

Lie algebras, vertex algebras and applications

1. Affine Kac-Moody algebras

Problem 1 (Structure of affine Kac-Moody algebras)**.** Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a simple Lie algebra with root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, and $\widehat{\mathfrak{g}}$ the *affine Kac-Moody algebra* defined by $\widehat{\mathfrak{g}} := \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$, with the commutation relations:

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}(x|y)K,$$
$$[K, \widehat{\mathfrak{g}}] = 0,$$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. Here $(\ |\ )$ is a nondegenerate invariant bilinear form of $\mathfrak{g}$ such that $(\theta|\theta) = 2$, where θ is the highest positive root.

(1) Explain how the Lie algebra $\widehat{\mathfrak{g}}$ can be presented by generators $(E_i)_{0\leqslant i\leqslant r}$, $(F_i)_{0\leqslant i\leqslant r}$, $(H_i)_{0\leqslant i\leqslant r}$, and relations

$$[H_i, H_j] = 0,$$
$$[E_i, F_j] = \delta_{i,j}H_i,$$
$$[H_i, E_j] = C_{i,j}E_j, \quad [H_i, F_j] = -C_{i,j}F_j,$$
$$(\operatorname{ad} E_i)^{1-C_{i,j}}E_j = 0, \quad (\operatorname{ad} F_i)^{1-C_{i,j}}F_j = 0 \text{ for } i \neq j,$$

where $\hat{C} = (C_{i,j})_{0\leqslant i\leqslant r}$ is an *affine Cartan matrix*, that is, $\hat{C}$ satisfies the following relations the following properties,

$$C_{i,i} = 2 \text{ for all } i,$$
$$C_{i,j} \in \mathbb{Z} \text{ and } C_{i,j} \leqslant 0 \text{ if } i \neq j,$$
$$C_{i,j} = 0 \text{ if and only if } C_{j,i} = 0,$$

all proper principal minors are strictly positive, i.e.,

$$\det\left((C_{i,j})_{0\leqslant i,j\leqslant s}\right) > 0 \quad \text{for} \quad 0 \leqslant s \leqslant r-1,$$

and $\det(\hat{C}) = 0$.

Note that one can choose the labelling $\{0, \dots, r\}$ so that the subalgebra generated by $(E_i)_{1\leqslant i\leqslant r}$, $(F_i)_{1\leqslant i\leqslant r}$, $(H_i)_{1\leqslant i\leqslant r}$ is isomorphic to $\mathfrak{g}$, that is, $(C_{i,j})_{1\leqslant i\leqslant r}$ is the Cartan matrix C of $\mathfrak{g}$.

Detail the construction for $\widehat{\mathfrak{sl}_2}$.

(2) By considering the *extended affine Lie algebra*:

$$\widetilde{\mathfrak{g}} := \widehat{\mathfrak{g}} \oplus \mathbb{C}D,$$

with commutation relations (apart from those of $\widehat{\mathfrak{g}}$),

$$[D, xt^m] = mxt^m, \qquad [D, K] = 0, \qquad x \in \mathfrak{g},\ m \in \mathbb{Z},$$

describe the *root system* $\hat{\Delta}$ of $\widetilde{\mathfrak{g}}$ associated with the Cartan subalgebra

$$\widetilde{\mathfrak{h}} := \widehat{\mathfrak{h}} \oplus \mathbb{C}D = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

Give a precise description for $\widehat{\mathfrak{sl}_2}$. What is the dimension of $\dim \widehat{\mathfrak{g}}_\alpha = \{x \in \widehat{\mathfrak{g}} \colon [h, x] = \alpha(h)x \text{ for all } h \in \widetilde{\mathfrak{h}}\}$ for $\alpha \in \hat{\Delta}$?

(References: [5, Chapter 7], [1, Appendix A].)

Problem 2 (Casimir operator, highest weight modules and the BGG category $\mathscr{O}$)**.** Let $\widetilde{\mathfrak{g}}$ be as in the previous problem: the bilinear form $(\,|\,)$ extends to an invariant nondegenerate bilinear form on $\widetilde{\mathfrak{g}}$ (see [5, Theorem 2.2]). Let $\hat{\rho} \in \widetilde{\mathfrak{h}}^*$ be such that

$$\hat{\rho}(\alpha_i^\vee) = 1 \quad \text{for} \quad i = 0, \dots, r.$$

For $\alpha \in \hat{\Delta}_+$, choose a basis $\{e_\alpha^{(1)}, \dots, e_\alpha^{(k)}\}$ of $\widehat{\mathfrak{g}}_\alpha$ such that $\{e_{-\alpha}^{(1)}, \dots, e_{-\alpha}^{(k)}\}$ is the basis of $\widehat{\mathfrak{g}}_\alpha$ dual of $\{e_\alpha^{(1)}, \dots, e_\alpha^{(k)}\}$ with respect to $(\,|\,)$. Let $\{u_i\}_{i\in I}$ and $\{u^i\}_{i\in I}$ be dual basis of $\widetilde{\mathfrak{h}}$ with respect to $(\,|\,)$. Set

$$\Omega := 2 \sum_{\alpha \in \hat{\Delta}_+} \sum_j e_{-\alpha}^{(j)} e_\alpha^{(j)} + 2\nu^{-1}(\hat{\rho}) + \sum_i u_i u^i,$$

where $\nu \colon \widetilde{\mathfrak{h}} \to \widetilde{\mathfrak{h}}^*$ is the isomorphism induced by $(\,|\,)$. ⚠ *Ω is not an element of $U(\widetilde{\mathfrak{g}})$ (infinite sum) in general.*

Let M be a representation of $\widetilde{\mathfrak{g}}$ such that:

(1) M is finitely generated as $U(\widetilde{\mathfrak{g}})$-module,
(2) M is $\widetilde{\mathfrak{h}}$-diagonalizable, that is, $M = \bigoplus_{\lambda \in \widetilde{\mathfrak{h}}^*} M_\lambda$, where

$$M_\lambda = \{m \in M \colon h.m = \lambda(h)m \text{ for all } h \in \widetilde{\mathfrak{h}}\},$$

(3) the action of $\widehat{\mathfrak{n}}_+$ on M is *locally finite*, that is, for all $m \in M$, $U(\widehat{\mathfrak{n}}_+).m$ is finite-dimensional.

The representations of $\widetilde{\mathfrak{g}}$ satisfying (1), (2), (3) are the objects of a full category of the category of $\widetilde{\mathfrak{g}}$-modules called the *BGG (Bernstein-Gelfand-Gelfand) category $\mathscr{O}$*, see [4, Chapter 1].

(1) Show that $\Omega|_M$ is a well-defined element of $\mathrm{End}(M)$.
(2) Show that $\Omega|_M$ commutes with the action of any $x \in \widetilde{\mathfrak{g}}$.
(3) Let $v \in M_\lambda$ be a singular vector. Show that

$$\Omega v = 2(\rho|\lambda)v + \left(\sum_{i\in I} u_i u^i\right) v = (\lambda + 2\rho|\lambda)v.$$

(4) Deduce that if $M = U(\widehat{\mathfrak{n}}_-).v$ with v as in (3), then

$$\Omega = (\lambda + 2\rho|\lambda)\,\mathrm{Id}_M.$$

(References: [5, Chapter 2], [4, Chapter 1].)

2. Vertex algebras

Problem 3 (commutative vertex algebras)**.** Show that a differential algebra R with a derivation ∂ carries a canonical commutative vertex algebra structure such that the vacuum vector is the unit, and

$$Y(a,z)b = \left(e^{z\partial}a\right)b = \sum_{n\geqslant 0} \frac{z^n}{n!}(\partial^n a)b \quad \text{for all} \quad a, b \in R.$$

Give various examples of commutative vertex algebras: coming from the *arc space of an affine variety*, from the graded algebra of a vertex algebra using the *Li filtration*, etc.
(Reference: [1, Part I, Sections 1 et 2 & Part II, Section 4].)

Problem 4 (Heisenberg vertex algebra and Fock space)**.** Let $\mathscr{B}$ be the unital associative algebra generated by elements b_n, for $n \in \mathbb{Z}$, with relations

$$[b_m, b_n] = m\delta_{m+n,0}, \quad m, n \in \mathbb{Z}.$$

(1) Verify that

$$[b(z), b(w)] = \partial_w \delta(z-w)$$

so that $b(z)$ is local to itself and

$$b(z)b(w) \sim \frac{1}{(z-w)^2}.$$

(2) For $\alpha \in \mathbb{C}$, set

$$L(z) = \frac{1}{2} {:}b(z)^2{:} + \alpha \partial_z b(z).$$

Show that $L(z)$ is local to $b(z)$ and itself, and prove the following OPEs:

$$L(z)b(w) \sim -\frac{2\alpha}{(z-w)^3} + \frac{b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{(z-w)},$$
$$L(z)L(w) \sim \frac{(1-12\alpha^2)/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)}.$$

(3) A $\mathscr{B}$-module M is called *smooth* if for each $m \in M$ there exits an integer N such that $b_n m = 0$ for $n > N$. If M is a smooth $\mathscr{B}$-module,

$$b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-1}$$

is a field on M.

Let M be a smooth $\mathscr{B}$-module. Show that the following correspondence gives the vertex algebra $\langle b(z) \rangle_M$ a $\mathscr{B}$-module structure:

$$\mathscr{B} \to \operatorname{End}(\langle b(z) \rangle_M) \quad b_n \mapsto b(z)_{(n)}$$

(4) Let

$$\pi = \mathbb{C}[b_{-1}, b_{-2}, \dots,].$$

Verify that π is a smooth $\mathscr{B}$-module on which b_n, $n \geq 0$, acts as $n\frac{\partial}{\partial b_{-n}}$, and b_{-n}, $n > 0$, acts as multiplication by b_{-n}. Define

$$T = \sum_{n>0} n b_{-n-1} \frac{\partial}{\partial b_{-n}} \in \operatorname{End} \pi.$$

Show that there is a unique vertex algebra structure on π such that 1 is the vacuum vector and $Y(b_{-1}, z) = b(z)$.

(5) Show that there is a surjective homomorphism $\pi \to \langle b(z) \rangle_M$ of vertex algerbas.

(6) Set $\omega = \frac{1}{2} b_{-1}^2 + \alpha b_{-2} \in \pi$, so that $L(z) = Y(\omega, z)$. Verify that the OPEs of the questions (2) are equivalent to the following relations:

$$b_0 \omega = 0, \quad b_1 \omega = b_{-1}, \quad b_2 \omega = 2\alpha.$$

(7) Show that $L_{-1} = T$ on π.

(8) Show that the vertex algebra π is simple, that is, there is no non-trivial ideal of π. This implies that the vertex algebra $\langle b(z) \rangle_M$ of local fields on *any* non-trivial smooth $\mathscr{B}$-module M is isomorphic to π.

(References: [1, Part I, Section 2], [6].)

Problem 5 (Sugawara construction)**.** *The aim of the problem is to give to the universal affine vertex $V^k(\mathfrak{g})$ at a non-critical level a conformal structure.*

(1) Let $V^\kappa(\mathfrak{h})$ be the Heisenberg vertex algebra associated with a r-dimensional commutative Lie algebra $\mathfrak{h}$ and a nondegenerate bilinear form κ.

1.1 Show that

$$T(z) = \frac{1}{2} \sum_{i=1}^{r} {:}x_i(z) x^i(z){:}$$

is a *stress tensor* (i.e., conformal vector) for $V^\kappa(\mathfrak{h})$ with central charge r, where $\{x_i\}_{1 \leqslant i \leqslant r}$ and $\{x^i\}_{1 \leqslant i \leqslant r}$ are dual basis of $\mathfrak{h}$ with respect to κ.

1.2 Observe that if $r = 1$, we have $V^\kappa(\mathfrak{h}) \cong \pi$. Then compare the construction of T with that of the previous problem.

(2) Let $V^k(\mathfrak{g})$ be the universal affine vertex algebra associated with $\mathfrak{g}$ simple, and the invariant nondegenerate bilinear form

$$(\,|\,) = \frac{1}{2h^\vee} \times \text{Killing form},$$

where $h^\vee$ is the *dual Coxeter number*. Set

$$S = \frac{1}{2}\sum_{i=1}^{\dim\mathfrak{g}} x_{i,(-1)}x^i_{(-1)}|0\rangle,$$

where $\{x_i\}_{1\leqslant i\leqslant \dim\mathfrak{g}}$ and $\{x^i\}_{1\leqslant i\leqslant \dim\mathfrak{g}}$ are dual basis with respect to $(\,|\,)$.

(3) For $k \neq -h^\vee$, show that $L = \dfrac{S}{k+h^\vee}$ is a stress tensor of $V^k(\mathfrak{g})$ with central charge

$$c(k) = \frac{k\dim\mathfrak{g}}{k+h^\vee}.$$

This construction is referred to as the *Sugawara construction*.

(4) Verify that

$$[L_m, x_{(n)}] = -nx_{(m+n)} \quad x\in\mathfrak{g},\ m,n\in\mathbb{Z}.$$

(References: [1, Part I, Section 2], [2, Section 3], [6, Theorem 5.7].)

Problem 6 (center of the universal affine vertex algebra)**.**

(1) Let W be a vertex subalgebra of a vertex algebra V. We set

$$\mathrm{Com}(W,V) = \{v\in V\colon [w_{(m)}, v_{(n)}] = 0 \text{ for all } w\in W,\ m,n\in\mathbb{Z}\}.$$

Show that

$$\mathrm{Com}(W,V) = \{v\in V\colon w_{(n)}v = 0 \text{ for all } w\in W, n\geqslant 0\}.$$

(2) Let V be a vertex algebra, and suppose that there exists a vertex algebra homomorphism $\phi\colon V^\kappa(\mathfrak{g})\to V$, so that V is a $\widehat{\mathfrak{g}}_\kappa$-module. Show that

$$\mathrm{Com}(\phi(V^\kappa(\mathfrak{g})), V) = V^{\mathfrak{g}[t]},$$

where $V^{\mathfrak{g}[t]} = \{v\in V\colon \mathfrak{g}[t]v = 0\}$. Show that we have the following isomorphism of commutative $\mathbb{C}$-algebras (the product on the commutative vertex algebra $Z(V^k(\mathfrak{g}))$ is the normally ordered product):

$$Z(V^k(\mathfrak{g})) \cong \mathrm{End}_{\widehat{\mathfrak{g}}}(V^k(\mathfrak{g})).$$

It is easily seen that $Z(V^k(\mathfrak{g})) = \mathbb{C}|0\rangle$ for $k\neq -h^\vee$ using the stress tensor L. For $k=-h^\vee$, the center

$$Z(V^{-h^\vee}(\mathfrak{g})) =: \mathfrak{z}(\widehat{\mathfrak{g}})$$

is "huge", and it is usually referred as the *Feigin-Frenkel center*: we have

$$\mathrm{gr}\,\mathfrak{z}(\widehat{\mathfrak{g}}) \cong \mathscr{O}(\mathscr{J}_\infty(\mathfrak{g}/\!/G)),$$

where $\mathfrak{g}/\!/G = \mathrm{Spec}\mathscr{O}(\mathfrak{g})^G$ and $\mathscr{J}_\infty(\mathfrak{g}/\!/G)$ is the arc space of $\mathfrak{g}/\!/G$.

(References: [1, Part I, Section 2], [2, Section 18].)

3. Associated varieties of vertex algebras

Problem 7 (simple affine vertex algebras associated with $\mathfrak{sl}_2$)**.** Let N_k be the proper maximal ideal of $V^k(\mathfrak{sl}_2)$ so that $L_k(\mathfrak{sl}_2) = V^k(\mathfrak{sl}_2)/N_k$. Let I_k be the image of N_k in $R_{V^k(\mathfrak{sl}_2)} = \mathbb{C}[\mathfrak{sl}_2]$. It is known that either N_k is trivial, that is, $V^k(\mathfrak{sl}_2)$ is simple, or N_k is generated by a singular vector v whose image $\overline{v}$ in I_k is nonzero ([7, 10]).

We assume in this exercise that N_k is non trivial. Thus, $N_k = U(\widehat{\mathfrak{sl}_2})v$.

(1) Using *Kostant's Separation Theorem* ([8]) show that, up to a nonzero scalar,

$$\overline{v} = \Omega^m e^n,$$

for some $m,n\in\mathbb{Z}_{>0}$, where $\Omega = 2ef + \frac{1}{2}h^2$ is the Casimir element of the symmetric algebra of $\mathfrak{sl}_2$.

(2) Deduce from this that $X_{L_k(\mathfrak{sl}_2)}$ is contained in the nilpotent cone $\mathscr{N}$ of $\mathfrak{sl}_2$.

It is known that N_k is nontrivial if and only if k is an admissible level *for $\mathfrak{sl}_2$, that is,*

$$k = -2 + p/q, \quad p,q\in\mathbb{Z}_{>0},\ (p,q)=1,\ p\geqslant 2,$$

or $k=-2$ is critical. Since $X_{L_k(\mathfrak{sl}_2)} = \{0\}$ if and only if $k\in\mathbb{Z}_{\geqslant 0}$, we get that $X_{L_k(\mathfrak{sl}_2)} = \mathscr{N}$ if and only if $k=-2$ or k is admissible and $k\notin\mathbb{Z}_{\geqslant 0}$.

Problem 8 (an explicit computation of an associated variety)**.** The aim of this exercise is to compute $X_{L_{-3/2}(\mathfrak{sl}_3)}$. It is known ([12]) that the proper maximal ideal of $V^{-3/2}(\mathfrak{sl}_3)$ is generated by the singular vector v given by:

$$v := \frac{1}{3}\left((h_1t^{-1})(e_{1,3}t^{-1})|0\rangle - (h_2t^{-1})(e_{1,3}t^{-1})|0\rangle\right) + (e_{1,2}t^{-1})(e_{2,3}t^{-1})|0\rangle - \frac{1}{2}e_{1,3}t^{-2}|0\rangle,$$

where $h_1 := e_{1,1} - e_{2,2}$, $h_2 := e_{2,2} - e_{3,3}$ and $e_{i,j}$ is the elementary matrix of the coefficient (i, j) in $\mathfrak{sl}_3$ identified with the set of traceless 3-size square matrices.

(1) Verify that v is indeed a singular vector for $\widehat{\mathfrak{sl}_3}$.
(2) Let $\mathfrak{h} := \mathbb{C}h_1 + \mathbb{C}h_2$ be the usual Cartan subalgebra of $\mathfrak{sl}_3$. Show that $X_{L_{-3/2}(\mathfrak{sl}_3)} \cap \mathfrak{h} = \{0\}$, and deduce from this that $X_{L_{-3/2}(\mathfrak{sl}_3)}$ is contained in the nilpotent cone of $\mathfrak{sl}_3$.
(3) Show that the nilpotent cone is not contained in $X_{L_{-3/2}(\mathfrak{sl}_3)}$.
(4) Denoting by $\mathbb{O}_{min}$ the minimal nilpotent orbit of $\mathfrak{sl}_3$, conclude that

$$X_{L_{-3/2}(\mathfrak{sl}_3)} = \overline{\mathbb{O}_{min}}.$$

References

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