$, is largest connected normal solvable subgroup of G.

Proposition 15.1 – an algebraic group is semisimple if and only if its radical is trivial

A connected algebraic group G is semisimple if and only if its radical $\mathrm{Rad}(G)$ is trivial.

EXERCISE 15.1. Prove the proposition.

A subgroup of a linear algebraic group is called ***unipotent*** if all its elements are unipotent, in the sense that there are unipotent [1] in some $\mathrm{GL}_n(\Bbbk)$. Remember that, by Chevalley's Theorem 2.2, G can always be embedded into some $\mathrm{GL}_n(\Bbbk)$.

Rigorously, we must verify that this notion of unipotent element does not depend on the embedding. It is the case, but it is not completely trivial, see [8, §15.3].

The largest connected normal unipotent subgroup of G, denoted by $\mathrm{Rad}_u(G)$, is called the ***unipotent radical*** of G.

Definition 15.2 – reductive group

Let G be a linear algebraic group. We say that G is ***reductive*** if G is connected, $G \neq \{e\}$ and its unipotent radical $\mathrm{Rad}_u(G)$ is trivial.

For example, any semisimple group is reductive.

1. A matrix $x \in \mathrm{GL}_n(\Bbbk)$ is ***unipotent*** if $I_n + x$ is nilpotent in $\mathscr{M}_n(\Bbbk)$.

Lemma 15.3

Let G be a connected nontrivial linear algebraic group, and V a faithful representation of G. Assume that V is a simple representation. Then G is reductive.

SKETCH OF PROOF. Because $\mathrm{Rad}_u(G)$ is normal in G, the subspace $V^{\mathrm{Rad}_u(G)}$ is a subrepresentation of V. Then $V^{\mathrm{Rad}_u(G)} \neq \varnothing$ because $\mathrm{Rad}_u(G)$ is a unipotent group (i.e., consisted of unipotent elements). This comes from an analogue to Engel's Theorem that we do not prove here. Therefore $V^{\mathrm{Rad}_u(G)} = V$, using the simplicity of V. In particular $\mathrm{Rad}_u(G)$ acts trivially on V. Because the representation is faithful, $\mathrm{Rad}_u(G) = \{e\}$ hence G is reductive. □

For example, the groups $\mathrm{GL}_n(\Bbbk)$, $\mathrm{SL}_n(\Bbbk)$, $\mathrm{SO}_n(\Bbbk)$, $\mathrm{Sp}_n(\Bbbk)$ are reductive because the standard representation is faithful and simple.

Note that $\mathrm{GL}_n(\Bbbk)$ is not semisimple, while the others are.

15.2. The flag variety

Let V be a $\Bbbk$-vector space of finite dimension $n \in \mathbb{N}^*$. Recall that a ***flag*** in V is a chain

$$0 \subset V_1 \subset \cdots \subset V_k \subset V$$

of subspaces of V, each properly included in the next one. A ***full flag*** of V is one for which $\dim V_i = i$ for all i. For example the flag encountered in Engel's Theorem 4.4 is full.

Denote by $\mathscr{F}(V)$ the collection of all full flags in V.

Lemma 15.4

The set $\mathscr{F}(V)$ has a structure of a projective variety, called the ***flag variety*** of V.

PROOF. It is known that one can give to the Cartesian product

$$\mathfrak{G}_1(V) \times \cdots \times \mathfrak{G}_n(V)$$

a structure of a projective variety, where $\mathfrak{G}_d(V)$ denotes the ***Grassmann variety***, that is, the collection of all d-dimensional subspaces of V: just embed $\mathfrak{G}_d(V)$ into the projective space $\mathbb{P}(\wedge^d V)$ via the map

$$\psi\colon \mathfrak{G}_d(V) \to \mathbb{P}(\wedge^d V)$$

sending W to the point in the projective space belonging to $\mathbb{P}(\wedge^d W)$.

Now $\mathscr{F}(V)$ identifies in an obvious way with a subset, which we need only show to be closed, of the above Cartesian product. To simplify, consider the case where the product is just

$$\mathfrak{G}_d(V) \times \mathfrak{G}_{d+1}(V).$$

It is enough to show that the set of pairs (W, W') such that $W \subset W'$ is closed, and we leave this to the reader. □

By definition, a variety X is ***complete*** if for all varieties Y, the projection map $X \times Y \to Y$ is closed.

Proposition 15.5 – properties of complete varieties

We list here some facts about complete variety and refer to [5] or [8] for more details.

(a) A closed subvariety of a complete (resp. projective) variety is complete (resp. projective).

(b) If $\varphi\colon X \to Y$ is a morphism of varieties, with X complete, then the image is closed in Y, and complete.

(c) A complete affine variety has dimension 0.

(d) Projective varieties are complete, and complete quasiprojective varieties are projective.

In particular, the flag variety $\mathscr{F}(V)$ of a finite dimensional vector space V is projective, hence complete by (d).

EXAMPLE 15.1. The group $\mathrm{GL}_n(\Bbbk)$ naturally acts on $\mathscr{F}(V)$, with $V = \Bbbk^n$, and the fixed point of the canonical full flag (like in Engel's Theorem 4.4) is the subgroup of upper triangular matrices $\mathrm{T}_n(\Bbbk)$. Hence the quotient

$$\mathrm{GL}_n(\Bbbk)/\mathrm{T}_n(\Bbbk) \cong \mathscr{F}(V)$$

is projective and identifies with the flag variety of $\Bbbk^n$.

One can generalize this construction.

Let G be any connected linear algebraic group.

Definition 15.6 – Borel subgroup

A ***Borel subgroup*** of a linear algebraic group G is a closed connected solvable subgroup, maximal for these property (the word “closed” being redundant).

For example, $\mathrm{T}_n(\Bbbk)$ is a Borel subgroup of $\mathrm{GL}_n(\Bbbk)$. A connected solvable of largest possible dimension in G is evidently a Borel subgroup. But it is not obvious that every Borel subgroup have the same dimension. This results from a stronger fact.

We admit the following result.

Theorem 15.7 – Fixed Point Theorem

Let G be a connected solvable algebraic group, and let X be a nonempty complete variety on which G acts. Then G has a fixed point on X.

As an application, we obtain an analogue to Lie's Theorem 4.10 for connected solvable group. Let G be a closed connected solvable subgroup of $\mathrm{GL}(V)$. Then G acts on the flag variety $\mathscr{F}(V)$, which is complete, so G fixes a flag

$$0 \subset V_1 \subset \cdots \subset V_n \subset V.$$

In other words, G is triangular for a suitable choice of basis in V.

There are more substantial applications of the Fixed Point Theorem 15.7.

Let $\mathscr{B}$ be the collection of all Borel subgroups of G. It is called the ***flag variety*** of G.

Theorem 15.8 – properties of Borel subgroups

Let B be any Borel subgroup of G. Then G/B is a projective variety, and all other Borel subgroups are conjugated to B. Moreover, the normalizer of B in G equals B, i.e.,

$$N_G(B) = B,$$

where $N_G(B) = \{x \in G \colon xBx^{-1} \subset B\}$, and $\mathscr{B}$ identifies with the projective variety G/B.

Using $N_G(B) = B$ and the Fixed Point Theorem 15.7, one can show the last assertion which says that $\mathscr{B}$ identifies with the projective variety G/B. Indeed, if $B' \in \mathscr{B}$, then B' has a fixed xB on G/B (i.e., $x^{-1}B'x = B$). If yB is any fixed point of B', then

$$x^{-1}B'x = B' = yB^{-1},$$

that is, $y^{-1}x \in N_G(B) = B$, whence $xB = yB$. This shows that the assignment $B' \mapsto xB$ is unambiguous. It is surjective since $xBx^{-1} \mapsto xB$ for arbitrary $x \in G$. Finally, it is injective, again using the self-normalizing property.

Under the above one-to-one correspondence, the natural action of G on $\mathscr{B}$, $B' \mapsto xB'x^{-1}$ evidently goes over into the natural action of G on G/B given by $yB \mapsto xyB$.

EXAMPLE 15.2. Assume that $G = \mathrm{SL}_2(\Bbbk)$ and let $B = \mathrm{T}_2(\Bbbk) \cap \mathrm{SL}_2(\Bbbk)$ be the set of upper triangular matrices with determinant one. Then we have the following isomorphism of varieties:

$$\psi\colon \begin{array}{ccc} G/B & \longrightarrow & \mathbb{P}^1 \\ \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] & \longmapsto & [a : c]. \end{array}$$

So, for $G = \mathrm{SL}_2(\Bbbk)$, the flag variety identifies with the projective line $\mathbb{P}^1$.

16

Characters and representations

Let G be a linear algebraic group.

16.1. The group of characters

Definition 16.1 – character

A ***character*** of G is a morphism of algebraic groups $\chi\colon G \to \mathbb{G}_m$.

For example, $\det\colon \mathrm{GL}_n(\mathbb{k}) \to \mathbb{G}_m$ is a character of $\mathrm{GL}_n(\mathbb{k})$.

If χ_1, χ_2 are two characters of G, their product is defined by

$$(\chi_1\chi_2)(x) = \chi_1(x)\chi_2(x)$$

for all $x \in G$. Therefore the set $X^*(G)$ of all characters of G has a commutative group structure. We call it the ***character group***.

EXERCISE 16.1 (some examples of character groups).

(1) Describe $X^*(\mathbb{G}_m)$ and $X^*(\mathrm{D}_n(\mathbb{k}))$.

(2) Show that the group of characters of $\mathbb{G}_a$ is trivial.

Hint: realize $\mathbb{G}_a$ as the set of matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $a \in \mathbb{C}$.

(3) Show that the group of characters $\mathrm{SL}_n(\mathbb{k})$ is trivial.

Hint: observe that $(\mathrm{SL}_n(\mathbb{k}), \mathrm{SL}_n(\mathbb{k})) = \mathrm{SL}_n(\mathbb{k})$.

Characters arise in connection with linear representations, as follows. Let G be a closed subgroup of $\mathrm{GL}(V)$. For each $\chi \in X^*(G)$, set

$$V_\chi = \{v \in V : x(v) = \chi(x)v \text{ for all } x \in G\}.$$

Evidently, V_χ is a G-stable subspace of V (possibly 0). Any nonzero element of V_χ is called a ***semi-invariant*** of G, of ***weight*** χ. Conversely, if v is any nonzero vector which spans a G-stable line in V, then it is clear that

$$x(v) = \chi(x)v$$

defines a character χ of G.

More generally, if $\varphi\colon G \to \mathrm{GL}(V)$ is a rational representation, then the ***semi-invariants*** of G in V are by definition those of $\varphi(G)$. Notice that the composition with φ induces an injective group homomorphism

$$X^*(\varphi(G)) \hookrightarrow X^*(G),$$

so that V_χ may be defined in the obvious way for any $\chi \in X^*(G)$ coming from a character of $\varphi(G)$:

$$V_\chi = \{v \in V \colon \varphi(x)v = \chi(x)v \text{ for all } x \in G\},$$

where $\chi = \tilde{\chi} \circ \varphi$ for some $\tilde{\chi} \in X^*(\varphi(G))$.

EXERCISE 16.2 (the subspaces of semi-invariants are linearly independant). Let $\varphi\colon G \to \mathrm{GL}(V)$ be a rational representation. Show that subspaces V_χ, for $\chi \in X^*(G)$, are linearly independent. In particular, there are only finitely many of them.

Definition 16.2 – torus

Call an algebraic group a ***torus*** if it is isomorphic to $\mathrm{D}_n(\Bbbk) \cong (\mathbb{G}_m)^n$ for some n.

For an algebraic torus T, there is a notion dual to character. Any morphism of algebraic groups $\lambda\colon \mathbb{G}_m \to G$ is called a ***one-parameter subgroup of*** G, abbreviated by ***1-psg***. The set of these morphisms is denoted by $X_*(T)$. It becomes an abelian group if we define a product by:

$$(\lambda\mu)(a) = \lambda(a)\mu(a).$$

Notice that the composite of a 1-psg λ with a character χ of T yields a morphism of algebraic groups $\mathbb{G}_m \to \mathbb{G}_m$, i.e., an element of $X^*(\mathbb{G}_m) \cong \mathbb{Z}$ (see Exercise 16.1). This allows us to define a natural pairing

$$X^*(T) \times X_*(T) \to \mathbb{Z}, \quad (\chi, \lambda) \mapsto \langle \chi, \lambda \rangle$$

under which $X^*(T)$ and $X_*(T)$ become dual $\mathbb{Z}$-modules.

Recall that $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is a morphism of algebraic group. So if T is a torus of G, then $\mathrm{Ad}(T)$ is a torus of $\mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$. Set for $\alpha \in X^*(T)$,

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \colon (\mathrm{Ad}\, t)x = \alpha(t)x \text{ for all } t \in T\}.$$

Then we have the weight decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{c}_{\mathfrak{g}}(T) \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\right), \tag{24}$$

where $\Phi = \Phi(G, T)$ is the set of $\alpha \in X^*(T)$ such that $\mathfrak{g}_\alpha \neq \{0\}$, that is the set of weights of $\mathrm{Ad}(T)$. The weights of $\mathrm{Ad}(T)$ are called the ***roots*** of (G, T).

We admit the following theorem.

Theorem 16.3

Let G be a connected linear algebraic group.

(i) Each semisimple element[a] of G lies in a maximal torus of G.

(ii) The maximal tori of G are those of the Borel subgroups of G, and they are all conjugate.

(iii) We have $C_G(T) = N_G(T)^\circ$, where $C_G(T)$ is the centralizer of of T in G and $N_G(T)^\circ$ is the identity component of $N_G(T)$. In particular, $C_G(T)$ is connected.

a. As for unipotent elements, this is defined using an embedding $\varphi\colon G \hookrightarrow \mathrm{GL}(V)$: say that $x \in G$ if ***semisimple*** if $\varphi(x)$ is.

Call the common dimension of the maximal tori of G the ***rank*** of G.

EXERCISE 16.3. Show that $\mathrm{D}_n(\Bbbk) \cap \mathrm{SL}_n(\Bbbk)$ is a maximal torus of $\mathrm{SL}_n(\Bbbk)$ so that the rank of $\mathrm{SL}_n(\Bbbk)$ is $n-1$.

REMARK 16.1. For an arbitrary algebraic group G, it is known that the identity component G° is a normal closed subgroup of G of finite index. Thus, in the context of Theorem 16.3, the quotient $N_G(T)/C_G(T)$ is finite.

16.2. Maximal tori in a semisimple group

We now focus on the case where G is a semisimple group.

Theorem 16.4

Let G be semisimple group, and T be a maximal torus. Then

$$T = C_G(T) = N_G(T)^\circ.$$

PROOF. We already know by Theorem 16.3, (iii), that $C_G(T) = N_G(T)^\circ$. Let $\mathfrak{h}$ be the Lie algebra of T. One can show that

$$\operatorname{Lie} C_G(T) = \mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{c}_{\mathfrak{g}}(\operatorname{Lie}(T)) = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}).$$

Then the decomposition (24) tells us that the elements of $\mathfrak{h}$ are semisimple because $\mathrm{d}_e\mathrm{Ad} = \mathrm{ad}$. Since T is a maximal torus (in particular connected), $\mathfrak{h}$ is then a maximal toral subalgebra of $\mathfrak{g}$. Then we know that by Proposition 8.4 and (17) that

$$\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}.$$

Since T is connected, we deduce that $T = C_G(T) = N_G(T)^\circ$ because $C_G(T)$ is connected by Theorem 16.3, (iii). □

According to Remark 16.1 and Theorem 16.4, the quotient $N_G(T)/T$ is finite. It is called the ***Weyl group*** of the pair (G, T). Denote it by $\mathcal{W}(G, T)$. Because all maximal tori are conjugated, all Weyl groups are isomorphic.

It may happen that $N_G(T) \neq T$. For example, if T is the maximal torus $\mathrm{D}_n(\Bbbk) \cap \mathrm{SL}_n(\Bbbk)$ of $G = \mathrm{SL}_n(\Bbbk)$, then $N_G(T)/T \cong \mathfrak{S}_n$, the symmetric group of order n.

If T is a maximal torus of G, recall that the roots of G relative to T are the nontrivial weights of $\operatorname{Ad}(T)$ in $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{c}_{\mathfrak{g}}(T) \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right),$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \colon \operatorname{Ad} t(x) = \alpha(t)x,\ t \in T\}$.

If G is semisimple, it follows from the proof of Theorem 16.4 that

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right), \tag{25}$$

and Φ is a root system of $\mathbb{R} \otimes_{\mathbb{Z}} X^*(T)$. In fact Φ is an abstract root system in V, isomorphic to the root system of $(\mathfrak{g}, \mathfrak{h})$. Moreover,

$$\mathcal{W}(\Phi) \cong \mathcal{W}(G, T).$$

Recall that if G is a semimple group, its Lie algebra $\mathfrak{g}$ is semisimple (almost by definition). Moreover,

$$G/Z(G) = G/\operatorname{Ker}\operatorname{Ad} \cong \operatorname{Aut}_e(\mathfrak{g}).$$

This already shows that if G, G' are two semisimple groups whose Lie algebras $\mathfrak{g}, \mathfrak{g}'$ are isomorphic then

$$G/Z(G) \cong G/Z(G').$$

Now the classification of semisimple Lie algebras is completely understood by Chapter 8: $\mathfrak{g}$ is determined by its root system. To complete the picture we need to take into account of the center. It turns out that the center $Z(G)$ is isomorphic to a subgroup of the ***fundamental group*** of the root system Φ attached to $\mathfrak{g}$, the quotient of the weight lattice Λ by the root lattice Λ_r (see Figure 3 for its description for each irreducible Φ).

To explain the classification, we assume for simplicity that G is ***simple***, that is, its Lie algebra $\mathfrak{g}$ is simple.

If G is simple as an algebraic group, then it is merely ***almost simple*** as an abstract group, that is, it is noncommutative with no proper closed connected normal subgroups.

For example, $\mathrm{SL}_n(\Bbbk)$ is simple as an algebraic group but it is only almost simple as an abstract group (its center is not trivial).

Lemma 16.5

Assume that G is semisimple. Then the center $Z(G)$ is the intersection of all maximal tori of G.

16.3. Finite-dimensional representations of semisimple groups

Assume in this section that G is semisimple. If $\sigma\colon G \to \mathrm{GL}(V)$ is a rational representation, the ***weights*** of σ are the images in $X^*(T)$ of the weights of $\rho(T)$ in V, via the canonical homomorphism

$$X^*(\rho(T)) \to X^*(T).$$

Of course, ρ has only finitely many weights, since V is finite dimensional. It is often convenient to view V directly as a G-module, so that a weight space V_λ is described as

$$\{v \in V : t.v = \lambda(t)v \text{ for all } t \in T\}.$$

We call $\dim V_\lambda$ the ***multiplicity*** of the weight λ. Notice that $\mathcal{W}(G,T)$ permutes the weights of σ. More precisely, if $n \in N_G(T)$ represents $s \in \mathcal{W}(G,T)$, then

$$n.V_\lambda = V_{s(\lambda)}.$$

so all weights in a $\mathcal{W}(G,T)$-orbit have the same multiplicity.

When $\rho = \mathrm{Ad}$, the weights are just the roots (each with multiplicity 1) and the trivial weight 0 (with multiplicity $\ell = \operatorname{rank} G$). So,

$$X^*(\mathrm{Ad}(T)) = \Phi \subset X^*(T).$$

Let us see what can be said about the weights of an arbitrary rational representation $\rho\colon G \to \mathrm{GL}(V)$. The group $\rho(G)$ is either semisimple or trivial. We have

$$\operatorname{Ker}\rho \subset Z(G) \subset T.$$

Let B be a Borel subgroup of G containing T whose Lie algebra is the Borel subalgebra $\mathfrak{b}^+$ as in Section 10.1. For each $\alpha \in \Phi$, let U_α be the unique connected subgroup of B whose Lie algebra is $\mathfrak{g}_\alpha$. For any $\alpha \in \Phi$, $\sigma(U_\alpha)$ maps a weight space V_λ into $\sum_{k\in\mathbb{N}} V_{\lambda+k\alpha}$. Set

$$T_\alpha := (\operatorname{Ker}\alpha)^\circ, \quad Z_\alpha = C_G(T_\alpha).$$

Then Z_α is a reductive group of rank one such that

$$\mathrm{Lie}(Z_\alpha) =: \mathfrak{z}_\alpha = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha},$$

and $\sigma(Z_\alpha)$ stabilizes $\sum_{k\in\mathbb{N}} V_{\lambda+k\alpha}$. In particular $s_\alpha(\lambda)$ is of the form $\lambda + k\alpha$. In the framework of abstract root system, s_α acts in $E = \mathbb{R} \otimes_{\mathbb{Z}} X^*(T)$ as the reflection

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda|\alpha)}{(\alpha|\alpha)}\alpha.$$

The conclusion is that the number

$$\frac{2(\lambda|\alpha)}{(\alpha|\alpha)} = \langle \lambda, \alpha^\vee \rangle$$

is an integer, and so by definition, λ is an ***integral weight***, see Section 11.1.

Recall here that weight lattice Λ of Φ is an ℓ-rank lattice in E, which contains the root lattice Λ_r.

Combining these observations, we conclude that all weights of rational representations are integral weights. Therefore, $X^*(T)$ consists of integral weights and the index of the root lattice in $X^*(T)$ is bounded by a constant depending only on Φ.

To summarize,

$$\Lambda_r \subset X^*(T) \subset \Lambda,$$

and for any rational representation $\rho\colon G \to \mathrm{GL}(V)$, we have

$$X^*(\rho(T)) \subset X^*(T) \subset \Lambda.$$

Moreover,

$$[X^*(T) : \Lambda_r] \leqslant [\Lambda : \Lambda_r].$$

EXAMPLE 16.1. If G is semisimple with Lie algebra $\mathfrak{sl}_2(\Bbbk)$, then the fundamental weight is $\varpi_1 = \alpha_1/2$ and $[\Lambda : \Lambda_r] = 2$. So there is only two possibilities for the position of $X^*(T)$.

We define the ***fundamental group*** of G by

$$\pi(G) := \Lambda/X^*(T).$$

When this is trivial, we say that G is ***simply connected***. At the extreme opposite, when $X^*(T) = \Lambda_r$ we say that G is of ***adjoint type***. In general, we have

$$X^*(Z(G)) \cong X^*(T)/\Lambda_r.$$

So a semisimple group of adjoint type has a trivial center.

For example, $\operatorname{Ad} G \cong G_{ad}$ is always of adjoint type for any semisimple G. In this language, $\mathrm{SL}_2(\Bbbk)$ is simply connected and $\mathrm{PSL}_2(\Bbbk)$ is adjoint.

EXERCISE 16.4. Determine the index in $X^*(T)$ of Λ_r for $\mathrm{SL}_n(\Bbbk)$ and $\mathrm{PSL}_n(\Bbbk)$. Draw the lattices for $n = 2$.

Theorem 16.6

Let G, G' be two simple groups, with corresponding root systems Φ, Φ'. If $\Phi \cong \Phi'$ as abstract root systems and if $\pi(G) \cong \pi(G')$, then $G \cong G'$ as algebraic groups.

We now aim to identify the simple highest weight representations of $\mathfrak{g}$ coming from (simple) rational representations of G.

First of all, if $\rho\colon G \to \mathrm{GL}(V)$ is a simple rational representation, then it induces a simple finite-dimensional representation $\sigma\colon \mathfrak{g} \to \mathfrak{gl}(V)$, and so it is a simple highest weight representation with highest weight $\lambda \in \Lambda_+$ (see Theorem 11.1). So if σ comes from a representation of G, we conclude that

$$\lambda \in X^*(T) \cap \Lambda_+.$$

Conversely, one can show (but this is less direct than in the case of semisimple Lie algebras) that to any $\lambda \in X^*(T) \cap \Lambda_+$ one can associate a simple rational representation of G with highest weight [1] λ.

Setting

$$X^*(T)_+ = X^*(T) \cap \Lambda_+,$$

the set of ***dominant integral weights*** of G, we have thus stated the following theorem.

Theorem 16.7 – simple rational representations of semisimple groups

Let $\lambda \in X^*(T)_+$ be a dominant integral weights for G. Then there exists an irreducible G-module of highest weight λ. Moreover, the simple rational representations of G (up to isomorphism) are parameterized by the set $X^*(T)_+$.

The purpose of next chapter is to give a geometrical construction of the simple rational representations $V(\lambda)$ associated to $\lambda \in X^*(T)_+$. This will prove in particular the hard part of Theorem 16.7.

EXAMPLE 16.2. Let $G = \mathrm{SL}_2(\Bbbk)$, and $T = \mathrm{D}_2(\Bbbk) \cap \mathrm{SL}_2(\Bbbk)$. We have

$$X^*(T) = \Lambda \cong \mathbb{Z},$$

and so the set of classes of irreducible rational representations of $\mathrm{SL}_2(\Bbbk)$ is parameterized by $\mathbb{N}$. One can describe this set explicitly. The group $\mathrm{SL}_2(\Bbbk)$ naturally acts on $\Bbbk^2$ and so on $A = \Bbbk[X,Y]$, the polynomial algebra in two variables. The subspace $\Bbbk[X,Y]_m$ of m-degree homogeneous polynomials is $\mathrm{SL}_2(\Bbbk)$-invariant and irreducible. By Remark 3.3, we know that it is also the space of an irreducible of $\mathfrak{sl}_2(\Bbbk)$.

Let now $G = \mathrm{PSL}_2(\Bbbk)$. Its Lie algebra is again $\mathfrak{sl}_2(\Bbbk)$, but here we have

$$X^*(\bar{T}) = \Lambda_r \cong 2\mathbb{Z},$$

where $\bar{T}$ is the maximal torus in $\mathrm{PSL}_2(\Bbbk)$ given by the image of T by the projection $\mathrm{SL}_2(\Bbbk) \twoheadrightarrow \mathrm{PSL}_2(\Bbbk) = \mathrm{SL}_2(\Bbbk)/\{\pm I_2\}$. We notice that $\Bbbk[X,Y]_m$ is a representation of $\mathrm{PSL}_2(\Bbbk)$ if and only if the center of $\mathrm{SL}_2(\Bbbk)$, which is just $\{\pm I_2\}$, acts trivially. But this happens if and only if m is even, and this is coherent with $X^*(\bar{T})_+ \cong 2\mathbb{N}$.

1. It is a highest weight representation for the Lie algebra $\mathfrak{g}$, but it is also a highest weight representation for G where the notion can be defined similarly.

17

Borel–Weil and Borel–Weil–Bott Theorems

17.1. The main statements

Let G be a semisimple, B a Borel subgroup of G, T un tore maximal de B, and N the unipotent radical of B. Let $X^*(T)$ the character lattice of T, and $X^*(T)_+$ the set of dominant weights, that is,

$$X^*(T)_+ = X^*(T) \cap \Lambda,$$

where Λ is the weight lattice associated with the root system Φ of (G, T). Let also $\mathfrak{g}, \mathfrak{b}^+, \mathfrak{h}, \mathfrak{n}_+$ be Lie algebras of G, B, T, N, respectively. Then we have the root decomposition (25).

Any character $\lambda\colon T \to \mathbb{G}_m$ can be extended to a character of B by setting $\lambda|_N = 1$ using the fact that $B = T \ltimes N$ (see for instance [8, Section 19]). Conversely any character of B is trivial on N because $(B, B) = N$ and induces a character of T. In other words,

$$X^*(B) = X^*(T).$$

Then we define an action of B on $\Bbbk$ by setting

$$b \cdot z = \lambda(b)z \quad \text{for all } z \in \Bbbk.$$

Denote by $\Bbbk_\lambda$ this B-representation.

By making B acting on the left on $\Bbbk_{-\lambda}$ (this means that B acts by $\lambda(b^{-1})$), we obtain an action of B on $G \times \Bbbk_{-\lambda}$, where B acts on G by the right multiplication:

$$b.(g, z) = (gb^{-1}, -\lambda(b)z).$$

Then one can consider the quotient

$$G \times_B \Bbbk_{-\lambda} := (G \times \Bbbk_{-\lambda})/B.$$

Since the action of B on G is free, the B-action on $G \times \Bbbk_{-\lambda}$ is free as well, so $G \times_B \Bbbk_{-\lambda}$ is an algebraic variety and the canonical projection

$$\pi\colon G \times_B \Bbbk_{-\lambda} \twoheadrightarrow G/B$$

is a line bundle on the flag variety G/B, with fiber $\Bbbk_{-\lambda}$.

We denote by $\mathcal{L}_\lambda$ this line bundle. This line bundle est G-equivariant, so that any cohomology $H^p(G/B, \mathcal{L}_\lambda)$ is naturally equipped with a G-module structure. In particular, the space $H^0(G/B, \mathcal{L}_\lambda)$ of global sections of $\mathcal{L}_\lambda$ is a G-module.

Next, we know from Section 16.3 that for any $\lambda \in X^*(T)_+$ there exists a finite-dimensional representation $V(\lambda)$ of G, with highest weight λ.

Theorem 17.1 – Borel–Weil

Let G be a semisimple group, B a Borel subgroup of G, T a maximal torus of B. Then for any dominant weights $\lambda \in X^*(T)_+$, we have an isomorphism of G-modules

$$H^0(G/B, \mathcal{L}_\lambda) = \Gamma(G/B, \mathcal{L}_\lambda) \cong V(\lambda)^*,$$

where $H^0(G/B, \mathcal{L}_\lambda) = \Gamma(G/B, \mathcal{L}_\lambda)$ is the set of global sections of $\mathcal{L}_\lambda$, and $V(\lambda)^*$ is the dual representation of the finite-dimensional G-module $V(\lambda)$.

REMARK 17.1. Recall from Exercise 11.5 that $V(\lambda)^* \cong V(-w_0\lambda)$ where w_0 is the longest element of the Weyl group.

So Theorem 17.2 gives a geometrical description of any finite-dimensional representation of G, and thus of $\mathfrak{g}$. Indeed, since $\mathfrak{g}$ is semisimple there is always a simply-connected semisimple algebraic group G_{ss} with Lie algebra $\mathfrak{g}$ (see [8, §33.6]). For such a G_{ss}, we have

$$X^*(T) = \Lambda,$$

and thus $X^*(T)_+ = \Lambda_+$ parameterized the finite-dimensional representations of $\mathfrak{g}$.

The statement holds for any reductive groups.

We have also a refinement of Theorem 17.2, due to Bott that we state for cultural purposes only.

Recall that the Weyl group $\mathcal{W} = \mathcal{W}(\Phi)$ acts on $\mathfrak{h}^*$ by the ***twisted action*** given by (see Definition 12.5 in Section 12.4):

$$w \circ \lambda = w(\lambda + \rho) - \rho,$$

for all $w \in \mathcal{W}$ and $\lambda \in \mathfrak{h}^*$ where $w(\lambda + \rho)$ is the usual action. For $w \in \mathcal{W}$, recall that $\ell(w)$ denotes the length of w (see the digression before Exercise 11.5).

Theorem 17.2 – Borel–Weil–Bott

Let G be a semisimple group, B a Borel subgroup of G, T a maximal torus of B. Let $\lambda \in X^*(T)_+$, and set

$$\mathcal{L}_{w\circ\lambda} := (G \times_B \Bbbk_{-w\circ\lambda} \to G/B)$$

the line bundle associated with $w \circ \lambda$. Then we have isomorphisms of G-modules for all $p \in \mathfrak{n}$:

$$H^p(G/B, \mathcal{L}_{w\circ\lambda}) \cong \begin{cases} V(\lambda)^* & \text{if } p = \ell(w), \\ 0 & \text{if } p \neq \ell(w). \end{cases}$$

17.2. Algebraic Peter–Weyl Theorem

Assume that G is semisimple, and consider the G-action on $\Bbbk[G]$ by right multiplication:

$$(g.f)(x) = f(xg), \quad g, x \in G, \ f \in \Bbbk[G].$$

EXERCISE 17.1. Show that $\Bbbk[G]$ as a right G-module is ***locally finite***, that is, for any $f \in \Bbbk[G]$ there is a finite-dimensional submodule of $\Bbbk[G]$ containing f.

Hint: remember that $f = (1 \otimes \varepsilon) \circ \Delta$, where $\varepsilon \colon \Bbbk[G] \to \Bbbk$ is the evaluation map et e and Δ is the comultiplication.

Denote by $\Bbbk[G]_{(\lambda)}$ the $V(\lambda)$-***isotypic component*** of $\Bbbk[G]$, that is, the sum of the simple submodules isomorphic to $V(\lambda)$ as G-modules. As a consequence of Exercise 17.1, we obtain the decomposition of G-modules:

$$\Bbbk[G] = \bigoplus_{\lambda \in X^*(T)_+} \Bbbk[G]_{(\lambda)}. \tag{26}$$

Lemma 17.3

Let (V, σ) be a simple rational G-modules. Then (V^*, σ^*) is simple and $V^* \otimes V$ is simple as a $G \times G$-modules.

PROOF. It is easy to see that V^* is simple if V is using the fact the orthogonal of a submodule in the dual is still a submodule.

Let

$$\theta = \sigma^* \otimes \sigma$$

be the $G \times G$-representation $V^* \otimes V$ induced by σ and σ^*. By Burnside's Theorem 17.4 (see below), the subalgebra of $\mathrm{End}(V)$ (resp. $\mathrm{End}(V^*)$) generated by $\sigma(G)$ (resp. $\sigma^*(G)$) is equal to $\mathrm{End}(V)$ (resp. $\mathrm{End}(V^*)$). Since

$$\mathrm{End}(V^* \otimes V) \cong \mathrm{End}(V^*) \otimes \mathrm{End}(V),$$

we deduce that the subalgebra of $\mathrm{End}(V^* \otimes V)$ generated by $\theta(G \times G)$ is $\mathrm{End}(V^* \otimes V)$ which proves that $V^* \otimes V$ is simple. Indeed, the vector subspace generated by $\theta(G \times G)v$ for any nonzero vector v in $V^* \otimes V$ contains the subalgebra of $\mathrm{End}(V^* \otimes V)$ generated by $\theta(G \times G)$. □

We now recall the Burnside Theorem ([12, Theorem 10.8.11]) used in the above proof.

Theorem 17.4 – Burnside

Let V be a finite dimensional $\Bbbk$-vector space and A be a subalgebra of $\mathrm{End}(V)$. If the only A-stable subspace of V are $\{0\}$ and V, then $A = \mathrm{End}(V)$.

The product $G \times G$ acts on $\Bbbk[G]$ by

$$((g_1, g_2).f)(x) = f(g_1^{-1} x g_2),$$

for $g_1, g_2, x \in G$ and $f \in \Bbbk[G]$.

Theorem 17.5 – Peter–Weyl

We have an isomorphism of $G \times G$-modules

$$\Bbbk[G] \cong \bigoplus_{\lambda \in X^*(T)_+} V(\lambda)^* \otimes V(\lambda),$$

induced by the assignment

$$\begin{array}{rccl} \phi_\lambda \colon V(\lambda)^* \otimes V(\lambda) & \longrightarrow & \Bbbk[G] \\ f \otimes v & \longmapsto & (g \mapsto f(gv)). \end{array}$$

***Fritz Peter**, 1899–1949, was a German mathematician who helped prove the Peter–Weyl theorem. He was a student of Hermann Weyl, and later became headmaster of a secondary school.*

! Peter–Weyl Theorem 17.5 holds for any reductive group.

EXERCISE 17.2. The aim is to prove Theorem 17.5. Set

$$\phi := \sum_{\lambda \in X^*(T)_+} \phi_\lambda.$$

(1) Verify that ϕ is a morphism of $G \times G$-modules, that is,

$$(g_1, g_1).\phi(f \otimes v) = \phi((g_1.f) \otimes (g_2.v)),$$

for all $(g_1, g_2) \in G \times G$, $f \in V(\lambda)^*$ and $v \in V(\lambda)$ for $\lambda \in X^*(T)_+$.

(2) Let $f \in V(\lambda)^* \setminus \{0\}$. Show that $f \otimes V(\lambda)$ is a simple G-module for the G-action given by $1 \otimes g$, isomorphic to $V(\lambda)$. Deduce that $\phi(f \otimes V(\lambda))$ is also isomorphic to $V(\lambda)$ and contained in $\Bbbk[G]_{(\lambda)}$ so that for all $\lambda \in X^*(T)_+$,

$$\Bbbk[G]_{(\lambda)} \neq 0.$$

(3) Let V be a simple submodule of $\Bbbk[G]$, isomorphic to $V(\lambda)$, and let $x \mapsto u_x$ a G-isomorphism $V(\lambda) \to V$. Define $f \in V(\lambda)^*$ by
$$f(x) := u_x(e), \quad x \in V(\lambda).$$
Observing that $\phi(f \otimes x) = u_x$ for all $x \in V(\lambda)$, get that
$$\phi(f \otimes V(\lambda)) = V,$$
and conclude using the question 2 that
$$\phi(V(\lambda)^* \otimes V(\lambda)) = \Bbbk[G]_{(\lambda)}.$$
This proves the surjectivity of ϕ by (26).

(4) Prove that ϕ is injective and conclude.

> Hint: note that if ϕ were not injective then ϕ_λ would be not injective for some λ and use Lemma 17.3.

17.3. Proof of Borel–Weil Theorem

This section is devoted to the proof of Borel–Weil Theorem 17.2.
Consider the line bundle $\mathcal{L} := \mathcal{L}_\lambda$ and the change of basis via $G \to G/B$:

$$\begin{array}{ccc} G \times \Bbbk_{-\lambda} & \longrightarrow & G \times_B \Bbbk_{-\lambda} \\ {\scriptstyle \hat{\pi}} \downarrow & & \downarrow {\scriptstyle \pi} \\ G & \longrightarrow & G/B \end{array}$$

Denote by $\widehat{\mathcal{L}} = \left(G \times \Bbbk_{-\lambda} \xrightarrow{\hat{\pi}} G\right)$ the resulting line bundle (it is the trivial line bundle on G).

Let us compare the sections of $\mathcal{L}$ and $\widehat{\mathcal{L}}$. The group G acts on $G \times \Bbbk_{-\lambda}$ and on $G \times_B \Bbbk_{-\lambda}$ by left multiplication on the left factor. This induces a G-action on the space of sections of $\mathcal{L}$ and $\widehat{\mathcal{L}}$. On the other hand, B acts on $G \times \Bbbk_{-\lambda}$ by
$$b.(g, z) = (gb, \lambda(b^{-1})z),$$
for $b \in B$, $g \in G$ and $z \in \mathbb{C}_{-\lambda}$ and this induces a B-action on space of sections of $\widehat{\mathcal{L}}$.

Lemma 17.6

We have an isomorphism of G-modules,
$$\Gamma(G/B, \mathcal{L}) \cong \Gamma(G, \widehat{\mathcal{L}})^B,$$
where $\Gamma(G/B, \mathcal{L})$ and $\Gamma(G, \widehat{\mathcal{L}})$ are the space of sections of $\mathcal{L}$ and $\widehat{\mathcal{L}}$, respectively.

EXERCISE 17.3. The aim is to prove Lemma 17.6.

(1) Show that
$$\Gamma(G, \widehat{\mathcal{L}}) \cong \Bbbk[G] \otimes \Bbbk_{-\lambda}$$
as G-modules, and describe the B-action on the right factor.

(2) Verify that the sections σ of $\mathcal{L}$ are of the form $\sigma(gB) = [b, f(g)]$ for some regular functions
$$f : G \to \Bbbk_{-\lambda}, \quad \text{i.e.,} \quad f \in \Bbbk[G] \otimes \Bbbk_{-\lambda},$$
and show that in order to have σ well-defined, f must be B invariant for the B-action of Question 1.

(3) Conclude that
$$\Gamma(G/B, \mathcal{L}) \cong \Gamma(G, \widehat{\mathcal{L}})^B.$$

So in order to prove Theorem 17.2 it remains to show that

$$\Gamma(G, \widehat{\mathcal{L}})^B \cong V(\lambda)^*$$

as a G-modules.

Note that $G \times B$ acts on $\Bbbk[G] \times \Bbbk_{-\lambda}$ by

$$(g, b).(f \otimes z) = (g, b).f \otimes \lambda(b^{-1})z,$$

for $g \in G$, $b \in B$, $f \in \Bbbk[G]$ and $z \in \Bbbk_{-\lambda}$. Since we have an isomorphism of G-modules

$$\Gamma(G, \widehat{\mathcal{L}}) \cong \Bbbk[G] \otimes \Bbbk_{-\lambda},$$

and since the G-action commutes with that of B, we get by Peter–Weyl Theorem 17.5,

$$\Gamma(G, \widehat{\mathcal{L}})^B \cong (\Bbbk[G] \otimes \Bbbk_{-\lambda})^B \cong \bigoplus_{\mu \in X^*(T)_+} \left(V(\mu)^* \otimes V(\mu) \otimes \Bbbk_{-\lambda}\right)^B.$$

It is clear that B acts on the last two factors only in the sum, whence

$$\Gamma(G, \widehat{\mathcal{L}})^B \cong \bigoplus_{\mu \in X^*(T)_+} V(\mu)^* \otimes \left(V(\mu) \otimes \Bbbk_{-\lambda}\right)^B.$$

EXERCISE 17.4. Let $\mu \in X^*(T)_+$.

(1) Using the triangular decomposition of $\mathfrak{g}$, observe that $V(\mu)$ has a unique B-stable line, generated by a highest weight vector v_μ with for all $t \in T$,

$$t.v_\mu = \mu(t)v_\mu.$$

(2) Show that

$$\left(V(\mu) \otimes \Bbbk_{-\lambda}\right)^B = \left(\Bbbk_\mu \otimes \Bbbk_{-\lambda}\right)^T.$$

Hint: for $v \otimes z \in \left(V(\mu) \otimes \Bbbk_{-\lambda}\right)^B$, note that the line generated by v must B-stable.

(3) Show that space $\Bbbk_\mu \otimes \Bbbk_{-\lambda}$ can have T-invariants only if $\mu = \lambda$, and that if this is the case then

$$\left(\Bbbk_\lambda \otimes \Bbbk_{-\lambda}\right)^T \cong \mathbb{C}.$$

(4) Conclude that

$$\left(V(\mu) \otimes \Bbbk_{-\lambda}\right)^B \cong \begin{cases} V(\lambda) & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$$

We conclude the proof the the theorem thanks to Exercise 17.4: we have

$$\Gamma(G, \widehat{\mathcal{L}})^B \cong \bigoplus_{\mu \in X^*(T)_+} V(\mu)^* \otimes \left(V(\mu) \otimes \Bbbk_{-\lambda}\right)^B \cong V(\lambda)^*.$$

EXAMPLE 17.1. Let $G = \mathrm{SL}_2(\mathbb{C})$ with Borel subgroup $B = \mathrm{T}_2(\mathbb{C}) \cap \mathrm{SL}_2(\mathbb{C})$ and $T = \mathrm{D}_2(\mathbb{C}) \cap \mathrm{SL}_2(\mathbb{C})$. Then an integral dominant weights $\lambda \in X^*(T)_+$ is of the form

$$\lambda \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^n, \quad n = \lambda(\alpha^\vee) \in \mathbb{N}.$$

We have

$$\Lambda_r = \mathbb{Z}\alpha, \quad X^*(T) = \Lambda = \mathbb{Z}\rho,$$

where $\rho = \dfrac{1}{2}\alpha$. In particular the irreducible (finite-dimensional) representations of $\mathrm{SL}_2(\mathbb{C})$ are in bijection with $\mathbb{N}$. As seen in Example 15.2, the flag variety G/B is isomorphic to $\mathbb{P}^1$. Let $\psi \colon G/B \to \mathbb{P}^1$ as in Example 15.2.

Let us recognize the line bundle $\Gamma(\mathbb{P}^1, \psi_*(\mathbb{C}_{-\lambda}))$ for $\lambda = n\rho$. Recall that the sections of $\Gamma(\mathbb{P}^1, \psi_*(\mathbb{C}_{-\lambda}))$ identify with the sections $f \colon \mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}_{-\lambda}$ such that

$$f(g) = \lambda(b^{-1})f(gb), \quad \text{i.e.,} \quad f(gb) = \lambda(b)f(g),$$

for $g \in G$ and $b \in B$. Consider the open affine subsets $U_0 = \{x_0 \neq 0\}$ and $U_1 = \{x_1 \neq 0\}$ of $\mathbb{P}^1$. A section $\sigma \in \Gamma(U_0, \psi_*(\mathbb{C}_{-\lambda}))$ identifies with a fonction $f\colon \pi^{-1}(U_0) \to \mathbb{C}_{-\lambda}$, where

$$\pi\colon \mathrm{SL}_2(\mathbb{C}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{C})/B \cong \mathbb{P}^1$$

is the canonical projection, such that

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = f\left(\begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = \lambda\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f\begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix}.$$

Since the coracine $\alpha^\vee$ of $\mathrm{SL}_2(\mathbb{C})$ corresponds to the 1-psg

$$\begin{array}{rcl} \alpha^\vee\colon \mathbb{G}_m & \longrightarrow & T \\ t & \longmapsto & \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \end{array}$$

we get

$$\lambda\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = n\rho\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \frac{n}{2}\alpha\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} == a^{\frac{n}{2}\langle \alpha, \alpha^\vee \rangle} = a^n.$$

This means that f, viewed as defined on U_0, verifies:

$$f([a:c]) = a^n f([1:ca^{-1}]$$

for all a, c, $a \neq 0$, that is,

$$f([tx:ty]) = t^n f([x:y]$$

for all x, y, $x \neq 0$. One can argue similarly for U_1. Hence $\psi_*(\mathbb{C}_{-\lambda})$ is isomorphic to the line bundle $\mathcal{O}_{\mathbb{P}^1}(n)$ of $\mathbb{P}^1$ whose sections are the n-degree homogenous polynomials.

In this case, Borel–Weil Theorem 17.2 gives

$$V(n\rho)^* \cong \Gamma(G/B, \mathbb{C}_{-n\rho}) \cong \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) = \mathbb{C}[X,Y]_n,$$

where $\mathbb{C}[X,Y]_n$ is the space of n-degree homogeneous polynomials in variables X and Y. Denoting $\mathbb{C}^2$ the standard representation of $\mathfrak{sl}_2(\mathbb{C})$ we know that (see Remark 3.3 and Example 16.2):

$$V_{n\rho} \cong V_{n\rho}^* \cong \mathbb{C}[X,Y]_n \cong \mathrm{Sym}_n^*(\mathbb{C}^2).$$

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