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# Filtration principale de Kostant et chemins dans des réseaux de poids

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# Contents

# Introduction

In this thesis, we are interested in several increasing filtrations on the Cartan subalgebra $\mathfrak{h}$ of a complex simple Lie algebra $\mathfrak{g}$ with triangular decomposition,

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

that come from different contexts. Here, by *increasing filtration* we mean an increasing sequence $(F^m\mathfrak{h})_{m\geqslant 0}$ of subspaces of $\mathfrak{h}$ such that $\mathfrak{h} = \bigcup_{m\geqslant 0} F^m\mathfrak{h}$. We usually require that for all $m, n$, $[F^m\mathfrak{h}, F^n\mathfrak{h}] \subset F^{m+n}\mathfrak{h}$ but since $\mathfrak{h}$ is abelian, this condition is superfluous.

The different filtrations we consider are described in the next paragraph: one is the *principal filtration* which comes from the Langlands dual of $\mathfrak{g}$ (cf. Definition 1), one is the *symmetric filtration* (cf. Definition 2) which comes from the symmetric algebra and the Chevalley projection, two other ones, the *enveloping filtrations* (cf. Definition 3), come from the enveloping algebra and the Harish-Chandra projections, and the last one is the *Clifford filtration* (cf. Definition 4) which comes from the Clifford algebra of $\mathfrak{g}$ associated with the inner product on $\mathfrak{g}$.

It is known that all these filtrations coincide. These results come from a combination of works of several authors [Roh08, AM12, Jos12a, Jos12b]. The remarkable connexion between the principal filtration and the filtration coming from the Clifford algebra was essentially conjectured by Kostant.

The goal of this thesis is to provide another proof of the existing relation between the symmetric filtration $(\mathscr{F}_S^{(m)}\mathfrak{h})_m$ and the enveloping filtration $(\mathscr{F}_U^{(m)}\mathfrak{h})_m$, when $\mathfrak{g}$ is the simple Lie algebra $\mathfrak{sl}_{r+1}$ (type $A_r$) or the simple Lie algebra $\mathfrak{sp}_{2r}$ (type $C_r$), using special paths in the weight lattice of the standard representation. Together with Rohr's result (cf. Theorem 1) it gives another proof of Joseph's theorem (cf. Theorem 2) in these cases. The strategy is outlined at the end of this introduction.

Let us now describe in more detail the several filtrations above mentioned.

## The principal filtration

Let $\Delta \subset \mathfrak{h}^*$ be the root system of $(\mathfrak{g}, \mathfrak{h})$, $\Pi = \{\beta_1, \dots, \beta_r\}$ the system of simple roots with respect to $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ and $\Delta_+$ the corresponding set of positive roots. The root system is realized in an Euclidean space $\mathbb{R}^N$ with standard basis $\mathcal{E} = (\varepsilon_1, \dots, \varepsilon_N)$. For $\alpha \in \Delta$, we denote by $\check{\alpha}$ its coroot. We fix a Chevalley basis $\{e_\alpha, e_{-\alpha}, \check{\beta}_i \,;\, \alpha \in \Delta_+, i = 1, \dots, r\}$ of $\mathfrak{g}$, where $e_\alpha$ is a nonzero $\alpha$-root vector. In particular, $[e_{\beta_i}, e_{-\beta_i}] = \check{\beta}_i$ for $i = 1, \dots, r$. Let $\varpi_1, \dots, \varpi_r$ be the fundamental weights, and $\check{\varpi}_1, \dots, \check{\varpi}_r$ the fundamental co-weights, associated with $\beta_1, \dots, \beta_r$, respectively.

Let $B_\mathfrak{g}$ be an invariant non-degenerate bilinear form on $\mathfrak{g} \times \mathfrak{g}$. It is a nonzero multiple of the Killing form of $\mathfrak{g}$. Moreover, its restriction to $\mathfrak{h} \times \mathfrak{h}$ is non-degenerate. Let $B_\mathfrak{g}^\sharp \colon \mathfrak{h}^* \to \mathfrak{h}$ be the induced isomorphism. For $x \in \mathfrak{h}^*$ we denote by $x^\sharp$ its image by $B_\mathfrak{g}^\sharp$. If $\mathfrak{g}$ is simply-laced, that is, $\mathfrak{g}$ is of type $A, D, E$, then for some nonzero scalar $\lambda$, $\varpi_i^\sharp = \lambda \check{\varpi}_i$ for all $i \in \{1, \dots, r\}$.

Let $(e, h, f)$ be a principal $\mathfrak{sl}_2$-triple of $\mathfrak{g}$ corresponding to the above triangular decomposition, that is,

$$e = \sum_{i=1}^r e_{\beta_i}, \qquad h = 2 \sum_{i=1}^r \check{\varpi}_i, \qquad f = \sum_{i=1}^r c_i e_{-\beta_i},$$

where $c_i$ is a nonzero complex number such that $h = [e, f]$. The elements $e, h, f$ are regular elements of $\mathfrak{g}$, which means that their centralizer in $\mathfrak{g}$ have minimal dimension $r$.

One defines an increasing filtration $(\mathscr{F}^{(m)}\mathfrak{h})_{m \geqslant 0}$ of $\mathfrak{h}$ by:

$$\mathscr{F}^{(m)}\mathfrak{h} := \{x \in \mathfrak{h} \mid (\operatorname{ad} e)^{m+1} x = 0\}.$$

Notice that the dimension of the spaces $\mathscr{F}^{(m)}\mathfrak{h}$ jumps at the exponents $m = m_1, \dots, m_r$ of $\mathfrak{g}$.

We can also describe the filtration $(\mathscr{F}^{(m)}\mathfrak{h})_m$ as follows. The algebra $\mathfrak{s} := \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f \cong \mathfrak{sl}_2$ acts on $\mathfrak{g}$ by the adjoint action. Let $\mathfrak{g} = \bigoplus_{i=1}^r V_i$ be the decomposition of $\mathfrak{g}$ into simple $\mathfrak{s}$-modules. We have $\dim V_i = 2m_i + 1$ and $\dim V_i \cap \mathfrak{h} = 1$ for any $i = 1, \dots, r$. Then

$$\mathscr{F}^{(m)}\mathfrak{h} = \bigoplus_{j,\, m_j \leqslant m} V_j \cap \mathfrak{h}.$$

If the exponents $m_1, \dots, m_r$ are pairwise distinct, we have for $i = 1, \dots, r$,

$$\mathscr{F}^{(m_i)}\mathfrak{h} = \bigoplus_{j=1}^{i} V_j \cap \mathfrak{h}.$$

In most cases, the exponents $m_1, \dots, m_r$ are all distinct. The exception is the case of the $D_r$ series, for even $r$ and $r \geqslant 4$, when there are two coincident exponents (equal to $r-1$).

Let $\check{\mathfrak{g}}$ be the Langlands dual of $\mathfrak{g}$ which is the simple Lie algebra defined by the dual root system $\check{\Delta} = \{\check{\alpha}\ ;\ \alpha \in \Delta\}$. One may identify a Cartan subalgebra $\check{\mathfrak{h}}$ of $\check{\mathfrak{g}}$ with $\mathfrak{h}^*$. Let $(\check{e}, \check{h}, \check{f})$ be the corresponding principal $\mathfrak{sl}_2$-triple of $\check{\mathfrak{g}}$, and let $\rho$ be the half-sum of positive roots. Note that

$$\rho = \sum_{i=1}^{r} \varpi_i = \frac{1}{2}\check{h}.$$

The principal $\mathfrak{sl}_2$-triple $(\check{e}, \check{h}, \check{f})$ defines an increasing filtration $(\mathscr{F}^{(m)}\check{\mathfrak{h}})_m$ of $\check{\mathfrak{h}}$:

$$\mathscr{F}^{(m)}\check{\mathfrak{h}} := \{x \in \check{\mathfrak{h}} \mid (\operatorname{ad} \check{e})^{m+1}x = 0\} \subset \check{\mathfrak{h}} \cong \mathfrak{h}^*.$$

Since $\check{\mathfrak{g}}$ has the same exponents as $\mathfrak{g}$, the dimension of $\mathscr{F}^{(m)}\check{\mathfrak{h}}$ jumps at the exponents $m = m_1, \dots, m_r$, too.

**Definition 1** (the principal filtration)**.** *The* principal filtration *of* $\mathfrak{h}$ *is the filtration* $(\check{\mathscr{F}}^{(m)}\mathfrak{h})_m$*, where*

$$\check{\mathscr{F}}^{(m)}\mathfrak{h} := \left(B_{\mathfrak{g}}^{\sharp}(\mathscr{F}^{(m)}\check{\mathfrak{h}})\right)_m.$$

Since $[\check{e}, [\check{e}, \check{h}]] = 0$, we have $\rho \in \mathscr{F}^{(1)}\check{\mathfrak{h}}$ and $\rho^\sharp = B_{\mathfrak{g}}^{\sharp}(\rho) \in \check{\mathscr{F}}^{(1)}\mathfrak{h}$.

# The symmetric filtration

Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$. We have $S(\mathfrak{g}) = S(\mathfrak{h}) \oplus (\mathfrak{n}_- + \mathfrak{n}_+)S(\mathfrak{g})$. Let $\mathrm{Ch}\colon S(\mathfrak{g}) \to S(\mathfrak{h})$ be the *Chevalley projection map* which is the corresponding projection onto $S(\mathfrak{h})$. Let $S(\mathfrak{g})^{\mathfrak{g}}$ be the space of $\mathfrak{g}$-invariant elements of $S(\mathfrak{g})$. It is well-known that the image of $S(\mathfrak{g})^{\mathfrak{g}}$ by Ch is $S(\mathfrak{h})^W$, with $W$ the Weyl group of $(\mathfrak{g},\mathfrak{h})$.

Let $V$ be a finite-dimensional simple $\mathfrak{g}$-module. Consider the map,

$$\mathrm{Ch}\otimes 1\colon S(\mathfrak{g})\otimes V\to S(\mathfrak{h})\otimes V.$$

We have $(\mathrm{Ch}\otimes 1)((S(\mathfrak{g})\otimes V)^{\mathfrak{g}})\subset S(\mathfrak{h})\otimes V_0$, where $(S(\mathfrak{g})\otimes V)^{\mathfrak{g}}$ is the subspace of invariants under the diagonal action of $\mathfrak{g}$, and $V_0=\{v\in V\mid h.v=0\ \text{ for all } h\in\mathfrak{h}\}$ is the zero-weight space of $V$.

In the case where $V=\mathfrak{g}$ is the adjoint representation, the free $S(\mathfrak{g})^{\mathfrak{g}}$-module $(S(\mathfrak{g})\otimes\mathfrak{g})^{\mathfrak{g}}$ is generated by the differentials $\mathrm{d}p_1,\dots,\mathrm{d}p_r$ of homogeneous generators $p_1,\dots,p_r$ of $S(\mathfrak{g})^{\mathfrak{g}}$. Such homogeneous generators are of degrees $m_1+1,\dots,m_r+1$, respectively, if we order them by increasing degrees. We also have in this case, $(\mathrm{Ch}\otimes 1)((S(\mathfrak{g})\otimes\mathfrak{g})^{\mathfrak{g}})\subset S(\mathfrak{h})\otimes\mathfrak{h}$, since the zero weight subspace of the adjoint representation $\mathfrak{g}$ is $\mathfrak{h}$.

Let $(S^m(\mathfrak{g}))_m$ be the standard filtration on $S(\mathfrak{g})$ induced by the degree of elements. The symmetric algebra $S(\mathfrak{h})$ is canonically isomorphic to $\mathbb{C}[\mathfrak{h}^*]$. Let $\mathrm{ev}_\rho$ denotes the evaluation map at $\rho$, that is,

$$\mathrm{ev}_\rho\colon S(\mathfrak{h})\cong\mathbb{C}[\mathfrak{h}^*]\to\mathbb{C},\quad x\mapsto\langle\rho,x\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$.

**Definition 2** (the symmetric filtration)**.** *Set for $m\in\mathbb{Z}_{\geqslant 0}$,*

$$\mathscr{F}_S^{(m)}\mathfrak{h}:=(\mathrm{ev}_\rho\otimes 1)\circ(\mathrm{Ch}\otimes 1)((S^m(\mathfrak{g})\otimes\mathfrak{g})^{\mathfrak{g}}).$$

*We refer $(\mathscr{F}_S^{(m)}\mathfrak{h})_m$ as the* symmetric filtration *of $\mathfrak{h}$.*

This is indeed a filtration of $\mathfrak{h}$ since a famous result of Kostant [Kos63] asserts that the elements $\mathrm{d}p_1(x),\dots,\mathrm{d}p_r(x)$ are linearly independent if and only if $x\in\mathfrak{g}^*$ is *regular*, that is, its stablizer in $\mathfrak{g}$ for the coadjoint action has minimal dimension $r$. But it is well-known that $\rho$ is regular.

More specifically, we have the following statement, proved by Rohr:

**Theorem 1** ([Roh08])**.** *For any $m\in\mathbb{Z}_{\geqslant 0}$, we have:*

$$\mathscr{F}_S^{(m)}\mathfrak{h}=\check{\mathscr{F}}^{(m)}\mathfrak{h}.$$

*In other words, the symmetric filtration coincides with the principal filtration.*

Rohr obtained a more precise result, and explicitly described an orthogonal basis with respect to $B_{\mathfrak{g}}$ in $\mathfrak{h}$ from the algebra of invariants $(S(\mathfrak{g}) \otimes \mathfrak{g})^{\mathfrak{g}}$, cf. [Roh08].

**Remark 1.** *We have a similar statement for any representation $V$ when $\mathfrak{g}$ is simply laced; see [Jos12b, §3.3].*

## The enveloping filtration(s)

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and let

$$\mathrm{hc}\colon U(\mathfrak{g}) \to U(\mathfrak{h}) = S(\mathfrak{h})$$

be the Harish-Chandra map which is the projection map from $U(\mathfrak{g})$ to $U(\mathfrak{h})$ with respect to the decomposition,

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}_{-}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_{+}).$$

Its restriction to

$$U(\mathfrak{g})^{\mathfrak{h}} = \{u \in U(\mathfrak{g}) \mid (\mathrm{ad}\, h)u = 0 \text{ for all } h \in \mathfrak{h}\}$$

is a morphism of associative algebras. Moreover, it is well-known that the restriction of hc to the center $Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}}$ of $U(\mathfrak{g})$ is an isomorphism of commutative algebras whose image is $S(\mathfrak{h})^{W_\circ}$ ([HC51]). Here, the $\circ$-action on $\mathfrak{h}^*$, which induces an action of $\mathbb{C}[\mathfrak{h}^*] \cong S(\mathfrak{h})$, is given by:

$$w \circ \lambda = w(\lambda + \rho) - \rho, \qquad \text{for all } \lambda \in \mathfrak{h}^*,\ w \in W,$$

and $S(\mathfrak{h})^{W_\circ}$ stands for the ring of $W$-invariants of $S(\mathfrak{h})$ for the the $\circ$-action.

In [KNV11], Khoroshkin, Nazarov and Vinberg established the following triangular decomposition:

$$U(\mathfrak{g}) \otimes V = (S(\mathfrak{h}) \otimes V) \oplus (\rho_L(\mathfrak{n}_{-})(U(\mathfrak{g}) \otimes V) + \rho_R(\mathfrak{n}_{+})(U(\mathfrak{g}) \otimes V)),$$

where $\rho_L$ and $\rho_R$ are the two commuting $\mathfrak{g}$-actions on $U(\mathfrak{g}) \otimes V$ given by: $\rho_L(x)(a \otimes b) = xa \otimes b$ and $\rho_R(x)(a \otimes b) = -ax \otimes b + a \otimes x.b$ for $x \in \mathfrak{g}$, respectively. Consider the *generalized Harish-Chandra projection,*

$$\widetilde{\mathrm{hc}}\colon U(\mathfrak{g})\otimes V\to S(\mathfrak{h})\otimes V,$$

with respect to the above triangular decomposition. Khoroshkin, Nazarov and Vinberg showed that the image of the invariant part $(U(\mathfrak{g})\otimes V)^{\mathfrak{g}}$ for the diagonal action $\rho=\rho_L+\rho_R$, which is contained in $S(\mathfrak{h})\otimes V_0$, is the space of invariant under all the *Zhelobenko operators* (cf. [KNV11, Theorem 1]).

We now consider again the special case where $V=\mathfrak{g}$ is the adjoint representation. Then we get that $(\mathrm{hc}\otimes 1))((U^m(\mathfrak{g})\otimes\mathfrak{g})^{\mathfrak{g}})\subset(S(\mathfrak{h})\otimes\mathfrak{g})^{\mathfrak{g}}\subset S(\mathfrak{h})\otimes\mathfrak{h}$ since, as noted before, $\mathfrak{h}$ is the zero weight subspace of the adjoint representation $\mathfrak{g}$. Let $(U^m(\mathfrak{g}))_m$ be the standard filtration of $U(\mathfrak{g})$.

**Definition 3** (the enveloping filtrations). *Set for $m\in\mathbb{Z}_{\geqslant 0}$,*

$$\mathscr{F}_U^{(m)}\mathfrak{h}:=((\mathrm{ev}_\rho\otimes 1)\circ(\mathrm{hc}\otimes 1))((U^m(\mathfrak{g})\otimes\mathfrak{g})^{\mathfrak{g}}),$$

$$\widetilde{\mathscr{F}}_U^{(m)}\mathfrak{h}:=((\mathrm{ev}_\rho\otimes 1)\circ\widetilde{\mathrm{hc}})((U^m(\mathfrak{g})\otimes\mathfrak{g})^{\mathfrak{g}}).$$

*We refer $(\mathscr{F}_U^{(m)}\mathfrak{h})_m$ and $(\widetilde{\mathscr{F}}_U^{(m)}\mathfrak{h})_m$ as the* enveloping filtration *and the* generalized enveloping filtration *of $\mathfrak{h}$, respectively.*

It is not a priori clear that the sets $\mathscr{F}_U^{(m)}\mathfrak{h}$, $m\geqslant 0$, exhaust $\mathfrak{h}$ since we do not know a priori whether $((\mathrm{ev}_\rho\otimes 1)\circ(\mathrm{hc}\otimes 1))((U(\mathfrak{g})\otimes\mathfrak{g})^{\mathfrak{g}})$ is equal to $\mathfrak{h}$. Similarly, we do not know a priori whether $((\mathrm{ev}_\rho\otimes 1)\circ\widetilde{\mathrm{hc}})((U(\mathfrak{g})\otimes\mathfrak{g})^{\mathfrak{g}})$ is equal to $\mathfrak{h}$. This is indeed the case. A stronger result is in fact true.

**Theorem 2** ([Jos12a, Jos12b]). *For any $m\in\mathbb{Z}_{\geqslant 0}$,*

$$\mathscr{F}_U^{(m)}\mathfrak{h}=\check{\mathscr{F}}^{(m)}\mathfrak{h}\qquad\text{and}\qquad\widetilde{\mathscr{F}}_U^{(m)}\mathfrak{h}=\check{\mathscr{F}}^{(m)}\mathfrak{h}.$$

*In other words, the enveloping filtration (resp. the generalized enveloping filtration) coincides with the principal filtration.*

Joseph's theorem is a highly non-trivial result, especially in the non-simply laced cases. Its proof is deeply based on the description by Khoroshkin, Nazarov and Vinberg of the invariant spaces in term of Zhelobenko operators. This is an extremely delicate proof.

## The Clifford filtration

We now turn to a filtration arising from the Clifford algebra. Let $\mathrm{Cl}(\mathfrak{g})$ be the Clifford algebra over $\mathfrak{g}$ associated with the bilinear form $B_{\mathfrak{g}}$. Recall that $\mathrm{Cl}(\mathfrak{g})$ is the quotient of the tensor algebra $T\mathfrak{g}$ by the bilateral ideal generated by elements $x \otimes x - B_{\mathfrak{g}}(x,x)$ for $x \in \mathfrak{g}$. It is a quantization of the exterior algebra $\bigwedge \mathfrak{g}$.

Let

$$\phi\colon (x,y,z) \mapsto B_{\mathfrak{g}}(x,[y,z]) \;\in\; \bigwedge^3 \mathfrak{g}$$

be the invariant differential Cartan 3-form of $\mathfrak{g}$. It belongs to the basis

$$\phi_i \in \bigwedge^{2m_i+1}(\mathfrak{g}), \qquad i = 1, \dots, r,$$

of the space of *primitive invariants* $P(\mathfrak{g}) \subset (\bigwedge \mathfrak{g})^{\mathfrak{g}}$. By the Hopf-Koszul-Samelson theorem, the algebra of invariants $(\bigwedge \mathfrak{g})^{\mathfrak{g}}$ is itself an exterior algebra with generators $\phi_1, \dots, \phi_r$. Kostant proved that, similarly, the algebra of Clifford invariants $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ is a Clifford algebra $\mathrm{Cl}(P(\mathfrak{g}))$ with respect to a scalar product induced from $B_{\mathfrak{g}}$, [Kos97].

Consider the direct decomposition of the Clifford algebra

$$\mathrm{Cl}(\mathfrak{g}) = \mathrm{Cl}(\mathfrak{h}) \oplus (\mathfrak{n}_- \mathrm{Cl}(\mathfrak{g}) + \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+),$$

where $\mathfrak{g}$ is viewed as a subalgebra of $\mathrm{Cl}(\mathfrak{g})$ using the canonical injection $\mathfrak{g} \hookrightarrow \mathrm{Cl}(\mathfrak{g})$. The corresponding projection $\mathrm{hc}_{\mathrm{odd}}\colon \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{h})$ is called the *odd Harish-Chandra projection*. By a non-trivial result of Bazlov [Baz09, §5.6] and Kostant (private communication), the odd Harish-Chandra projection $\mathrm{hc}_{\mathrm{odd}}$ maps the invariant algebra $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \cong \mathrm{Cl}(P(\mathfrak{g}))$ onto $\mathfrak{h} \subset \mathrm{Cl}(\mathfrak{h})$.

**Definition 4** (the Clifford filtration)**.** *Set for $m \in \mathbb{Z}_{\geqslant 0}$,*

$$\mathscr{F}_{\mathrm{Cl}}^{(m)}\mathfrak{h} := \mathrm{hc}_{\mathrm{odd}}(q(P^{(2m+1)}(\mathfrak{g}))),$$

*where $(P^{2m+1}(\mathfrak{g}))_m$ is the natural filtration on $P(\mathfrak{g})$ induced from the degrees of generators, and $q : \bigwedge\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ is the quantisaton map. We refer $(\mathscr{F}_{\mathrm{Cl}}^{(m)}\mathfrak{h})_m$ as the* Clifford filtration *of $\mathfrak{h}$.*

It is a well-defined filtration on $\mathfrak{h}$ thanks to the above quoted result of Bazlov-Kostant.

**Theorem 3** ([AM12]). *For any $m \in \mathbb{N}$, we have*

$$\mathscr{F}_U^{(m)}\mathfrak{h} = \mathscr{F}_{\mathrm{Cl}}^{(m)}\mathfrak{h} \quad \text{and} \quad \widetilde{\mathscr{F}}_U^{(m)}\mathfrak{h} = \mathscr{F}_{\mathrm{Cl}}^{(m)}\mathfrak{h}.$$

*In other words, the Clifford filtration coincides with the enveloping filtration and with the generalized enveloping filtration.*

We have $\mathrm{hc}_{\mathrm{odd}}(q(\phi)) = B_{\mathfrak{g}}^{\sharp}(\rho)$ with $\phi = \phi_1$. Kostant conjectured that the images of the higher generators $q(\phi_i)$, $i = 2, \dots, r$, by the odd Harish-Chandra projection can be described using the principal filtration.

Next theorem positively answers his conjecture:

**Theorem 4** (Kostant's conjecture [Baz03, Jos12a, Jos12b, AM12]). *For any $m \in \mathbb{N}^*$, we have:*

$$\mathscr{F}_{\mathrm{Cl}}^{(m)}\mathfrak{h} = \check{\mathscr{F}}^{(m)}\mathfrak{h}.$$

The conjecture was proved in type $A$ by Bazlov [Baz03] in 2003, and then in full generality combining the works of Joseph [Jos12a, Jos12b] and Alekseev-Moreau [AM12] (see Theorems 2 and 3).

## Main results and strategy

In this thesis we establish the equality between the symmetric filtration (Definition 2) and the enveloping filtration (Definition 3) for $\mathfrak{g}$ of type $A$ or $C$:

**Theorem 5.** *Assume that $\mathfrak{g}$ is $\mathfrak{sl}_{r+1}$ or $\mathfrak{sp}_{2r}$. Then for any $m \in \mathbb{Z}_{\geqslant 0}$,*

$$\mathscr{F}_S^{(m)}\mathfrak{h} = \mathscr{F}_U^{(m)}\mathfrak{h}.$$

Together with Rohr's result (Theorem 1) and Alekseev-Moreau theorem (Theorem 3), Theorem 5 gives another proof of Kostant's conjecture for the types $A$ and $C$, while avoiding Joseph's result (Theorem 2). As mentioned above, Joseph's proof for the non-simply laced cases (e.g., type $C$) is particular involved. Since the symmetric filtration and the enveloping filtration are similarly defined, our approach is quite natural. Our proof is based on explicit description of invariants: in the case where $\mathfrak{g}$ is $\mathfrak{sl}_{r+1}$ or $\mathfrak{sp}_{2r}$, homogeneous generators of $S(\mathfrak{g})^{\mathfrak{g}}$ can be described from the standard representation. Our approach is then to describe the images by $\mathrm{Ch} \otimes 1$ and $\mathrm{hc} \otimes 1$ of their differentials and of the symmetrization of their differentials, respectively, using *weighted paths* in the crystal graph of the standard representation. It is very different from both Rohr and Joseph approaches. We refer the reader to next chapter for more details about weighted paths.

We hope that our method can be adapted to other contexts, for instance to affine Lie algebras in types $A$ and $C$ or to other particular representations different from the adjoint representation. Remember that the symmetric filtration and the enveloping filtration are both defined for the special case where $V = \mathfrak{g}$ is the adjoint representation. One may ask if an analogue result can be extended to some other finite-dimensional simple $\mathfrak{g}$-module $V$. Namely, one can ask whether the equality $\mathscr{F}_U^{(m)} V_0 = \mathscr{F}_S^{(m)} V_0$ holds for any $m$, where $\mathscr{F}_S^{(m)} V_0$ stands for $(\mathrm{ev}_\rho \otimes 1) \circ (\mathrm{Ch} \otimes 1))((S^m(\mathfrak{g}) \otimes V)^{\mathfrak{g}})$, and $\mathscr{F}_U^{(m)} V_0$ for $(\mathrm{ev}_\rho \otimes 1) \circ (\mathrm{hc} \otimes 1))((U^m(\mathfrak{g}) \otimes V)^{\mathfrak{g}})$. It is known that the equality does not hold for all simple $\mathfrak{g}$-module $V$ (a counter-example was found by Anton Alekseev [Ale]), but it would be interesting to know which are the representations for which it does; see also [Jos12b, §3.3] for related topics. Alekseev's counter-example suggests it is a hard problem, and general arguments cannot be applied.

We summarize in Figure 1 how the connections between all filtrations described above were established by different researchers.

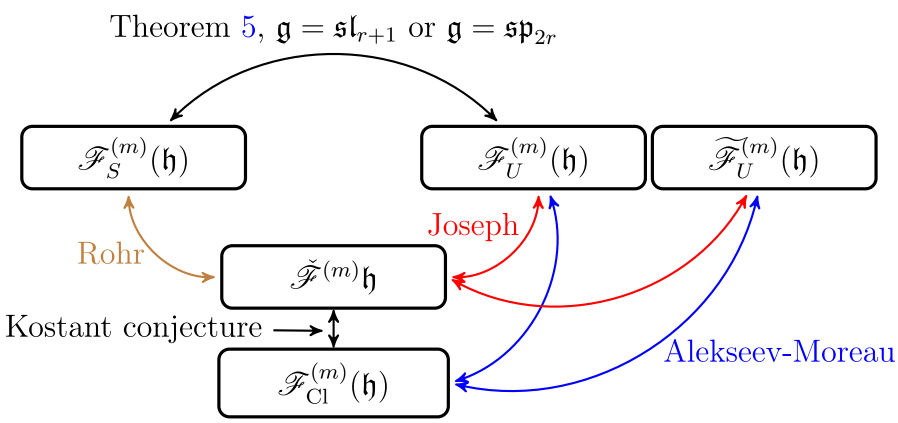



**FIGURE 1** – Connections between several filtrations on the Cartan subalgebra $\mathfrak{h}$.

## Strategy

We outline below our strategy to prove Theorem 5.

Assume that $\mathfrak{g} = \mathfrak{sl}_{r+1}$ or $\mathfrak{g} = \mathfrak{sp}_{2r}$. Then the exponents are $m_1 = 1, m_2 = 2, \ldots, m_r = r$ if $\mathfrak{g} = \mathfrak{sl}_{r+1}$, and $m_1 = 1, m_2 = 3, \ldots, m_r = 2r-1$ if $\mathfrak{g} = \mathfrak{sp}_{2r}$. One can choose homogeneous generators of $S(\mathfrak{g})^{\mathfrak{g}}$ as follows. Set $n = r+1$ if $\mathfrak{g} = \mathfrak{sl}_{r+1}$ and $n = 2r$ if $\mathfrak{g} = \mathfrak{sp}_{2r}$. Let $(\pi, \mathbb{C}^n)$ be the standard representation of $\mathfrak{g}$. It is the finite-dimensional irreducible representation with highest weight $\varpi_1$. Set for $m \in \mathbb{Z}_{\geqslant 0}$,

$$p_m := \frac{1}{(m+1)!} \mathrm{tr} \circ \pi^{m+1},$$

that is, $p_m(x) = \dfrac{1}{(m+1)!}\mathrm{tr}(\pi^{m+1}(x))$ for any $x \in \mathfrak{g}$. It is an element of $\mathbb{C}[\mathfrak{g}] \cong S(\mathfrak{g}^*)$. Identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ through the inner product $B_{\mathfrak{g}}$, $p_m$ becomes an element of $S(\mathfrak{g})$. The elements $p_{m_1}, \ldots, p_{m_r}$ are homogeneous generators of $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \cong S(\mathfrak{g})^{\mathfrak{g}}$ of degree $m_1+1, \ldots, m_r+1$, respectively. Their differentials, $\mathrm{d}p_1, \ldots, \mathrm{d}p_m$, are homogeneous free generators of the free module $(S(\mathfrak{g}) \otimes \mathfrak{g})^{\mathfrak{g}}$ over $S(\mathfrak{g})^{\mathfrak{g}}$. Moreover $(\beta \otimes 1)(\mathrm{d}p_{m_1}), \ldots, (\beta \otimes 1)(\mathrm{d}p_{m_r})$ are homogeneous free generators of the free module $(U(\mathfrak{g}) \otimes \mathfrak{g})^{\mathfrak{g}}$ over $Z(\mathfrak{g}) \cong U(\mathfrak{g})^{\mathfrak{g}}$ (cf. Proposition 1.2), where

$$\beta \colon S(\mathfrak{g}) \to U(\mathfrak{g}), \quad x_1 \cdots x_k \mapsto \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} x_{\sigma(1)} \cdots x_{\sigma(k)}$$

is the symmetrization map. Here, $\mathfrak{S}_k$ denotes the symmetric group of order $k$. We present more general constructions of invariants in Section 1.1.

Let $m \in \mathbb{Z}_{>0}$. Since $\mathrm{d}p_m$ and $(\beta \otimes 1)(\mathrm{d}p_m)$ are $\mathfrak{g}$-invariant,

$$\overline{\mathrm{d}p}_m := (\mathrm{Ch} \otimes 1)(\mathrm{d}p_m) \quad \text{and} \quad \widehat{\mathrm{d}p}_m := ((\mathrm{hc} \otimes 1) \circ (\beta \otimes 1))(\mathrm{d}p_m)$$

lie in $S(\mathfrak{h}) \otimes \mathfrak{h}$. Define the elements $\overline{\mathrm{d}p}_{m,k}$ and $\widehat{\mathrm{d}p}_{m,k}$ of $S(\mathfrak{h})$ for $k \in \{1, \ldots, r\}$ by:

$$\overline{\mathrm{d}p}_m = \frac{1}{m!} \sum_{k=1}^{r} \overline{\mathrm{d}p}_{m,k} \otimes \varpi_k^\sharp, \qquad \widehat{\mathrm{d}p}_m = \frac{1}{m!} \sum_{k=1}^{r} \widehat{\mathrm{d}p}_{m,k} \otimes \varpi_k^\sharp.$$

Our main results are the following:

**Theorem 6.** *Assume that $\mathfrak{g}$ is $\mathfrak{sl}_{r+1}$ or $\mathfrak{sp}_{2r}$. Let $m \in \mathbb{Z}_{>0}$. For some polynomial $\bar{Q}_m \in \mathbb{C}[X]$ of degree $m-1$ we have $\mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m,k}) = \bar{Q}_m(k)$ for $k = 1, \ldots, r$ so that*

$$\mathrm{ev}_\rho(\overline{\mathrm{d}p}_m) = \frac{1}{m!} \sum_{k=1}^{r} \bar{Q}_m(k) \varpi_k^\sharp.$$

*Moreover* $\bar{Q}_1 = 1$ *if* $\mathfrak{g} = \mathfrak{sl}_{r+1}$, *and* $\bar{Q}_1 = 2$ *if* $\mathfrak{g} = \mathfrak{sp}_{2r}$.

**Remark 2.** *For* $m > 1$, *the polynomial* $\bar{Q}_m$ *is explicitly described by formula* (2.4) *for* $\mathfrak{sl}_{r+1}$, *and by formula* (2.7) *for* $\mathfrak{sp}_{2r}$.

**Theorem 7.** *Assume that* $\mathfrak{g}$ *is* $\mathfrak{sl}_{r+1}$ *or* $\mathfrak{sp}_{2r}$. *Let* $m \in \mathbb{Z}_{>0}$. *For some polynomial* $\hat{Q}_m \in \mathbb{C}[X]$ *of degree at most* $m-1$ *we have* $\mathrm{ev}_\rho(\widehat{\mathrm{d}p}_{m,k}) = \hat{Q}_m(k)$ *for* $k = 1, \ldots, r$ *so that*

$$\mathrm{ev}_\rho(\widehat{\mathrm{d}p}_m) = \frac{1}{m!}\sum_{k=1}^{r} \hat{Q}_m(k)\varpi_k^\sharp.$$

*Moreover* $\hat{Q}_1 = 1$ *if* $\mathfrak{g} = \mathfrak{sl}_{r+1}$, *and* $\hat{Q}_1 = 2$ *if* $\mathfrak{g} = \mathfrak{sp}_{2r}$.

**Remark 3.** *For* $m > 1$, *there is no nice general description of the polynomial* $\hat{Q}_m$ *in term of* $r$ *and* $m$ *as for the symmetric case (see Remark 2). The polynomials* $\hat{Q}_m$*'s are defined inductively by formula* (3.5) *for* $\mathfrak{sl}_{r+1}$, *and by formula* (4.19) *for* $\mathfrak{sp}_{2r}$.

We claim that Theorem 6 and Theorem 7 are sufficient to prove the main theorem:

**Proposition 1.** *Theorem 6 and Theorem 7 imply Theorem 5.*

*Proof.* Assume that $\mathfrak{g}$ is $\mathfrak{sl}_{r+1}$ or $\mathfrak{sp}_{2r}$. First of all, we observe that the exponents $m_1, \ldots, m_r$ are pairwise distinct in both cases. On the other hand, by a result of Kostant mentioned above, the elements $\mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m_1}), \ldots, \mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m_r})$ are linearly independent in $\mathfrak{h}$ since $\rho$ is regular [Kos63]. Thus, for any $j \in \{1, \ldots, r\}$, the vector space generated by $\mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m_1}), \ldots, \mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m_j})$ has dimension $j$. We denote it by $\bar{V}_j$. Set for $j \in \mathbb{Z}_{\geqslant 0}$,

$$w_j = \sum_{k=1}^{r} k^j \varpi_k^\sharp \quad \text{and} \quad W_j := \mathrm{span}_{\mathbb{C}}\{w_0, \ldots, w_{j-1}\}.$$

We have $\dim W_j \leqslant j$.

As a first step, let us show that for all $j \in \{1, \ldots, r\}$,

$$\bar{V}_j = W_{m_j}.$$

By Theorem 6, the inclusion $\bar{V}_j \subset W_{m_j}$ holds. Assume that there is $w \in W_{m_j} \setminus \bar{V}_j$. Write $w$ in the basis $\mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m_1}), \ldots, \mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m_r})$:

$$w = \sum_{i=1}^{r} a_i\, \mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m_i}),$$

and let $q$ be the maximal integer $i \in \{1, \ldots, r\}$ such that $a_i \neq 0$. Since $w \notin \bar{V}_j$, $q > j$. But then $w$ is in $W_{m_q}$ and not in $W_{m_q - 1}$ by Theorem 6, which contradicts the fact that $w \in W_{m_j}$ because $m_j \leqslant m_q - 1$. We have shown the expected equality of vector spaces.

Let us now denote by $\hat{V}_j$ the vector space generated by $\mathrm{ev}_\rho(\widehat{\mathrm{d}p_{m_1}}), \ldots, \mathrm{ev}_\rho(\widehat{\mathrm{d}p_{m_j}})$ for $j \in \{1, \ldots, r\}$. Our aim is to show that $\hat{V}_j = \bar{V}_j$ for all $j \in \{1, \ldots, r\}$. By the first step, it suffices to establish that $\hat{V}_j = W_{m_j}$ for all $j \in \{1, \ldots, r\}$. According to Theorem 7, we have the inclusion $\hat{V}_j \subset W_{m_j}$ for all $j \in \{1, \ldots, r\}$, and $\dim W_{m_j}$ has dimension $j$ by the first step. Theorem 3 implies that $\hat{V}_r = \mathfrak{h}$, and so the elements $\mathrm{ev}_\rho(\widehat{\mathrm{d}p_{m_1}}), \ldots, \mathrm{ev}_\rho(\widehat{\mathrm{d}p_{m_r}})$ are linearly independent. In particular, each $\hat{V}_j$ has dimension $j$, hence the expected equality $\hat{V}_j = W_{m_j}$ for all $j \in \{1, \ldots, r\}$. This finishes the proof. □

The rest of this thesis is devoted to the proofs of Theorem 6 and Theorem 7.

The proof of Theorem 6 is relatively easy and is completed in Chapter 2. In contrast, the proof of Theorem 7 is much more involved and requires technical results on *weighted paths* (see §3.1, §4.1) and an equivalence relation on the set of heights of paths (§3.2, §4.2). The proof of Theorem 7 will be achieved in §3.3 for the type $A$ and in §4.3 for the type $C$.

The thesis will be organized as follows. Chapter 1 is about generalities on invariants coming from representations. We introduce in this chapter our central notion of weighted paths. Chapter 2 is devoted to the proof of Theorem 6. From Chapter 3 we focus on the proof of Theorem 7. In Chapter 3, we prove Theorem 7 in the case where $\mathfrak{g} = \mathfrak{sl}_{r+1}$. We start with some technical results on weighted paths and invariants in the enveloping algebra. Then we prove the theorem using equivalent classes on weighted paths. In Chapter 4, we prove Theorem 7 in the case where $\mathfrak{g} = \mathfrak{sp}_{2r}$. We follow for the $\mathfrak{sp}_{2r}$ case the general strategy of Chapter 3. However new phenomenons appear. Consequently, the proof is much more technical and new tools are needed.

## Notations

Unless otherwise specified, we keep the notations used in the introduction further on in the thesis.

# Chapter 1

# General setting and weighted paths in crystal graphs

As it is explain in the introduction, our strategy is to use an explicit description of generating invariants of $(S(\mathfrak{g}) \otimes \mathfrak{g})^{\mathfrak{g}}$ and $(U(\mathfrak{g}) \otimes \mathfrak{g})^{\mathfrak{g}}$ which can be done using the standard representation for both $\mathfrak{sl}_{r+1}$ and $\mathfrak{sp}_{2r}$. In this chapter, we start with a more general context: we will see that (not necessarily generating) invariants can be constructed from any finite-dimensional irreducible representation. In the case of *minuscule representations* (cf. Definition 1.4), and more generally irreducible highest weight representations whose all nonzero weight spaces have dimension one, one can introduce the notion of *weighted paths* (see Subsection 1.2). This notion will occupy a central place in the rest of the thesis. We will apply all these facts to the special case where the simple highest weight representation is the standard representation of $\mathfrak{sl}_{r+1}$ and $\mathfrak{sp}_{2r}$, that is, the simple highest weight representation associated with the first fundamental weight $\varpi_1$ (in both cases).

## 1.1 Invariant polynomial functions coming from representations

Recall that $\Delta$ is the root system of $(\mathfrak{g}, \mathfrak{h})$, that $\Pi = \{\beta_1, \ldots, \beta_r\}$ is the basis with respect to $\mathfrak{h} \oplus \mathfrak{n}_+$ and $\Delta_+$ is the corresponding set of positive roots. Let $Q = \sum_{i=1}^{r} \mathbb{Z}\beta_i$

be the root lattice and set

$$Q_+ := \sum_{i=1}^{r} \mathbb{Z}_{\geqslant 0}\beta_i.$$

We define a partial order on $\mathfrak{h}^*$ by :

$$\mu \succcurlyeq \lambda \iff \mu - \lambda \in Q_+.$$

For $(\lambda, \mu) \in (\mathfrak{h}^*)^2$, we denote by $[\![\lambda, \mu]\!]$ the set of $\nu \in \mathfrak{h}^*$ such that $\lambda \preccurlyeq \nu \preccurlyeq \mu$. Let also

$$P = \sum_{i=1}^{r} \mathbb{Z}\varpi_i \quad \text{ and } \quad P_+ := \sum_{i=1}^{r} \mathbb{Z}_{\geqslant 0}\varpi_i$$

be the weight lattice and the set of integral dominant weights, respectively. Note that any $\mu \in P$ is written as

$$\mu = \sum_{i=1}^{r} \langle \mu, \check{\beta}_i \rangle \varpi_i.$$

Let $\lambda \in P_+$. We write $(\pi_\lambda, V(\lambda))$ for the unique, up to isomorphism, finite-dimensional irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$. For $\mu \in \mathfrak{h}^*$,

$$V(\lambda)_\mu := \{v \in V(\lambda) \mid \pi_\lambda(x)v = \mu(x)v \text{ for all } x \in \mathfrak{h}\}$$

is the $\mu$-weight space of $V(\lambda)$. The set of nonzero weights of $V(\lambda)$ will be denoted by $P(\lambda)$:

$$P(\lambda) = \{\mu \in P \mid V(\lambda)_\mu \neq \{0\}\}.$$

Fix $m \in \mathbb{Z}_{>0}$, and let

$$p_m^{(\lambda)} = \frac{1}{(m+1)!} \mathrm{tr} \circ \pi_\lambda^{m+1}.$$

Thus $p_m^{(\lambda)}$ is the element of $\mathbb{C}[\mathfrak{g}] \cong S(\mathfrak{g}^*)$ defined by

$$p_m^{(\lambda)}(y) = \frac{1}{(m+1)!} \mathrm{tr}(\pi_\lambda(y)^{m+1}), \qquad y \in \mathfrak{g}.$$

Furthermore, $p_m^{(\lambda)}$ is a $\mathfrak{g}$-invariant element of $\mathbb{C}[\mathfrak{g}]$ of degree $m+1$, cf. [TY05, Lemma 31.2.3].

Let $\mathscr{B} = (b_1, \ldots, b_d)$ be a basis of $\mathfrak{g}$, and let $\mathscr{B}^* := (b_1^*, \ldots, b_d^*)$ be its dual basis.

Choose $y \in \mathfrak{g}$. Then $y = \sum_{i=1}^{d} b_i^*(y) b_i$ and so

$$\pi_\lambda(y)^{m+1} = \sum_{1 \leqslant i_1, \dots, i_{m+1} \leqslant d} \pi_\lambda(b_{i_1}) \circ \cdots \circ \pi_\lambda(b_{i_{m+1}}) b_{i_1}^*(y) \dots b_{i_{m+1}}^*(y),$$

which implies the following lemma:

**Lemma 1.1.** *We have:*

$$p_m^{(\lambda)} = \frac{1}{(m+1)!} \sum_{1 \leqslant i_1, \dots, i_{m+1} \leqslant d} \operatorname{tr}(\pi_\lambda(b_{i_1}) \circ \cdots \circ \pi_\lambda(b_{i_{m+1}})) b_{i_1}^* \dots b_{i_{m+1}}^*.$$

Identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ through $B_\mathfrak{g}$, $p_m^{(\lambda)}$ becomes an element of $\mathbb{C}[\mathfrak{g}^*]^\mathfrak{g} \cong S(\mathfrak{g})^\mathfrak{g}$ of degree $m+1$. Moreover, its differential $\mathrm{d}p_m^{(\lambda)}$, defined by

$$\mathrm{d}p_m^{(\lambda)} = \sum_{k=1}^{d} \frac{\partial p_m^{(\lambda)}}{\partial b_k^*} \otimes b_k^*,$$

is an element of $(S(\mathfrak{g}) \otimes \mathfrak{g})^\mathfrak{g}$.

Recall that $\beta \colon S(\mathfrak{g}) \to U(\mathfrak{g})$ is the symmetrization map. Let $Z(\mathfrak{g}) \cong U(\mathfrak{g})^\mathfrak{g}$ be the center of the enveloping algebra. The first part of the following proposition is due to Kostant [Kos63].

**Proposition 1.2.** *Assume that $p_{m_1}^{(\lambda)}, \dots, p_{m_r}^{(\lambda)}$ are homogeneous generators of $S(\mathfrak{g})^\mathfrak{g}$. Then $\mathrm{d}p_{m_1}^{(\lambda)}, \dots, \mathrm{d}p_{m_r}^{(\lambda)}$ are free homogeneous generators of the free module $(S(\mathfrak{g}) \otimes \mathfrak{g})^\mathfrak{g}$ over $S(\mathfrak{g})^\mathfrak{g}$, and $(\beta \otimes 1)(\mathrm{d}p_{m_1}^{(\lambda)}), \dots, (\beta \otimes 1)(\mathrm{d}p_{m_r}^{(\lambda)})$ are free homogeneous generators of the free module $(U(\mathfrak{g}) \otimes \mathfrak{g})^\mathfrak{g}$ over $Z(\mathfrak{g})$.*

*Proof.* It suffices to prove the second part. Our arguments is adapted from Diximer's [Dix77]. Denoting $\hat{\beta} = \beta \otimes 1$, we first observe that $\hat{\beta}((S(\mathfrak{g}) \otimes \mathfrak{g})^\mathfrak{g})$ is equal to $(U(\mathfrak{g}) \otimes \mathfrak{g})^\mathfrak{g}$, since $\hat{\beta}$ is an isomorphism which commutes with the diagonal action of $\mathfrak{g}$. Then it sends invariants to invariants.

Let $M$ be the sub-$Z(\mathfrak{g})$-module of $(U(\mathfrak{g}) \otimes \mathfrak{g})^\mathfrak{g}$ generated by $\hat{\beta}(\mathrm{d}p_{m_1}), \dots, \hat{\beta}(\mathrm{d}p_{m_r})$. Then the graded module associated with the filtration induced on $M$ is a sub-$S(\mathfrak{g})^\mathfrak{g}$-module of $(S(\mathfrak{g}) \otimes \mathfrak{g})^\mathfrak{g}$, since $S(\mathfrak{g})^\mathfrak{g}$ is the graded space associated with the filtration induced on $Z(\mathfrak{g})$. It contains $\mathrm{d}p_{m_1} \dots, \mathrm{d}p_{m_r}$, and so it is equal to $(S(\mathfrak{g}) \otimes \mathfrak{g})^\mathfrak{g}$. On the other side, the graded space of $(U(\mathfrak{g}) \otimes \mathfrak{g})^\mathfrak{g}$ is contained in $(S(\mathfrak{g}) \otimes \mathfrak{g})^\mathfrak{g}$. Hence we get that $M = (U(\mathfrak{g}) \otimes \mathfrak{g})^\mathfrak{g}$.

For the freeness, let $K$ be the module of relations between $\hat{\beta}(\mathrm{d}p_{m_1}), \dots, \hat{\beta}(\mathrm{d}p_{m_r})$ on $Z(\mathfrak{g})$, so that

$$K \subset Z(\mathfrak{g})^r.$$

The filtration on $Z(\mathfrak{g})$ induces a filtration on $K$ and the graded module associated with this filtration is contained in the module of relations on $S(\mathfrak{g})^{\mathfrak{g}}$ between $\mathrm{d}p_{m_1} \dots, \mathrm{d}p_{m_r}$, which is zero. Hence $K = 0$. □

*Example* 1.3. In Table 1.1, we give examples where $p_{m_1}^{(\lambda)}, \dots, p_{m_r}^{(\lambda)}$ are homogeneous generators of $S(\mathfrak{g})^{\mathfrak{g}}$ following [Meh88].

| Type | $\lambda$ | $\dim V(\lambda)$ | $N$ | decomposition in the basis $\mathcal{E}$ |
|---|---|---|---|---|
| $A_r$ | $\varpi_1$ | $r+1$ | $r+1$ | $\varpi_1 = \varepsilon_1 - \frac{1}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1})$ |
| $B_r$ | $\varpi_1$ | $2r+1$ | $r$ | $\varpi_1 = \varepsilon_1$ |
| $C_r$ | $\varpi_1$ | $2r$ | $r$ | $\varpi_1 = \varepsilon_1$ |
| $D_r$, odd $r$ | $\varpi_1$ | $2r$ | $r$ | $\varpi_r = \varepsilon_1$ |
| $G_2$ | $\varpi_1$ | 7 | 3 | $\varpi_1 = -\varepsilon_2 + \varepsilon_3$ |
| $F_4$ | $\varpi_4$ | 26 | 4 | $\varpi_4 = \varepsilon_1$ |
| $E_6$ | $\varpi_1$ | 27 | 8 | $\varpi_1 = \frac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6)$ |

**TABLE 1.1** – Examples of $\lambda$ for which Proposition 1.2 holds.

**Definition 1.4.** *The representation $V(\lambda)$ is called* minuscule *if it is not trivial and if $P(\lambda) = W.\lambda$, where $W.\lambda$ denotes the orbit of the highest weight $\lambda$ under the action of the Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$. In that case, we say that the dominant weight $\lambda$ is* minuscule.

*Remark* 1.5. The minuscule weights are fundamental weights and if $\lambda$ is minuscule, then each nonzero weight space in $V(\lambda)$ has dimension 1. Moreover, $\dim V(\lambda)_0 = 0$. The minuscule weights are given in Table 1.2.

| Type | minuscule weights | $N$ | decomposition in the basis $\mathcal{E}$ |
|---|---|---|---|
| $A_r$ | $\varpi_1, \dots, \varpi_r$ | $r+1$ | $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1})$ |
| $B_r$ | $\varpi_r$ | $r$ | $\varpi_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r)$ |
| $C_r$ | $\varpi_1$ | $r$ | $\varpi_1 = \varepsilon_1$ |
| $D_r$ | $\varpi_1, \varpi_{r-1}, \varpi_r$ | $r$ | $\varpi_1 = \varepsilon_1, \varpi_{n+t} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n) + t\varepsilon_t$, $t \in \{-1, 0\}$ |
| $E_6$ | $\varpi_1, \varpi_6$ | 8 | $\varpi_1 = \frac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6), \varpi_6 = \frac{1}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_5$ |
| $E_7$ | $\varpi_7$ | 8 | $\varpi_7 = \varepsilon_6 + \frac{1}{2}(\varepsilon_8 - \varepsilon_7)$ |

**TABLE 1.2** – Minuscule weights

We fix from now on $\delta \in P_+$ such that each nonzero weight space $V(\delta)_\mu$, $\mu \in P(\delta)$, has dimension one. Choose for any $\mu \in P(\delta)$ a nonzero vector $v_\mu \in V(\delta)_\mu$. By our assumption, the set $\{v_\mu \mid \mu \in P(\delta)\}$ forms a basis of $V(\delta)$. For $(\lambda, \mu) \in P(\delta)^2$ and $b \in \mathscr{B}$, define the scalar $a^{(b)}_{\lambda,\mu}$ by:

$$\pi_\delta(b) v_\mu = \sum_{\lambda \in P(\delta)} a^{(b)}_{\lambda,\mu} v_\lambda. \tag{1.1}$$

Next lemma follows immediately from Lemma 1.1.

**Lemma 1.6.** *For any $m > 0$, we have:*

$$p^{(\delta)}_m = \frac{1}{(m+1)!} \sum_{1 \leqslant i_1,\dots,i_{m+1} \leqslant d} \ \sum_{\substack{(\mu_{j_1},\dots,\mu_{j_{m+1}}) \\ \in P(\delta)^{m+1}}} a^{(b_{i_1})}_{\mu_{j_1},\mu_{j_2}} \dots a^{(b_{i_{m+1}})}_{\mu_{j_{m+1}},\mu_{j_1}} b^*_{i_1} \dots b^*_{i_{m+1}}.$$

Assume now that $\mathscr{B} = \{e_\alpha, \check{\beta}_i \mid \alpha \in \Delta, i = 1, \dots, r\}$ is the Chevalley basis of $\mathfrak{g}$ (in a fixed order), so that

$$\mathscr{B}^* = \{c_\alpha e_{-\alpha}, \varpi_i^\sharp \mid \alpha \in \Delta, i = 1, \dots, r\},$$

with $c_\alpha \neq 0$ for $\alpha \in \Delta$. We denote by $\overline{\mathrm{d}p}^{(\delta)}_m$ and $\widehat{\mathrm{d}p}^{(\delta)}_m$ the images of $\mathrm{d}p^{(\delta)}_m$ by $\mathrm{Ch} \otimes 1$ and $(\mathrm{hc} \otimes 1) \circ \hat{\beta}$, respectively, where $\hat{\beta}$ is $\beta \otimes 1$ as in the proof of Proposition 1.2. When there will be no ambiguity about $\delta$, we will simply denote by $p_m$, $\mathrm{d}p_m$, $\overline{\mathrm{d}p}_m$, $\widehat{\mathrm{d}p}_m$ the corresponding elements.

Let $m > 0$. Since $\mathrm{d}p^{(\delta)}_m$ and $\hat{\beta}(\mathrm{d}p^{(\delta)}_m)$ are $\mathfrak{g}$-invariant, $\overline{\mathrm{d}p}^{(\delta)}_m$ and $\widehat{\mathrm{d}p}^{(\delta)}_m$ lie in $S(\mathfrak{h}) \otimes \mathfrak{h}$. Define the elements $\overline{\mathrm{d}p}^{(\delta)}_{m,k}$ and $\widehat{\mathrm{d}p}^{(\delta)}_{m,k}$ of $S(\mathfrak{h})$, for $k \in \{1, \dots, r\}$, by:

$$\overline{\mathrm{d}p}^{(\delta)}_m = \frac{1}{m!} \sum_{k=1}^r \overline{\mathrm{d}p}^{(\delta)}_{m,k} \otimes \varpi_k^\sharp, \qquad \widehat{\mathrm{d}p}^{(\delta)}_m = \frac{1}{m!} \sum_{k=1}^r \widehat{\mathrm{d}p}^{(\delta)}_{m,k} \otimes \varpi_k^\sharp.$$

For $k \in \{1, \dots, r\}$, we set

$$P(\delta)_k := \{\mu \in P(\delta) \mid \langle \mu, \check{\beta}_k \rangle \neq 0\}. \tag{1.2}$$

**Lemma 1.7.** *We have*

$$(1) \quad \overline{\mathrm{d}p}^{(\delta)}_{m,k} = \sum_{\mu \in P(\delta)_k} \ \sum_{1 \leqslant i_1,\dots,i_m \leqslant r} \langle \mu, \check{\beta}_{i_1} \rangle \dots \langle \mu, \check{\beta}_{i_m} \rangle \langle \mu, \check{\beta}_k \rangle \, \varpi^\sharp_{i_1} \dots \varpi^\sharp_{i_m},$$

(2) $$\widehat{\mathrm{d}p}_{m,k}^{(\delta)} = \sum_{\substack{\mu_{j_1},\dots,\mu_{j_m} \in P(\delta) \\ \mu_{j_1} \in P(\delta)_k}} \sum_{1 \leqslant i_1,\dots,i_m \leqslant r} a^{(b_{i_1})}_{\mu_{j_1},\mu_{j_2}} \dots a^{(b_{i_m})}_{\mu_{j_m},\mu_{j_1}} \langle \mu_{j_1}, \check{\beta}_k \rangle \mathrm{hc}(b^*_{i_1} \dots b^*_{i_m}).$$

*Proof.* First of all, by Lemma 1.6, we have

$$\mathrm{d}p_m^{(\delta)} = \frac{1}{m!} \sum_{k=1}^{d} \sum_{\substack{\mu_{j_1},\dots,\mu_{j_{m+1}} \in P(\delta) \\ \mu_{j_1} \in P(\delta)_k}} \sum_{1 \leqslant i_1,\dots,i_m \leqslant d} a^{(b_{i_1})}_{\mu_{j_1},\mu_{j_2}} \dots a^{(b_{i_m})}_{\mu_{j_m},\mu_{j_{m+1}}} a^{(b_k)}_{\mu_{j_{m+1}},\mu_{j_1}} b^*_{i_1} \dots b^*_{i_m} \otimes b^*_k.$$

(1) For $\lambda, \mu \in P(\delta)$ and $i \in \{1, \dots, r\}$,

$$a^{(\check{\beta}_i)}_{\lambda,\mu} = \begin{cases} \langle \mu, \check{\beta}_i \rangle & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$$

Since $\mathrm{d}p_m^{(\delta)}$ is $\mathfrak{g}$-invariant, its image by $\mathrm{Ch} \otimes 1$ belongs to $S(\mathfrak{h}) \otimes \mathfrak{h}$, and we have:

$$\begin{aligned} \overline{\mathrm{d}p}_m^{(\delta)} &= (\mathrm{Ch} \otimes 1)(\mathrm{d}p_m^{(\delta)}) \\ &= \frac{1}{m!} \sum_{k=1}^{r} \sum_{\mu \in P(\delta)} \sum_{1 \leqslant i_1,\dots,i_m \leqslant r} \langle \mu, \check{\beta}_{i_1} \rangle \dots \langle \mu, \check{\beta}_{i_m} \rangle \langle \mu, \check{\beta}_k \rangle \, \varpi^\sharp_{i_1} \dots \varpi^\sharp_{i_m} \otimes \varpi^\sharp_k, \end{aligned}$$

whence the statement.

(2) The image of $\mathrm{d}p_m^{(\delta)}$ by the map $\hat{\beta}$ is the following element of $(U(\mathfrak{g}) \otimes \mathfrak{g})^{\mathfrak{g}}$:

$$\begin{aligned} &\hat{\beta}(\mathrm{d}p_m^{(\delta)}) \\ &= \frac{1}{m!} \sum_{k=1}^{d} \sum_{1 \leqslant i_1,\dots,i_m \leqslant d} \sum_{\substack{\mu_{j_1},\dots,\mu_{j_{m+1}} \in P(\delta) \\ \mu_{j_1} \in P(\delta)_k}} a^{(b_{i_1})}_{\mu_{j_1},\mu_{j_2}} \dots a^{(b_{i_m})}_{\mu_{j_m},\mu_{j_{m+1}}} a^{(b_k)}_{\mu_{j_{m+1}},\mu_{j_1}} b^*_{i_1} \dots b^*_{i_m} \otimes b^*_k. \end{aligned}$$

Since $(\beta \otimes 1)(\mathrm{d}p_m^{(\delta)})$ is $\mathfrak{g}$-invariant, its image by $\mathrm{hc} \otimes 1$ belongs to $S(\mathfrak{h}) \otimes \mathfrak{h}$ and we have:

$$\begin{aligned} \widehat{\mathrm{d}p}_m^{(\delta)} &= ((\mathrm{hc} \otimes 1) \circ \hat{\beta})(\mathrm{d}p_m^{(\delta)}) \\ &= \frac{1}{m!} \sum_{k=1}^{r} \sum_{1 \leqslant i_1,\dots,i_m \leqslant d} \sum_{\substack{\mu_{j_1},\dots,\mu_{j_m} \in P(\delta) \\ \mu_{j_1} \in P(\delta)_k}} a^{(b_{i_1})}_{\mu_{j_1},\mu_{j_2}} \dots a^{(b_{i_m})}_{\mu_{j_m},\mu_{j_1}} \langle \mu_{j_1}, \check{\beta}_k \rangle \mathrm{hc}(b^*_{i_1} \dots b^*_{i_m}) \otimes \varpi^\sharp_k, \end{aligned}$$

whence the statement. ∎

## 1.2 Weighted paths

Let $\delta \in P_+$. Continue to assume that each nonzero weight space of $V(\delta)$ has dimension one.

In this section we introduce paths in the *crystal graph* of $\delta$, labelled by roots. They will serve us to describe the invariants $\widehat{\mathrm{d}p}_m^{(\delta)}$, $m \in \mathbb{Z}_{>0}$.

The *crystal graph* of the integral dominant weight $\delta$, denoted by $\mathscr{C}(\delta)$, is defined as follows :

- $\mathscr{C}(\delta)$ contain $\#P(\delta)$ vertices,
- its arrows are labeled by the simple roots $\beta_i, i \in 1, \dots, r$,
- an arrow $\delta_i \xrightarrow{\beta_j} \delta_k$ exists when $\delta_k = \delta_i - \beta_j$.

*Example* 1.8. For $\mathfrak{g} = \mathfrak{sl}_{r+1}$ and $\mathfrak{g} = \mathfrak{sp}_{2r}$, $V(\varpi_1)$ is the standard representation. The crystal graph of $\delta = \varpi_1$ is described in §2.1.1 for $\mathfrak{g} = \mathfrak{sl}_{r+1}$, and in §2.1.2 for $\mathfrak{g} = \mathfrak{sp}_{2r}$. The reader is referred to these paragraphs for all notations relative to these examples.

Let $m \in \mathbb{Z}_{>0}$ and $(\mu, \nu) \in P(\delta)^2$. Let us introduce the set of paths of length $m$ starting at $\mu$ and ending at $\nu$ with steps in $P(\delta)$.

**Definition 1.9** (path). *Let $\mathscr{P}_m(\mu, \nu)$ be the set of sequences $\underline{\mu} = (\mu^{(1)}, \dots, \mu^{(m+1)})$ in $P(\delta)$ such that $\mu^{(1)} = \mu$, $\mu^{(m+1)} = \nu$ and for all $i = 1, \dots, m$, $\mu^{(i)} - \mu^{(i+1)} \in \Delta \cup \{0\}$. The elements of $\mathscr{P}_m(\mu, \nu)$ are called the* paths of length $m$ starting at $\mu$ and ending at $\nu$. *When $\mu = \nu$, we write $\mathscr{P}_m(\mu)$ for $\mathscr{P}_m(\mu, \mu)$.*

*Remark* 1.10. For $\mathfrak{g} = \mathfrak{sl}_{r+1}$ or $\mathfrak{g} = \mathfrak{sp}_{2r}$ with $\delta = \varpi_1$ (cf. Example 1.8), the difference of two different weights is always a root. So the condition $\mu^{(i)} - \mu^{(i+1)} \in \Delta \cup \{0\}$ is automatically satisfied. Note that, furthermore, in these cases each root $\alpha \in \Delta$ can be written as a difference of two weights.

Such paths have been considered in a more general situation in [LLP12] in connection with Kashiwara's crystal basis theory [Kas95].

Recall that the *height* of a positive roots $\alpha$ is

$$\mathrm{ht}(\alpha) := \sum_{i=1}^{r} n_i,$$

if $\alpha = \sum_{i=1}^{r} n_i \beta_i \in \Delta_+$. For $\alpha \in -\Delta_+$, we define its height by $\mathrm{ht}(\alpha) := -\mathrm{ht}(-\alpha)$. We adopt the convention that $\mathrm{ht}(0) := 0$.

**Definition 1.11** (height of a path). *Let* $\underline{\mu} \in \mathscr{P}_m(\mu,\nu)$. *For any* $i \in \{1,\dots,m\}$, *set*

$$\mathrm{ht}(\underline{\mu})_i := \mathrm{ht}(\mu^{(i)} - \mu^{(i+1)}).$$

*Then we define the* height *of* $\underline{\mu}$ *by*

$$\mathrm{ht}(\underline{\mu}) := (\mathrm{ht}(\underline{\mu})_1, \dots, \mathrm{ht}(\underline{\mu})_m) \in \mathbb{Z}^m.$$

For $\underline{\mu} \in \mathscr{P}_m(\mu)$, we have $\sum_{i=1}^m \mathrm{ht}(\underline{\mu})_i = 0$.

**Definition 1.12** (weighted path). *Let* $\hat{\mathscr{P}}_m(\mu,\nu)$ *be the set of pairs* $(\underline{\mu}, \underline{\alpha})$ *where* $\underline{\mu} \in \mathscr{P}_m(\mu,\nu)$ *and* $\underline{\alpha} := (\alpha^{(1)}, \dots, \alpha^{(m)})$ *is a sequence of roots satisfying for any* $j \in \{1,\dots,m\}$ *the following conditions:*

1. *if* $\mathrm{ht}(\underline{\mu})_j \neq 0$ *then* $\alpha^{(j)} = \mu^{(j)} - \mu^{(j+1)}$,
2. *if* $\mathrm{ht}(\underline{\mu})_j = 0$ *then* $\alpha^{(j)} \in \Pi$ *and* $\langle \mu^{(j)}, \check{\alpha}^{(j)} \rangle \neq 0$.

*We call the elements of* $\hat{\mathscr{P}}_m(\mu,\nu)$ *the* weighted paths of length $m$ starting at $\mu$ and ending at $\nu$. *When* $\mu = \nu$, *we write* $\hat{\mathscr{P}}_m(\mu)$ *for* $\hat{\mathscr{P}}_m(\mu,\mu)$. *We denote by* $\mathbf{1}_\mu$ *the trivial path* $(\mu, \varnothing)$. *It has by convention length* 0.

We represent a weighted path $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu,\nu)$ by a colored and oriented graph as follows. The vertices are the weights $\mu^{(1)}, \dots, \mu^{(m+1)}$ and the oriented arrow from $\mu^{(j)}$ to $\mu^{(j+1)}$ for $j \in \{1,\dots,m\}$ is labelled by the root $\alpha^{(j)}$.

*Example* 1.13. Assume that $\mathfrak{g} = \mathfrak{sl}_6$ and $\delta = \varpi_1$ (cf. Example 1.8). For $i = 1,\dots,6$, let $\varepsilon_i \in \mathfrak{h}^*$ be the linear map defined by $\varepsilon_i(h) = h_i$ if $h = \mathrm{diag}(h_1,\dots,h_6)$. Let $\beta_i = \varepsilon_i - \varepsilon_{i+1}$, $i \in \{1,\dots,5\}$ and $\delta_i = \varepsilon_i - \frac{1}{6}(\varepsilon_1 + \cdots + \varepsilon_6)$, with $i = 1,\dots,6$ (see §2.1.1 for more general notations about $\mathfrak{sl}_{r+1}$).
Let $\mu = \delta_3$. Consider the weighted path $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)$ with $\underline{\mu} = (\delta_3, \delta_4, \delta_6, \delta_5, \delta_5, \delta_5, \delta_3)$ and $\underline{\alpha} = (\beta_3, \beta_4 + \beta_5, -\beta_5, \beta_5, \beta_4, -\beta_3 - \beta_4)$. We have:

1. $\mu^{(1)} = \delta_3$, $\mathrm{ht}(\underline{\mu})_1 = \mathrm{ht}(\delta_3 - \delta_4) = \mathrm{ht}(\varepsilon_3 - \varepsilon_4) = \mathrm{ht}(\beta_3) = 1$, and $\alpha^{(1)} = \mu^{(1)} - \mu^{(2)} = \delta_3 - \delta_4 = \beta_3$,
2. $\mu^{(2)} = \delta_4, \mathrm{ht}(\underline{\mu})_2 = \mathrm{ht}(\delta_4 - \delta_6) = \mathrm{ht}(\varepsilon_4 - \varepsilon_6) = \mathrm{ht}(\beta_4 + \beta_5) = 2$, and $\alpha^{(2)} = \mu^{(2)} - \mu^{(3)} = \beta_4 + \beta_5$,
3. $\mu^{(3)} = \delta_6$, $\mathrm{ht}(\underline{\mu})_3 = \mathrm{ht}(\delta_6 - \delta_5) = \mathrm{ht}(\varepsilon_6 - \varepsilon_5) = \mathrm{ht}(-\beta_5) = -1$, and $\alpha^{(3)} = \mu^{(3)} - \mu^{(4)} = -\beta_5$,

4. $\mu^{(4)} = \delta_5$, $\mathrm{ht}(\underline{\mu})_4 = \mathrm{ht}(\delta_5 - \delta_5) = 0$; we observe that $\langle \mu^{(4)}, \check{\alpha} \rangle \neq 0$ for $\alpha \in \Pi$ if and only if $\alpha = \beta_4$ or $\alpha = \beta_5$; our choice is $\alpha^{(4)} = \beta_5$,

5. $\mu^{(5)} = \delta_5$, $\mathrm{ht}(\underline{\mu})_5 = \mathrm{ht}(\delta_5 - \delta_5) = 0$; we observe that $\langle \mu^{(5)}, \check{\alpha} \rangle \neq 0$ for $\alpha \in \Pi$ and only if $\alpha = \beta_4$ or $\alpha = \beta_5$; our choice is $\alpha^{(5)} = \beta_4$,

6. $\mu^{(6)} = \delta_5$, $\mathrm{ht}(\underline{\mu})_6 = \mathrm{ht}(\delta_5 - \delta_3) = \mathrm{ht}(\varepsilon_5 - \varepsilon_3) = \mathrm{ht}(-\beta_4 - \beta_3) = -2$, and $\alpha^{(6)} = -\beta_3 - \beta_4$.

We have $\mathrm{ht}(\underline{\mu}) = (1, 2, -1, 0, 0, -2)$ and $\sum_{i=1}^{6} \mathrm{ht}(\underline{\mu})_i = 0$. We represent in Figure 1.1 this weighted path.

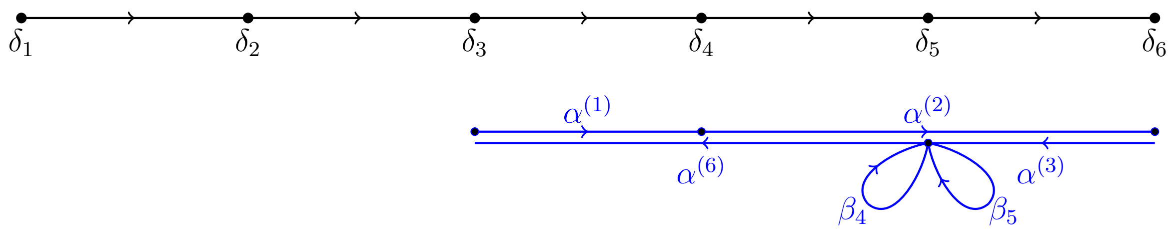



**FIGURE 1.1** – An example of weighted path in $\mathfrak{sl}_6$.

Recall that $\mathscr{B}$ is the Chevalley basis $\{e_\alpha, \check{\beta}_i \mid \alpha \in \Delta, i = 1, \dots, r\}$ of $\mathfrak{g}$ and that $\mathscr{B}^*$ is its dual basis, identified with a basis of $\mathfrak{g}$ through the inner product $B_{\mathfrak{g}}$. Namely, $\mathscr{B}^* = \{c_\alpha e_{-\alpha}, \varpi_i^\sharp \mid \alpha \in \Delta, i = 1, \dots, r\}$, with $c_\alpha \neq 0$. For $\beta \in \Pi$, we write $\varpi_\beta$ the fundamental weight corresponding to $\beta$. Thus $\varpi_{\beta_i} = \varpi_i$, $i = 1, \dots, r$, but it will be convenient to have both notations.

Pick $\mu, \nu \in P(\delta)$.

**Definition 1.14.** *Let $m \in \mathbb{Z}_{>0}$ and $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu, \nu)$. For $j \in \{1, \dots, m\}$, we define the element $b_{(\underline{\mu},\underline{\alpha}),j}$ of $\mathscr{B}$ as follows:*

1. *if $\mathrm{ht}(\underline{\mu})_j \neq 0$, set $b_{(\underline{\mu},\underline{\alpha}),j} := c_{\alpha^{(j)}} e_{-\alpha^{(j)}}$ so that $b^*_{(\underline{\mu},\underline{\alpha}),j} := e_{\alpha^{(j)}}$,*

2. *if $\mathrm{ht}(\underline{\mu})_j = 0$, set $b_{(\underline{\mu},\underline{\alpha}),j} := \check{\alpha}^{(j)}$ so that $b^*_{(\underline{\mu},\underline{\alpha}),j} = \varpi^\sharp_{\alpha^{(j)}}$.*

**Definition 1.15.** *Let $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu, \nu)$. We define the element $b^*_{\underline{\mu},\underline{\alpha}}$ of $U(\mathfrak{g})$ by*

$$b^*_{\underline{\mu},\underline{\alpha}} := a_{\underline{\mu},\underline{\alpha}} b^*_{(\underline{\mu},\underline{\alpha}),1} \cdots b^*_{(\underline{\mu},\underline{\alpha}),m},$$

*where*

$$a_{\underline{\mu},\underline{\alpha}} := a_{\mu^{(m+1)},\mu^{(m)}}^{(b_{(\underline{\mu},\underline{\alpha}),m})} \dots a_{\mu^{(2)},\mu^{(1)}}^{(b_{(\underline{\mu},\underline{\alpha}),1})}$$

In the case where $\mu = \nu$, we observe that $b^*_{\underline{\mu},\underline{\alpha}}$ belongs to $U(\mathfrak{g})^{\mathfrak{h}}$.

**Definition 1.16** (weight). *Assume that* $\mu = \nu$. *We define the* weight *of* $(\underline{\mu}, \underline{\alpha})$ *to be the complex number,*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) := (\mathrm{ev}_\rho \circ \mathrm{hc})(b^*_{\underline{\mu},\underline{\alpha}}).$$

*We adopt the convention that* $\mathrm{wt}(\mathbf{1}_\mu) = 1$.

*Example* 1.17. Assume that $\mathfrak{g} = \mathfrak{sl}_6$ and $\delta = \varpi_1$. Choose the the weighted path as in Example 1.13. We have:

1. $\mathrm{ht}(\underline{\mu})_1 = 1 \neq 0$ and $\alpha^{(1)} = \beta_3 = \varepsilon_3 - \varepsilon_4$ then $b_{(\underline{\mu},\underline{\alpha}),1} = e_{\varepsilon_4 - \varepsilon_3}$ and $b^*_{(\underline{\mu},\underline{\alpha}),1} = e_{\varepsilon_3 - \varepsilon_4}$,
2. $\mathrm{ht}(\underline{\mu})_2 = 2 \neq 0$ and $\alpha^{(2)} = \beta_4 + \beta_5 = \varepsilon_4 - \varepsilon_6$ then $b_{(\underline{\mu},\underline{\alpha}),2} = e_{\varepsilon_6 - \varepsilon_4}$ and $b^*_{(\underline{\mu},\underline{\alpha}),2} = e_{\varepsilon_4 - \varepsilon_6}$,
3. $\mathrm{ht}(\underline{\mu})_3 = -1 \neq 0$ and $\alpha^{(3)} = -\beta_5 = \varepsilon_6 - \varepsilon_5$ then $b_{(\underline{\mu},\underline{\alpha}),3} = e_{\varepsilon_5 - \varepsilon_6}$ and $b^*_{(\underline{\mu},\underline{\alpha}),3} = e_{\varepsilon_6 - \varepsilon_5}$,
4. $\mathrm{ht}(\underline{\mu})_4 = 0$ and $\alpha^{(4)} = \beta_5$ then $b_{(\underline{\mu},\underline{\alpha}),4} = \check{\beta}_5$ and $b^*_{(\underline{\mu},\underline{\alpha}),4} = \check{\varpi}_5$,
5. $\mathrm{ht}(\underline{\mu})_5 = 0$ and $\alpha^{(5)} = \beta_4$ then $b_{(\underline{\mu},\underline{\alpha}),5} = \check{\beta}_4$ and $b^*_{(\underline{\mu},\underline{\alpha}),5} = \check{\varpi}_4$,
6. $\mathrm{ht}(\underline{\mu})_6 = -2 \neq 0$ and $\alpha^{(6)} = -\beta_3 - \beta_4 = \varepsilon_5 - \varepsilon_3$ then $b_{(\underline{\mu},\underline{\alpha}),6} = e_{\varepsilon_3 - \varepsilon_5}$ and $b^*_{(\underline{\mu},\underline{\alpha}),6} = e_{\varepsilon_5 - \varepsilon_3}$.

Moreover,

$$\begin{aligned} a_{\underline{\mu},\underline{\alpha}} &= a_{\mu^{(7)},\mu^{(6)}}^{(b_{(\underline{\mu},\underline{\alpha}),6})} a_{\mu^{(6)},\mu^{(5)}}^{(b_{(\underline{\mu},\underline{\alpha}),5})} \dots a_{\mu^{(2)},\mu^{(1)}}^{(b_{(\underline{\mu},\underline{\alpha}),1})} \\ &= a_{\varepsilon_3,\varepsilon_5}^{(e_{\varepsilon_3-\varepsilon_5})} a_{\varepsilon_5,\varepsilon_5}^{(\check{\beta}_4)} a_{\varepsilon_5,\varepsilon_5}^{(\check{\beta}_5)} a_{\varepsilon_5,\varepsilon_6}^{(e_{\varepsilon_5-\varepsilon_6})} a_{\varepsilon_6,\varepsilon_4}^{(e_{\varepsilon_6-\varepsilon_4})} a_{\varepsilon_4,\varepsilon_3}^{(e_{\varepsilon_4-\varepsilon_3})} = \langle \delta_5, \check{\beta}_4 \rangle \langle \delta_5, \check{\beta}_5 \rangle = -1. \end{aligned}$$

and

$$b^*_{\underline{\mu},\underline{\alpha}} = a_{\underline{\mu},\underline{\alpha}} b^*_{\underline{\mu},\underline{\alpha},1} \dots b^*_{\underline{\mu},\underline{\alpha},6} = -e_{\varepsilon_3-\varepsilon_4} e_{\varepsilon_4-\varepsilon_6} e_{\varepsilon_6-\varepsilon_5} \check{\varpi}_5 \check{\varpi}_4 e_{\varepsilon_5-\varepsilon_3}$$

*Example* 1.18. Assume that $\mathfrak{g} = \mathfrak{sp}_8$ and $\delta = \varpi_1$ (cf. Example 1.8). For any $i = 1, \ldots, 4$, let $\varepsilon_i \in \mathfrak{h}^*$ be the linear map defined by $\varepsilon_i(h) = h_i$ if $h = \operatorname{diag}(h_1, \ldots, h_4, -h_1, \ldots, -h_4) \in \mathfrak{h}$. Let $\beta_i := \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, 3$ and $\beta_4 = 2\varepsilon_4$, and $\delta_i = \varepsilon_i$, $\overline{\delta}_i = -\varepsilon_i$, for $i = 1, \ldots, 4$ (see §2.1.2 for more general notations about $\mathfrak{sp}_{2r}$). We represent in Figure 1.2 the weighted path $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_8(\delta_1)$ of length 7 starting and ending at $\delta_1$ with

$$\underline{\mu} = (\delta_1, \delta_4, \overline{\delta}_4, \overline{\delta}_4, \overline{\delta}_1, \overline{\delta}_1, \overline{\delta}_3, \delta_1),$$
$$\underline{\alpha} = (\varepsilon_1 - \varepsilon_4, 2\varepsilon_4, \beta_3, \varepsilon_1 - \varepsilon_4, \beta_1, \varepsilon_3 - \varepsilon_1, -\varepsilon_1 - \varepsilon_3), \quad \text{and}$$
$$\operatorname{ht}(\underline{\mu}) = (3, 1, 0, 3, 0, -2, -5).$$

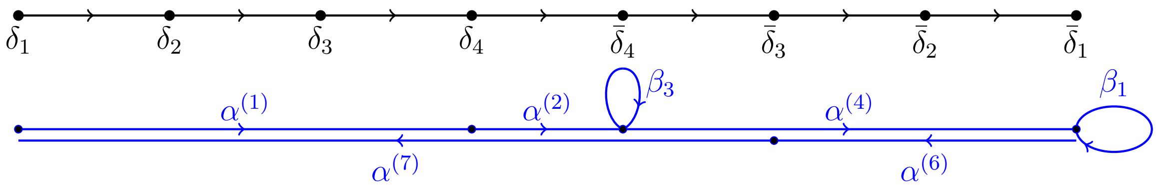



**FIGURE 1.2** – An example of weighted path in $\mathfrak{sp}_8$.

Thus, we have

$$\begin{aligned}
b^*_{(\underline{\mu},\underline{\alpha}),1} &= e_{\varepsilon_1-\varepsilon_4} \quad \text{and} \quad b_{(\underline{\mu},\underline{\alpha}),1} = e_{\varepsilon_4-\varepsilon_1},\\
b^*_{(\underline{\mu},\underline{\alpha}),2} &= e_{2\varepsilon_4} \quad \text{and} \quad b_{(\underline{\mu},\underline{\alpha}),2} = 2e_{-2\varepsilon_4},\\
b^*_{(\underline{\mu},\underline{\alpha}),3} &= \varpi^\sharp_{\beta_3} = \check{\varpi}_3 \quad \text{and} \quad b_{(\underline{\mu},\underline{\alpha}),3} = \check{\beta}_3,\\
b^*_{(\underline{\mu},\underline{\alpha}),4} &= e_{\varepsilon_1-\varepsilon_4} \quad \text{and} \quad b_{(\underline{\mu},\underline{\alpha}),1} = e_{\varepsilon_4-\varepsilon_1},\\
b^*_{(\underline{\mu},\underline{\alpha}),5} &= \varpi^\sharp_{\beta_1} = \check{\varpi}_1 \quad \text{and} \quad b_{(\underline{\mu},\underline{\alpha}),1} = \check{\beta}_1,\\
b^*_{(\underline{\mu},\underline{\alpha}),6} &= e_{\varepsilon_3-\varepsilon_1} \quad \text{and} \quad b_{(\underline{\mu},\underline{\alpha}),1} = e_{\varepsilon_1-\varepsilon_3},\\
b^*_{(\underline{\mu},\underline{\alpha}),7} &= e_{-\varepsilon_1-\varepsilon_3} \quad \text{and} \quad b_{(\underline{\mu},\underline{\alpha}),1} = e_{\varepsilon_1+\varepsilon_3}.
\end{aligned}$$

Hence,

$$\begin{aligned}
a_{\underline{\mu},\underline{\alpha}} &= a^{(b_{(\underline{\mu},\underline{\alpha}),7})}_{\mu^{(8)},\mu^{(7)}} a^{(b_{(\underline{\mu},\underline{\alpha}),6})}_{\mu^{(7)},\mu^{(6)}} \ldots a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}}\\
&= a^{(e_{\varepsilon_1+\varepsilon_3})}_{\varepsilon_1,-\varepsilon_3} a^{(e_{\varepsilon_1-\varepsilon_3})}_{-\varepsilon_3,-\varepsilon_1} a^{(\check{\beta}_1)}_{-\varepsilon_1,-\varepsilon_1} a^{(e_{\varepsilon_4-\varepsilon_1})}_{-\varepsilon_1,-\varepsilon_4} a^{(\check{\beta}_3)}_{-\varepsilon_4,-\varepsilon_4} a^{(2e_{-2\varepsilon_4})}_{-\varepsilon_4,\varepsilon_4} a^{(e_{\varepsilon_4-\varepsilon_1})}_{\varepsilon_4,\varepsilon_1}
\end{aligned}$$

$$= 1 \times (-1) \times \langle \overline{\delta}_1, \check{\beta}_1 \rangle \times (-1) \times \langle \overline{\delta}_4, \check{\beta}_3 \rangle \times 2 \times 1$$
$$= -2,$$

and

$$b^*_{\underline{\mu},\underline{\alpha}} = a_{\underline{\mu},\underline{\alpha}} b^*_{(\underline{\mu},\underline{\alpha}),1} \dots b^*_{(\underline{\mu},\underline{\alpha}),7} = -2 e_{\varepsilon_1-\varepsilon_4} e_{2\varepsilon_4} \check{\varpi}_3 e_{\varepsilon_1-\varepsilon_4} \check{\varpi}_1 e_{\varepsilon_3-\varepsilon_1} e_{-\varepsilon_1-\varepsilon_3}.$$

Recall (1.2) that for $k \in \{1, \dots, r\}$:

$$P(\delta)_k := \{\mu \in P(\delta) \mid \langle \mu, \check{\beta}_k \rangle \neq 0\}.$$

We are now in a position to express the elements $\overline{\mathrm{d}p}_{m,k}$ and $\widehat{\mathrm{d}p}_{m,k}$ in term of *weighted paths* as follows:

**Lemma 1.19.** *Let $m \in \mathbb{Z}_{>0}$ and $k \in \{1, \dots, r\}$. We have:*

$$\overline{\mathrm{d}p}_{m,k} = \sum_{\mu \in P(\delta)_k} \sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu) \\ \mathrm{ht}(\underline{\mu}) = \underline{0}}} \langle \mu, \check{\beta}_k \rangle b^*_{\underline{\mu},\underline{\alpha}},$$
$$\widehat{\mathrm{d}p}_{m,k} = \sum_{\mu \in P(\delta)_k} \sum_{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)} \langle \mu, \check{\beta}_k \rangle \mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}}).$$

*Therefore, we have:*

$$\mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m,k}) = \sum_{\mu \in P(\delta)_k} \sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu) \\ \mathrm{ht}(\underline{\mu}) = \underline{0}}} \mathrm{wt}(\underline{\mu}, \underline{\alpha}) \langle \mu, \check{\beta}_k \rangle,$$
$$\mathrm{ev}_\rho(\widehat{\mathrm{d}p}_{m,k}) = \sum_{\mu \in P(\delta)_k} \sum_{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)} \mathrm{wt}(\underline{\mu}, \underline{\alpha}) \langle \mu, \check{\beta}_k \rangle.$$

*Proof.* Recall the standard fact that for $\alpha \in \Delta$ and $\mu \in P(\delta)$,

$$\pi_\delta(e_\alpha) v_\mu \in V(\delta)_{\mu+\alpha} = \mathbb{C} v_{\mu+\alpha},$$

with the convention that $v_{\mu+\alpha} = 0$ if $\mu + \alpha \notin P(\delta)$. The equality $V(\delta)_{\mu+\alpha} = \mathbb{C} v_{\mu+\alpha}$ holds because of our assumption that all weight spaces have dimension one. Therefore,

$$a^{(e_\alpha)}_{\mu,\nu} \neq 0 \implies \big(\mu = \nu + \alpha \quad \text{and} \quad \nu + \alpha \in P(\delta)\big).$$

For $i \in \{1, \dots, r\}$ and $\mu \in P(\delta)$,

$$\pi_\delta(\check{\beta}_i) v_\mu = \langle \mu, \check{\beta}_i \rangle v_\mu.$$

Therefore,

$$a^{(\check{\beta}_i)}_{\mu,\nu} \neq 0 \iff \big(\nu = \mu \quad \text{and} \quad \langle \mu, \check{\beta}_i \rangle \neq 0\big).$$

According to Lemma 1.7, we have:

$$\widehat{\mathrm{d}p}_{m,k} = \sum_{\substack{(\mu_{j_1},\dots,\mu_{j_m}) \\ \in P(\delta)^m}} \sum_{1 \leqslant i_1,\dots,i_m \leqslant r} a^{(b_{i_1})}_{\mu_{j_1},\mu_{j_2}} \dots a^{(b_{i_m})}_{\mu_{j_m},\mu_{j_1}} \langle \mu_{j_1}, \check{\beta}_k \rangle \mathrm{hc}(b^*_{i_1} \dots b^*_{i_m}),$$

and it is enough to sum over $\mu_{\underline{j}} := (\mu_{j_1}, \dots, \mu_{j_m}) \in P(\delta)^m$ and $\underline{i} := (i_1, \dots, i_m) \in \{1, \dots, r\}^m$ such that

$$a^{(b_{i_1})}_{\mu_{j_1},\mu_{j_2}} \dots a^{(b_{i_m})}_{\mu_{j_m},\mu_{j_1}} \neq 0 \quad \text{and} \quad \langle \mu_{j_1}, \check{\beta}_k \rangle \neq 0.$$

Fix such $(\underline{i}, \mu_{\underline{j}})$ and set for $j \in \{1, \dots, m\}$,

$$\mu^{(1)} := \mu_{j_1},\ \mu^{(2)} := \mu_{j_m},\ \mu^{(3)} := \mu_{j_{m-1}} \quad \dots, \quad \mu^{(m)} := \mu_{j_2}, \quad \mu^{(m+1)} := \mu_{j_1},$$
$$\alpha^{(j)} := \begin{cases} \alpha & \text{if} \quad b_{i_j} = c_\alpha e_{-\alpha}, \quad \alpha \in \Delta, \\ \beta & \text{if} \quad b_{i_j} = \check{\beta}, \quad \beta \in \Pi, \end{cases}$$

and

$$\mu := \mu_{j_1}.$$

Then $\mu \in P(\delta)_k$ and $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)$. Moreover, following Definition 1.15, we get:

$$b_{(\underline{\mu},\underline{\alpha}),j} = b_{i_{m-j+1}} \quad \text{for} \quad j = 1, \dots, m,$$
$$a_{\underline{\mu},\underline{\alpha}} = a^{(b_{(\underline{\mu},\underline{\alpha}),m})}_{\mu^{(m+1)},\mu^{(m)}} \dots a^{(b_{(\underline{\mu},\underline{\alpha}),2})}_{\mu^{(3)},\mu^{(2)}} a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}} \quad \text{and} \quad b^*_{\underline{\mu},\underline{\alpha}} = a_{\underline{\mu},\underline{\alpha}} b^*_{(\underline{\mu},\underline{\alpha}),1} \dots b^*_{(\underline{\mu},\underline{\alpha}),m-1} b^*_{(\underline{\mu},\underline{\alpha}),m},$$

whence the expected formula for $\widehat{\mathrm{d}p}_{m,k}$:

$$\widehat{\mathrm{d}p}_{m,k} = \sum_{\mu \in P(\delta)_k} \sum_{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)} \langle \mu, \check{\beta}_k \rangle \mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}}).$$

Then the formula for $\mathrm{ev}_\rho(\widehat{\mathrm{d}p}_{m,k})$ is obvious by definition of $\mathrm{wt}(\underline{\mu}, \underline{\alpha})$ (cf.Definition 1.16).

Let us now turn to $\overline{\mathrm{d}p}_{m,k}^{(\delta)}$ and $\mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m,k})$. By Lemma 1.7, we have:

$$\overline{\mathrm{d}p}_{m,k}^{(\delta)} = \sum_{\mu\in P(\delta)_k} \sum_{1\leqslant i_1,\ldots,i_m\leqslant r} \langle\mu,\check{\beta}_{i_1}\rangle\ldots\langle\mu,\check{\beta}_{i_m}\rangle\langle\mu,\check{\beta}_k\rangle\,\varpi_{i_1}^\sharp\ldots\varpi_{i_m}^\sharp.$$

To each $\underline{i} := (i_1,\ldots,i_m)\in\{1,\ldots,r\}^m$ such that

$$\langle\mu,\check{\beta}_{i_1}\rangle\ldots\langle\mu,\check{\beta}_{i_m}\rangle\neq 0$$

we attach the weighted paths $(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\mu)$ with

$$\mu^{(1)}=\cdots=\mu^{(m)}=\mu \quad\text{and}\quad \alpha^{(j)}=\check{\beta}_{i_j} \quad\text{for}\quad j=1,\ldots,m.$$

Since all weighted paths $(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\mu)$ with $\mathrm{ht}(\underline{\mu})=\underline{0}$ are of this form, we get the desired statement. Indeed, for such paths, $b^*_{\underline{\mu},\underline{\alpha}}\in U(\mathfrak{h})=S(\mathfrak{h})$ thus $\mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}})=b^*_{\underline{\mu},\underline{\alpha}}$.

Note that for paths $(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\mu)$ such that $\mathrm{ht}(\underline{\mu})=\underline{0}$, we have

$$\mathrm{wt}(\underline{\mu},\underline{\alpha})=\langle\mu,\check{\alpha}^{(1)}\rangle\ldots\langle\mu,\check{\alpha}^{(m)}\rangle\langle\rho,\varpi^\sharp_{\alpha^{(1)}}\rangle\ldots\langle\rho,\varpi^\sharp_{\alpha^{(m)}}\rangle,$$

and the $\alpha^{(j)}$'s run through the set

$$\Pi_\mu := \{\beta\in\Pi \mid \langle\mu,\check{\beta}\rangle\neq 0\}.$$

Hence we get,

$$\mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m,k}) = \sum_{\mu\in P(\delta)_k}\sum_{\underline{\alpha}\in(\Pi_\mu)^m}\langle\mu,\check{\alpha}^{(1)}\rangle\ldots\langle\mu,\check{\alpha}^{(m)}\rangle\langle\rho,\varpi^\sharp_{\alpha^{(1)}}\rangle\ldots\langle\rho,\varpi^\sharp_{\alpha^{(m)}}\rangle\langle\mu,\check{\beta}_k\rangle. \tag{1.3}$$

□

## 1.3 Operations on weighted paths

We keep the notations of previous sections. We study in this section some useful operations on the set of weighted paths. Let $(m,n)\in(\mathbb{Z}_{>0})^2$ and $(\lambda,\mu,\nu)\in P(\delta)^3$.

**Definition 1.20** (concatenation)**.** *Let* $(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\lambda,\mu)$ *and* $(\underline{\mu}',\underline{\alpha}')\in\hat{\mathscr{P}}_n(\mu,\nu)$.

*We define the sequences* $\underline{\mu} \star \underline{\mu}'$ *and* $\underline{\alpha} \star \underline{\alpha}'$ *by:*

$$\underline{\mu} \star \underline{\mu}' := (\mu^{(1)}, \dots, \mu^{(m+1)} = \mu'^{(1)}, \dots, \mu'^{(n+1)}), \quad \underline{\alpha} \star \underline{\alpha}' := (\alpha^{(1)}, \dots, \alpha^{(m)}, \alpha'^{(1)}, \dots, \alpha'^{(n)}).$$

*The pair* $(\underline{\mu}, \underline{\alpha}) \star (\underline{\mu}', \underline{\alpha}') := (\underline{\mu} \star \underline{\mu}', \underline{\alpha} \star \underline{\alpha}')$ *defines a weighted path of* $\hat{\mathscr{P}}_{m+n}(\lambda, \nu)$ *that we call the* concatenation of the paths $(\underline{\mu}, \underline{\alpha})$ and $(\underline{\mu}', \underline{\alpha}')$.

**Lemma 1.21.** *Let* $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\lambda, \mu)$ *and* $(\underline{\mu}', \underline{\alpha}') \in \hat{\mathscr{P}}_n(\mu, \nu)$.

1. *We have*
$$b^*_{(\underline{\mu},\underline{\alpha})\star(\underline{\mu}',\underline{\alpha}')} = b^*_{\underline{\mu},\underline{\alpha}}\, b^*_{\underline{\mu}',\underline{\alpha}'}.$$

2. *If* $\lambda = \mu = \nu$,
$$\mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})\star(\underline{\mu}',\underline{\alpha}')}) = \mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}})\mathrm{hc}(b^*_{\underline{\mu}',\underline{\alpha}'})$$
*and*
$$\mathrm{wt}(\underline{\mu} \star \underline{\mu}', \underline{\alpha} \star \underline{\alpha}') = \mathrm{wt}(\underline{\mu}, \underline{\alpha})\, \mathrm{wt}(\underline{\mu}', \underline{\alpha}').$$

*Proof.* (1) From Definition 1.15, we have

$$b^*_{\underline{\mu},\underline{\alpha}} = a^{(b_{(\underline{\mu},\underline{\alpha}),m})}_{\mu^{(m+1)},\mu^{(m)}} \dots a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}} b^*_{(\underline{\mu},\underline{\alpha}),1} \dots b^*_{(\underline{\mu},\underline{\alpha}),m},$$
$$b^*_{\underline{\mu}',\underline{\alpha}'} = a^{(b_{(\underline{\mu}',\underline{\alpha}'),n})}_{\mu'^{(n+1)},\mu'^{(n)}} \dots a^{(b'_{(\underline{\mu}',\underline{\alpha}'),1})}_{\mu'^{(2)},\mu'^{(1)}} b^*_{(\underline{\mu}',\underline{\alpha}'),1} \dots b^*_{(\underline{\mu}',\underline{\alpha}'),n}.$$

Thus,

$$\begin{aligned} b^*_{\underline{\mu},\underline{\alpha}} b^*_{\underline{\mu}',\underline{\alpha}'} &= a^{(b_{(\underline{\mu},\underline{\alpha}),m})}_{\mu^{(m+1)},\mu^{(m)}} \dots a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}} b^*_{(\underline{\mu},\underline{\alpha}),1} \dots b^*_{(\underline{\mu},\underline{\alpha}),m} a^{(b_{(\underline{\mu}',\underline{\alpha}'),n})}_{\mu'^{(n+1)},\mu'^{(n)}} \dots a^{(b'_{(\underline{\mu}',\underline{\alpha}'),1})}_{\mu'^{(2)},\mu'^{(1)}} b^*_{(\underline{\mu}',\underline{\alpha}'),1} \dots b^*_{(\underline{\mu}',\underline{\alpha}'),n} \\ &= a^{(b_{(\underline{\mu}',\underline{\alpha}'),n})}_{\mu'^{(n+1)},\mu'^{(n)}} \dots a^{(b'_{(\underline{\mu}',\underline{\alpha}'),1})}_{\mu'^{(2)},\mu'^{(1)}} a^{(b_{(\underline{\mu},\underline{\alpha}),m})}_{\mu'^{(1)},\mu^{(m)}} \dots a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}} b^*_{(\underline{\mu},\underline{\alpha}),1} \dots b^*_{(\underline{\mu},\underline{\alpha}),m} b^*_{(\underline{\mu}',\underline{\alpha}'),1} \dots b^*_{(\underline{\mu}',\underline{\alpha}'),n}, \end{aligned}$$

and

$$b^*_{\underline{\mu}\star\underline{\mu}',\underline{\alpha}\star\underline{\alpha}'} = a^{(b_{(\underline{\mu}',\underline{\alpha}'),n})}_{\mu'^{(n+1)},\mu'^{(n)}} \dots a^{(b'_{(\underline{\mu}',\underline{\alpha}'),1})}_{\mu'^{(2)},\mu'^{(1)}} a^{(b_{(\underline{\mu},\underline{\alpha}),m})}_{\mu'^{(1)},\mu^{(m)}} \dots a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}} b^*_{(\underline{\mu},\underline{\alpha}),1} \dots b^*_{(\underline{\mu},\underline{\alpha}),m} b^*_{(\underline{\mu}',\underline{\alpha}'),1} \dots b^*_{(\underline{\mu}',\underline{\alpha}'),n}.$$

Since $\mu^{(m+1)} = \mu'^{(1)}$ then we have the desired equality.

(2) Assume $\lambda = \mu$ then $\mu^{(m+1)} = \mu^{(1)}$. Let $b^*_{\underline{\mu},\underline{\alpha}} := a_{\underline{\mu},\underline{\alpha}} b^*_{(\underline{\mu},\underline{\alpha}),1} \dots b^*_{(\underline{\mu},\underline{\alpha}),m}$. For any $h \in \mathfrak{h}$, we have

$$(\mathrm{ad}\ h) b^*_{\underline{\mu},\underline{\alpha}} = \sum_{j=1}^{m} a_{\underline{\mu},\underline{\alpha}} b^*_{(\underline{\mu},\underline{\alpha}),1} \dots [h, b^*_{(\underline{\mu},\underline{\alpha}),j}] \dots b^*_{(\underline{\mu},\underline{\alpha}),m}$$

$$= \sum_{j=1}^{m} \alpha^{(j)}(h) b^*_{\underline{\mu},\underline{\alpha}} = \sum_{j=1}^{m} (\mu^{(j)} - \mu^{(j+1)})(h) b^*_{\underline{\mu},\underline{\alpha}} = (\mu^{(1)} - \mu^{(m+1)})(h) b^*_{\underline{\mu},\underline{\alpha}} = 0.$$

Thus, $b^*_{\underline{\mu},\underline{\alpha}} \in U(\mathfrak{g})^{\mathfrak{h}}$. Similarly we have $b^*_{\underline{\mu}',\underline{\alpha}'} \in U(\mathfrak{g})^{\mathfrak{h}}$. The restrictions to $U(\mathfrak{g})^{\mathfrak{h}}$ of hc and of $(\mathrm{ev}_\rho \circ \mathrm{hc})$ are homomorphisms of algebras. Therefore,

$$\mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})\star(\underline{\mu}',\underline{\alpha}')}) = \mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}} b^*_{\underline{\mu}',\underline{\alpha}'}) = \mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}}) \mathrm{hc}(b^*_{\underline{\mu}',\underline{\alpha}'}),$$

and

$$\mathrm{wt}(\underline{\mu} \star \underline{\mu}', \underline{\alpha} \star \underline{\alpha}') = (\mathrm{ev}_\rho \circ \mathrm{hc})(b^*_{(\underline{\mu},\underline{\alpha})\star(\underline{\mu}',\underline{\alpha}')}) = \mathrm{wt}(\underline{\mu}, \underline{\alpha})\, \mathrm{wt}(\underline{\mu}', \underline{\alpha}').$$

□

**Definition 1.22** (Cutting a vertex)**.** *Let $m \in \mathbb{Z}_{>1}$, $(\mu, \nu) \in P(\delta)^2$ and $i \in \{2, \dots, m\}$. Let $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\lambda, \mu)$ and assume that $(\underline{\mu}, \underline{\alpha})$ is a weighted path with no ramification*[1]*. We define the sequences $\underline{\mu}^{\#i}$ and $\underline{\alpha}^{\#i}$ as follows:*

1. *If $\mathrm{ht}(\underline{\mu})_{i-1} \neq 0$, $\mathrm{ht}(\underline{\mu})_i \neq 0$ and $\alpha^{(i-1)} + \alpha^{(i)} \neq 0$, then we set:*

$$\underline{\mu}^{\#i} := (\mu^{(1)}, \dots, \mu^{(i-1)}, \mu^{(i+1)}, \dots, \mu^{(m+1)}),$$
$$\underline{\alpha}^{\#i} := (\alpha^{(1)}, \dots, \alpha^{(i-1)} + \alpha^{(i)}, \alpha^{(i+1)}, \dots, \alpha^{(m)}).$$

   *The pair $(\underline{\mu}, \underline{\alpha})^{\#i} := (\underline{\mu}^{\#i}, \underline{\alpha}^{\#i})$ defines an element of $\hat{\mathscr{P}}_{m-1}(\lambda, \mu)$.*

2. *If $\mathrm{ht}(\underline{\mu})_{i-1} \neq 0$, $\mathrm{ht}(\underline{\mu})_i \neq 0$ and $\alpha^{(i-1)} + \alpha^{(i)} = 0$, then we set:*

$$\underline{\mu}^{\#i} := (\mu^{(1)}, \dots, \mu^{(i-1)} = \mu^{(i+1)}, \dots, \mu^{(m+1)}),$$
$$\underline{\alpha}^{\#i} := (\alpha^{(1)}, \dots, \alpha^{(i-2)}, \alpha^{(i+1)}, \dots, \alpha^{(m)}).$$

   *The pair $(\underline{\mu}, \underline{\alpha})^{\#i} := (\underline{\mu}^{\#i}, \underline{\alpha}^{\#i})$ defines an element of $\hat{\mathscr{P}}_{m-2}(\lambda, \mu)$.*

3. *If $\mathrm{ht}(\underline{\mu})_i = 0$, then we set:*

$$\underline{\mu}^{\#i} := (\mu^{(1)}, \dots, \mu^{(i-1)}, \mu^{(i+1)}, \dots, \mu^{(m+1)}),$$
$$\underline{\alpha}^{\#i} := (\alpha^{(1)}, \dots, \alpha^{(i-1)}, \alpha^{(i+1)}, \dots, \alpha^{(m)}).$$

[1] By path with no ramification, it means that any vertex $\mu^{(i)}$ is related by an arrow to at most two other vertices.

*The pair* $(\underline{\mu},\underline{\alpha})^{\#i} := (\underline{\mu}^{\#i},\underline{\alpha}^{\#i})$ *defines an element of* $\hat{\mathscr{P}}_{m-1}(\lambda,\mu)$.

Our definition cannot be applied for a vertex $\mu^{(i)}$ such that $\mathrm{ht}(\underline{\mu})_{i-1} = 0$ and $\mathrm{ht}(\underline{\mu})_i \neq 0$. So we cannot cut such a vertex $\mu^{(i)}$. In such situation $\mu^{(i-1)} = \mu^{(i)}$, so we can cut the vertex $\mu^{(i-1)}$ instead. We illustrate in Figure 1.3 the operation of "cutting the vertex $\mu^{(3)}$" in the three above situations.

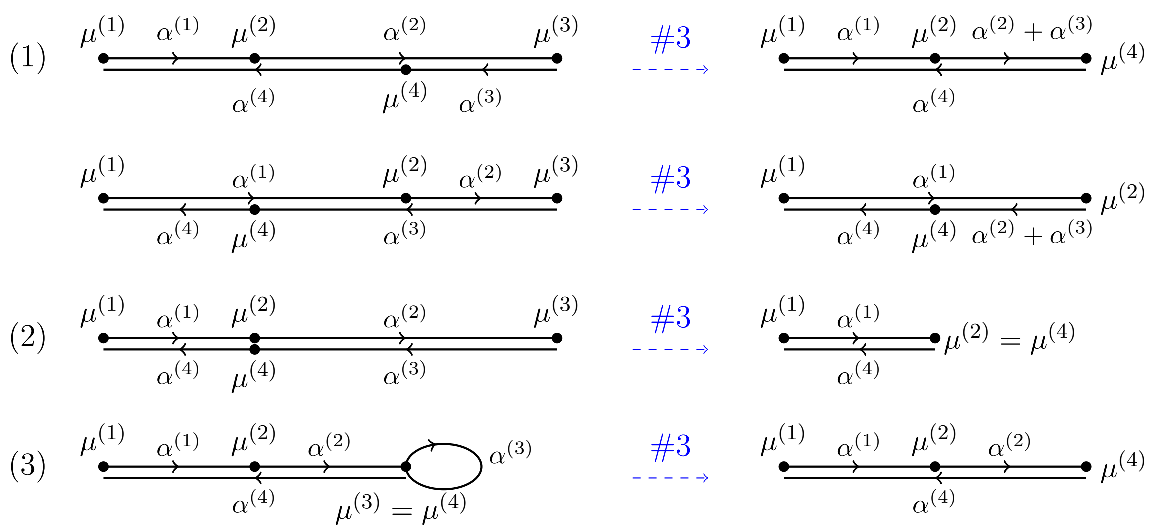



**FIGURE 1.3** – Cutting the vertex $\mu^{(3)}$

Let $m \in \mathbb{Z}_{>1}$ and $(\mu,\nu) \in P(\delta)^2$. For $\underline{i} := (i_1,\dots,i_m) \in \mathbb{Z}^m$, set

$$\hat{\mathscr{P}}_m(\mu,\nu)_{\underline{i}} := \{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu,\nu) \mid \mathrm{ht}(\underline{\mu}) = \underline{i}\ \}.$$

The elements of $\hat{\mathscr{P}}_m(\mu,\nu)_{\underline{i}}$ share the same vertices and only the labels of the "loops" (i.e. the labels $\alpha^{(j)}$ such that $\mu^{(j)} = \mu^{(j+1)}$) may differ. We obtain the following partition of $\hat{\mathscr{P}}_m(\mu,\nu)$:

$$\hat{\mathscr{P}}_m(\mu,\nu) = \bigsqcup_{\underline{i}\in\mathbb{Z}^m} \hat{\mathscr{P}}_m(\mu,\nu)_{\underline{i}}.$$

The set $\hat{\mathscr{P}}_m(\mu,\nu)_{\underline{i}}$ is empty for almost all $\underline{i}$. We endow the set $\mathbb{Z}^m$ with the lexicographical order $\preccurlyeq$. The zero element is $\underline{0} = (0,\dots,0)$. We denote by $\mathbb{Z}^m_{\succcurlyeq\underline{0}}$ (respectively, $\mathbb{Z}^m_{\succ\underline{0}}$) the set of elements in $\mathbb{Z}^m$ greater (respectively, strictly greater) than $\underline{0}$ for the lexicographical order.

Note that if $\underline{i} \in \mathbb{Z}^m_{\prec\underline{0}}$ then $\mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}}) = 0$ and $\mathrm{wt}(\underline{\mu},\underline{\alpha}) = 0$.

**Definition 1.23** (turning back)**.** *Let $\underline{i} \in \mathbb{Z}^m_{\succ 0}$ such that $\sum_{j=1}^m i_j = 0$. Let $q(\underline{i})$ be the smallest integer $q$ of $\{1, \ldots, m\}$ such that $\underline{i}_q < 0$, and let $p(\underline{i})$ be the largest integer $p$ of $\{1, \ldots, q(\underline{i})\}$ such that $\underline{i}_{p-1} > 0$. Thus $\underline{i}$ is as follows:*

$$\underline{i} = (\underbrace{i_1, \ldots, i_{p(\underline{i})-2}}_{\geqslant 0}, \underbrace{i_{p(\underline{i})-1}}_{>0}, 0, \ldots, 0, \underbrace{i_{q(\underline{i})}}_{<0}, i_{q(\underline{i})+1}, \ldots, i_m).$$

*For a path $\underline{\mu}$ of height $\underline{i}$, we will say that $\mu^{(p(\underline{i}))}$* is the position of the first turning back.

Note that $p(\underline{i})$ is always strictly greater than 1 for such $\underline{i}$.

**Lemma 1.24.** *Let $m \in \mathbb{Z}_{>1}$, $(\mu, \nu) \in P(\delta)^2$, $i \in \{2, \ldots, m\}$ and $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\lambda, \mu)$.*

1. *If $\mathrm{ht}(\underline{\mu})_{i-1} \neq 0$, $\mathrm{ht}(\underline{\mu})_i \neq 0$ and $\alpha^{(i-1)} + \alpha^{(i)} = 0$, then:*

$$a_{\underline{\mu},\underline{\alpha}} = (c_{\alpha^{(i-1)}})^2 a_{(\underline{\mu},\underline{\alpha})^{\# i}}.$$

   *with $c_{\alpha^{(i-1)}}$ as in Definition 1.14.*

2. *If $\mathrm{ht}(\underline{\mu})_i = 0$, then:*

$$a_{\underline{\mu},\underline{\alpha}} = \langle \mu^{(i)}, \check{\alpha}^{(i)} \rangle a_{(\underline{\mu},\underline{\alpha})^{\# i}}.$$

*Proof.* Recall that $(\pi_\delta, V(\delta))$ is a representation with highest weight $\delta$ and $(\nu_\lambda)_{\lambda \in P(\delta)}$ is a fixed basis of $V(\delta)$. For $\lambda, \mu \in P(\delta)$ and $b \in \mathfrak{h}$,

$$\pi_\delta(b) v_\mu = \sum_{\lambda \in P(\delta)} a^{(b)}_{\lambda,\mu} v_\lambda.$$

1. If $\alpha^{(i-1)} + \alpha^{(i)} = 0$ then $\alpha^{(i-1)} = -\alpha^{(i)}$ and $\mu^{(i-1)} = \mu^{(i+1)}$. We have,

$$\underline{\mu}^{\# i} := (\mu^{(1)}, \ldots, \mu^{(i-1)} = \mu^{(i+1)}, \ldots, \mu^{(m+1)}),$$
$$\underline{\alpha}^{\# i} := (\alpha^{(1)}, \ldots, \alpha^{(i-2)}, \alpha^{(i+1)}, \ldots, \alpha^{(m)}).$$

   Thus,

$$a_{\underline{\mu},\underline{\alpha}} = a^{(b_{(\underline{\mu},\underline{\alpha}),m})}_{\mu^{(m+1)},\mu^{(m)}} \cdots a^{(c_{\alpha^{(i)}} e_{-\alpha^{(i)}})}_{\mu^{(i+1)},\mu^{(i)}} a^{(c_{\alpha^{(i-1)}} e_{-\alpha^{(i-1)}})}_{\mu^{(i)},\mu^{(i-1)}} \cdots a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}}$$
$$a_{(\underline{\mu},\underline{\alpha})^{\# i}} = a^{(b_{(\underline{\mu},\underline{\alpha}),m})}_{\mu^{(m+1)},\mu^{(m)}} \cdots a^{(b_{(\underline{\mu},\underline{\alpha}),i+1})}_{\mu^{(i+2)},\mu^{(i+1)}} a^{(b_{(\underline{\mu},\underline{\alpha}),i-2})}_{\mu^{(i-1)},\mu^{(i-2)}} \cdots a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}}.$$

On the other hand,

$$\begin{aligned} a^{(e_{-\alpha^{(i)}})}_{\mu^{(i+1)},\mu^{(i)}} a^{(e_{-\alpha^{(i-1)}})}_{\mu^{(i)},\mu^{(i-1)}} v_{\mu^{(i+1)}} &= \pi_\delta(e_{-\alpha^{(i)}})\pi_\delta(e_{-\alpha^{(i-1)}})v_{\mu^{(i-1)}} \\ &= \pi_\delta([e_{-\alpha^{(i)}}, e_{-\alpha^{(i-1)}}])v_{\mu^{(i-1)}} + \pi_\delta(e_{-\alpha^{(i-1)}}e_{-\alpha^{(i)}})v_{\mu^{(i-1)}}. \end{aligned}$$

Assume that $\mu^{(i-1)} - \alpha^{(i)} \in P(\delta)$. Then

$$\mu^{(i-1)} - \alpha^{(i)} = \mu^{(i-1)} - (\mu^{(i)} - \mu^{(i-1)}) = 2\mu^{(i-1)} - \mu^{(i)},$$

since $\alpha^{(i)} = -\alpha^{(i-1)}$ and $\mu^{(i-1)} = \mu^{(i+1)}$. Note that $\mu^{(i-1)} - \alpha^{(i)} = 2\mu^{(i-1)} - \mu^{(i)} \in P(\delta)$ if and only if $\mu^{(i-1)} = \mu^{(i)}$. Thus we get $\alpha^{(i-1)} = 0$, which is a contradiction with the condition $\alpha^{(i-1)} \neq 0$. Hence $\mu^{(i-1)} - \alpha^{(i)} \notin P(\delta)$ and

$$\begin{aligned} a^{(e_{-\alpha^{(i)}})}_{\mu^{(i+1)},\mu^{(i)}} a^{(e_{-\alpha^{(i-1)}})}_{\mu^{(i)},\mu^{(i-1)}} v_{\mu^{(i+1)}} &= \pi_\delta([e_{-\alpha^{(i)}}, e_{-\alpha^{(i-1)}}])v_{\mu^{(i-1)}} \\ &= \pi_\delta(\check{\alpha}^{(i-1)})v_{\mu^{(i-1)}} \\ &= \langle \mu^{(i-1)}, \check{\alpha}^{(i-1)} \rangle v_{\mu^{(i+1)}}. \end{aligned}$$

Therefore,

$$\begin{aligned} a_{\underline{\mu},\underline{\alpha}} &= a^{(b_{(\underline{\mu},\underline{\alpha}),m})}_{\mu^{(m+1)},\mu^{(m)}} \cdots a^{(c_{\alpha^{(i)}}e_{-\alpha^{(i)}})}_{\mu^{(i+1)},\mu^{(i)}} a^{(c_{\alpha^{(i-1)}}e_{-\alpha^{(i-1)}})}_{\mu^{(i)},\mu^{(i-1)}} \cdots a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}} \\ &= c_{\alpha^{(i)}} c_{\alpha^{(i-1)}} \langle \mu^{(i-1)}, \check{\alpha}^{(i-1)} \rangle a^{(b_{(\underline{\mu},\underline{\alpha}),m})}_{\mu^{(m+1)},\mu^{(m)}} \cdots a^{(b_{(\underline{\mu},\underline{\alpha}),i+1})}_{\mu^{(i+2)},\mu^{(i+1)}} a^{(b_{(\underline{\mu},\underline{\alpha}),i-2})}_{\mu^{(i-1)},\mu^{(i-2)}} \cdots a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}} \\ &= (c_{\alpha^{(i-1)}})^2 \langle \mu^{(i-1)}, \check{\alpha}^{(i-1)} \rangle a_{(\underline{\mu},\underline{\alpha})^{\# i}} \\ &= (c_{\alpha^{(i-1)}})^2 a_{(\underline{\mu},\underline{\alpha})^{\# i}}, \end{aligned}$$

since $\langle \mu^{(i-1)}, \check{\alpha}^{(i-1)} \rangle = 1$.

2. If $\mathrm{ht}(\underline{\mu})_i = 0$, we have $\mu^{(i)} = \mu^{(i+1)}$ and,

$$\begin{aligned} \underline{\mu}^{\# i} &:= (\mu^{(1)}, \dots, \mu^{(i-1)}, \mu^{(i+1)}, \dots, \mu^{(m+1)}), \\ \underline{\alpha}^{\# i} &:= (\alpha^{(1)}, \dots, \alpha^{(i-1)}, \alpha^{(i+1)}, \dots, \alpha^{(m)}). \end{aligned}$$

Then

$$\begin{aligned} a_{\underline{\mu},\underline{\alpha}} &= a^{(b_{(\underline{\mu},\underline{\alpha}),m})}_{\mu^{(m+1)},\mu^{(m)}} \cdots a^{(\check{\alpha}^{(i)})}_{\mu^{(i+1)},\mu^{(i)}} \cdots a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}} \\ a_{(\underline{\mu},\underline{\alpha})^{\# i}} &= a^{(b_{(\underline{\mu},\underline{\alpha}),m})}_{\mu^{(m+1)},\mu^{(m)}} \cdots a^{(b_{(\underline{\mu},\underline{\alpha}),i+1})}_{\mu^{(i+2)},\mu^{(i+1)}} a^{(b_{(\underline{\mu},\underline{\alpha}),i-1})}_{\mu^{(i)},\mu^{(i-1)}} \cdots a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}}. \end{aligned}$$

On the other hand,

$$a^{(\check{\alpha}^{(i)})}_{\mu^{(i+1)},\mu^{(i)}} v_{\mu^{(i+1)}} = \pi_\delta(\check{\alpha}^{(i)}) v_{\mu^{(i)}} = \langle \mu^{(i)}, \check{\alpha}^{(i)} \rangle v_{\mu^{(i)}}.$$

Thus

$$\begin{aligned} a_{\underline{\mu},\underline{\alpha}} &= a^{(b_{(\underline{\mu},\underline{\alpha}),m})}_{\mu^{(m+1)},\mu^{(m)}} \dots a^{(\check{\alpha}^{(i)})}_{\mu^{(i+1)},\mu^{(i)}} \dots a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}} \\ &= a^{(b_{(\underline{\mu},\underline{\alpha}),m})}_{\mu^{(m+1)},\mu^{(m)}} \dots a^{(b_{(\underline{\mu},\underline{\alpha}),i+1})}_{\mu^{(i+2)},\mu^{(i+1)}} \langle \mu^{(i)}, \check{\alpha}^{(i)} \rangle a^{(b_{(\underline{\mu},\underline{\alpha}),i-1})}_{\mu^{(i)},\mu^{(i-1)}} \dots a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}} \\ &= \langle \mu^{(i)}, \check{\alpha}^{(i)} \rangle a_{(\underline{\mu},\underline{\alpha})^{\# i}}. \end{aligned}$$

□

# Chapter 2

## Proof of Theorem 6

In this chapter we first collect useful notations and data about the roots and weights in types $A$ and $C$ (cf. Section 2.1). Using this, we prove Theorem 6 for $\mathfrak{g} = \mathfrak{sl}_{r+1}$ (cf. Section 2.2), and for $\mathfrak{g} = \mathfrak{sp}_{2r}$ (cf. Section 2.3). The preliminary results used for the proof in both cases are important not only for Theorem 6 itself, but will be also crucial in the proof of Theorem 7 in the study of weighted paths *with loops* (see §3.2.4 and §4.2.4). Lastly, we introduce in Section 2.4 the notion of *admissible triple* for the type $C$. This notion will be useful in Chapter 4 to prove Theorem 7 for $\mathfrak{g} = \mathfrak{sp}_{2r}$.

## 2.1 Roots and weights

### 2.1.1 In type $A$

Assume that $\mathfrak{g} = \mathfrak{sl}_{r+1}$. We may realize $\mathfrak{g}$ as the set of $(r+1)$-size square traceless matrices. Let $\mathfrak{h}$ be the Cartan subalgebra consisted of all diagonal matrices of $\mathfrak{g}$, and $\tilde{\mathfrak{h}}$ be the set of all $(r+1)$-size diagonal matrices. Let $e = \mathrm{diag}(1, \ldots, 1) \in \tilde{\mathfrak{h}}$. One may identify $\mathfrak{h}^*$ with $\{\lambda \in (\tilde{\mathfrak{h}})^* | \; \lambda(e) = 0\}$. Let $\widetilde{B}$ the scalar product on $\tilde{\mathfrak{h}}$ defined by $\widetilde{B}(X, Y) = \mathrm{tr}(XY)$ and denote by $B$ its restriction to $\mathfrak{h}$. $\widetilde{B}$ induces an isomorphism $\widetilde{B}^\sharp : (\tilde{\mathfrak{h}})^* \to \tilde{\mathfrak{h}}$. Moreover, it induces an isomorphism $\{\lambda \in (\tilde{\mathfrak{h}})^* | \; \lambda(e) = 0\} \to \mathfrak{h}$, which is equal to $B^\sharp$ with the identification $\{\lambda \in (\tilde{\mathfrak{h}})^* | \; \lambda(e) = 0\} \cong \mathfrak{h}^*$.

We define a scalar product $\widetilde{B}^*$ on $\mathfrak{h}^*$ by $\widetilde{B}^*(\lambda, \mu) = \widetilde{B}(\widetilde{B}^\sharp(\lambda), \widetilde{B}^\sharp(\mu))$. It is clear that $\widetilde{B}^*(\lambda, \mu) = \langle \lambda, \widetilde{B}^\sharp(\mu) \rangle$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$. As there is no risk of confusion, we will also denote in the sequel by $\langle \cdot, \cdot \rangle$ the scalar product on $\tilde{\mathfrak{h}}$, $\mathfrak{h}$, $\tilde{\mathfrak{h}}^*$ and $\mathfrak{h}^*$.

Set $\{E_{1,1}, \ldots, E_{r+1,r+1}\}$ is a basis of $\tilde{\mathfrak{h}}$, where $E_{i,j}$ denotes the elementary matrix associated with the coefficient $(i,j)$, and $\{\varepsilon_1, \ldots, \varepsilon_{r+1}\}$ its dual basis. We have $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$, and $\widetilde{B}^\sharp(\varepsilon_i) = E_{i,i}$. Then the root system of $(\mathfrak{g}, \mathfrak{h})$ is $\Delta = \{\varepsilon_i - \varepsilon_j \mid i \neq j\} \subset \{\lambda \in (\tilde{\mathfrak{h}})^* | \ \lambda(e) = 0\}$, the dual system is $\check{\Delta} = \{E_{i,i} - E_{j,j} \mid i \neq j\}$, and we make the standard choice of $\Delta_+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$ for the set of positive roots. The set of simple roots is $\Pi = \{\beta_1, \ldots, \beta_r\}$, where $\beta_1 = \varepsilon_1 - \varepsilon_2, \ldots, \beta_r = \varepsilon_r - \varepsilon_{r+1}$. The fundamental weights are

$$\varpi_i = (\varepsilon_1 + \cdots + \varepsilon_i) - \frac{i}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1}), \qquad i = 1, \ldots, r.$$

The fundamental co-weights are

$$\check{\varpi}_i = (E_{1,1} + \cdots + E_{i,i}) - \frac{i}{r+1}(E_{1,1} + \cdots + E_{r+1,r+1}), \qquad i = 1, \ldots, r.$$

We have $B^\sharp(\varepsilon_i - \varepsilon_j) = E_{i,i} - E_{j,j}$ and $B^\sharp(\varpi_i) = \check{\varpi}_i$, $i = 1, \ldots, r$. The $(\varepsilon_i - \varepsilon_j)$-root space of $\mathfrak{g}$ is spanned by the root vector $e_{\varepsilon_i - \varepsilon_j} := E_{i,j}$. Take $B_\mathfrak{g}$ the invariant nondegenerate bilinear form on $\mathfrak{g}$ defined by $B_\mathfrak{g}(X,Y) = \mathrm{tr}(X,Y)$. Its restriction to $\mathfrak{h}$ is equal to $B$. Since $\mathrm{tr}(E_{i,j}E_{k,l}) = \delta_{j,k}\delta_{i,l}$, in the notation of Section 1.1 (after Lemma 1.6), we have:

$$c_{\varepsilon_i - \varepsilon_j} = 1, \qquad i \neq j.$$

The irreducible representation $V := V(\varpi_1)$ is the standard representation

$$\mathfrak{g} \to \mathrm{End}(\mathbb{C}^{r+1}), \quad X \mapsto (v \mapsto Xv)\,.$$

The nonzero weights of $V(\varpi_1)$ are $\{\delta_1, \ldots, \delta_{r+1}\}$ with

$$\delta_i = \varepsilon_i - \frac{1}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1}), \qquad i = 1, \ldots, r+1.$$

Moreover, $V_{\delta_i} = \mathbb{C}v_i$, for $i = 1, \ldots, r+1$, where $(v_1, \ldots, v_{r+1})$ is the canonical basis of $\mathbb{C}^{r+1}$. We have

$$e_{\varepsilon_i - \varepsilon_j} v_k = E_{i,j} v_k = \delta_{j,k} v_i.$$

In other words,

$$e_{\varepsilon_i - \varepsilon_j} v_k = a_{\varepsilon_i, \varepsilon_k}^{(e_{\varepsilon_i - \varepsilon_j})} v_i = \delta_{j,k} v_i.$$

Hence

$$a_{\varepsilon_i,\varepsilon_k}^{(e_{\varepsilon_l-\varepsilon_j})} = \delta_{i,l}\delta_{j,k} \quad \text{and so} \quad a_{\varepsilon_i,\varepsilon_k}^{(e_{\varepsilon_i-\varepsilon_k})} = 1.$$

We represent in Figure 2.1 the crystal graph $\mathscr{C}(\varpi_1)$.

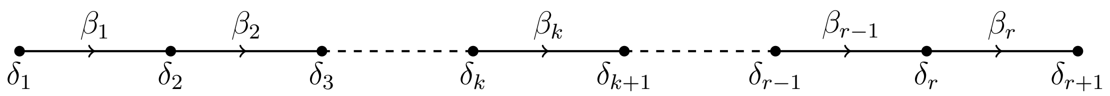



**FIGURE 2.1** – Crystal graph of $\delta = \varpi_1$ for $\mathfrak{g} = \mathfrak{sl}_{r+1}$.

### 2.1.2 In type $C$

Assume that $\mathfrak{g} = \mathfrak{sp}_{2r}$. We may realize $\mathfrak{g}$ as the set of matrices $A$ of $\mathscr{M}_{2r}(\mathbb{C})$ such that $JA + A^t J = 0$, where

$$J = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix},$$

and for $n \in \mathbb{Z}_{\geqslant 0}$, $\mathscr{M}_n(\mathbb{C})$ stands for the set of $n$-size square matrices. Thus an element $x \in \mathfrak{g}$ is a matrix of the form :

$$\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^t \end{pmatrix},$$

where $Z_i \in \mathscr{M}_r(\mathbb{C})$ and $Z_2, Z_3$ are symmetric. Let $\mathfrak{h}$ be the Cartan subalgebra consisted of all diagonal matrices of $\mathfrak{g}$. Write $E_i = \begin{pmatrix} E_{i,i} & 0 \\ 0 & -E_{i,i} \end{pmatrix}$, for $i = 1, \dots, r$. Set $\{E_i\}_{1\leqslant i\leqslant r}$ is a basis of $\mathfrak{h}$, and $\{\varepsilon_i, \dots, \varepsilon_r\}$ its dual basis. Take for $B_{\mathfrak{g}}$ the non-degenerate invariant bilinear form $(X, Y) \mapsto \dfrac{1}{2}\mathrm{tr}(XY)$. The set of roots of $(\mathfrak{g}, \mathfrak{h})$ is $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_k \,|\, 1 \leqslant i < j \leqslant r,\ 1 \leqslant k \leqslant r\}$. We make the standard choice of $\Delta_+ = \{\varepsilon_i \pm \varepsilon_j, 2\varepsilon_k \mid 1 \leqslant i < j \leqslant r,\ 1 \leqslant k \leqslant r\}$, with basis $\Pi = \{\beta_1, \dots, \beta_r\}$, where $\beta_i := \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \dots, r-1$ and $\beta_r = 2\varepsilon_r$. Denote by $\check{\beta}_i$ the coroot of the simple root $\beta_i$. Using the non-degenerate invariant bilinear form $B_{\mathfrak{g}}$, one may identify, as for $\mathfrak{sl}_{r+1}$, the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with $\mathfrak{h}^*$ and we get

$$\check{\beta}_1 = \varepsilon_1 - \varepsilon_2, \quad \dots, \quad \check{\beta}_{r-1} = \varepsilon_{r-1} - \varepsilon_r, \quad \check{\beta}_r = \varepsilon_r.$$

The fundamental weights are:

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i, \qquad i = 1, \ldots, r.$$

The fundamental co-weights are:

$$\check{\varpi}_i = \varepsilon_1 + \cdots + \varepsilon_i, \quad \text{for} \quad i = 1, \ldots, r-1$$

$$\text{and} \quad \check{\varpi}_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r).$$

Note that the half-sum of positive roots is

$$\rho = \varpi_1 + \cdots + \varpi_r = \sum_{i=1}^{r}(\varepsilon_1 + \cdots + \varepsilon_i) = \sum_{i=1}^{r}(r - i + 1)\varepsilon_i.$$

For $\alpha \in \Delta$, the $\alpha$-root space of $\mathfrak{g}$ is spanned by the root vector $e_\alpha$ as defined below [Car05] :

$$\begin{aligned}
&e_{\varepsilon_i - \varepsilon_j} = E_{i,j} - E_{j+r,i+r}, \quad 1 \leqslant i \neq j \leqslant r,\\
&e_{\varepsilon_i + \varepsilon_j} = E_{i,j+r} + E_{j,i+r}, \quad 1 \leqslant i < j \leqslant r,\\
&e_{-\varepsilon_i - \varepsilon_j} = E_{i+r,j} + E_{j+r,i}, \quad 1 \leqslant i < j \leqslant r,\\
&e_{2\varepsilon_i} = E_{i,i+r}, \quad 1 \leqslant i \leqslant r,\\
&e_{-2\varepsilon_i} = E_{i+r,i}, \quad 1 \leqslant i \leqslant r.
\end{aligned}$$

The constant structures are the following (we write only the nonzero ones):

$$\begin{aligned}
&[e_{\varepsilon_i - \varepsilon_j}, e_{\varepsilon_j - \varepsilon_k}] = e_{\varepsilon_i - \varepsilon_k} \quad (i \neq k),\\
&[e_{\varepsilon_i - \varepsilon_j}, e_{\varepsilon_k - \varepsilon_i}] = -e_{\varepsilon_k - \varepsilon_j} \quad (j \neq k),\\
&[e_{\varepsilon_i - \varepsilon_j}, e_{\varepsilon_j + \varepsilon_k}] = e_{\varepsilon_i + \varepsilon_k} \quad (k \neq i),\\
&[e_{\varepsilon_i - \varepsilon_j}, e_{\varepsilon_j + \varepsilon_i}] = 2e_{2\varepsilon_i},\\
&[e_{\varepsilon_i - \varepsilon_j}, e_{-\varepsilon_i - \varepsilon_j}] = -2e_{-2\varepsilon_j},\\
&[e_{\varepsilon_i - \varepsilon_j}, e_{-\varepsilon_i - \varepsilon_k}] = -e_{-\varepsilon_j - \varepsilon_k} \quad (k \neq j),\\
&[e_{\varepsilon_i - \varepsilon_j}, e_{2\varepsilon_j}] = e_{\varepsilon_i + \varepsilon_j},\\
&[e_{\varepsilon_i - \varepsilon_j}, e_{-2\varepsilon_i}] = -e_{-\varepsilon_i - \varepsilon_j},\\
&[e_{\varepsilon_i + \varepsilon_j}, e_{-\varepsilon_j - \varepsilon_k}] = e_{\varepsilon_i - \varepsilon_k} \quad (k \neq j),
\end{aligned} \tag{2.1}$$

$$[e_{\varepsilon_i+\varepsilon_j}, e_{-2\varepsilon_j}] = e_{\varepsilon_i-\varepsilon_j},$$
$$[e_{2\varepsilon_i}, e_{-\varepsilon_i-\varepsilon_j}] = e_{\varepsilon_i-\varepsilon_j} \quad (j \neq i).$$

In the notation of Section 1.1 (after Lemma 1.6), we have:

$$c_{\varepsilon_i-\varepsilon_j} = 1, \qquad i \neq j,$$
$$c_{\varepsilon_i+\varepsilon_j} = 1 \qquad i \neq j,$$
$$c_{-\varepsilon_i-\varepsilon_j} = 1 \qquad i \neq j,$$
$$c_{2\varepsilon_i} = 2,$$
$$c_{-2\varepsilon_i} = 2.$$

The irreducible representation $V := V(\varpi_1)$ is the standard representation

$$\mathfrak{g} \to \mathrm{End}(\mathbb{C}^{2r}), \quad X \mapsto (v \mapsto Xv)\,.$$

The nonzero weights of $V(\varpi_1)$ are $\{\delta_1, \dots, \delta_r, \overline{\delta}_1, \dots, \overline{\delta}_r\}$, where

$$\delta_i = \varepsilon_i, \qquad \overline{\delta}_i = -\varepsilon_i, \quad i = 1, \dots, r.$$

Moreover,

$$V_{\varepsilon_i} = \mathbb{C}v_i, \quad i = 1, \dots, r, \quad \text{and} \quad V_{-\varepsilon_i} = \mathbb{C}v_{i+r}, \quad i = 1, \dots, r,$$

where $(v_1, \dots, v_r, v_{1+r}, \dots, v_{2r})$ is the canonical basis of $\mathbb{C}^{2r}$.

We have, for $k = 1, \dots, r$ and $i \neq j$,

$$e_{\varepsilon_i-\varepsilon_j} v_k = (E_{i,j} - E_{j+r,i+r}) v_k = \delta_{j,k} v_i,$$
$$e_{\varepsilon_i-\varepsilon_j} v_{k+r} = -\delta_{i,k} v_{j+r},$$
$$e_{\varepsilon_i+\varepsilon_j} v_{k+r} = \delta_{j,k} v_i + \delta_{i,k} v_j,$$
$$e_{\varepsilon_i+\varepsilon_j} v_k = 0,$$
$$e_{-\varepsilon_i-\varepsilon_j} v_k = \delta_{k,j} v_{i+r} + \delta_{i,k} v_{j+r},$$
$$e_{-\varepsilon_i-\varepsilon_j} v_{k+r} = 0,$$
$$e_{2\varepsilon_i} v_{k+r} = \delta_{i,k} v_i,$$
$$e_{2\varepsilon_i} v_k = 0,$$
$$e_{-2\varepsilon_i} v_k = \delta_{i,k} v_{i+r},$$

$$e_{-2\varepsilon_i} v_{k+r} = 0.$$

Hence

$$
\begin{aligned}
a_{\varepsilon_i,\varepsilon_k}^{(e_{\varepsilon_l-\varepsilon_j})} &= \delta_{i,l}\delta_{j,k}, & a_{-\varepsilon_j,-\varepsilon_k}^{(e_{\varepsilon_i-\varepsilon_l})} &= -\delta_{j,l}\delta_{i,k},\\
a_{\varepsilon_l,-\varepsilon_k}^{(e_{\varepsilon_i+\varepsilon_j})} &= \delta_{i,l}\delta_{j,k} + \delta_{j,l}\delta_{i,k}, & a_{-\varepsilon_l,\varepsilon_k}^{(e_{-\varepsilon_i-\varepsilon_j})} &= \delta_{i,l}\delta_{j,k} + \delta_{j,l}\delta_{i,k},\\
a_{\varepsilon_i,-\varepsilon_i}^{(e_{2\varepsilon_k})} &= \delta_{i,k}, & a_{-\varepsilon_i,\varepsilon_i}^{(e_{-2\varepsilon_k})} &= \delta_{i,k}.
\end{aligned}
$$

In particular,

$$a_{\varepsilon_i,\varepsilon_k}^{(e_{\varepsilon_i-\varepsilon_k})} = 1,\ a_{-\varepsilon_j,-\varepsilon_i}^{(e_{\varepsilon_i-\varepsilon_j})} = -1,\ a_{\varepsilon_i,-\varepsilon_j}^{(e_{\varepsilon_i+\varepsilon_j})} = 1,\ a_{-\varepsilon_i,\varepsilon_j}^{(e_{-\varepsilon_i-\varepsilon_j})} = 1,\ a_{\varepsilon_i,-\varepsilon_i}^{(e_{2\varepsilon_i})} = 1,\ a_{-\varepsilon_i,\varepsilon_i}^{(e_{-2\varepsilon_i})} = 1. \tag{2.2}$$

We represent in Figure 2.2 the crystal graph $\mathscr{C}(\varpi_1)$.

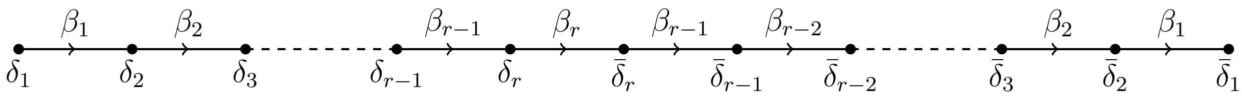



**FIGURE 2.2** – Crystal graph of $\delta = \varpi_1$ for $\mathfrak{g} = \mathfrak{sp}_{2r}$.

Recall that $B_{\mathfrak{g}}^{\sharp} \colon \mathfrak{h}^* \to \mathfrak{h}$ is the isomorphism induced from the non-degenerate bilinear from $B_{\mathfrak{g}}$. We have:

$$\varpi_i^{\sharp} = \check{\varpi}_i, \quad i = 1, \dots, r-1, \quad \text{and} \quad \varpi_r^{\sharp} = 2\check{\varpi}_r.$$

## 2.2 Proof of Theorem 6 for $\mathfrak{g} = \mathfrak{sl}_{r+1}$

We assume in this section that $\mathfrak{g} = \mathfrak{sl}_{r+1}$. We keep the notations of §2.1.1.

Let $\delta = \varpi_1$. Then $P(\delta) = \{\delta_1, \cdots, \delta_{r+1}\}$. For $k \in \{1, \dots, r\}$, set

$$P(\delta)_k := \{\mu \in P(\delta) | \langle \mu, \check{\beta}_k \rangle \neq 0\}.$$

For $k \in \{1, \dots, r\}$, we have (cf. Figure 2.1): $\langle \mu, \check{\beta}_k \rangle \neq 0 \Leftrightarrow \mu = \delta_k$ or $\mu = \delta_{k+1}$.

Hence,

$$P(\delta)_k = \{\delta_k, \delta_{k+1}\}.$$

Note that $\langle \delta_k, \check{\beta}_k \rangle = 1$ and $\langle \delta_{k+1}, \check{\beta}_k \rangle = -1$. For $\mu \in P(\delta)$, set

$$\Pi_\mu := \{\beta \in \Pi \mid \langle \mu, \check{\beta} \rangle \neq 0\}.$$

We have:

$$\Pi_{\delta_k} = \{\beta_{k-1}, \beta_k\}, \quad k = 2, \dots, r, \qquad \Pi_{\delta_1} = \{\beta_1\}, \qquad \Pi_{\delta_{r+1}} = \{\beta_r\}.$$

According to Lemma 1.19 and (1.3), we get

$$\begin{aligned} &\mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m,k}) \\ &= \sum_{\underline{\alpha} \in (\Pi_{\delta_k})^m} \prod_{i=1}^m \langle \delta_k, \check{\alpha}^{(i)} \rangle \langle \rho, \check{\varpi}_{\alpha^{(i)}} \rangle - \sum_{(\underline{\alpha} \in (\Pi_{\delta_{k+1}})^m} \prod_{i=1}^m \langle \delta_{k+1}, \check{\alpha}^{(i)} \rangle \langle \rho, \check{\varpi}_{\alpha^{(i)}} \rangle \end{aligned} \tag{2.3}$$

since $\varpi_i^\sharp = \check{\varpi}_i$ for all $i = 1, \dots, r$, $\mathfrak{g}$ being simply laced.

**Lemma 2.1.** *1. For any $j \in \{1, \dots, r+1\}$,*

$$\langle \rho, \varepsilon_j \rangle = \frac{r}{2} - j + 1.$$

*2. For $k \in \{1, \dots, r+1\}$,*

$$\langle \rho, \check{\varpi}_k \rangle = \frac{k}{2}(r - k + 1), \quad \langle \rho, \check{\varpi}_k - \check{\varpi}_{k-1} \rangle = \frac{r}{2} - k + 1,$$

*where by convention $\varpi_0 = \varpi_{r+1} = 0$.*

*Proof.* Recall that the half-sum $\rho$ of positive roots is equal to $\varpi_1 + \ldots + \varpi_r$.

(1) We have

$$\langle \varpi_k, \varepsilon_j \rangle = \langle \varepsilon_1 + \cdots + \varepsilon_k - \frac{k}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1}), \varepsilon_j \rangle = \begin{cases} -\frac{k}{r+1} & \text{if } k < j \\ 1 - \frac{k}{r+1} & \text{if } k \geqslant j. \end{cases}$$

Hence,

$$\langle \rho, \varepsilon_j \rangle = \langle \sum_{k=1}^r \varpi_k, \varepsilon_j \rangle = \sum_{k=1}^{j-1} \frac{-k}{r+1} + \sum_{k=j}^r \left(1 - \frac{k}{r+1}\right)$$

$$
\begin{aligned}
&= \frac{-1}{r+1} - \frac{2}{r+1} - \cdots - \frac{j-1}{r+1} \\
&\quad + \left(1 - \frac{j}{r+1}\right) + \left(1 - \frac{j+1}{r+1}\right) + \cdots + \left(1 - \frac{r}{r+1}\right) \\
&= r - j + 1 - \frac{1}{r+1}\sum_{k=1}^{r} k \\
&= r - j + 1 - \frac{1}{r+1}\frac{r(r+1)}{2} = \frac{r}{2} - j + 1.
\end{aligned}
$$

(2) We have

$$
\begin{aligned}
\langle \rho, \breve{\varpi}_k \rangle &= \langle \rho, \varepsilon_1 + \cdots + \varepsilon_k - \frac{k}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1}) \rangle \\
&= \langle \rho, \varepsilon_1 + \cdots + \varepsilon_k \rangle - \langle \rho, \frac{k}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1}) \rangle \\
&= \sum_{j=1}^{k} \left(\frac{r}{2} - j + 1\right) - \frac{k}{r+1}\sum_{j=1}^{r+1}\left(\frac{r}{2} - j + 1\right) \\
&= \frac{k(\frac{r}{2} + \frac{r}{2} - k + 1)}{2} - \frac{k}{r+1}\frac{(r+1)(\frac{r}{2} + \frac{r}{2} - r - 1 + 1)}{2} \\
&= \frac{k}{2}(r - k + 1).
\end{aligned}
$$

Then for $k \in \{2, \ldots, r\}$, we have

$$
\begin{aligned}
\langle \rho, \breve{\varpi}_k - \breve{\varpi}_{k-1} \rangle &= \langle \rho, \breve{\varpi}_k \rangle - \langle \rho, \breve{\varpi}_{k-1} \rangle \\
&= \frac{k}{2}(r - k + 1) - \frac{(k-1)}{2}(r - k + 2) = \frac{r}{2} - k + 1.
\end{aligned}
$$

For $k = 1$, we have

$$
\langle \rho, \breve{\varpi}_1 - \breve{\varpi}_0 \rangle = \frac{1}{2} - 1 + 1 = \frac{r}{2}.
$$

On the other hand,

$$
\langle \rho, \breve{\varpi}_1 - \breve{\varpi}_0 \rangle = \langle \rho, \breve{\varpi}_1 \rangle = \frac{r}{2}(r - 1 + 1) = \frac{r}{2}.
$$

For $k = r + 1$, we have

$$
\langle \rho, \breve{\varpi}_{r+1} - \breve{\varpi}_r \rangle = \frac{r}{2} - (r+1) + 1 = -\frac{r}{2}.
$$

On the other hand,

$$\langle \rho, \check{\varpi}_{r+1} - \check{\varpi}_r \rangle = -\langle \rho, \check{\varpi}_r \rangle = -\frac{r}{2}(r - r + 1) = -\frac{r}{2}.$$

Therefore the equality still holds for $k = 1$ and $k = r + 1$. □

**Lemma 2.2.** *For some polynomial $T_m \in \mathbb{C}[X]$ of degree $m$,*

$$\sum_{\underline{\alpha} \in (\Pi_{\delta_k})^m} \prod_{i=1}^{m} \langle \delta_k, \check{\alpha}^{(i)} \rangle \langle \rho, \check{\varpi}_{\alpha^{(i)}} \rangle = T_m(k), \qquad \forall\, k = 1, \ldots, r+1.$$

*Moreover, the leading term of $T_m$ is $(-X)^m$.*

*Proof.* Assume first that $k \in \{2, \ldots, r\}$. Then $\Pi_{\delta_k} = \{\beta_{k-1}, \beta_k\}$ and $\langle \delta_k, \check{\beta}_k \rangle = -\langle \delta_k, \check{\beta}_{k-1} \rangle = 1$. So by Lemma 2.1,

$$\begin{aligned}
\sum_{\underline{\alpha} \in (\Pi_{\delta_k})^m} \prod_{i=1}^{m} \langle \delta_k, \check{\alpha}^{(i)} \rangle \langle \rho, \check{\varpi}_{\alpha^{(i)}} \rangle &= \sum_{i=0}^{m} \binom{m}{i} (-1)^i \langle \rho, \check{\varpi}_{k-1} \rangle^i \langle \rho, \check{\varpi}_k \rangle^{m-i} \\
&= (-\langle \rho, \check{\varpi}_{k-1} \rangle + \langle \rho, \check{\varpi}_k \rangle)^m \\
&= (\langle \rho, \check{\varpi}_k - \check{\varpi}_{k-1} \rangle)^m = \left(\frac{r}{2} - k + 1\right)^m.
\end{aligned}$$

If $k = 1$, then $\Pi_{\delta_1} = \{\beta_1\}$ and $\langle \delta_1, \check{\beta}_1 \rangle = 1$. So, by Lemma 2.1,

$$\sum_{\underline{\alpha} \in (\Pi_{\delta_1})^m} \prod_{i=1}^{m} \langle \delta_1, \check{\alpha}^{(i)} \rangle \langle \rho, \check{\varpi}_{\alpha^{(i)}} \rangle = \langle \rho, \check{\varpi}_1 \rangle^m = \left(\frac{r}{2}\right)^m = \left(\frac{r}{2} - k + 1\right)^m$$

with $k = 1$.

If $k = r + 1$, then $\Pi_{\delta_{r+1}} = \{\beta_r\}$ and $\langle \delta_{r+1}, \check{\beta}_r \rangle = -1$. So, by Lemma 2.1,

$$\begin{aligned}
\sum_{\underline{\alpha} \in (\Pi_{\delta_r})^m} \prod_{i=1}^{m} \langle \delta_r, \check{\alpha}^{(i)} \rangle \langle \rho, \check{\varpi}_{\alpha^{(i)}} \rangle &= (-1)^m \langle \rho, \check{\varpi}_r \rangle^m \\
&= (-1)^m \left(\frac{r}{2}\right)^m = \left(\frac{r}{2} - k + 1\right)^m
\end{aligned}$$

with $k = r + 1$.

Hence, setting $T_m(X) := (\frac{r}{2} - X + 1)^m$ we get the statement. □

We are now in a position to prove Theorem 6 for $\mathfrak{g} = \mathfrak{sl}_{r+1}$.

*Proof of Theorem 6 for* $\mathfrak{g} = \mathfrak{sl}_{r+1}$. Let $m \in \{1, \dots, r\}$. By Lemma 2.2 and (2.3), we have for any $k \in \{1, \dots, r\}$,

$$\begin{aligned}
\mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m,k}) &= T_m(k) - T_m(k+1) \\
&= \left(\frac{r}{2} - k + 1\right)^m - \left(\frac{r}{2} - k\right)^m \\
&= \sum_{i=0}^{m} \binom{m}{i} \left(\frac{r}{2} + 1\right)^{m-i} (-k)^i - \sum_{i=0}^{m} \binom{m}{i} \left(\frac{r}{2}\right)^{m-i} (-k)^i \\
&= (-k)^m - (-k)^m + m\left(\frac{r}{2} + 1\right)(-k)^{m-1} - m\left(\frac{r}{2}\right)(-k)^{m-1} \\
&\quad + \sum_{i=0}^{m-2} \binom{m}{i} (-k)^i \left(\left(\frac{r}{2} + 1\right)^{m-i} - \left(\frac{r}{2}\right)^{m-i}\right) \\
&= m(-k)^{m-1} + \sum_{i=0}^{m-2} \binom{m}{i} (-k)^i \left(\left(\frac{r}{2} + 1\right)^{m-i} - \left(\frac{r}{2}\right)^{m-i}\right).
\end{aligned}$$

Hence,

$$\bar{Q}_m(X) := m(-X)^{m-1} + \sum_{i=0}^{m-2} \binom{m}{i} (-X)^i \left(\left(\frac{r}{2} + 1\right)^{m-i} - \left(\frac{r}{2}\right)^{m-i}\right) \tag{2.4}$$

is a polynomial of degree $m - 1$, and we get

$$\begin{aligned}
\mathrm{ev}_\rho(\overline{\mathrm{d}p}_m) &= \mathrm{ev}_\rho\left(\frac{1}{m!}\sum_{k=1}^{r} \overline{\mathrm{d}p}_{m,k} \otimes \varpi_k^\sharp\right) \\
&= \frac{1}{m!}\sum_{k=1}^{r} \mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m,k})\check{\varpi}_k = \frac{1}{m!}\sum_{k=1}^{r} \bar{Q}_m(k)\check{\varpi}_k.
\end{aligned}$$

Moreover, $\bar{Q}_1 = 1$. □

## 2.3 Proof of Theorem 6 for $\mathfrak{g} = \mathfrak{sp}_{2r}$

We assume in this section that $\mathfrak{g} = \mathfrak{sp}_{2r}$. We keep the notations of §2.1.2.

Set

$$P(\delta)_k := \{\mu \in P(\delta) \mid \langle \mu, \check{\beta}_k \rangle \neq 0\}.$$

Note that, for $k \in \{1, \dots, r-1\}$, we have: $\langle \mu, \check{\beta}_k \rangle \neq 0 \Leftrightarrow \mu = \delta_k$, $\mu = \delta_{k+1}$, $\mu = \overline{\delta}_k$, or $\mu = \overline{\delta}_{k+1}$. For $k = r$, we have: $\langle \mu, \check{\beta}_r \rangle \neq 0 \Leftrightarrow \mu = \delta_r$ or $\mu = \overline{\delta}_r$. Thus, we have

$$P(\delta)_k = \{\delta_k, \delta_{k+1}, \overline{\delta}_k, \overline{\delta}_{k+1}\}, \quad k = 1, \ldots, r-1, \quad \text{and} \quad P(\delta)_r = \{\delta_r, \overline{\delta}_r\}.$$

For $\mu \in P(\delta)$, set

$$\Pi_\mu := \{\beta \in \pi \mid \langle \mu, \check{\beta} \rangle \neq 0\}.$$

Let $\mu = \delta_k$. Note that, for $k = 2, \ldots, r$,

$$\Pi_{\delta_k} = \{\beta_{k-1}, \beta_k\} \quad \text{and} \quad \Pi_{\overline{\delta}_k} = \{\beta_{k-1}, \beta_k\}.$$

For $k = 1$,

$$\Pi_{\delta_1} = \{\beta_1\} \quad \text{and} \quad \Pi_{\overline{\delta}_1} = \{\beta_1\}.$$

According to Lemma 1.19 and (1.3), we get for $k \in \{1, \ldots, r-1\}$,

$$\begin{aligned} &\mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m,k}) \\ &\quad = \sum_{\underline{\alpha} \in (\Pi_{\delta_k})^m} \prod_{i=1}^m \langle \delta_k, \check{\alpha}^{(i)} \rangle \langle \rho, \varpi^\sharp_{\alpha^{(i)}} \rangle - \sum_{(\underline{\alpha} \in (\Pi_{\delta_{k+1}})^m} \prod_{i=1}^m \langle \delta_{k+1}, \check{\alpha}^{(i)} \rangle \langle \rho, \varpi^\sharp_{\alpha^{(i)}} \rangle \\ &\quad + \sum_{\underline{\alpha} \in (\Pi_{\overline{\delta}_{k+1}})^m} \prod_{i=1}^m \langle \overline{\delta}_{k+1}, \check{\alpha}^{(i)} \rangle \langle \rho, \varpi^\sharp_{\alpha^{(i)}} \rangle - \sum_{(\underline{\alpha} \in (\Pi_{\overline{\delta}_k})^m} \prod_{i=1}^m \langle \overline{\delta}_k, \check{\alpha}^{(i)} \rangle \langle \rho, \varpi^\sharp_{\alpha^{(i)}} \rangle. \end{aligned} \tag{2.5}$$

For $k = r$,

$$\begin{aligned} &\mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m,r}) \\ &\quad = \sum_{\underline{\alpha} \in (\Pi_{\delta_r})^m} \prod_{i=1}^m \langle \delta_r, \check{\alpha}^{(i)} \rangle \langle \rho, \varpi^\sharp_{\alpha^{(i)}} \rangle - \sum_{(\underline{\alpha} \in (\Pi_{\overline{\delta}_r})^m} \prod_{i=1}^m \langle \overline{\delta}_r, \check{\alpha}^{(i)} \rangle \langle \rho, \varpi^\sharp_{\alpha^{(i)}} \rangle. \end{aligned} \tag{2.6}$$

**Lemma 2.3.** *1. For any $j \in \{1, \ldots, r\}$, $\langle \rho, \varepsilon_j \rangle = r - j + 1$.*

*2. For $k \in \{1, \ldots, r\}$,*

$$\langle \rho, \varpi^\sharp_k \rangle = \frac{k}{2}(2r - k + 1), \qquad \langle \rho, \varpi^\sharp_k - \varpi^\sharp_{k-1} \rangle = r - k + 1,$$

*where by convention $\varpi^\sharp_0 = 0$.*

*Proof.* 1. We have

$$\langle \varpi_k, \varepsilon_j \rangle = \langle \varepsilon_1 + \cdots + \varepsilon_k, \varepsilon_j \rangle = \begin{cases} 0 & \text{if } k < j \\ 1 & \text{if } k \geqslant j. \end{cases}$$

Hence,

$$\langle \rho, \varepsilon_j \rangle = \langle \sum_{k=1}^{r} \varpi_k, \varepsilon_j \rangle = \sum_{k=j}^{r} 1 = r - j + 1.$$

2. For $k \in \{1, \ldots, r-1\}$,

$$\begin{aligned} \langle \rho, \varpi_k^\sharp \rangle = \langle \rho, \check{\varpi}_k \rangle = \langle \rho, \varepsilon_1 + \cdots + \varepsilon_k \rangle &= \sum_{j=1}^{k} r - j + 1 \\ = \frac{k}{2}(r + (r - k + 1) &= \frac{k}{2}(2r - k + 1). \end{aligned}$$

For $k = r$

$$\begin{aligned} \langle \rho, \varpi_k^\sharp \rangle &= 2\langle \rho, \check{\varpi}_r \rangle = 2\langle \rho, \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r) \rangle \\ &= \langle \rho, \varepsilon_1 + \cdots + \varepsilon_k \rangle = \frac{r}{2}(r+1) = \frac{r}{2}(2r - r + 1). \end{aligned}$$

We see the equality still holds for $k = r$.

Then for $k \in \{2, \ldots, r-1\}$, we have

$$\begin{aligned} \langle \rho, \varpi_k^\sharp - \varpi_{k-1}^\sharp \rangle &= \langle \rho, \varpi_k^\sharp \rangle - \langle \rho, \varpi_{k-1}^\sharp \rangle \\ &= \frac{k}{2}(2r - k + 1) - \frac{(k-1)}{2}(2r - (k-1) + 1) = r - k + 1. \end{aligned}$$

For $k = 1$, we have

$$\langle \rho, \varpi_1^\sharp - \varpi_0^\sharp \rangle = \langle \rho, \varpi_1^\sharp \rangle = \frac{1}{2}(2r - 1 + 1) = r = r - 1 + 1.$$

For $k = r$, we have

$$\begin{aligned} \langle \rho, \varpi_r^\sharp - \varpi_{r-1}^\sharp \rangle &= \langle \rho, \varpi_r^\sharp \rangle - \langle \rho, \varpi_{r-1}^\sharp \rangle \\ &= \frac{r}{2}(r+1) - \frac{r-1}{2}(2r - (r-1) + 1) \end{aligned}$$

$$= \frac{r}{2}(r+1) - \frac{r-1}{2}(r+2) = 1 = r - r + 1.$$

Therefore the equality still holds for $k = 1$ and $k = r$. □

**Lemma 2.4.** *For some polynomial $T_m \in \mathbb{C}[X]$ of degree $m$, we have*

$$\sum_{\underline{\alpha} \in (\Pi_{\delta_k})^m} \prod_{i=1}^{m} \langle \delta_k, \check{\alpha}^{(i)} \rangle \langle \rho, \varpi^{\sharp}_{\alpha^{(i)}} \rangle = T_m(k)$$

*and*

$$\sum_{\underline{\alpha} \in (\Pi_{\overline{\delta}_k})^m} \prod_{i=1}^{m} \langle \overline{\delta}_k, \check{\alpha}^{(i)} \rangle \langle \rho, \varpi^{\sharp}_{\alpha^{(i)}} \rangle = (-1)^m T_m(k)$$

*for all $k \in \{1, \ldots, r\}$.*

*Proof.* Assume first that $k \in \{2, \ldots, r\}$, then $\Pi_{\delta_k} = \{\beta_{k-1}, \beta_k\}$, $\Pi_{\overline{\delta}_k} = \{\beta_{k-1}, \beta_k\}$ and $\langle \delta_k, \check{\beta}_k \rangle = -\langle \delta_k, \check{\beta}_{k-1} \rangle = -\langle \overline{\delta}_k, \check{\beta}_k \rangle = \langle \overline{\delta}_k, \check{\beta}_{k-1} \rangle = 1$.
We have,

$$\begin{aligned}
\sum_{\underline{\alpha} \in (\Pi_{\delta_k})^m} \prod_{i=1}^{m} \langle \delta_k, \check{\alpha}^{(i)} \rangle \langle \rho, \varpi^{\sharp}_{\alpha^{(i)}} \rangle &= \sum_{i=0}^{m} \binom{m}{i} (-1)^i \langle \rho, \varpi^{\sharp}_{k-1} \rangle^i \langle \rho, \varpi^{\sharp}_{k} \rangle^{m-i} \\
&= (-\langle \rho, \varpi^{\sharp}_{k-1} \rangle + \langle \rho, \varpi^{\sharp}_{k} \rangle)^m \\
&= (\langle \rho, \check{\varpi}_k - \varpi^{\sharp}_{k-1} \rangle)^m = (r - k + 1)^m,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\underline{\alpha} \in (\Pi_{\overline{\delta}_k})^m} \prod_{i=1}^{m} \langle \overline{\delta}_k, \check{\alpha}^{(i)} \rangle \langle \rho, \varpi^{\sharp}_{\alpha^{(i)}} \rangle &= \sum_{i=0}^{m} \binom{m}{i} (-1)^i \langle \rho, \varpi^{\sharp}_{k} \rangle^i \langle \rho, \varpi^{\sharp}_{k-1} \rangle^{m-i} \\
&= (\langle \rho, \varpi^{\sharp}_{k-1} \rangle - \langle \rho, \varpi^{\sharp}_{k} \rangle)^m \\
&= (\langle \rho, \varpi^{\sharp}_{k-1} - \check{\varpi}_k \rangle)^m \\
&= (-(r - k + 1))^m = (-1)^m (r - k + 1)^m.
\end{aligned}$$

If $k = 1$, then $\Pi_{\delta_1} = \{\beta_1\}, \Pi_{\overline{\delta}_1} = \{\beta_1\}$ and $\langle \delta_1, \check{\beta}_1 \rangle = -\langle \overline{\delta}_1, \check{\beta}_1 \rangle = 1$. So,

$$\sum_{\underline{\alpha} \in (\Pi_{\delta_1})^m} \prod_{i=1}^{m} \langle \delta_1, \check{\alpha}^{(i)} \rangle \langle \rho, \check{\varpi}_{\alpha^{(i)}} \rangle = \langle \rho, \check{\varpi}_1 \rangle^m = (r)^m = (r - k + 1)^m,$$

with $k=1$, and

$$\sum_{\underline{\alpha}\in(\Pi_{\bar{\delta}_1})^m}\prod_{i=1}^{m}\langle\delta_1,\check{\alpha}^{(i)}\rangle\langle\rho,\check{\varpi}_{\alpha^{(i)}}\rangle=(-\langle\rho,\check{\varpi}_1\rangle)^m=(-r)^m=(-1)^m(r-k+1)^m,$$

with $k=1$. Hence, setting $T_m(X):=(r-X+1)^m$ we get the statement. □

We are now in a position to prove Theorem 6 for $\mathfrak{g}=\mathfrak{sp}_{2r}$.

*Proof of Theorem 6 for* $\mathfrak{g}=\mathfrak{sp}_{2r}$. It easily seen that $p_m=0$ if $m$ is even, then there is no loss of generality assuming that $m\in\{1,3,\dots,2r-1\}$. By Lemma 2.4 and (2.5) and (2.6), we have for any $k\in\{1,\dots,r-1\}$,

$$\begin{aligned}
\mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m,k})&=T_m(k)-T_m(k+1)+(-1)^mT_m(k+1)-(-1)^mT_m(k)\\
&=(1-(-1)^m)T_m(k)+((-1)^m-1)T_m(k+1)^m\\
&=(1-(-1)^m)(T_m(k)-T_m(k+1))\\
&=2(T_m(k)-T_m(k+1))\\
&=2\left((r-k+1)^m-(r-(k+1)+1)^m\right)\\
&=2\left((r-k+1)^m-(r-k)^m\right)\\
&=2\left(\sum_{i=0}^{m}\binom{m}{i}(r+1)^{m-i}(-k)^i-\sum_{i=0}^{m}\binom{m}{i}(r)^{m-i}(-k)^i\right)\\
&=2\left(m(-k)^{m-1}+\sum_{i=0}^{m-2}\binom{m}{i}(r+1)^{m-i}(-k)^i-\sum_{i=0}^{m-2}\binom{m}{i}(r)^{m-i}(-k)^i\right)
\end{aligned}$$

since $m$ is odd.

For $k=r$,

$$\begin{aligned}
\mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m,r})&=T_m(r)-(-1)^mT_m(r)\\
&=(1-(-1)^m)T_m(r)=2(1)^m=2,
\end{aligned}$$

again since $m$ is odd. On the other hand,

$$\mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m,r})=2=2(T_m(r)-T_m(r+1)).$$

Hence,

$$\bar{Q}_m(X):=2(T_m(X)-T_m(X+1))$$

$$= 2\left(m(-X)^{m-1} + \sum_{i=0}^{m-2} \binom{m}{i} (r+1)^{m-i}(-X)^i - \sum_{i=0}^{m-2} \binom{m}{i} (r)^{m-i}(-X)^i\right) \quad (2.7)$$

is a polynomial of degree $m-1$, and we get

$$\begin{aligned} \mathrm{ev}_\rho(\overline{\mathrm{d}p}_m) &= \mathrm{ev}_\rho(\frac{1}{m!}\sum_{k=1}^{r} \overline{\mathrm{d}p}_{m,k} \otimes \varpi_k^\sharp) \\ &= \frac{1}{m!}\sum_{k=1}^{r} \mathrm{ev}_\rho(\overline{\mathrm{d}p}_{m,k})\varpi_k^\sharp = \frac{1}{m!}\sum_{k=1}^{r} \bar{Q}_m(k)\varpi_k^\sharp. \end{aligned}$$

Moreover, $\bar{Q}_1 = 2$. □

## 2.4 Admissible triples

We continue to assume that $\mathfrak{g} = \mathfrak{sp}_{2r}$, and we continue to use the notations of §2.1.2. We introduce in this section the notion of *admissible triple* in order to classify roots and weights with respect to the crystal graph of $\delta = \varpi_1$.

**Definition 2.5** (admissible triple). *A triple $(\alpha, \mu, \nu)$ is called* admissible *if $(\mu, \nu) \in (P(\delta))^2$, $\alpha \in \Delta_+$ and $\alpha = \mu - \nu$. Such triples are classified by different types as follows:*

**Type I** *: for some $i \in \{1, \dots, r\}$,*

$$(\alpha, \mu, \nu) = (2\varepsilon_i, \delta_i, \bar{\delta}_i),$$

**Type II** *: for some $i, j \in \{1, \dots, r\}$, $i \neq j$,*

$$(\alpha, \mu, \nu) = (\varepsilon_i + \varepsilon_j, \delta_i, \bar{\delta}_j),$$

**Type III a** *: for some $i, j \in \{1, \dots, r\}$, $i \neq j$,*

$$(\alpha, \mu, \nu) = (\varepsilon_i - \varepsilon_j, \delta_i, \delta_j),$$

**Type III b** *: for some $i, j \in \{1, \dots, r\}$, $i \neq j$,*

$$(\alpha, \mu, \nu) = (\varepsilon_i - \varepsilon_j, \bar{\delta}_j, \bar{\delta}_i).$$

We illustrate in Figure 2.3 the types of admissible triples in $\mathfrak{g}$.

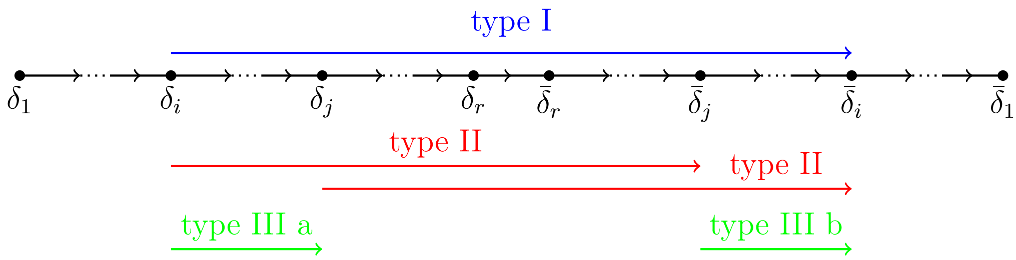



**FIGURE 2.3** – The types of admissible triple $(\alpha, \mu, \nu)$ in $\mathfrak{g} = \mathfrak{sp}_{2r}$.

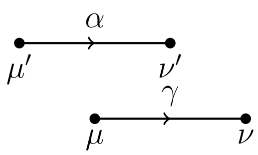
*Remark* 2.6. If $\alpha$ is a long root then there is unique $\mu \in P(\delta)$ such that $\mu - \alpha \in P(\delta)$, and if $\alpha$ is a short root then there are exactly two weights $\mu \in P(\delta)$ such that $\mu - \alpha \in P(\delta)$. In other words, if $\alpha$ is short root then there are two admissible triples $(\alpha, \mu, \nu)$ containing $\alpha$.

For $\gamma$ in the root lattice $Q$, we say that $\gamma$ has sign $+$ (respectively, $-$, $0$) if $\gamma \succ 0$ (respectively, $\gamma \prec 0$, $\gamma = 0$). The following lemma will be particularly useful in the proof of Theorem 4.1.

**Lemma 2.7.** *Let $(\gamma, \mu, \nu)$ be an admissible triple. Let us consider an admissible triple $(\alpha, \mu', \nu')$ which verifies the following conditions:*

*(i) either $\alpha - \gamma \in \Delta$, or $\alpha = \gamma$,*

*(ii) $\nu' \succ \nu$.*




*The triples which satisfy the above conditions (i) and (ii) are the following, depending on the type of $(\gamma, \mu, \nu)$:*

1. *(Type I) If $(\gamma, \mu, \nu) = (2\varepsilon_i, \delta_i, \bar{\delta}_i)$ with $i \in \{1, \dots, r\}$. Then the triples $(\alpha, \mu', \nu')$ satisfying the conditions (i) and (ii) are:*

| $(\alpha, \mu', \nu')$ | *condition* | $\alpha - \gamma$ | *sign of* $\alpha - \gamma$ |
|---|---|---|---|
| $(\varepsilon_i + \varepsilon_k, \delta_i, \overline{\delta}_k)$ | $i < k$ | $\varepsilon_k - \varepsilon_i$ | $-$ |
| $(\varepsilon_i - \varepsilon_k, \delta_i, \delta_k)$ | $i < k$ | $-(\varepsilon_i + \varepsilon_k)$ | $-$ |

2. *(Type II) If $(\gamma, \mu, \nu) = (\varepsilon_i + \varepsilon_j, \delta_i, \overline{\delta}_j)$, with $i, j \in \{1, \dots, r\}$, $i \neq j$. Then the triples $(\alpha, \mu', \nu')$ satisfying the conditions (i) and (ii) are:*

| $(\alpha, \mu', \nu')$ | *condition* | $\alpha - \gamma$ | *sign of* $\alpha - \gamma$ |
|---|---|---|---|
| $(2\varepsilon_i, \delta_i, \overline{\delta}_i)$ | $j < i$ | $\varepsilon_i - \varepsilon_j$ | $-$ |
| $(\varepsilon_i + \varepsilon_k, \delta_i, \overline{\delta}_k)$ | $j < k, k \neq i$ | $\varepsilon_k - \varepsilon_j$ | $-$ |
| $(\varepsilon_j + \varepsilon_k, \delta_j, \overline{\delta}_k)$ | $j < k < i$ | $\varepsilon_k - \varepsilon_i$ | $+$ |
| $(\varepsilon_j + \varepsilon_k, \delta_j, \overline{\delta}_k)$ | $i < k, j < k$ | $\varepsilon_k - \varepsilon_i$ | $-$ |
| $(\varepsilon_j + \varepsilon_i, \delta_j, \overline{\delta}_i)$ | $j < i$ | $0$ | $0$ |
| $(\varepsilon_k + \varepsilon_i, \delta_k, \overline{\delta}_i)$ | $k < j < i$ | $\varepsilon_k - \varepsilon_j$ | $+$ |
| $(\varepsilon_k + \varepsilon_i, \delta_k, \overline{\delta}_i)$ | $k \neq i, j < i, j < k$ | $\varepsilon_k - \varepsilon_j$ | $-$ |
| $(\varepsilon_i - \varepsilon_k, \delta_i, \delta_k)$ | $i < k$ | $-(\varepsilon_k + \varepsilon_j)$ | $-$ |
| $(\varepsilon_j - \varepsilon_k, \delta_j, \delta_k)$ | $j < k$ | $-(\varepsilon_k + \varepsilon_j)$ | $-$ |

3. *(Type III a) If $(\gamma, \mu, \nu) = (\varepsilon_i - \varepsilon_j, \delta_i, \delta_j)$, with $i, j \in \{1, \dots, r\}$, $i < j$. Then the only triple $(\alpha, \mu', \nu')$ satisfying the conditions (i) and (ii) is:*

| $(\alpha, \mu', \nu')$ | *condition* | $\alpha - \gamma$ | *sign of* $\alpha - \gamma$ |
|---|---|---|---|
| $(\varepsilon_i - \varepsilon_k, \delta_i, \delta_k)$ | $i < k < j$ | $\varepsilon_j - \varepsilon_k$ | $-$ |

4. *(Type III b) If $(\gamma, \mu, \nu) = (\varepsilon_i - \varepsilon_j, \overline{\delta}_j, \overline{\delta}_i)$, with $i, j \in \{1, \dots, r\}$, $i < j$. Then the triples $(\alpha, \mu', \nu')$ satisfying the conditions (i) and (ii) are:*

| $(\alpha, \mu', \nu')$ | *condition* | $\alpha - \gamma$ | *sign of* $\alpha - \gamma$ |
|---|---|---|---|
| $(\varepsilon_i + \varepsilon_k, \delta_i, \overline{\delta}_k)$ | $i < k$ | $\varepsilon_k + \varepsilon_j$ | $+$ |
| $(\varepsilon_i - \varepsilon_k, \delta_i, \delta_k)$ | $i < j < k$ | $\varepsilon_j - \varepsilon_k$ | $+$ |
| $(\varepsilon_i - \varepsilon_k, \delta_i, \delta_k)$ | $i < k < j$ | $\varepsilon_j - \varepsilon_k$ | $-$ |
| $(\varepsilon_i - \varepsilon_j, \delta_i, \delta_j)$ | $i < j$ | $0$ | $0$ |
| $(\varepsilon_k - \varepsilon_j, \delta_k, \delta_j)$ | $k < i < j$ | $\varepsilon_k - \varepsilon_i$ | $+$ |
| $(\varepsilon_k - \varepsilon_j, \delta_k, \delta_j)$ | $i < k < j$ | $\varepsilon_k - \varepsilon_i$ | $-$ |
| $(\varepsilon_k - \varepsilon_i, \overline{\delta}_j, \overline{\delta}_k)$ | $i < k < j$ | $\varepsilon_k - \varepsilon_i$ | $-$ |

*Proof.* 1. Let $(\gamma, \mu, \nu)$ has type I, $\gamma = \delta_i - \overline{\delta}_i = 2\varepsilon_i$ with $i \in \{1, \dots, r\}$. We will search for positive roots $\alpha$ where the conditions (i) and (ii) hold as follows:

- Assume $(\alpha, \mu', \nu')$ has type I, $\alpha = \delta_k - \overline{\delta}_k = 2\varepsilon_k$. Then either $\alpha - \gamma \notin \Delta$ or $\alpha = \gamma$. If $\alpha = \gamma$ then the condition (ii) is not satisfied. Hence there is no $(\alpha, \mu', \nu')$ of type I that satisfies the conditions of the lemma if $(\gamma, \mu, \nu)$ has type I.
- Assume $(\alpha, \mu', \nu')$ has type II. Write $\alpha = \delta_k - \overline{\delta}_l$, with $k \neq l$. Hence

  $$\alpha - \gamma = \delta_k - \overline{\delta}_l - \delta_i + \overline{\delta}_i$$

  is a root if and only if $\delta_k = \delta_i$ or $\overline{\delta}_l = \overline{\delta}_i$.
  $*$ If $\delta_k = \delta_i$ then $\alpha = \delta_i - \overline{\delta}_l$ satisfies the condition of the lemma if $i < l$. In this case, $\alpha - \gamma = \overline{\delta}_i - \overline{\delta}_l \in -\Delta_+$.
  $*$ If $\overline{\delta}_l = \overline{\delta}_i$ then $\alpha = \delta_k - \overline{\delta}_i$ does not hold condition (ii).
- Assume $(\alpha, \mu', \nu')$ has type III (a). Write $\alpha = \delta_k - \delta_l$ with $k < l$. Hence

  $$\alpha - \gamma = \delta_k - \delta_l - \delta_i + \overline{\delta}_i$$

  is a root if and only if $\delta_k = \delta_i$. If it is so, then $\alpha - \gamma = \overline{\delta}_i - \delta_l \in -\Delta_+$.
- Assume $(\alpha, \mu', \nu')$ has type III (b). Write $\alpha = \overline{\delta}_k - \overline{\delta}_l$, with $k > l$. Hence

  $$\alpha - \gamma = \overline{\delta}_k - \overline{\delta}_l - \delta_i + \overline{\delta}_i$$

  is a root if and only if $\overline{\delta}_l = \overline{\delta}_i$. We have $\alpha = \overline{\delta}_k - \overline{\delta}_i$ does not satisfy condition (iii). Hence there is no $(\alpha, \mu', \nu')$ of type III (b) that satisfies the conditions of the lemma if $(\gamma, \mu, \nu)$ has type I.

2. Let $(\gamma, \mu, \nu)$ has type I, that is $\gamma = \delta_i - \overline{\delta}_j = \varepsilon_i + \varepsilon_j$, with $i, j \in \{1, \ldots, r\}$, $i \neq j$. We will search for positive roots $\alpha$ where the conditions (i) and (ii) hold as follows:

   - Assume $(\alpha, \mu', \nu')$ has type I, $\alpha = \delta_k - \overline{\delta}_k$. Hence

     $$\alpha - \gamma = \delta_k - \overline{\delta}_k - \delta_i + \overline{\delta}_j$$

     is a root if and only if $\delta_k = \delta_i$ or $\overline{\delta}_k = \overline{\delta}_j$.
     $*$ If $\delta_k = \delta_i$ then $\alpha = \delta_k - \overline{\delta}_k$ satisfies the condition of the lemma if $l < k$. If it is so, then $\alpha - \gamma = \overline{\delta}_j - \overline{\delta}_k \in -\Delta_+$.
     $*$ If $\overline{\delta}_k = \overline{\delta}_i$ then $\alpha = \overline{\delta}_k - \overline{\delta}_j$ does not satisfy condition (ii).

- Assume $(\alpha, \mu', \nu')$ has type II. Write $\alpha = \delta_k - \overline{\delta}_l = \varepsilon_k + \varepsilon_l$. Hence

$$\alpha - \gamma = \varepsilon_k + \varepsilon_l - \varepsilon_i - \varepsilon_j$$

  is a root if and only if $\varepsilon_k = \varepsilon_i$, $\varepsilon_k = \varepsilon_j$, $\varepsilon_l = \varepsilon_i$, $\varepsilon_l = \varepsilon_j$.

  $*$ If $\varepsilon_k = \varepsilon_i$, we have $\alpha = \varepsilon_i + \varepsilon_l = \delta_i - \overline{\delta}_l$ satisfies the condition of the lemma if $j < l$. In this case $\alpha - \gamma = \varepsilon_l - \varepsilon_j \in -\Delta_+$.

  $*$ If $\varepsilon_k = \varepsilon_j$ is the case, then $\alpha = \varepsilon_j + \varepsilon_l = \delta_j - \overline{\delta}_l$ satisfies the condition of the lemma if $j < l$, and $\alpha - \gamma = \varepsilon_l - \varepsilon_i \in \Delta_+$ for $j < l < i$, and $\alpha - \gamma \in -\Delta_+$ for $i < j, j < l$.

  $*$ If $\varepsilon_l = \varepsilon_i$ is the case, then $\alpha = \varepsilon_k + \varepsilon_i = \delta_k - \overline{\delta}_i$ satisfies the condition of the lemma if $j < i$. In this case, $\alpha - \gamma = \varepsilon_k - \varepsilon_j \in \Delta_+$ for $k < j < i$, and $\alpha - \gamma \in -\Delta_+$ for $j < i, j < k$.

  $*$ If $\varepsilon_l = \varepsilon_j$ then $\alpha = \varepsilon_k + \varepsilon_j = \delta_k - \overline{\delta}_j$ does not satisfy condition (ii).

  Moreover, $\alpha = \delta_j - \overline{\delta}_i$ with $i > j$ also satisfies the conditions of the lemma. If it is so, then $\alpha = \gamma$.

- Assume $(\alpha, \mu', \nu')$ has type III (a). Write $\alpha = \delta_k - \delta_l = \varepsilon_k - \varepsilon_l$ with $k < l$. Hence

$$\alpha - \gamma = \varepsilon_k - \varepsilon_l - \varepsilon_i - \varepsilon_j$$

  is a root if and only if $\varepsilon_k = \varepsilon_i$ or $\varepsilon_k = \varepsilon_j$.

  $*$ If $\varepsilon_k = \varepsilon_i$, we have $\alpha = \varepsilon_i - \varepsilon_l = \delta_i - \delta_l$ with $i < l$. In this case $\alpha - \gamma = -(\varepsilon_l + \varepsilon_j) \in -\Delta_+$.

  $*$ If $\varepsilon_k = \varepsilon_j$, then $\alpha = \varepsilon_j - \varepsilon_l = \delta_j - \delta_l$ with $j < l$. In this case $\alpha - \gamma = -(\varepsilon_l + \varepsilon_i) \in -\Delta_+$.

- Assume $(\alpha, \mu', \nu')$ has type III (b). Write $\alpha = \overline{\delta}_k - \overline{\delta}_l$ with $k > l$. Hence

$$\alpha - \gamma = \overline{\delta}_k - \overline{\delta}_l - \delta_i + \overline{\delta}_j$$

  is a root if and only if $\overline{\delta}_l = \overline{\delta}_j$. If it is so, then $\alpha = \overline{\delta}_k - \overline{\delta}_j$ does not satisfy (ii).

3. Let $(\gamma, \mu, \nu)$ has type I, that is $\gamma = \delta_i - \delta_j = \varepsilon_i - \varepsilon_j$, with $i, j \in \{1, \ldots, r\}$, $i < j$. The admissible triples of root $\alpha = \mu' - \nu'$ that satisfy the conditions of the lemma could not be from type I, type II or type III (b) since $\nu'$ is less than $\delta_j$. And so $(\alpha, \mu', \nu')$ has type III (a).

Assume $\alpha = \delta_k - \delta_l = \varepsilon_k - \varepsilon_l$ with $k < l < j$ Hence

$$\alpha - \gamma = \varepsilon_k - \varepsilon_l - \varepsilon_i + \varepsilon_j$$

is a root if and only if $\varepsilon_k = \varepsilon_i$ or $\varepsilon_l = \varepsilon_j$.

$*$ If $\varepsilon_k = \varepsilon_i$ then $\alpha = \varepsilon_i - \varepsilon_l$ satisfies the condition of lemma if $i < l < j$. In this case, $\alpha - \gamma = \varepsilon_j - \varepsilon_l \in -\Delta_+$.

$*$ If $\varepsilon_l = \varepsilon_j$ then $\alpha = \varepsilon_k - \varepsilon_j$ does not satisfy the condition (ii).

4. Let $(\gamma, \mu, \nu)$ has type I, that is $\gamma = \overline{\delta}_j - \overline{\delta}_i = \varepsilon_i - \varepsilon_j$, with $i, j \in \{1, \ldots, r\}$, $i < j$. We will search for positive roots $\alpha$ where the conditions (i) and (ii) hold as follows:

   - Assume $(\alpha, \mu', \nu')$ has type I, $\alpha = \delta_k - \overline{\delta}_k = 2\varepsilon_k$, then

     $$\alpha - \gamma = 2\varepsilon_k - \varepsilon_i + \varepsilon_j$$

     is a root if and only if $\varepsilon_k = \varepsilon_i$. If it is so, then $\alpha = \delta_i - \overline{\delta}_i$ does not satisfy condition (ii).

   - Assume $(\alpha, \mu', \nu')$ has type II. Write $\alpha = \delta_k - \overline{\delta}_l = \varepsilon_k + \varepsilon_l$ with $l > i$. Hence

     $$\alpha - \gamma = \varepsilon_k + \varepsilon_l - \varepsilon_j + \varepsilon_i$$

     is a root if and only if $\varepsilon_k = \varepsilon_i$ or $\varepsilon_l = \varepsilon_i$.

     $*$ If $\varepsilon_k = \varepsilon_i$, we have $\alpha = \varepsilon_i + \varepsilon_l = \delta_i - \overline{\delta}_l$ satisfies the lemma if $l > i$. In this case $\alpha - \gamma = \varepsilon_l + \varepsilon_j \in \Delta_+$.

     $*$ If $\varepsilon_l = \varepsilon_i$ is the case, then $\alpha = \varepsilon_k + \varepsilon_i = \delta_k - \overline{\delta}_i$ does not satisfy condition (ii).

   - Assume $(\alpha, \mu', \nu')$ has type III (b). Write $\alpha = \varepsilon_k - \varepsilon_l$ with $k < l$. Hence

     $$\alpha - \gamma = \varepsilon_k - \varepsilon_l - \varepsilon_i + \varepsilon_j$$

     is a root if and only if $\varepsilon_k = \varepsilon_i$ or $\varepsilon_l = \varepsilon_j$.

     $*$ If $\varepsilon_k = \varepsilon_i$, we have $\alpha = \varepsilon_i - \varepsilon_l = \delta_i - \delta_l$ with $i < l$. In this case $\alpha - \gamma = \varepsilon_j - \varepsilon_l \in \Delta_+$ for $i < j < l$, and $\alpha - \gamma \in -\Delta_+$ for $i < l < j$.

     $*$ If $\varepsilon_l = \varepsilon_j$ is the case, then $\alpha = \varepsilon_k - \varepsilon_i = \delta_k - \delta_i$ with $k < i$. In this case, $\alpha - \gamma = \varepsilon_k - \varepsilon_i \in \Delta_+$ for $k < i < j$, and $\alpha - \gamma \in -\Delta_+$ for $i < k < j$.

Moreover, $\alpha = \delta_i - \delta_j$ with $i < j$ also satisfies the conditions of the lemma. If it is so, then $\alpha = \gamma$.

- Assume $(\alpha, \mu', \nu')$ has type III (b). Write $\alpha = \overline{\delta}_k - \overline{\delta}_l$ with $k > l$. Hence

$$\alpha - \gamma = \overline{\delta}_k - \overline{\delta}_l - \overline{\delta}_j + \overline{\delta}_i$$

is a root if and only if $\overline{\delta}_k = \overline{\delta}_j$ or $\overline{\delta}_l - \overline{\delta}_i$.

$*$ If $\overline{\delta}_k = \overline{\delta}_j$ then $\alpha = \overline{\delta}_j = \overline{\delta}_l$ is satisfies the condition of the lemma if $i < l < j$. In this case $\alpha - \gamma = \overline{\delta}_i - \overline{\delta}_l \in -\Delta_+$.

$*$ If $\overline{\delta}_l = \overline{\delta}_i$ is the case, then $\alpha = \overline{\delta}_k - \overline{\delta}_i$ does not satisfy condition (ii).

□

# Chapter 3

## Proof of Theorem 7 for $\mathfrak{sl}_{r+1}$

This chapter is devoted to the proof of Theorem 7. The reader is referred to Chapter 2, §2.1.1, for all relative notations to $\mathfrak{g} = \mathfrak{sl}_{r+1}$, $r \geqslant 2$ and $\delta = \varpi_1$. We start with some technical results on weighted paths (cf. Section 3.1) that will be used to prove Theorem 3.1 below. This theorem is important since it allows us to make an important reduction in the sequel. Namely, it will be enough in many situations to consider only weighted paths $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)$ such that $\mu^{(i)} \preccurlyeq \mu$ for any $i \in \{1, \dots, m\}$, that is, entirely located "on the right hand side of $\mu$". We introduce in Section 3.2 equivalent classes (cf. Definition 3.6) on the set of weighted paths: §3.2.2 deals with paths *without zero* while §3.2.4 deals with paths *with zeroes* (see Definition 3.8 for the notion of path with, or without, zeroes). The proof of Theorem 7 is achieved in Section 3.3.

The results of this chapter will play an essential role in the next one, too, when dealing with the $\mathfrak{sp}_{2r}$ case since we will roughly follow the same strategy.

Throughout this chapter, it is assumed that $\mathfrak{g} = \mathfrak{sl}_{r+1}, r \geqslant 2$ and $\delta = \varpi_1$. We retain all relative notations from previous chapters.

## 3.1 A preliminary result

The goal of this section is to prove the following result that will allow us to consider only certain weighted paths. It is a very important step.

**Theorem 3.1.** *Let $m \in \mathbb{Z}_{>0}$, $\mu \in P(\delta)$ and $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)$. Assume that for some $i \in \{1, \dots, m\}$, $\mu^{(i)} \succ \mu$. Then* $\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = 0$.

According to Theorem 3.1, it will be enough in many situations to consider only weighted paths $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)$ such that $\mu^{(i)} \preccurlyeq \mu$ for any $i \in \{1, \dots, m\}$.

### 3.1.1 Some reduction lemmas

We establish in this paragraph reduction results in order to show Theorem 3.1 in §3.1.2.

**Lemma 3.2.** *Let $m \in \mathbb{Z}_{>0}$, $(\mu, \nu) \in P(\delta)^2$, $\gamma \in \Delta_+$ and $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu, \nu)$ such that $\mathrm{ht}(\underline{\mu})_i \geqslant 0$ for any $i$. Assume that $\gamma = \mu' - \nu'$, with $\mu', \nu' \in P(\delta)$, and that $\nu' \prec \mu^{(i)}$ for all $i \in \{1, \dots, m+1\}$. Note that $\mu' \succ \nu'$ since $\gamma \in \Delta_+$.*

1. *Let $i \in \{1, \dots, m+1\}$. If $\mathrm{ht}(\underline{\mu})_i > 0$ then either $\alpha^{(i)} - \gamma \notin \Delta$ or $\alpha^{(i)} - \gamma \in -\Delta_+$. Moreover, if $\alpha^{(i)} - \gamma \in -\Delta_+$, then $(\underline{\mu}, \underline{\alpha})$ and $\gamma' := \gamma - \alpha^{(i)}$ still satisfy the above conditions with $\gamma'$ in place of $\gamma$.*

2. *For all $u \in U(\mathfrak{g})$, we have $\mathrm{hc}(b^*_{\underline{\mu}, \underline{\alpha}} e_{-\gamma} u) = 0$.*

*Proof.* 1. Write $\gamma = \varepsilon_j - \varepsilon_k$, with $j < k$. The hypothesis says that for all $i \in \{1, \dots, m\}$ such that $\mathrm{ht}(\underline{\mu})_i > 0$ then $\alpha^{(i)} = \varepsilon_{j_i} - \varepsilon_{k_i}$ with $j_i < k_i < k$. Hence

$$\alpha^{(i)} - \gamma = \varepsilon_{j_i} - \varepsilon_{k_i} - \varepsilon_j + \varepsilon_k$$

is a root if and only if $j_i = j$. If it is so, then $\alpha^{(i)} - \gamma = \varepsilon_k - \varepsilon_{k_i}$ is a negative root since $k > k_i$. Moreover, $\gamma' := \varepsilon_{k_i} - \varepsilon_k$ still verifies the condition of the lemma.

2. We prove the assertion by induction on $m$. Set,

$$a := a^{(b_{(\underline{\mu}, \underline{\alpha}), m})}_{\mu^{(m)}, \mu^{(m+1)}}.$$

Assume $m = 1$. If $\mathrm{ht}(\underline{\mu})_1 > 0$ then by Part 1 either $\alpha^{(1)} - \gamma \notin \Delta$, and

$$\mathrm{hc}(b^*_{\underline{\mu}, \underline{\alpha}} e_{-\gamma} u) = \mathrm{hc}(a e_{\alpha^{(1)}} e_{-\gamma} u) = \mathrm{hc}(a e_{-\gamma} e_{\alpha^{(1)}} u) = 0,$$

or $\alpha^{(1)} - \gamma = -\gamma'$, with $\gamma' \in \Delta_+$, and

$$\mathrm{hc}(b^*_{\underline{\mu}, \underline{\alpha}} e_{-\gamma} u) = \mathrm{hc}(a e_{\alpha^{(1)}} e_{-\gamma} u) = \mathrm{hc}(a e_{-\gamma} e_{\alpha^{(1)}} u) + \mathrm{hc}(a a' e_{-\gamma'} u) = 0,$$

where $a' \in \mathbb{C}$.

If $\mathrm{ht}(\underline{\mu})_1 = 0$, then

$$\mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}} e_{-\gamma} u) = \mathrm{hc}(a\check{\varpi}_{\alpha^{(1)}} e_{-\gamma} u) = \mathrm{hc}(a e_{-\gamma} \check{\varpi}_{\alpha^{(1)}} u - a\langle \gamma, \check{\varpi}_{\alpha^{(1)}} \rangle e_{-\gamma} u) = 0.$$

In both cases, we obtain the statement.

Let $m \geqslant 2$ and assume the statement true for any $m' \in \{1, \dots, m-1\}$. Write $(\underline{\mu}, \underline{\alpha}) = (\underline{\mu}', \underline{\alpha}') \star ((\mu^{(m)}, \mu^{(m+1)}), \alpha^{(m)})$, where $(\underline{\mu}', \underline{\alpha}')$ has length $m-1$. Note that the weighted $(\underline{\mu}', \underline{\alpha}')$ and $\gamma$ satisfy the conditions of the lemma.

There are two cases:

- $\mathrm{ht}(\underline{\mu})_m > 0$.

  By Part 1 either $\alpha^{(m)} - \gamma \notin \Delta$, then by induction hypothesis, we get

  $$\mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}} e_{-\gamma} u) = \mathrm{hc}(a b^*_{\underline{\mu}',\underline{\alpha}'} e_{\alpha^{(m)}} e_{-\gamma} u) = \mathrm{hc}(a b^*_{\underline{\mu}',\underline{\alpha}'} e_{-\gamma} e_{\alpha^{(m)}} u) = 0,$$

  or $\alpha^{(m)} - \gamma = -\gamma'$ with $\gamma' \in \Delta_+$, and by induction,

  $$\begin{aligned}\mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}} e_{-\gamma} u) &= \mathrm{hc}(a b^*_{\underline{\mu}',\underline{\alpha}'} e_{\alpha^{(m)}} e_{-\gamma} u) \\ &= \mathrm{hc}(a b^*_{\underline{\mu}',\underline{\alpha}'} e_{-\gamma} e_{\alpha^{(m)}} u) + \mathrm{hc}(a a' b^*_{\underline{\mu}',\underline{\alpha}'} e_{-\gamma'} u) = 0,\end{aligned}$$

  where $a' \in \mathbb{C}$, since the path $(\underline{\mu}', \underline{\alpha}')$ and $\gamma'$ still satisfy the conditions of the lemma by (1).

- If $\mathrm{ht}(\underline{\mu})_m = 0$, then by induction hypothesis, we get

  $$\begin{aligned}\mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}} e_{-\gamma} u) &= \mathrm{hc}(a b^*_{\underline{\mu}',\underline{\alpha}'} \check{\varpi}_{\alpha^{(m)}} e_{-\gamma} u) \\ &= \mathrm{hc}(a b^*_{\underline{\mu}',\underline{\alpha}'} e_{-\gamma} \check{\varpi}_{\alpha^{(m)}} u - a\langle \gamma, \check{\varpi}_{\alpha^{(m)}} \rangle b^*_{\underline{\mu}',\underline{\alpha}'} e_{-\gamma} u) = 0,\end{aligned}$$

  whence the statement.

□

**Lemma 3.3.** *Let $\mu \in P(\delta)$, $m \in \mathbb{Z}_{>1}$, $\underline{i} \in \mathbb{Z}^m_{\succ \underline{0}}$ and $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)_{\underline{i}}$. Set $p := p(\underline{i})$ and $q := q(\underline{i})$.*

1. *Assume $p = q$ and $\alpha^{(p-1)} + \alpha^{(p)} \neq 0$. Then*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\# p}.$$

*2. Assume $p = q$ and $\alpha^{(p-1)} + \alpha^{(p)} = 0$.*

*(a) If $i_1 = \cdots = i_{p-2} = 0$, or if $p = 2$, then*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \mathrm{ht}(\check{\alpha}^{(p-1)})\,\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p}.$$

*(b) Otherwise,*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (\mathrm{ht}(\check{\alpha}^{(p-1)}) + 1)\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p}.$$

*3. Assume $p < q$. Then $i_p = 0$ and $\alpha^{(p)} \in \Pi_{\mu^{(p)}} = \{\beta \in \Pi \mid \langle \mu^{(p)}, \check{\beta} \rangle \neq 0\}$.*

*(a) If $\langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle = 1$, then*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \langle \rho, \check{\varpi}_{\alpha^{(p)}} \rangle \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p}.$$

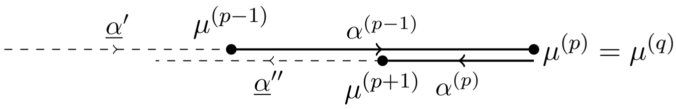
*(b) If $\langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle = -1$, then*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (-\langle \rho, \check{\varpi}_{\alpha^{(p)}} \rangle + 1)\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p}.$$

*Proof.* 1. Note that $\alpha^{(p-1)} + \alpha^{(p)} \neq 0$ implies $i_{p-1} + i_p \neq 0$. Write

$$(\underline{\mu}, \underline{\alpha}) = (\underline{\mu}', \underline{\alpha}') \star ((\mu^{(p-1)}, \mu^{(p)}), \alpha^{(p-1)}) \star ((\mu^{(p)}, \mu^{(p+1)}), \alpha^{(p)}) \star (\underline{\mu}'', \underline{\alpha}''),$$

where $(\underline{\mu}', \underline{\alpha}')$ and $(\underline{\mu}'', \underline{\alpha}'')$ have length $p-2$ and $m-p$, respectively.




Since $\mathfrak{g} = \mathfrak{sl}_{r+1}$ and $\delta = \varpi_1$, $a^{(e_{-\alpha^{(i)}})}_{\mu^{(i+1)},\mu^{(i)}}$, $a^{(e_{-\alpha^{(i-1)}})}_{\mu^{(i)},\mu^{(i-1)}}$, $a^{(e_{-\alpha^{(i-1)}-\alpha^{(i)}})}_{\mu^{(i+1)},\mu^{(i-1)}}$ and $n_{\alpha^{(p-1)},\alpha^{(p)}}$ are all equal to 1, and so

$$\begin{aligned} b^*_{(\underline{\mu},\underline{\alpha})} &= b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p-1)}} e_{\alpha^{(p)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\ &= (b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} + n_{\alpha^{(p-1)},\alpha^{(p)}} b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p-1)}+\alpha^{(p)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}) \\ &= b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} + b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}. \end{aligned}$$

The weighted path $(\underline{\mu}', \underline{\alpha}')$ and the positive root $\gamma = -\alpha^{(p)}$ verify the conditions of

Lemma 3.2. Hence by Lemma 3.2, we get

$$\mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')}e_{\alpha^{(p)}}e_{\alpha^{(p-1)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')}) = 0,$$

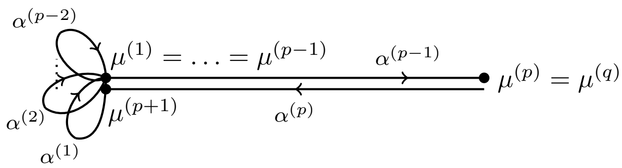
and so

$$\mathrm{wt}(\underline{\mu},\underline{\alpha}) = \mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p}.$$

2. (a) Assume first that $i_1 = \cdots = i_{p-2} = 0$ and $p \neq 2$ (that is, $p > 2$). Write

$$(\underline{\mu},\underline{\alpha}) = (\underline{\mu}',\underline{\alpha}') \star ((\mu^{(p-1)},\mu^{(p)}),\alpha^{(p-1)}) \star ((\mu^{(p)},\mu^{(p+1)}),\alpha^{(p)}) \star (\underline{\mu}'',\underline{\alpha}'')$$

as in Part 1. Here $(\underline{\mu}',\underline{\alpha}')$ is a concatenation of loops.




In particular $b^*_{\underline{\mu}',\underline{\alpha}'}$ is in $S(\mathfrak{h})$. Then, because $a^{(e_{-\alpha^{(p)}})}_{\mu^{(p+1)},\mu^{(p)}} = a^{(e_{-\alpha^{(p-1)}})}_{\mu^{(p)},\mu^{(p-1)}} = 1$, we have

$$\begin{aligned} b^*_{\underline{\mu},\underline{\alpha}} &= b^*_{(\underline{\mu}',\underline{\alpha}')}e_{\alpha^{(p-1)}}e_{-\alpha^{(p-1)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\ &= b^*_{(\underline{\mu}',\underline{\alpha}')}e_{-\alpha^{(p-1)}}e_{\alpha^{(p-1)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')} + b^*_{(\underline{\mu}',\underline{\alpha}')}\check{\alpha}^{(p-1)}b^*_{(\underline{\mu}'',\underline{\alpha}'')} \end{aligned}$$

since $\alpha^{(p)} = -\alpha^{(p-1)}$. Since $b^*_{\underline{\mu}',\underline{\alpha}'}$ is in $S(\mathfrak{h})$,

$$\mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')}e_{-\alpha^{(p-1)}}e_{\alpha^{(p-1)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')}) = b^*_{(\underline{\mu}',\underline{\alpha}')}\mathrm{hc}(e_{-\alpha^{(p-1)}}e_{\alpha^{(p-1)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')}) = 0.$$

Therefore,

$$\mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}}) = \check{\alpha}^{(p-1)}\mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')}b^*_{(\underline{\mu}'',\underline{\alpha}'')}) = \langle \mu^{(p-1)}, \check{\alpha}^{(p-1)}\rangle \check{\alpha}^{(p-1)}\mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}).$$

Since

$$\mathrm{ev}_\rho(\check{\alpha}^{(p-1)}) = \langle \rho, \check{\alpha}^{(p-1)}\rangle = \mathrm{ht}(\check{\alpha}^{(p-1)}),$$

we get the expected equality:

$$\mathrm{wt}(\underline{\mu},\underline{\alpha}) = \mathrm{ht}(\check{\alpha}^{(p-1)})\,\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p}.$$

If $p = 2$, we have

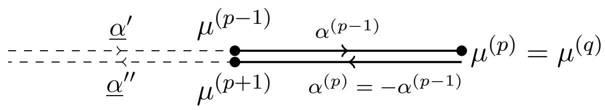
$$b^*_{\underline{\mu},\underline{\alpha}} = e_{\alpha^{(1)}} e_{-\alpha^{(1)}} b^*_{\underline{\mu}'',\underline{\alpha}''} = (e_{-\alpha^{(1)}} e_{\alpha^{(1)}} + \check{\alpha}^{(1)}) b^*_{\underline{\mu}'',\underline{\alpha}''},$$

where $(\underline{\mu}'', \underline{\alpha}'')$ is a weighted path of length $m-2$. Then we conclude as in the first situation.

(b) Assume that we are not in one of the situations of (a). Write

$$(\underline{\mu}, \underline{\alpha}) = (\underline{\mu}', \underline{\alpha}') \star ((\mu^{(p-1)}, \mu^{(p)}), \alpha^{(p-1)}) \star ((\mu^{(p)}, \mu^{(p+1)}), \alpha^{(p)}) \star (\underline{\mu}'', \underline{\alpha}'')$$

as in Part 1.




Note that $a^{(-e_{\alpha^{(p-1)}})}_{\mu^{(p)},\mu^{(p-1)}} a^{(-e_{\alpha^{(p)}})}_{\mu^{(p+1)},\mu^{(p)}} = 1$. Then we have

$$\begin{aligned} b^*_{(\underline{\mu},\underline{\alpha})} &= b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p-1)}} e_{\alpha^{(p)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\ &= b^*_{(\underline{\mu}',\underline{\alpha}')} e_{-\alpha^{(p-1)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} + b^*_{(\underline{\mu}',\underline{\alpha}')} \check{\alpha}^{(p-1)} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \end{aligned} \tag{3.1}$$

since $\alpha^{(p)} = -\alpha^{(p-1)}$. The weighted path $(\underline{\mu}', \underline{\alpha}')$ and the positive root $\gamma = \alpha^{(p-1)}$ satisfy the conditions of Lemma 3.2. Hence by Lemma 3.2, we get

$$\mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})}) = \mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')} \check{\alpha}^{(p-1)} b^*_{(\underline{\mu}'',\underline{\alpha}'')}).$$

Let $\alpha^{(s)} \in \underline{\alpha}'$ such that $i_s > 0$ and $\langle \alpha^{(s)}, \check{\alpha}^{(p-1)} \rangle \neq 0$. Observe that

$$\langle \alpha^{(s)}, \check{\alpha}^{(p-1)} \rangle = \langle \mu^{(s)} - \mu^{(s+1)}, \check{\alpha}^{(p-1)} \rangle \neq 0$$

if and only if $\mu^{(s+1)} = \mu^{(p-1)}$, thus $\alpha^{(s)}$ is unique (see Figure 3.1).

For all other roots $\alpha^{(t)} \in \underline{\alpha}'$ with $t \neq s$, if $i_t > 0$ then $\langle \alpha^{(t)}, \check{\alpha}^{(p-1)} \rangle = 0$, and so

$$b^*_{(\underline{\mu},\underline{\alpha}),t}\ \check{\alpha}^{(p-1)} = \check{\alpha}^{(p-1)}\ b^*_{(\underline{\mu},\underline{\alpha}),t}.$$

Otherwise, $b^*_{(\underline{\mu},\underline{\alpha}),t} = \varpi^\sharp_{(\alpha^{(t)})} \in \mathfrak{h}$ thus we also get $b^*_{(\underline{\mu},\underline{\alpha}),t}\ \check{\alpha}^{(p-1)} = \check{\alpha}^{(p-1)}\ b^*_{(\underline{\mu},\underline{\alpha}),t}$. We see that $\check{\alpha}^{(p-1)}$ commutes with all roots in $\underline{\alpha}'$, except with $\alpha^{(s)}$.

Write

$$(\underline{\mu}', \underline{\alpha}') = (\underline{\mu}'_1, \underline{\alpha}'_1) \star ((\mu^{(s)}, \mu^{(s+1)}), \alpha^{(s)}) \star (\underline{\mu}'_2, \underline{\alpha}'_2),$$

where $(\underline{\mu}'_2, \underline{\alpha}'_2)$ is a concatenation of loops and $(\underline{\mu}'_1, \underline{\alpha}'_1)$ has length $s-1$. Note that the weighted path $(\underline{\mu}'_1, \underline{\alpha}'_1)$ may be trivial.

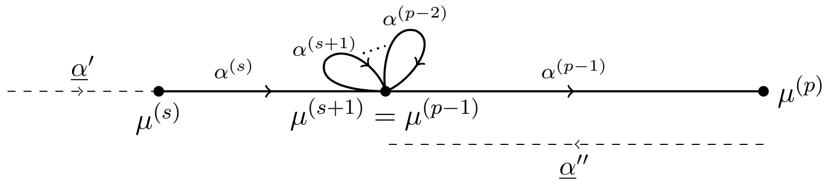



**FIGURE 3.1** – Path in case (2) (b)

Note that $a^{(-e_{\alpha^{(s)}})}_{\mu^{(s+1)},\mu^{(s)}} = 1$, and we get

$$\begin{aligned}
b^*_{(\underline{\mu}',\underline{\alpha}')}\check{\alpha}^{(p-1)}b^*_{(\underline{\mu}'',\underline{\alpha}'')} &= b^*_{(\underline{\mu}'_1,\underline{\alpha}'_1)}e_{\alpha^{(s)}}b^*_{(\underline{\mu}'_2,\underline{\alpha}'_2)}\check{\alpha}^{(p-1)}b^*_{(\underline{\mu}'',\underline{\alpha}'')}\\
&= b^*_{(\underline{\mu}'_1,\underline{\alpha}'_1)}e_{\alpha^{(s)}}\check{\alpha}^{(p-1)}b^*_{(\underline{\mu}'_2,\underline{\alpha}'_2)}b^*_{(\underline{\mu}'',\underline{\alpha}'')}\\
&= b^*_{(\underline{\mu}'_1,\underline{\alpha}'_1)}\left(\check{\alpha}^{(p-1)}e_{\alpha^{(s)}} - \langle\alpha^{(s)},\check{\alpha}^{(p-1)}\rangle e_{\alpha^{(s)}}\right)b^*_{(\underline{\mu}'_2,\underline{\alpha}'_2)}b^*_{(\underline{\mu}'',\underline{\alpha}'')}\\
&= \check{\alpha}^{(p-1)}b^*_{(\underline{\mu}'_1,\underline{\alpha}'_1)}e_{\alpha^{(s)}}b^*_{(\underline{\mu}'_2,\underline{\alpha}'_2)}b^*_{(\underline{\mu}'',\underline{\alpha}'')} - \langle\alpha^{(s)},\check{\alpha}^{(p-1)}\rangle b^*_{(\underline{\mu}'_1,\underline{\alpha}'_1)}e_{\alpha^{(s)}}b^*_{(\underline{\mu}'_2,\underline{\alpha}'_2)}b^*_{(\underline{\mu}'',\underline{\alpha}'')},
\end{aligned}$$

since $\check{\alpha}^{(p-1)}$ commutes with all roots of $\underline{\alpha}'_1$ and $\underline{\alpha}'_2$. Observe from Figure 3.1, $\langle\alpha^{(s)},\check{\alpha}^{(p-1)}\rangle = -1$ and the weighted path

$$(\underline{\mu}'_1, \underline{\alpha}'_1) \star ((\mu^{(s)},\mu^{(s+1)}),\alpha^{(s)}) \star (\underline{\mu}'_2, \underline{\alpha}'_2) \star (\underline{\mu}'', \underline{\alpha}'') = (\underline{\mu}', \underline{\alpha}') \star (\underline{\mu}'', \underline{\alpha}'')$$

is nothing but $(\underline{\mu}, \underline{\alpha})^{\#p}$. Thus,

$$b^*_{(\underline{\mu}',\underline{\alpha}')}\check{\alpha}^{(p-1)}b^*_{(\underline{\mu}'',\underline{\alpha}'')} = (\check{\alpha}^{(p-1)} - \langle\alpha^{(s)},\check{\alpha}^{(p-1)}\rangle)b^*_{(\underline{\mu},\underline{\alpha})^{\#p}} = (\check{\alpha}^{(p-1)}+1)b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}.$$

Remark that

$$\langle\rho,\check{\alpha}\rangle = \mathrm{ht}(\check{\alpha}) = \mathrm{ht}(\check{\alpha}^{(p-1)}).$$

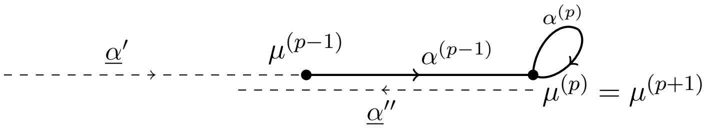
Hence,

$$\mathrm{wt}(\underline{\mu},\underline{\alpha}) = (\mathrm{ht}(\check{\alpha}^{(p-1)})+1)\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p}.$$

3. Write

$$(\underline{\mu},\underline{\alpha}) = (\underline{\mu}',\underline{\alpha}') \star ((\mu^{(p-1)},\mu^{(p)}),\alpha^{(p-1)}) \star ((\mu^{(p)},\mu^{(p+1)}),\alpha^{(p)}) \star (\underline{\mu}'',\underline{\alpha}''),$$

where $(\underline{\mu}',\underline{\alpha}')$ and $(\underline{\mu}'',\underline{\alpha}'')$ have length $p-2$ and $m-p$, respectively.




Let supp$(\alpha)$, the support of $\alpha \in \Delta$, denotes the set of $\beta \in \Pi$ such that $\langle \alpha, \check{\varpi}_\beta \rangle \neq 0$. We have

$$\begin{aligned} b^*_{(\underline{\mu},\underline{\alpha})} &= a^{(-e_{\alpha^{(p-1)}})}_{\mu^{(p-1)},\mu^{(p)}} \langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p-1)}} \check{\varpi}_{\alpha^{(p)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\ &= a^{(-e_{\alpha^{(p-1)}})}_{\mu^{(p-1)},\mu^{(p)}} \langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle (b^*_{(\underline{\mu}',\underline{\alpha}')} \check{\varpi}_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} - \langle \alpha^{(p-1)}, \check{\varpi}_{\alpha^{(p)}} \rangle b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}) \\ &= \langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle (\check{\varpi}_{\alpha^{(p)}} - \langle \alpha^{(p-1)}, \check{\varpi}_{\alpha^{(p)}} \rangle) b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}, \end{aligned}$$

since

$$a^{(-e_{\alpha^{(p-1)}})}_{\mu^{(p-1)},\mu^{(p)}} b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} = b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}$$

and $\check{\varpi}_{\alpha^{(p)}}$ commutes with all roots in $\underline{\alpha}'$. Indeed, the support of $\alpha'^{(j)}$, for $j = 1, \dots, p-2$, does not contain the simple root $\alpha^{(p)}$.

If $\langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle = 1$ then $\langle \alpha^{(p-1)}, \check{\varpi}_{\alpha^{(p)}} \rangle = 0$, and so

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \langle \rho, \check{\varpi}_{\alpha^{(p)}} \rangle \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p}.$$

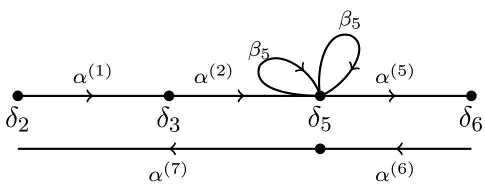
If $\langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle = -1$ then $\langle \alpha^{(p-1)}, \check{\varpi}_{\alpha^{(p)}} \rangle = 1$, and so

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (-\langle \rho, \check{\varpi}_{\alpha^{(p)}} \rangle + 1) \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p}.$$

□

*Example* 3.4. Assume that $\mathfrak{g} = \mathfrak{sl}_6$, $\delta = \varpi_1$, $\underline{i} \in \mathbb{Z}^m_{\succ \underline{0}}$. Let $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_7(\delta_2)_{\underline{i}}$ with

$$\begin{aligned} \underline{\mu} &= (\delta_2, \delta_3, \delta_5, \delta_5, \delta_5, \delta_6, \delta_5, \delta_2) = (\varepsilon_2, \varepsilon_3, \varepsilon_5, \varepsilon_5, \varepsilon_5, \varepsilon_6, \varepsilon_5, \varepsilon_2), \\ \underline{\alpha} &= (\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)} = \alpha^{(p-1)}, \alpha^{(6)} = \alpha^{(p)}, \alpha^{(7)}) \\ &= (\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_5, \beta_5, \beta_5, \varepsilon_5 - \varepsilon_6, \varepsilon_6 - \varepsilon_5, \varepsilon_5 - \varepsilon_2) \\ \underline{i} &= (1, 2, 0, 0, 1, -1, -3). \end{aligned}$$



In this path $\alpha^{(p-1)} + \alpha^{(p)} = 0$. Thus,

$$\begin{aligned} a_{\underline{\mu},\underline{\alpha}} &= a^{(e_{\varepsilon_3-\varepsilon_2})}_{\varepsilon_2,\varepsilon_3} a^{(e_{\varepsilon_2-\varepsilon_5})}_{\varepsilon_2,\varepsilon_5} a^{(e_{\varepsilon_5-\varepsilon_6})}_{\varepsilon_5,\varepsilon_6} a^{(e_{\varepsilon_6-\varepsilon_5})}_{\varepsilon_6,\varepsilon_5} a^{(\check{\beta}_5)}_{\varepsilon_5,\varepsilon_5} a^{(\check{\beta}_5)}_{\varepsilon_5,\varepsilon_5} a^{(e_{\varepsilon_5-\varepsilon_3})}_{\varepsilon_5,\varepsilon_3} a^{(e_{\varepsilon_3-\varepsilon_2})}_{\varepsilon_3,\varepsilon_2} \\ &= \langle \delta_5, \check{\beta}_5 \rangle \langle \delta_5, \check{\beta}_5 \rangle = 1, \\ a_{(\underline{\mu},\underline{\alpha})^{\#p}} &= a^{(e_{\varepsilon_3-\varepsilon_2})}_{\varepsilon_2,\varepsilon_3} a^{(e_{\varepsilon_2-\varepsilon_5})}_{\varepsilon_2,\varepsilon_5} a^{(\check{\beta}_5)}_{\varepsilon_5,\varepsilon_5} a^{(\check{\beta}_5)}_{\varepsilon_5,\varepsilon_5} a^{(e_{\varepsilon_5-\varepsilon_3})}_{\varepsilon_5,\varepsilon_3} a^{(e_{\varepsilon_3-\varepsilon_2})}_{\varepsilon_3,\varepsilon_2} = \langle \delta_5, \check{\beta}_5 \rangle \langle \delta_5, \check{\beta}_5 \rangle = 1. \end{aligned}$$

By Lemma 3.2 we have,

$$\begin{aligned} \mathrm{hc}\big(b^*_{(\underline{\mu},\underline{\alpha})}\big) &= a_{\underline{\mu},\underline{\alpha}} \mathrm{hc}\big(e_{\alpha^{(1)}} e_{\alpha^{(2)}} \check{\varpi}_{\alpha^{(4)}} \check{\varpi}_{\alpha^{(5)}} e_{\alpha^{(5)}} e_{\alpha^{(6)}} e_{\alpha^{(7)}}\big) \\ &= \mathrm{hc}\big(e_{\alpha^{(1)}} e_{\alpha^{(2)}} \check{\varpi}_5 \check{\varpi}_5 e_{\alpha^{(6)}} e_{\alpha^{(5)}}\big) + \mathrm{hc}\big(e_{\alpha^{(1)}} e_{\alpha^{(2)}} \check{\varpi}_5 \check{\varpi}_5 \check{\beta}_5 e_{\alpha^{(7)}}\big) \\ &= \mathrm{hc}\big(e_{\alpha^{(1)}} \check{\beta}_5 e_{\alpha^{(2)}} \check{\varpi}_5 \check{\varpi}_5 e_{\alpha^{(7)}}\big) - \langle \alpha^{(2)}, \check{\beta}_5 \rangle \mathrm{hc}\big(e_{\alpha^{(1)}} e_{\alpha^{(2)}} \check{\varpi}_5 \check{\varpi}_5 e_{\alpha^{(7)}}\big) \\ &= \big(\check{\beta}_5 - \langle \alpha^{(2)}, \check{\beta}_5 \rangle\big) \mathrm{hc}\big(e_{\alpha^{(1)}} e_{\alpha^{(2)}} \check{\varpi}_5 \check{\varpi}_5 e_{\alpha^{(7)}}\big), \end{aligned}$$

since $\mathrm{hc}\big(e_{\alpha^{(1)}} e_{\alpha^{(2)}} \check{\varpi}_5 \check{\varpi}_5 e_{\alpha^{(6)}} e_{\alpha^{(5)}}\big) = 0$ and $\check{\beta}_5$ commutes with $\alpha^{(1)}$. Note that $\check{\beta}_5 = \check{\alpha}^{(5)}$, $\langle \alpha^{(2)}, \check{\beta}_5 \rangle = 1$ and $e_{\alpha^{(1)}} e_{\alpha^{(2)}} \check{\varpi}_5 \check{\varpi}_5 e_{\alpha^{(7)}} = b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}$ . Hence,

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (\mathrm{ht}(\check{\alpha}^{(5)}) + 1) \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p}.$$

### 3.1.2 Proof of Theorem 3.1

As a direct consequence of Lemma 3.3, we get the following result.

**Proposition 3.5.** *Let $\mu \in P(\delta)$, $m \in \mathbb{Z}_{>1}$, $\underline{i} \in \mathbb{Z}^m_{\succ \underline{0}}$ and $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)_{\underline{i}}$. In particular, $\sum_{j=1}^m i_j = 0$. Set $p := p(\underline{i})$ and $q := q(\underline{i})$. Then for some scalar $K_{\underline{\mu},\underline{\alpha}}$, we have:*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K_{\underline{\mu},\underline{\alpha}} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p}.$$

*In particular,* $\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = 0$ *if* $\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} = 0$.

We are now in a position to prove Theorem 3.1.

*Proof of Theorem 3.1.* Let $(\underline{\mu}, \underline{\alpha})$ be as in the theorem and set $\underline{i} := \mathrm{ht}(\underline{\mu})$. First of all, we observe that if $\underline{i} \in \mathbb{Z}^m_{\prec \underline{0}}$, then $\mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}}) = 0$ and so the statement is clear.

We prove the statement by induction on $m$. Necessarily, $m \geqslant 2$. If $m = 2$, then the hypothesis implies that $\underline{i} \in \mathbb{Z}^m_{\prec \underline{0}}$ and so the statement is true.

Assume $m \geqslant 3$ and that for all weighted paths $(\underline{\mu}', \underline{\alpha}') \in \hat{\mathscr{P}}_{m'}(\mu)$, with $m' < m$, such that for some $i' \in \{1, \ldots, m'\}$, $\mu'^{(i')} \not\succcurlyeq \mu$, we have $\mathrm{wt}(\underline{\mu}', \underline{\alpha}') = 0$. If $\underline{i} \in \mathbb{Z}^m_{\prec \underline{0}}$ the statement is true. So we can assume that $\underline{i} \in \mathbb{Z}^m_{\succcurlyeq \underline{0}}$. Then necessarily $\underline{i} \in \mathbb{Z}^m_{\succ \underline{0}}$. Let $p, q$ be as in Proposition 3.5. We observe that the weighted path $(\underline{\mu}, \underline{\alpha})^{\#p}$ satisfies the hypothesis of the theorem and it is not empty. Hence by our induction hypothesis and Proposition 3.5, we get the statement. Notice that for $m = 3$, $\underline{i}$ is necessarily of the form $(i_1, i_2, i_3)$ with $i_1 > 0$, $i_2 < 0$, $i_3 > 0$ and $i_1 + i_2 + i_3 = 0$ so $(\underline{\mu}, \underline{\alpha})^{\#p}$ has length 2, and we can indeed apply the induction hypothesis. □

## 3.2 An equivalence relation on the set of weighted paths

For $\varepsilon \in \{-1, 0, 1\}$, denote by $\varepsilon\mathbb{Z}_{\succcurlyeq \underline{0}}$ the set $\{\varepsilon n \mid n \in \mathbb{Z}_{\succcurlyeq \underline{0}}\}$. Denote by $\mathscr{P}_m$ (respectively, $\hat{\mathscr{P}}_m$) the union of all sets $\mathscr{P}_m(\mu)$ (respectively, $\hat{\mathscr{P}}_m(\mu)$) for $\mu$ running through $P(\delta)$.

### 3.2.1 Definitions and first properties

**Definition 3.6.** *We define an equivalence relation $\sim$ on $\hat{\mathscr{P}}_m$ by induction on $m$ as follows.*

1. *If $m = 1$, there is only one equivalence class represented by the trivial path of length 0.*

2. *For $m = 2$, we say that two paths $(\underline{\mu}, \underline{\alpha})$ and $(\underline{\mu}', \underline{\alpha}')$ in $\hat{\mathscr{P}}_m$, with $\mathrm{ht}(\underline{\mu}) = \underline{i}$ and $\mathrm{ht}(\underline{\mu}') = \underline{i}'$, are* equivalent, *if there is $(\varepsilon_1, \varepsilon_2) \in \{0, 1\}^2$ such that $\underline{i} \in \varepsilon_1\mathbb{Z}_{\succcurlyeq \underline{0}} \times \varepsilon_2\mathbb{Z}_{\succcurlyeq \underline{0}}$ and $\underline{i}' \in \varepsilon_1\mathbb{Z}_{\succcurlyeq \underline{0}} \times \varepsilon_2\mathbb{Z}_{\succcurlyeq \underline{0}}$.*

3. *For $m \geqslant 3$, we say that we say that two paths $(\underline{\mu}, \underline{\alpha})$ and $(\underline{\mu}', \underline{\alpha}')$ in $\hat{\mathscr{P}}_m$, with $\mathrm{ht}(\underline{\mu}) = \underline{i}$ and $\mathrm{ht}(\underline{\mu}') = \underline{i}'$, are equivalent, if the following conditions are satisfied:*

   *(a) there is $(\varepsilon_1, \ldots, \varepsilon_m) \in \{-1, 0, 1\}^m$ such that $\underline{i} \in \prod_{i=1}^m \varepsilon_i\mathbb{Z}_{\succcurlyeq \underline{0}}$ and $\underline{i}' \in \prod_{i=1}^m \varepsilon_i\mathbb{Z}_{\succcurlyeq \underline{0}}$,*

   *(b) the paths $(\underline{\mu}, \underline{\alpha})^{\#p(\underline{i})}$ and $(\underline{\mu}', \underline{\alpha}')^{\#p(\underline{i}')}$ are equivalent, where $p(\underline{i})$ is as in Definition 1.23.*

*For $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m$, denote by $[(\underline{\mu}, \underline{\alpha})]$ the equivalence class of $(\underline{\mu}, \underline{\alpha})$ in $\hat{\mathscr{P}}_m$ with respect to $\sim$, and denote by $\mathscr{E}_m$ the set of equivalence classes.*

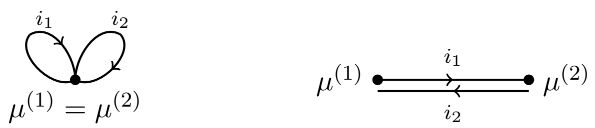
We observe that the equivalence class of $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m$ only depends on the sequence $\mathrm{ht}(\underline{\mu})$. Hence, by abuse of notation we will often write $[\mathrm{ht}(\underline{\mu})]$ for the class of $(\underline{\mu}, \underline{\alpha})$ (this will be not anymore the case in type $C$).

*Example* 3.7.

1. Assume $m = 2$. We have only two equivalence classes:

$$[0,0] \qquad \text{and} \qquad [1,-1].$$

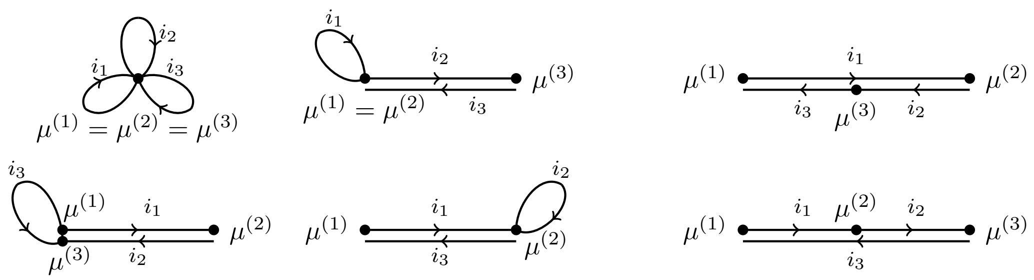
   We represent below weighted paths whose heights are in $[0,0]$ and $[1,-1]$ respectively:



2. Assume $m = 3$. We have six equivalence classes:

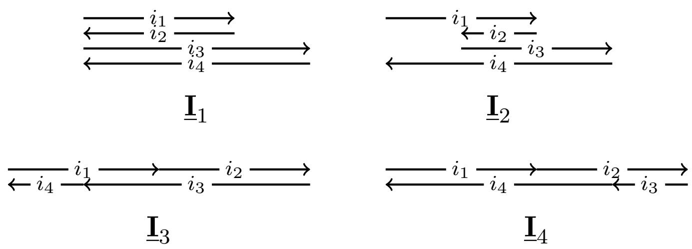
$$[0,0,0], \quad [0,1,-1], \quad [2,-1,-1], \quad [1,-1,0], \quad [1,0,-1], \quad [1,1,-2].$$

   We represent below weighted paths whose heights are in the above respective classes.



   For $m = 3$, the equivalence classes are determined only by condition (a) of Definition 3.6. This does not hold anymore from $m = 4$.

3. Assume $m = 4$. The following four weighted paths have pairwise not equivalent heights. The heights of $\underline{\mathbf{I}}_1$ and $\underline{\mathbf{I}}_2$ (also $\underline{\mathbf{I}}_3$ and $\underline{\mathbf{I}}_4$) satisfy condition (a) of Definition 3.6 but not the condition (b), so they are not equivalent. Here, $\underline{\mathbf{I}}_i$ refers as the equivalent class of the corresponding path.




The equivalence classes in $\mathbb{Z}^4$ are

$$[0,0,0,0],\ [0,0,1,-1],\ [0,1,0,-1],\ [0,1,-1,0],\ [1,-1,0,0], [1,0,-1,0],$$
$$[1,0,0,-1],\ [2,-1,-1,0],\ [2,-1,0,-1],\ [1,1,0,-2],\ [1,0,1,-2],\ [1,1,-2,0],$$
$$[0,2,-1,-1],\ [0,1,1,-2],\ [2,0,-1,-1],\ [1,1,-1,-1],\ [1,-1,1,-1],\ [1,1,1,-3],$$
$$[1,2,-1,2],\ [2,1,-2,-1],\ [2,-1,1,-2],\ [3,-1,-1,-1].$$

They are indeed the only pairwise non-equivalent classes. The verifications are left to the reader.

**Definition 3.8** (paths with zeroes and without zero)**.** *Let $m \in \mathbb{Z}_{>0}$ and $\underline{\mathbf{I}} \in \mathscr{E}_m$. The number $n$ of zero values of $\underline{i} := \mathrm{ht}(\underline{\mu})$ does not depend on $(\underline{\mu}', \underline{\alpha}')$ in $\underline{\mathbf{I}}$. We will say that $\underline{\mathbf{I}}$* has $n$ zeroes. *The positions of the zeroes only depend on $\underline{\mathbf{I}}$. If $n = 0$, we will say that $\underline{\mathbf{I}}$* has no zero.

By definition, the position $p(\underline{i})$ of the first turning back does not depend on $\underline{i} \in \underline{\mathbf{I}}$. Similarly, the integer $q(\underline{i})$ does not depend on $\underline{i} \in \underline{\mathbf{I}}$. Denote by $p(\underline{\mathbf{I}})$ and $q(\underline{\mathbf{I}})$ these integers. Furthermore, the class of $(\underline{\mu}, \underline{\alpha})^{\# p(\underline{i})}$ only depends on $\underline{\mathbf{I}}$. Denote by $\underline{\mathbf{I}}^{\#}$ this equivalence class.

For $m' \in \mathbb{Z}_{>0}$, denote by $\ell(\underline{\mathbf{I}}') := m'$ the length of $\underline{\mathbf{I}}'$ for some equivalence class $\underline{\mathbf{I}}' \in \mathscr{E}_{m'}$ . We have $\ell(\underline{\mathbf{I}}) = m$, $\ell(\underline{\mathbf{I}}^{\#}) = m - 1$ if $i_p + i_{p-1} \neq 0$, $\ell(\underline{\mathbf{I}}^{\#}) = m - 2$ if $i_p + i_{p-1} = 0$, etc.

### 3.2.2 Elements of $\mathscr{E}_m$ without zero

We study in this subsection the elements of $\mathscr{E}_m$ without zero. If $\underline{\mathbf{I}}$ has no zero, note that, necessarily, $p(\underline{\mathbf{I}}) = q(\underline{\mathbf{I}})$ and $\underline{\mathbf{I}}^{\#}$ has no zero, too.

*Remark* 3.9. If $\underline{\mathbf{I}}$ has no zero then for any weighted path $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)$, $\mu \in P(\delta)$, such that $\mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}$, we have

$$\mathrm{ht}(\check{\alpha}^{(j)}) = i_j \quad \text{for all} \quad j = 1, \ldots, m,$$

since $\mathfrak{g}$ has type $A$, where $\mathrm{ht}(\underline{\mu}) = (i_1, \ldots, i_m)$.

Remark 3.9 will be used repeatedly in the sequel.

*Example* 3.10. For $m = 4$, there are 7 equivalence classes without zero:

$$[3,-1,-1,-1], \quad [1,-1,1,-1], \quad [2,-1,1,-2], \quad [1,1,-1,-1],$$
$$[1,2,-1,-2], \quad [2,1,-2,-1], \quad [1,1,1,-3].$$

**Lemma 3.11.** *Let $\underline{\mathbf{I}} \in \mathscr{E}_m$ without zero, and set $p := p(\underline{\mathbf{I}})$.*

1. *There is a polynomial $A_{\underline{\mathbf{I}}} \in \mathbb{C}[X_1, \ldots, X_{m-1}]$ of total degree $\leqslant \lfloor \frac{m}{2} \rfloor$ such that for all $\underline{i} = (i_1, \ldots, i_m) \in \underline{\mathbf{I}}$ and for all $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m$ with $\mathrm{ht}(\underline{\mu}) = \underline{i}$,*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = A_{\underline{\mathbf{I}}}(i_1, \ldots, i_{m-1}).$$

   *(Here, $\lfloor x \rfloor$ denotes the largest integer $\leqslant x$.) Moreover, $A_{\underline{\mathbf{I}}}$ is a sum of monomials of the form $X_{j_1} \ldots X_{j_l}$, $1 \leqslant j_1 < \cdots < j_l < m$.*

2. *The polynomial $A_{\underline{\mathbf{I}}}$ is defined by induction as follows. We have*

$$A_{[1,-1]}(X_1) = X_1,$$
$$A_{[2,-1,-1]}(X_1, X_2) = X_1 + X_2, \qquad A_{[1,1,-2]}(X_1, X_2) = X_1.$$

   *Assume $m \geqslant 4$.*

   *(a) If $i_{p-1} + i_p \neq 0$, then*

$$A_{\underline{\mathbf{I}}}(X_1, \ldots, X_{m-1}) = A_{\underline{\mathbf{I}}^{\#}}(X_1, \ldots, X_{p-2}, X_{p-1} + X_p, \ldots, X_{m-1}).$$

   *(b) If $i_{p-1} + i_p = 0$ and $p = 2$, then*

$$A_{\underline{\mathbf{I}}}(X_1, \ldots, X_{m-1}) = X_{p-1} A_{\underline{\mathbf{I}}^{\#}}(X_1, \ldots, X_{p-2}, X_{p+1}, \ldots, X_{m-1}).$$

   *(c) If $i_{p-1} + i_p = 0$ and $p > 2$, then*

$$A_{\underline{\mathbf{I}}}(X_1, \ldots, X_{m-1}) = (X_{p-1} + 1) A_{\underline{\mathbf{I}}^{\#}}(X_1, \ldots, X_{p-2}, X_{p+1}, \ldots, X_{m-1}).$$

*Example* 3.12. For $m = 4$, we have:

$$A_{[3,-1,-1,-1]} = X_1 + X_2 + X_3, \quad A_{[1,-1,1,-1]} = X_1 X_3, \quad A_{[2,-1,1,-2]} = X_1 + X_2,$$

$$A_{[1,1,-1,-1]} = X_1(X_2+1), \quad A_{[2,2,-1,-3]} = X_1, \quad A_{[1,1,1,-3]} = X_1,$$
$$A_{[2,2,-3,-1]} = X_1 + X_2 + X_3.$$

*Proof.* We prove all the statements together by induction on $m$.

Assume $m = 2$. The only equivalence class in without zero is $[1,-1]$; $p([1,-1]) = 2$ and $i_1 + i_2 = 0$. Let $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_2$ with $\mathrm{ht}(\underline{\mu}) \sim (1,-1)$. By Lemma 3.3 (2) (a) we have $\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \mathrm{ht}(\check{\alpha}^{(1)}) = i_1$. Hence,

$$A_{[1,-1]}(X_1) = X_1$$

satisfies the conditions of the lemma.

Assume $m = 3$. There are two equivalence classes without zero: $[2,-1,-1]$ and $[1,1,-2]$. By Lemma 3.3 (1), for any $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_3$ with $\mathrm{ht}(\underline{\mu}) \sim (2,-1,-1)$, we have $\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = i_1 + i_2$ and for any $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_3$ with $\mathrm{ht}(\underline{\mu}) \sim (1,1,-2)$, we have $\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = i_1$. Hence

$$A_{[2,-1,-1]}(X_1, X_2) = X_1 + X_2 \quad \text{and} \quad A_{[1,1,-2]}(X_1, X_2) = X_1$$

satisfy the conditions of the lemma.

Let $m \geqslant 4$ and assume the proposition true for any $m' \in \{2, \ldots, m-1\}$ and any $\underline{\mathbf{I}}' \in \mathscr{E}_{m'}$. Let $\underline{\mathbf{I}} \in \mathscr{E}_m$ and $(\underline{\mu}, \underline{\alpha}) \in \underline{\mathbf{I}}$.

Assume that $i_{p-1} + i_p \neq 0$. Then $\mathrm{ht}(\underline{\mu})^{\#p}$ is in $\underline{\mathbf{I}}^{\#}$ and by Lemma 3.3 (1), $\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \mathrm{wt}\left((\underline{\mu}, \underline{\alpha})^{\#p}\right)$. By our induction hypothesis, there is a polynomial $A_{\underline{\mathbf{I}}^{\#}} \in \mathbb{C}[Y_1, \ldots, Y_{m-2}]$ of total degree $\leqslant \lfloor \frac{m-1}{2} \rfloor$ such that

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \mathrm{wt}\left((\underline{\mu}, \underline{\alpha})^{\#p(\underline{\mathbf{I}})}\right) = A_{\underline{\mathbf{I}}^{\#}}(i_1, \ldots, i_{p-1} + i_p, \ldots, i_{m-1}).$$

Hence the polynomial

$$A_{\underline{\mathbf{I}}}(X_1, \ldots, X_{m-1}) := A_{\underline{\mathbf{I}}^{\#}}(X_1, \ldots, X_{p-2}, X_{p-1} + X_p, \ldots, X_{m-1})$$

satisfies the conditions of the lemma.

Assume that $i_{p-1} + i_p = 0$. Then $\mathrm{ht}(\underline{\mu})^{\#p(\underline{\mathbf{I}})}$ is in $\underline{\mathbf{I}}^{\#}$.

$*$ If $p = 2$, then by Lemma 3.3 (2)(a), we have:

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = i_{p-1} \mathrm{wt}((\underline{\mu}, \underline{\alpha})^{\#p(\underline{\mathbf{I}})}).$$

By our induction hypothesis, there is a polynomial $A_{\underline{\mathbf{I}}^\#} \in \mathbb{C}[Y_1, \ldots, Y_{m-3}]$ of total degree $\leqslant \lfloor \frac{m-2}{2} \rfloor$ such that

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = i_{p-1}\mathrm{wt}\left((\underline{\mu}, \underline{\alpha})^{\#p(\underline{\mathbf{I}})}\right) = i_{p-1}A_{\underline{\mathbf{I}}^\#}(i_1, \ldots, i_{p-2}, i_{p+1}, \ldots, i_{m-1}).$$

Hence the polynomial

$$A_{\underline{\mathbf{I}}}(X_1, \ldots, X_{m-1}) := X_{p-1}A_{\underline{\mathbf{I}}^\#}(X_1, \ldots, X_{p-2}, X_{p+1}, \ldots, X_{m-1})$$

satisfies the conditions of the lemma.

$*$ If $p > 2$, then by Lemma 3.3(2)(b), we have:

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (i_{p-1} + 1)\mathrm{wt}((\underline{\mu}, \underline{\alpha})^{\#p(\underline{\mathbf{I}})}).$$

By our induction hypothesis, there is a polynomial $A_{\underline{\mathbf{I}}^\#} \in \mathbb{C}[Y_1, \ldots, Y_{m-2}]$ of total degree $\leqslant \lfloor \frac{m-2}{2} \rfloor$ such that

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (i_{p-1} + 1)\mathrm{wt}\left((\underline{\mu}, \underline{\alpha})^{\#p(\underline{\mathbf{I}})}\right) = (i_{p-1} + 1)A_{\underline{\mathbf{I}}^\#}(i_1, \ldots, i_{p-2}, i_{p+1}, \ldots, i_{m-1}).$$

Hence the polynomial

$$A_{\underline{\mathbf{I}}}(X_1, \ldots, X_{m-1}) := (X_{p-1} + 1)A_{\underline{\mathbf{I}}^\#}(X_1, \ldots, X_{p-2}, X_{p+1}, \ldots, X_{m-1})$$

satisfies the conditions of the lemma. □

### 3.2.3 A key lemma

The following result will be used in the proof of Lemma 3.14. Since it is a very classical fact, we omit the proof.

**Lemma 3.13.** *Let $d \in \mathbb{Z}_{\geqslant 0}$ and $N \in \mathbb{Z}_{>0}$. There is a polynomial $S_d \in \mathbb{C}[X]$ of degree $d + 1$ such that*

$$\sum_{i=1}^{N} i^d = S_d(N).$$

*Namely, if $B_1, B_2, B_3, \ldots$ are the Bernouilli numbers, then $S_d$ is given by:*

$$S_d(X) = \frac{1}{d+1} \sum_{j=0}^{d} \binom{d+1}{j} \tilde{B}_j X^{d+1-j},$$

*where* $\tilde{B}_0 = 1$, $\tilde{B}_1 = \frac{1}{2}$ *and* $\tilde{B}_j = B_j$ *for* $j \geqslant 2$. *In particular, the leading term of* $S_d(X)$ *is* $\dfrac{X^{d+1}}{d+1}$ *and* $S_d(0) = 0$.

**Lemma 3.14.** *Let* $m \in \mathbb{Z}_{>0}$, $d \in \mathbb{Z}_{\geqslant 0}$, *and* $\underline{\mathbf{I}} \in \mathscr{E}_m$ *without zero. Set* $p := p(\underline{\mathbf{I}})$. *Let* $\underline{d} = (d_1, \ldots, d_{m-1})$ *with* $d_1 + \cdots + d_{m-1} = d$. *Then for some polynomial* $T_{\underline{d},\underline{\mathbf{I}}} \in \mathbb{C}[X]$ *of degree* $\leqslant d + m - \deg A_{\underline{\mathbf{I}}}$, *we have*

$$\forall k \in \{1, \ldots, r+1\}, \qquad \sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k),\\ \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!],\, \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}}} i_1^{d_1} \cdots i_{m-1}^{d_{m-1}} = T_{\underline{d},\underline{\mathbf{I}}}(k).$$

*where the integer* $i_j$, *for* $j = 1, \ldots, m-1$, *denotes* $\mathrm{ht}(\underline{\mu})_j$. *In particular, if for some* $k \in \{1, \ldots, r+1\}$, *the set* $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!], \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}\}$ *is empty, we have* $T_{\underline{d},\underline{\mathbf{I}}}(k) = 0$.

By Lemma 3.11, we get

$$A_{\underline{\mathbf{I}}}(i_1, \cdots, i_{m-1}) = \sum_{\underline{d}' = (d'_1, \ldots, d'_{m-1})} C_{\underline{d}'}\, i_1^{d'_1} \cdots i_{m-1}^{d'_{m-1}},$$

where $|\underline{d}'| \leqslant \deg A_{\underline{\mathbf{I}}}$. Thus,

$$\begin{aligned}
\sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k),\\ \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!],\\ \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}}} i_1^{d_1} \ldots i_{m-1}^{d_{m-1}} A_{\underline{\mathbf{I}}}(i_1, \ldots, i_{m-1}) &= \sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k),\\ \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!],\\ \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}}} i_1^{d_1} \ldots i_{m-1}^{d_{m-1}} \Big( \sum_{\underline{d}'} C_{\underline{d}'}\, i_1^{d'_1} \cdots i_{m-1}^{d'_{m-1}} \Big) \\
&= \sum_{\underline{d}'} C_{\underline{d}'} \sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k),\\ \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!],\\ \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}}} i_1^{d_1 + d'_1} \ldots i_{m-1}^{d_{m-1} + d'_{m-1}} \\
&= \sum_{\underline{d}'} C_{\underline{d}'}\, T_{\underline{d}+\underline{d}',\underline{\mathbf{I}}}(k),
\end{aligned}$$

where $T_{\underline{d}+\underline{d}',\underline{\mathbf{I}}}$ is a polynomial of degree $\leqslant |\underline{d}| + |\underline{d}'| + m - \deg A_{\underline{\mathbf{I}}} \leqslant d + m$. Hence the lemma implies that

$$\sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k),\\ \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!],\\ \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}}} i_1^{d_1} \ldots i_{m-1}^{d_{m-1}} A_{\underline{\mathbf{I}}}(i_1, \ldots, i_{m-1})$$

is a polynomial on $k$ of degree $\leqslant d + m$.

*Proof.* We prove the lemma by induction on $m$. More precisely, we prove by induction on $m$ the following:

*For all* $\underline{\mathbf{I}} \in \mathscr{E}_m$ *without zero and all* $d \in \mathbb{Z}_{\geqslant 0}$ *with* $d_1 + \cdots + d_{m-1} = d$. *Set*

$$\forall k \in \{1, \dots, r+1\}, \quad \tilde{T}_{\underline{d}, \underline{\mathbf{I}}}(k) := \sum_{\substack{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k), \\ \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!],\, \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}}} i_1^{d_1} \cdots i_{m-1}^{d_{m-1}}.$$

*Then there exists some polynomial* $T_{\underline{d}, \underline{\mathbf{I}}} \in \mathbb{C}[X]$ *of degree at most* $d + m - \deg A_{\underline{\mathbf{I}}}$ *such that*

$$T_{\underline{d}, \underline{\mathbf{I}}}(k) = \tilde{T}_{\underline{d}, \underline{\mathbf{I}}}(k),$$

*for all* $k \in \{1, \dots, r+1\}$. *In particular, if for some* $k \in \{1, \dots, r+1\}$, *the set* $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!], \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}\}$ *is empty, we have* $T_{\underline{d}, \underline{\mathbf{I}}}(k) = 0$.

The case $m = 1$ is empty. Assume $m = 2$, and let $\underline{\mathbf{I}} \in \mathscr{E}_2$ without zero.

The only equivalence class in $\mathscr{E}_2$ without zero is $[1, -1]$, so $\underline{\mathbf{I}} = [1, -1]$ and $p([1, -1]) = 2$. Let $k \in \{1, \dots, r\}$. Then the set $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_2(\delta_k) \mid \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!], \mathrm{ht}(\underline{\mu}) \sim (1, -1)\}$ is nonempty. It is empty for $k = r + 1$.

Let $d = d_1 \in \mathbb{Z}_{\geqslant 0}$.

We get

$$\sum_{\substack{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_2(\delta_k) \\ \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!],\, \mathrm{ht}(\underline{\mu}) \sim (1,-1)}} i_1^{d_1} = \sum_{i_1 = 1}^{r+1-k} i_1^d = S_d(r + 1 - k).$$

By Lemma 3.13, the polynomial

$$T_{d,[1,-1]}(X) := S_d(r + 1 - X),$$

has degree $d + 1 = d + 2 - \deg A_{[1,-1]}$ since $A_{[1,-1]}(X_1) = X_1$. Hence, for any $k \in \{1, \dots, r\}$,

$$\sum_{\substack{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_2(\delta_k) \\ \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!],\, \mathrm{ht}(\underline{\mu}) \sim (1,-1)}} i_1^{d_1} = T_{d,[1,-1]}(k).$$

Moreover, the set $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_2(\delta_k) \mid \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!], \mathrm{ht}(\underline{\mu}) \sim (1, -1)\}$ is empty if and only if $k = r + 1$, but $T_{d,[1,-1]}(r + 1) = S_d(0) = 0$. Therefore the equality still holds. This proves the claim for $m = 2$.

Assume $m \geqslant 3$ and the claim proven for any $m' \in \{1, \dots, m - 1\}$.

Let $\underline{\mathbf{I}} \in \mathscr{E}_m$ without zero, set $p := p(\underline{\mathbf{I}})$, and let $\underline{d} = (d_1, \dots, d_{p-1})$ with $d_1 + \cdots + d_{p-1} = d$. Let $k \in \{1, \dots, r + 1\}$ such that the set $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k), \mid \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!]$ and $\mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}\}$ is nonempty. Then the set $\{(\underline{\mu}', \underline{\alpha}') \in \hat{\mathscr{P}}_{m-1}(\delta_k), \mid \underline{\mu}' \in [\![\delta_{r+1}, \delta_k]\!]$ and $\mathrm{ht}(\underline{\mu}') \in \underline{\mathbf{I}}^{\#}\}$ is nonempty, too.

$*$ Assume that $i_{p-1} + i_p > 0$.
Let $\underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!]$ such that $\underline{i} := \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}$.

$$\underline{i} = (i_1, \ldots, i_{p-1}, i_p, i_{p+1}, \ldots, i_m).$$

Set $\underline{i}' := \mathrm{ht}(\underline{\mu}^{\#p})$,

$$\underline{i}' = (i'_1, \ldots, i'_{p-2}, i'_{p-1}, i'_p, \ldots, i'_{m-1})$$

where $i'_{p-1} = i_{p-1} + i_p > 0$. Then $i_{p-1} > i'_{p-1}$, and so $i_{p-1} = i'_{p-1} + i$ with $i$ runs through $\{1, \ldots, r+1-k-i'_1-\cdots-i'_{p-1}\}$. Hence, there are precisely $r+1-k-(i'_1+\cdots+i'_{p-1})$ elements $\underline{i} \in \mathbb{Z}^m$ such that $\underline{i} \in \underline{\mathbf{I}}$ and $\underline{i}^{\#p} = \mathrm{ht}(\underline{\mu}^{\#p})$. The heights $\underline{i} := \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}$ can be expressed in term of $\underline{i}'$ as follows:

$$\begin{aligned} &i_1 = i'_1, \quad \ldots, \quad i_{p-2} = i'_{p-2}, \\ &i_{p-1} = i'_{p-1} + i, \qquad i_p = -i, \\ &i_{p+1} = i'_p, \quad \ldots, \quad i_{m-1} = i'_{m-2}, \end{aligned}$$

where $i$ runs through $\{1, \ldots, r+1-k-i'_1-\cdots-i'_{p-1}\}$ (see Figure 3.2 for an illustration).

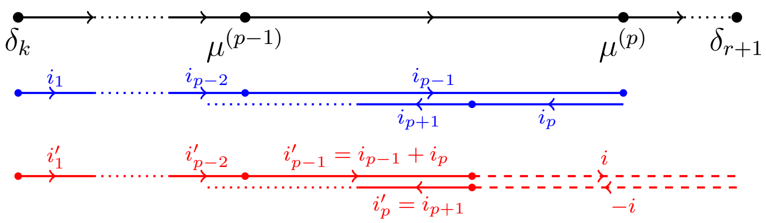



**FIGURE 3.2** – $\underline{i} = \mathrm{ht}(\underline{\mu})$ and $\underline{i}' := \mathrm{ht}(\underline{\mu}^{\#p})$ for the case $i_{p-1} + i_p > 0$

By Lemma 3.13 we get,

$$\tilde{T}_{\underline{d},\underline{\mathbf{I}}}(k) = \sum_{\substack{(\underline{\mu}',\underline{\alpha}') \in \hat{\mathscr{P}}_{m-1}(\delta_k) \\ \underline{\mu} \in [\![\delta_{r+1},\delta_k]\!] \\ \mathrm{ht}(\underline{\mu}') \in \underline{\mathbf{I}}^{\#}}} \sum_{\substack{1 \leqslant i \leqslant r+1-k \\ -i'_1-\cdots-i'_{p-1}}} (i'_1)^{d_1} \cdots (i'_{p-2})^{d_{p-2}} (i'_{p-1}+i)^{d_{p-1}} (-i)^{d_p} \cdots (i'_{m-2})^{d_{m-1}}$$

$$= \sum_{\substack{(\underline{\mu}',\underline{\alpha}')\in\hat{\mathscr{P}}_{m-1}(\delta_k)\\ \underline{\mu}\in[\![\delta_{r+1},\delta_k]\!]\\ \mathrm{ht}(\underline{\mu}')\in\underline{\mathbf{I}}^{\#}}} \sum_{\substack{1\leqslant i\leqslant r+1-k\\ -i_1'-\cdots-i_{p-1}'}} (i_1')^{d_1}\cdots(i_{p-2}')^{d_{p-2}} \left(\sum_{j=0}^{d_{p-1}}\binom{d_{p-1}}{j}(i_{p-1}')^{d_{p-1}-j} i^j\right)(-i)^{d_p}\cdots(i_{m-2}')^{d_{m-1}}$$

$$= \sum_{\substack{(\underline{\mu}',\underline{\alpha}')\in\hat{\mathscr{P}}_{m-1}(\delta_k)\\ \underline{\mu}\in[\![\delta_{r+1},\delta_k]\!]\\ \mathrm{ht}(\underline{\mu}')\in\underline{\mathbf{I}}^{\#}}} (-1)^{d_p}(i_1')^{d_1}\cdots(i_{p-2}')^{d_{p-2}}(i_p')^{d_{p+1}}\cdots(i_{m-2}')^{d_{m-1}}$$

$$\times \sum_{j=0}^{d_{p-1}}\binom{d_{p-1}}{j}(i_{p-1}')^{d_{p-1}-j}S_{d_p+j}(r+1-k-i_1'-\cdots-i_{p-1}')$$

$$= \sum_{\substack{(\underline{\mu}',\underline{\alpha}')\in\hat{\mathscr{P}}_{m-1}(\delta_k)\\ \underline{\mu}\in[\![\delta_{r+1},\delta_k]\!]\\ \mathrm{ht}(\underline{\mu}')\in\underline{\mathbf{I}}^{\#}}} (-1)^{d_p}(i_1')^{d_1}\cdots(i_{p-2}')^{d_{p-2}}(i_p')^{d_{p+1}}\cdots(i_{m-2}')^{d_{m-1}}\sum_{j=0}^{d_{p-1}}\binom{d_{p-1}}{j}(i_{p-1}')^{d_{p-1}-j}$$

$$\times \frac{1}{d_p+j+1}\sum_{l=0}^{d_p+j}\binom{d_p+j+1}{l}\tilde{B}_l(r+1-k-i_1'-\cdots-i_{p-1}')^{d_p+j+1-l}$$

$$= \sum_{j=0}^{d_{p-1}}\sum_{l=0}^{d_p+j}\frac{1}{d_p+j+1}\binom{d_{p-1}}{j}\binom{d_p+j+1}{l}\tilde{B}_l \sum_{\substack{(\underline{\mu}',\underline{\alpha}')\in\hat{\mathscr{P}}_{m-1}(\delta_k)\\ \underline{\mu}\in[\![\delta_{r+1},\delta_k]\!]\\ \mathrm{ht}(\underline{\mu}')\in\underline{\mathbf{I}}^{\#}}} \sum_{\substack{\underline{q}\in\mathbb{N}^p\\ |\underline{q}|=d_p+j+1-l}} \frac{(d_p+j+1-l)!}{q_1!\ldots q_p!}$$

$$\times (-1)^{2d_p+j+1-l-q_p}(r+1-k)^{q_p}(i_1')^{d_1+q_1}\cdots(i_{p-1}')^{d_{p-1}+q_{p-1}-j}(i_p')^{d_{p+1}}\cdots(i'_{m-2})^{d_{m-1}}$$

$$= \sum_{j=0}^{d_{p-1}}\sum_{l=0}^{d_p+j}\frac{1}{d_p+j+1}\binom{d_{p-1}}{j}\binom{d_p+j+1}{l}\tilde{B}_l \sum_{\substack{\underline{q}\in\mathbb{N}^p\\ |\underline{q}|=d_p+j+1-l}} \frac{(d_p+j+1-l)!}{q_1!\ldots q_p!}(-1)^{2d_p+j+1-l-q_p}$$

$$\times (r+1-k)^{q_p} \sum_{\substack{(\underline{\mu}',\underline{\alpha}')\in\hat{\mathscr{P}}_{m-1}(\delta_k)\\ \underline{\mu}\in[\![\delta_{r+1},\delta_k]\!]\\ \mathrm{ht}(\underline{\mu}')\in\underline{\mathbf{I}}^{\#}}} (i_1')^{d_1+q_1}\cdots(i_{p-1}')^{d_{p-1}+q_{p-1}-j}(i_p')^{d_{p+1}}\cdots(i_{m-2}')^{d_{m-1}}.$$

Set $\underline{d}' = (d_1+q_1,\ldots,d_{p-1}+q_{p-1}-j,d_{p+1},\ldots,d_{m-1})$, with $|\underline{d}'| = d-d_p+d_p+j+1-l-q_p-j = d+1-l-q_p \leqslant d+1-q_p$, and

$$\tilde{T}_{\underline{d}',\underline{\mathbf{I}}^{\#}}(k) = \sum_{\substack{(\underline{\mu}',\underline{\alpha}')\in\hat{\mathscr{P}}_{m-1}(\delta_k)\\ \underline{\mu}\in[\![\delta_{r+1},\delta_k]\!]\\ \mathrm{ht}(\underline{\mu}')\in\underline{\mathbf{I}}^{\#}}} (i_1')^{d_1+q_1}\cdots(i_{p-1}')^{d_{p-1}+q_{p-1}-j}(i_p')^{d_{p+1}}\cdots(i_{m-2}')^{d_{m-1}}.$$

Then

$$\tilde{T}_{\underline{d},\underline{\mathbf{I}}}(k) = \sum_{j=0}^{d_{p-1}}\sum_{l=0}^{d_p+j}\frac{1}{d_p+j+1}\binom{d_{p-1}}{j}\binom{d_p+j+1}{l}\tilde{B}_l$$

$$\times \sum_{\substack{\underline{q}\in\mathbb{N}^p \\ |\underline{q}|=d_p+j+1-l}} \frac{(d_p+j+1-l)!}{q_1!\dots q_p!}(-1)^{2d_p+j+1-l-q_p}(r+1-k)^{q_p}\tilde{T}_{\underline{d}',\underline{\mathbf{I}}^\#}(k). \quad (3.2)$$

By the induction hypothesis applied to $m-1$ and $\underline{\mathbf{I}}^\#$, there exists a polynomial $T_{\underline{d}',\underline{\mathbf{I}}^\#}$ of degree $\leqslant d+1-q_p+(m-1)-\deg A_{\underline{\mathbf{I}}^\#} = d-q_p+m-\deg A_{\underline{\mathbf{I}}}$, since $\deg A_{\underline{\mathbf{I}}^\#} = \deg A_{\underline{\mathbf{I}}}$ by Lemma 3.11, such that $T_{\underline{d}',\underline{\mathbf{I}}^\#}(k) = \tilde{T}_{\underline{d}',\underline{\mathbf{I}}^\#}(k)$ for all $k$ such that $\{(\underline{\mu}',\underline{\alpha}') \in \hat{\mathscr{P}}_{m-1}(\delta_k),\ |\ \underline{\mu}' \in [\![\delta_{r+1},\delta_k]\!], \mathrm{ht}(\underline{\mu}') \in \underline{\mathbf{I}}^\#\}$ is nonempty.

Set

$$T_{\underline{d},\underline{\mathbf{I}}}(X) = \sum_{j=0}^{d_{p-1}} \sum_{l=0}^{d_p+j} \frac{1}{d_p+j+1}\binom{d_{p-1}}{j}\binom{d_p+j+1}{l}\tilde{B}_l$$
$$\times \sum_{\substack{\underline{q}\in\mathbb{N}^{p+1} \\ |\underline{q}|=d_p+j+1-l}} \frac{(d_p+j+1-l)!}{q_1!\dots q_p!}(-1)^{2d_p+j+1-l-q_p}(r+1-k)^{q_p}T_{\underline{d}',\underline{\mathbf{I}}^\#}(X).$$

Then by the induction hypothesis and (3.2), we have $T_{\underline{d},\underline{\mathbf{I}}}(k) = \tilde{T}_{\underline{d},\underline{\mathbf{I}}}(k)$ and $T_{\underline{d},\underline{\mathbf{I}}}$ is a polynomial of degree $\leqslant d+m-\deg A_{\underline{\mathbf{I}}}$ for all $k$ such that $\{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k),\ |\ \underline{\mu} \in [\![\delta_{r+1},\delta_k]\!], \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}\}$ is nonempty.

It remains to verify that $T_{\underline{d},\underline{\mathbf{I}}}(k) = 0$ when the set

$$\{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\delta_{r+1},\delta_k]\!] \text{ and } \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}\} \quad (3.3)$$

is empty. In this case, $\tilde{T}_{\underline{d},\underline{\mathbf{I}}}(k) = 0$ by the definition. The Set (3.3) is empty if $\{(\underline{\mu}',\underline{\alpha}') \in \hat{\mathscr{P}}_{m-1}(\delta_k) \mid \underline{\mu}' \in [\![\delta_{r+1},\delta_k]\!] \text{ and } \mathrm{ht}(\underline{\mu}') \in \underline{\mathbf{I}}^\#\} = \varnothing$. But our induction hypothesis says that, in this case,

$$T_{\underline{d},\underline{\mathbf{I}}}(k) = T_{\underline{d}',\underline{\mathbf{I}}^\#}(k) = 0,$$

for any $\underline{d}' \in \mathbb{Z}_{\geqslant 0}^{p-1}$. Otherwise, this means that the set $\{(\underline{\mu}',\underline{\alpha}') \in \hat{\mathscr{P}}_{m-1}(\delta_k) \mid \underline{\mu}' \in [\![\delta_{r+1},\delta_k]\!] \text{ and } \mathrm{ht}(\underline{\mu}') \in \underline{\mathbf{I}}^\#\}$ is nonempty and so for any $\underline{i}' \in \underline{\mathbf{I}}^\#$ and for any $(\underline{\mu}',\underline{\alpha}') \in \hat{\mathscr{P}}_{m-1}(\delta_k)$ such that $\mathrm{ht}(\underline{\mu}') = \underline{i}'$, we have $i'_1+\dots+i'_{p-1} = r+1-k$. In that event, $T_{\underline{d},\underline{\mathbf{I}}}(k) = 0$ by the construction.

$*$ Assume that $i_{p-1}+i_p < 0$.

Let $\underline{\mu} \in [\![\delta_{r+1},\delta_k]\!]$ such that $\underline{i} := \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}$.

$$\underline{i} = (i_1,\dots,i_{p-1},i_p,i_{p+1},\dots,i_m).$$

Set $\underline{i}' := \mathrm{ht}(\underline{\mu}^{\#p})$,

$$\underline{i}' = (i'_1, \ldots, i'_{p-2}, i'_{p-1}, i'_p, \ldots, i'_{m-1})$$

where $i'_{p-1} = i_{p-1} + i_p < 0$. We have $i_p < i'_{p-1} < 0$, and so $i_p = i'_{p-1} - i$ with $i$ runs through $\{1, \ldots, r+1-k-i'_1 - \cdots - i'_{p-1}\}$. Hence, the heights $\underline{i} := \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}$ can be expressed in term of $\underline{i}'$ as follows:

$$\begin{aligned}
&i_1 = i'_1, \quad \ldots, \quad i_{p-2} = i'_{p-2},\\
&i_{p-1} = i, \quad i_p = i'_{p-1} - i,\\
&i_{p+1} = i'_p, \quad \ldots, \quad i_{m-1} = i'_{m-2},
\end{aligned}$$

where $i$ runs through $\{1, \ldots, r+1-k-i'_1 - \cdots - i'_{p-1}\}$ (see Figure 3.3 for an illustration).

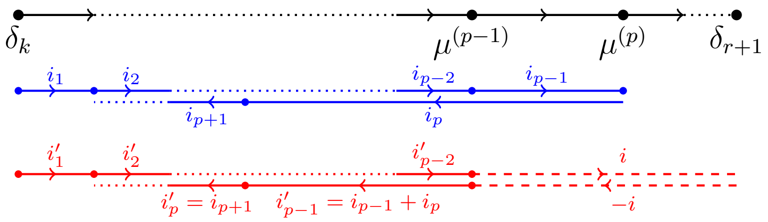



**FIGURE 3.3** – $\underline{i} = \mathrm{ht}(\underline{\mu})$ and $\underline{i}' := \mathrm{ht}(\underline{\mu}^{\#p})$ for the case $i_{p-1} + i_p < 0$

Hence,

$$\tilde{T}_{\underline{d},\underline{\mathbf{I}}}(k) = \sum_{\substack{(\underline{\mu}',\underline{\alpha}') \in \hat{\mathscr{P}}_{m-1}(\delta_k)\\ \underline{\mu} \in [\![\delta_{r+1},\delta_k]\!]\\ \mathrm{ht}(\underline{\mu}') \in \underline{\mathbf{I}}^{\#}}} \sum_{\substack{1 \leqslant i \leqslant r+1-k\\ -i'_1 - \cdots - i'_{p-1}}} (i'_1)^{d_1} \cdots (i'_{p-2})^{d_{p-2}} i^{d_{p-1}} (i'_{p-1} - i)^{d_p} (i'_p)^{d_{p+1}} \cdots (i'_{m-2})^{d_{m-1}}$$

Then we conclude exactly as in the first case.

To verify that $T_{\underline{d},\underline{\mathbf{I}}}(k) = 0$ when

$$\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!] \text{ and } \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}\} = \varnothing,$$

the arguments are the same as for the case $i_{p-1} + i_p > 0$ so we omit details.

$*$ Assume that $i_{p-1} + i_p = 0$.

Since $\underline{\mathbf{I}}$ has no zero, we necessarily have $m \geqslant 4$. Let $\underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!]$ such that $\underline{i} := \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}$.

$$\underline{i} = (i_1, \ldots, i_{p-1}, i_p, i_{p+1}, \ldots, i_m).$$

Set $\underline{i}' := \mathrm{ht}(\underline{\mu}^{\#p})$,

$$\underline{i}' = (i'_1, \ldots, i'_{p-2}, i'_{p-1}, i'_p, \ldots, i'_{m-2})$$

where $i'_{p-1} = i_{p+1}$. Set $i_{p-1} = i$ then $i_p = -i$ with $i$ runs through $\{1, \ldots, r+1-k-i'_1 - \cdots - i'_{p-2}\}$. Hence, the heights $\underline{i} := \mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}$ can be expressed in term of $\underline{i}'$ as follows:

$$\begin{aligned}
&i_1 = i'_1, \quad \ldots, \quad i_{p-2} = i'_{p-2},\\
&i_{p-1} = i, \quad i_p = -i,\\
&i_{p+1} = i'_{p-1}, \quad \ldots, \quad i_{m-1} = i'_{m-3},
\end{aligned}$$

where $i$ runs through $\{1, \ldots, r+1-k-i'_1 - \cdots - i'_{p-2}\}$ (see Figure 3.4 for an illustration).

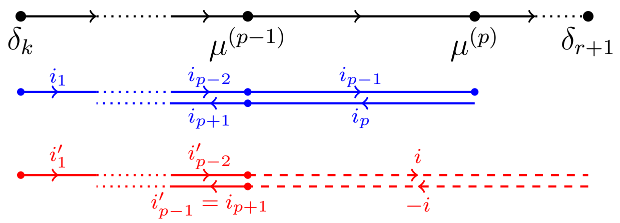



**FIGURE 3.4** – $\underline{i} = \mathrm{ht}(\underline{\mu})$ and $\underline{i}' := \mathrm{ht}(\underline{\mu}^{\#p})$ for the case $i_{p-1} + i_p = 0$

By Lemma 3.13 we get,

$$\begin{aligned}
\tilde{T}_{\underline{d},\underline{\mathbf{I}}}(k) &= \sum_{\substack{(\underline{\mu}',\underline{\alpha}') \in \mathscr{P}_{m-1}(\delta_k)\\ \underline{\mu} \in [\![\delta_{r+1},\delta_k]\!]\\ \mathrm{ht}(\underline{\mu}') \in \underline{\mathbf{I}}^{\#}}} \sum_{\substack{1 \leqslant i \leqslant r+1-k\\ -i'_1 - \cdots - i'_{p-2}}} (i'_1)^{d_1} \cdots (i'_{p-2})^{d_{p-2}} (i)^{d_{p-1}} (-i)^{d_p} (i'_p)^{d_{p+1}} \cdots (i'_{m-3})^{d_{m-1}}\\
&= \sum_{\substack{(\underline{\mu}',\underline{\alpha}') \in \mathscr{P}_{m-1}(\delta_k)\\ \underline{\mu} \in [\![\delta_{r+1},\delta_k]\!]\\ \mathrm{ht}(\underline{\mu}') \in \underline{\mathbf{I}}^{\#}}} (-1)^{d_p} (i'_1)^{d_1} \cdots (i'_{p-2})^{d_{p-2}} (i'_p)^{d_{p+1}} \cdots (i'_{m-3})^{d_{m-1}} \sum_{\substack{1 \leqslant i \leqslant r+1-k\\ -i'_1 - \cdots - i'_{p-2}}} (i)^{d_{p-1}+d_p}
\end{aligned}$$

$$= \sum_{\substack{(\underline{\mu}',\underline{\alpha}')\in\hat{\mathscr{P}}_{m-1}(\delta_k)\\ \underline{\mu}\in[\![\delta_{r+1},\delta_k]\!]\\ \mathrm{ht}(\underline{\mu}')\in\underline{\mathbf{I}}^{\#}}} (-1)^{d_p}(i'_1)^{d_1}\cdots(i'_{p-2})^{d_{p-2}}\cdots(i'_{m-3})^{d_{m-1}} S_{d_{p-1}+d_p}(r+1-k-i'_1-\cdots-i'_{p-2})$$

$$= \sum_{\substack{(\underline{\mu}',\underline{\alpha}')\in\hat{\mathscr{P}}_{m-1}(\delta_k)\\ \underline{\mu}\in[\![\delta_{r+1},\delta_k]\!]\\ \mathrm{ht}(\underline{\mu}')\in\underline{\mathbf{I}}^{\#}}} (-1)^{d_p}(i'_1)^{d_1}\cdots(i'_{p-2})^{d_{p-2}}\cdots(i'_{m-3})^{d_{m-1}}$$

$$\times \frac{1}{d_{p-1}+d_p+1}\sum_{j=0}^{d_{p-1}+d_p}\binom{d_{p-1}+d_p+1}{j}\tilde{B}_j(r+1-k-i'_1-\cdots-i'_{p-2})^{d_{p-1}+d_p+1-j}$$

$$= \frac{1}{d_{p-1}+d_p+1}\sum_{j=0}^{d_{p-1}+d_p}\binom{d_{p-1}+d_p+1}{j}\tilde{B}_j \sum_{\substack{\underline{q}\in\mathbb{N}^{p-1}\\ |\underline{q}|=d_{p-1}+d_p+1-j}} \frac{(d_{p-1}+d_p+1-j)!}{q_1!\dots q_{p-1}!}$$

$$\times(-1)^{d_{p-1}+2d_p+1-j-q_{p-1}}(r+1-k)^{q_{p-1}} \sum_{\substack{(\underline{\mu}',\underline{\alpha}')\in\hat{\mathscr{P}}_{m-1}(\delta_k)\\ \underline{\mu}\in[\![\delta_{r+1},\delta_k]\!]\\ \mathrm{ht}(\underline{\mu}')\in\underline{\mathbf{I}}^{\#}}} (i'_1)^{d_1+q_1}\cdots(i'_{p-2})^{d_{p-2}+q_{p-2}}\cdots(i'_{m-3})^{d_{m-1}}.$$

Set $\underline{d}' = (d_1+q_1,\dots,d_{p-2}+q_{p-2},d_{p+1},\dots,d_{m-1})$, with $|\underline{d}'| = d-d_{p-1}-d_p+d_{p-1}+d_p+1-j-q_{p-1}\leqslant d+1-q_{p-1}$, and

$$\tilde{T}_{\underline{d}',\underline{\mathbf{I}}^{\#}}(k) = \sum_{\substack{(\underline{\mu}',\underline{\alpha}')\in\hat{\mathscr{P}}_{m-1}(\delta_k)\\ \underline{\mu}\in[\![\delta_{r+1},\delta_k]\!]\\ \mathrm{ht}(\underline{\mu}')\in\underline{\mathbf{I}}^{\#}}} (i'_1)^{d_1+q_1}\cdots(i'_{p-2})^{d_{p-2}+q_{p-2}}\cdots(i'_{m-3})^{d_{m-1}}.$$

Then

$$\tilde{T}_{\underline{d},\underline{\mathbf{I}}}(k) = \frac{1}{d_{p-1}+d_p+1}\sum_{j=0}^{d_{p-1}+d_p}\binom{d_{p-1}+d_p+1}{j}\tilde{B}_j \sum_{\substack{\underline{q}\in\mathbb{N}^{p-1}\\ |\underline{q}|=d_{p-1}+d_p+1-j}} \frac{(d_{p-1}+d_p+1-j)!}{q_1!\dots q_{p-1}!}$$
$$(-1)^{d_{p-1}+2d_p+1-j-q_{p-1}}(r+1-k)^{q_{p-1}}\tilde{T}_{\underline{d}',\underline{\mathbf{I}}^{\#}}(k). \tag{3.4}$$

By the induction hypothesis applied to $m-2$ and $\underline{\mathbf{I}}^{\#}$, there exists a polynomial $T_{\underline{d}',\underline{\mathbf{I}}^{\#}}$ of degree $\leqslant d+1-q_{p-1}+(m-2)-\deg A_{\underline{\mathbf{I}}^{\#}} = d-q_{p-1}+m-\deg A_{\underline{\mathbf{I}}}$, since $\deg A_{\underline{\mathbf{I}}^{\#}} = \deg A_{\underline{\mathbf{I}}}-1$ if $i_{p-1}+i_p=0$ by Lemma 3.11, such that $T_{\underline{d}',\underline{\mathbf{I}}^{\#}}(k) = \tilde{T}_{\underline{d}',\underline{\mathbf{I}}^{\#}}(k)$ for all $k$ such that $\{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\delta_k),\ |\ \underline{\mu}\in[\![\delta_{r+1},\delta_k]\!],\mathrm{ht}(\underline{\mu})\in\underline{\mathbf{I}}\}$ is nonempty.

Set

$$T_{\underline{d},\underline{\mathbf{I}}}(X) = \frac{1}{d_{p-1}+d_p+1}\sum_{j=0}^{d_{p-1}+d_p}\binom{d_{p-1}+d_p+1}{j}\tilde{B}_j \sum_{\substack{\underline{q}\in\mathbb{N}^{p-1}\\ |\underline{q}|=d_{p-1}+d_p+1-j}} \frac{(d_{p-1}+d_p+1-j)!}{q_1!\dots q_{p-1}!}$$
$$\times(-1)^{d_{p-1}+2d_p+1-j-q_{p-1}}(r+1-k)^{q_{p-1}}T_{\underline{d}',\underline{\mathbf{I}}^{\#}}(X).$$

Then by the induction hypothesis and (3.4) we have $T_{\underline{d},\mathbf{I}}(k) = \tilde{T}_{\underline{d},\mathbf{I}}(k)$ and $T_{\underline{d},\mathbf{I}}$ is a polynomial of degree $\leqslant d + m - \deg A_{\mathbf{I}}$ for all $k$ such that $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k), \mid \underline{\mu} \in [\![\delta_{r+1}, \delta_k]\!], \mathrm{ht}(\underline{\mu}) \in \mathbf{I}\}$ is nonempty.

It remains to verify that $T_{\underline{d},\mathbf{I}}(k) = 0$ when the Set (3.3) is empty. In this case, $\tilde{T}_{\underline{d},\mathbf{I}}(k) = 0$ by definition. The Set (3.3) is empty if $\{(\underline{\mu}', \underline{\alpha}') \in \hat{\mathscr{P}}_{m-1}(\delta_k) \mid \underline{\mu}' \in [\![\delta_{r+1}, \delta_k]\!]$ and $\mathrm{ht}(\underline{\mu}') \in \mathbf{I}^{\#}\} = \varnothing$. But our induction hypothesis says that, in this case,

$$T_{\underline{d},\mathbf{I}}(k) = T_{\underline{d}',\mathbf{I}^{\#}}(k) = 0$$

for any $\underline{d}' \in \mathbb{Z}_{\geqslant 0}^{p-1}$. Otherwise, this means that for any $\underline{i}' \in \mathbf{I}^{\#}$ and for any $(\underline{\mu}', \underline{\alpha}') \in \hat{\mathscr{P}}_{m-2}(\delta_k)$ such that $\mathrm{ht}(\underline{\mu}') = \underline{i}'$, we have $i'_1 + \cdots + i'_{p-2} = r + 1 - k$. In that event, $T_{\underline{d},\mathbf{I}}(k) = 0$ by the construction. □

### 3.2.4 Elements of $\mathscr{E}_m$ with zeroes

We consider in this subsection the elements of $\mathscr{E}_m$ with zeroes.

Let $(m, n) \in (\mathbb{Z}_{>0})^2$, with $n \in \{0, \dots, m\}$, and $\mathbf{I} \in \mathscr{E}_m$ with $n$ zeroes in positions $j_1 < \cdots < j_n$. This means that $\mu^{(j_l)} = \mu^{(j_l+1)}$ for $l = 1, \dots, n$. Let $\underline{i} \in \mathbf{I}$, and let $\tilde{\underline{i}}$ be the sequence of $\mathbb{Z}^{m-n}$ obtained from $\underline{i}$ by removing all zeroes. Denote by $\tilde{\mathbf{I}}$ the equivalence class of $\tilde{\underline{i}}$ in $\mathbb{Z}^{m-n}$. This class only depends on $\mathbf{I}$ and has no zero.

Let $(\tilde{\underline{\mu}}, \tilde{\underline{\alpha}}) \in \hat{\mathscr{P}}_{m-n}$ with $\mathrm{ht}(\tilde{\underline{\mu}}) \in \tilde{\mathbf{I}}$. Thus $(\tilde{\underline{\mu}}, \tilde{\underline{\alpha}})$ has no loop. Define weighted paths whose height is in $\mathbf{I}$ from $(\tilde{\underline{\mu}}, \tilde{\underline{\alpha}})$ as follows. Set $\underline{j} := (j_1, \dots, j_n)$ and let $\underline{\beta} := (\beta^{(1)}, \dots, \beta^{(n)})$ be in

$$\Pi_{\underline{\mu}^{(\underline{j})}} := \Pi_{\mu^{(j_1)}} \times \cdots \times \Pi_{\mu^{(j_n)}}.$$

Define $(\tilde{\underline{\mu}}, \tilde{\underline{\alpha}})_{\underline{j};\underline{\beta}}$ to be the weighted path of length $m$ obtained from $(\tilde{\underline{\mu}}, \tilde{\underline{\alpha}})$ by "gluing the loop" labelled by $\beta^{(l)}$ at the vertex $\mu^{(j_l)}$ for $l = 1, \dots, n$.

Thus for such a $\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j})}}$ the height of the weighted path $(\tilde{\underline{\mu}}, \tilde{\underline{\alpha}})_{\underline{j};\underline{\beta}}$ is in $\mathbf{I}$. Moreover, all $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m$ with $\mathrm{ht}(\underline{\mu}) \in \mathbf{I}$ are of this form.

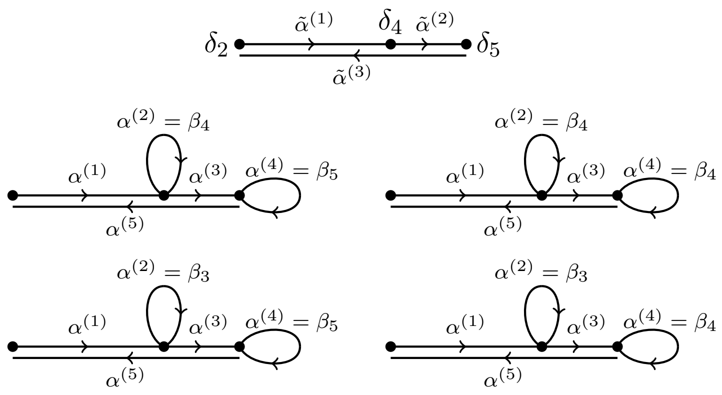
*Example* 3.15. Remember from Section 2.2 that

$$\Pi_{\delta_k} = \{\beta_{k-1}, \beta_k\},\ k = 2, \dots, r, \quad \Pi_{\delta_1} = \{\beta_1\}, \quad \Pi_{\delta_{r+1}} = \{\beta_r\}.$$

Assume that $r > 4$. Let $\mathbf{I} \in \mathscr{E}_5$ be the class $[1, 0, 1, 0, -2]$. Then $\mathbf{I}$ has 2 zeros in positions 2 and 4 and $\tilde{\mathbf{I}} = [1, 1, -2]$. We represent below the weighted path

$$(\tilde{\underline{\mu}}, \tilde{\underline{\alpha}}) = ((\delta_2, \delta_4, \delta_5, \delta_2), (\beta_2 + \beta_3, \beta_4, -\beta_2 - \beta_3 - \beta_4)) \in \hat{\mathscr{P}}_3(\delta_2)$$

whose height $(2,1,-3)$ is in $\tilde{\underline{\mathbf{I}}}$, and the four weighted paths $(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{(2,4);(\beta_4,\beta_5)}$, $(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{(2,4);(\beta_4,\beta_4)}$, $(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{(2,4);(\beta_3,\beta_5)}$ and $(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{(2,4);(\beta_3,\beta_4)}$ whose height is in $\underline{\mathbf{I}}$ obtained from it:




**Lemma 3.16.** *Let $m \in \mathbb{Z}_{>0}$ and $\underline{\mathbf{I}} \in \mathscr{E}_m$ with $n$ zeroes in positions $j_1,\ldots,j_n$ ($n \leqslant m$). There is a polynomial $B_{\underline{\mathbf{I}}}$ of degree $n$ such that for all $(\underline{\tilde{\mu}},\underline{\tilde{\alpha}}) \in \hat{\mathscr{P}}_{m-n}(\delta_k)$, $k \in \{1,\ldots,r+1\}$, such that $\mathrm{ht}(\underline{\tilde{\mu}}) \in \tilde{\underline{\mathbf{I}}}$,*

$$\sum_{\underline{\beta}\in\Pi_{\underline{\mu}(\underline{j})}} \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = B_{\underline{\mathbf{I}}}(k)A_{\tilde{\underline{\mathbf{I}}}}(\tilde{\imath}_1,\ldots,\tilde{\imath}_{m-n-1}),$$

*where $\mathrm{ht}(\underline{\tilde{\mu}}) = (\tilde{\imath}_1,\ldots,\tilde{\imath}_{m-n-1})$. Moreover, we have*

$$B_{\underline{\mathbf{I}}}(X) = \sum_{j=0}^{n} C_{\underline{\mathbf{I}}}^{(n-j)}(\tilde{\imath}_1,\ldots,\tilde{\imath}_{m-n-1})X^j$$

*where $C_{\underline{\mathbf{I}}}^{(0)} = (-1)^n$ and $C_{\underline{\mathbf{I}}}^{(j)} \in \mathbb{C}[X_1,\ldots,X_{m-n-1}]$ has total degree $< j$ for $j = 1,\ldots,n$. In particular, if $n=0$, we have $B_{\underline{\mathbf{I}}}(X) = 1$.*

*Proof.* First of all, observe that if $n=m$ then the result is known by Lemma 2.2. At the extreme opposite, if $n=0$, then the result is known by Lemma 3.11.

We prove the lemma by induction on $m$. By the above observation, the lemma is true for $m=1$ and $m=2$. Let $m \geqslant 3$ and assume the lemma true for any $m' \in \{1,\ldots,m-1\}$.

Set $p := p(\underline{\mathbf{I}})$ and $q := q(\underline{\mathbf{I}})$.

$*$ First case: $i_p = 0$. Then $p = j_l$ for some $l \in \{1,\ldots,n\}$. Assume that $\mu^{(1)} = \delta_k$ and $\mu^{(p)} = \delta_s$, then $s = k + \sum_{t=1}^{p-1} i_t$.
Set $\underline{j} := (j_1,\ldots,j_n)$ and let $\underline{\beta} := (\beta^{(1)},\ldots,\beta^{(n)})$ be in $\Pi_{\underline{\mu}(\underline{j})}$. Assume first that

$s \in \{2, \ldots, r\}$, we have

$$\begin{aligned}\Pi_{\underline{\mu}^{(\underline{j})}} &= \Pi_{\mu^{(j_1)}} \times \cdots \times \Pi_{\mu^{(j_{l-1})}} \times \Pi_{\mu^{(p)}} \times \cdots \times \Pi_{\mu^{(j_n)}} \\ &= \Pi_{\mu^{(j_1)}} \times \cdots \times \Pi_{\mu^{(j_{l-1})}} \times \Pi_{\delta_s} \times \cdots \times \Pi_{\mu^{(j_n)}} \\ &= \Pi_{\mu^{(j_1)}} \times \cdots \times \Pi_{\mu^{(j_{l-1})}} \times \{\beta_{s-1}, \beta_s\} \times \cdots \times \Pi_{\mu^{(j_n)}}.\end{aligned}$$

Choose $\underline{\beta}_1 = (\beta^{(1)}, \ldots, \beta_{s-1}, \ldots, \beta^{(n)})$ and $\underline{\beta}_2 = (\beta^{(1)}, \ldots, \beta_s, \ldots, \beta^{(n)})$ for some $\beta^{(i)} \in \Pi_{\mu^{(j_i)}}$. Then by Lemma 3.3 (3) and Lemma 2.1 (2), we have

$$\begin{aligned}\mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}_1}\right) &= (-\langle \rho, \breve{\varpi}_{\alpha^{(p)}} \rangle + 1)\mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right) \\ &= (-\langle \rho, \breve{\varpi}_{s-1} \rangle + 1)\mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right), \\ \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}_2}\right) &= \langle \rho, \breve{\varpi}_{\alpha^{(p)}} \rangle\mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right) \\ &= \langle \rho, \breve{\varpi}_{s} \rangle\mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right).\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}_1}\right) + \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}_2}\right) \\ &\quad = ((-\langle \rho, \breve{\varpi}_{\alpha^{s-1}} \rangle + 1) + \langle \rho, \breve{\varpi}_{\alpha^{s}} \rangle)\,\mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right) \\ &\quad = \left(-\left(\frac{s-1}{2}(r-(s-1)+1)\right) + 1 + \frac{s}{2}(r-s+1)\right)\mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right) \\ &\quad = \left(\frac{r}{2} - s + 2\right)\mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right) \\ &\quad = \left(\frac{r}{2} - k - \sum_{t=1}^{p-1} i_t + 2\right)\mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right).\end{aligned}$$

If $s = r+1$, then

$$\Pi_{\underline{\mu}^{(\underline{j})}} = \Pi_{\mu^{(j_1)}} \times \cdots \times \{\beta_{s-1}\} \times \cdots \times \Pi_{\mu^{(j_n)}}.$$

For some $\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j})}}$ and Then by Lemma 3.3 (3) and Lemma 2.1 (2), we have

$$\begin{aligned}\mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) &= (-\langle \rho, \breve{\varpi}_{s-1} \rangle + 1)\mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right) \\ &= \left(-\frac{r}{2} + 1\right)\mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right) \\ &= \left(\frac{r}{2} - r + 1\right)\mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right)\end{aligned}$$

$$= \left(\frac{r}{2} - k - \sum_{t=1}^{p-1} i_t + 2\right) \mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right).$$

Hence

$$\sum_{\underline{\beta}\in\Pi_{\underline{\mu}}(\underline{j})} \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = \sum_{\underline{\beta}\in\Pi_{\underline{\mu}}(\underline{j}')} \left(\frac{r}{2} - k - \sum_{t=1}^{p-1} i_t + 2\right) \mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right),$$

where $\underline{j}' := (j_1, \dots, j_{l-1}, j_{l+1}, \dots, j_n)$. So, the induction hypothesis applied to $\underline{\mathbf{I}}^{\#}$ gives,

$$\sum_{\underline{\beta}\in\Pi_{\underline{\mu}}(\underline{j})} \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = \left(\frac{r}{2} - k - \sum_{t=1}^{p-1} i_t + 2\right) B_{\underline{\mathbf{I}}^{\#}}(k) A_{\widetilde{\underline{\mathbf{I}}^{\#}}}(\tilde{\imath}_1, \dots, \tilde{\imath}_{m-n-1}).$$

Note that in this case $\widetilde{\underline{\mathbf{I}}^{\#}} = \tilde{\underline{\mathbf{I}}}$ and

$$\sum_{t=1}^{p-1} i_t = \sum_{t=1}^{\tilde{p}-1} \tilde{\imath}_t.$$

Thus we get,

$$\sum_{\underline{\beta}\in\Pi_{\underline{\mu}}(\underline{j})} \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = \left(\frac{r}{2} - k - \sum_{t=1}^{\tilde{p}-1} \tilde{\imath}_t + 2\right) B_{\underline{\mathbf{I}}^{\#}}(k) A_{\tilde{\underline{\mathbf{I}}}}(\tilde{\imath}_1, \dots, \tilde{\imath}_{m-n-1}).$$

Set

$$B_{\underline{\mathbf{I}}}(X) := \left(\frac{r}{2} - X - \sum_{t=1}^{\tilde{p}-1} \tilde{\imath}_t + 2\right) B_{\underline{\mathbf{I}}^{\#}}(X).$$

By our induction hypothesis applied to $\underline{\mathbf{I}}^{\#}$, $B_{\underline{\mathbf{I}}}$ has degree $n$, its leading term is $(-1)^n X^n$ since $\underline{\mathbf{I}}^{\#}$ has $n-1$ zeros, and the coefficient of $B_{\underline{\mathbf{I}}}(X)$ in $X^j$, $j \leqslant n$, is a polynomial in the variable $\tilde{\imath}_1, \dots, \tilde{\imath}_{m-n-1}$ of total degree $\leqslant n - j$. This proves the statement in this case.

∗ Second case: $p = q$ and and $i_{p-1} + i_p \neq 0$. Then necessarily, $n < m$. Then by Lemma 3.3(1), we get

$$\sum_{\underline{\beta}\in\Pi_{\underline{\mu}}(\underline{j})} \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = \sum_{\underline{\beta}\in\Pi_{\underline{\mu}}(\underline{j})} \mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right).$$

Let $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k)$ such that $\mathrm{ht}(\underline{\mu}) \in \underline{\mathbf{I}}$. Observe that the class of $\mathrm{ht}(\widetilde{\underline{\mu}^{\#p}})$ does not depend on such a $(\underline{\mu}, \underline{\alpha})$. Moreover, for any $\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j})}}$,

$$((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p} = (\widetilde{\underline{\mu}^{\#p}}, \widetilde{\underline{\alpha}^{\#p}})_{\underline{j}';\underline{\beta}}$$

where $\underline{j}' = (j'_1, \ldots, j'_n)$ is the sequence of positions of zeroes of $\mathrm{ht}(\underline{\mu}^{\#p})$. Therefore by our induction hypothesis and Lemma 3.11, we get

$$\begin{aligned}\sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right) &= \sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j}')}}} \mathrm{wt}\left((\widetilde{\underline{\mu}^{\#p}}, \widetilde{\underline{\alpha}^{\#p}})_{\underline{j}';\underline{\beta}}\right) \\ &= B_{\underline{\mathbf{I}}^{\#}}(k) A_{\widetilde{\underline{\mathbf{I}}^{\#}}}(\tilde{\imath}_1, \ldots, \tilde{\imath}_{p(\underline{\mathbf{I}}^{\#})-1} + \tilde{\imath}_{p(\underline{\mathbf{I}}^{\#})}, \tilde{\imath}_{m-1-n}) \\ &= B_{\underline{\mathbf{I}}^{\#}}(k) A_{\tilde{\underline{\mathbf{I}}}}(\tilde{\imath}_1, \ldots, \tilde{\imath}_{m-n-1}).\end{aligned}$$

Since $\underline{\mathbf{I}}^{\#}$ and $\underline{\mathbf{I}}$ have the same number $n$ of zeroes, by setting

$$B_{\underline{\mathbf{I}}} := B_{\underline{\mathbf{I}}^{\#}},$$

we get the statement by induction hypothesis.

$*$ Third case: $p = q$ and $i_{p-1} + i_p = 0$. First assume that $i_1 = \cdots = i_{p-2} = 0$ or that $p = 2$. Then by Lemma 3.3(2)(a), we get

$$\sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = \sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j})}}} i_{p-1} \mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right).$$

where $\underline{j}'$ is the sequence of positions of the zeroes of $\mathrm{ht}(\underline{\mu}^{\#p})$. So, by induction hypothesis Lemma 3.11, we obtain, arguing as in the second case, that

$$\begin{aligned}\sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left(((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right) &= \sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j}')}}} \mathrm{wt}\left(\widetilde{\underline{\mu}^{\#p}}, \widetilde{\underline{\alpha}^{\#p}})_{\underline{j}';\underline{\beta}}\right) \\ &= i_{p-1} B_{\underline{\mathbf{I}}^{\#}}(k) A_{\widetilde{\underline{\mathbf{I}}^{\#}}}(\tilde{\imath}_1, \ldots, \tilde{\imath}_{p(\underline{\mathbf{I}}^{\#})-2}, \tilde{\imath}_{p(\underline{\mathbf{I}}^{\#})+1}, \tilde{\imath}_{m-1-n}) \\ &= B_{\underline{\mathbf{I}}^{\#}}(k) A_{\tilde{\underline{\mathbf{I}}}}(\tilde{\imath}_1, \ldots, \tilde{\imath}_{m-n-1}).\end{aligned}$$

Since $\underline{\mathbf{I}}^{\#}$ and $\underline{\mathbf{I}}$ have the same number $n$ of zeroes, by setting

$$B_{\underline{\mathbf{I}}} := B_{\underline{\mathbf{I}}^{\#}},$$

we get the statement by induction hypothesis.

If we are not in one of the above situations, then by Lemma 3.3 (2)(b) and Lemma 3.11 and the induction hypothesis we conclude similarly. Namely, here we get that

$$\begin{aligned}\sum_{\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left(((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}})^{\#p}\right) &= \sum_{\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j}')}}} \mathrm{wt}\left((\underline{\tilde{\mu}}^{\#p},\underline{\tilde{\alpha}}^{\#p})_{\underline{j}';\underline{\beta}}\right)\\ &= (i_{p-1}+1)B_{\underline{\mathbf{I}}^{\#}}(k)A_{\underline{\tilde{\mathbf{I}}}^{\#}}(\tilde{\imath}_1,\dots,\tilde{\imath}_{p(\underline{\mathbf{I}}^{\#})-2},\tilde{\imath}_{p(\underline{\mathbf{I}}^{\#})+1},\tilde{\imath}_{m-1-n})\\ &= B_{\underline{\mathbf{I}}^{\#}}(k)A_{\underline{\tilde{\mathbf{I}}}}(\tilde{\imath}_1,\dots,\tilde{\imath}_{m-n-1}),\end{aligned}$$

and we set $B_{\underline{\mathbf{I}}} := B_{\underline{\mathbf{I}}^{\#}}$. Since $\underline{\mathbf{I}}$ and $\underline{\mathbf{I}}^{\#}$ have the same number $n$ of zeroes, we get the statement by induction hypothesis. □

*Example* 3.17. Let $\underline{\mathbf{I}} \in \mathscr{E}_5$ be the class $[1,0,1,0,-2]$ that has 2 zeroes in positions 2 and 4 as in Example 3.15. Then $\underline{\tilde{\mathbf{I}}} = [1,1,-2]$. Let $(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})$ be the weighted path,

$$(\underline{\tilde{\mu}},\underline{\tilde{\alpha}}) = ((\delta_k,\delta_i,\delta_j,\delta_k),(\delta_k-\delta_i,\delta_i-\delta_j,\delta_j-\delta_k)) \in \hat{\mathscr{P}}_3(\delta_k),$$

whose height is in $\underline{\tilde{\mathbf{I}}}$. Set $\underline{j} = \{2,4\}$ and let $\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j})}}$ with $\Pi_{\underline{\mu}^{(\underline{j})}} = \{\beta_{i-1},\beta_i\}\times\{\beta_{j-1},\beta_j\}$. We have

$$\begin{aligned}\sum_{\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) &= \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{(2,4);(\beta_{i-1},\beta_{j-1})}\right) + \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{(2,4);(\beta_{i-1},\beta_{j})}\right)\\ &\quad + \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{(2,4);(\beta_{i},\beta_{j-1})}\right) + \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{(2,4);(\beta_{i-1},\beta_{j})}\right)\end{aligned}$$

With the same arguments as Lemma 3.3 (3), we have

$$\begin{aligned}\mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{(2,4);\{\beta_{i-1},\beta_{i-1}\}}\right) &= (-\langle\rho,\breve{\varpi}_{i-1}\rangle+1)(-\langle\rho,\breve{\varpi}_{j-1}\rangle+1)\mathrm{wt}(\underline{\tilde{\mu}},\underline{\tilde{\alpha}});\\ \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{(2,4);\{\beta_{i-1},\beta_{j}\}}\right) &= (-\langle\rho,\breve{\varpi}_{i-1}\rangle+1)(\langle\rho,\breve{\varpi}_{j}\rangle)\mathrm{wt}(\underline{\tilde{\mu}},\underline{\tilde{\alpha}});\\ \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{(2,4);\{\beta_{i},\beta_{j-1}\}}\right) &= (\langle\rho,\breve{\varpi}_{i}\rangle)(-\langle\rho,\breve{\varpi}_{j-1}\rangle+1)\mathrm{wt}(\underline{\tilde{\mu}},\underline{\tilde{\alpha}});\\ \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{(2,4);\{\beta_{i},\beta_{j}\}}\right) &= (\langle\rho,\breve{\varpi}_{i}\rangle)(\langle\rho,\breve{\varpi}_{j}\rangle)\mathrm{wt}(\underline{\tilde{\mu}},\underline{\tilde{\alpha}}).\end{aligned}$$

Hence by Lemma 2.1 and Lemma 3.11, we get

$$\sum_{\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = (\langle\rho,\breve{\varpi}_i-\breve{\varpi}_{i-1}\rangle+1)\,(\langle\rho,\breve{\varpi}_j-\breve{\varpi}_{j-1}\rangle+1)\,\mathrm{wt}(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})$$

$$
\begin{aligned}
&= \Big( (\frac{r}{2} - i + 2)(\frac{r}{2} - j + 2) \Big) \operatorname{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) \\
&= (r - k - \tilde{\imath}_1 + 2)\,(r - k - \tilde{\imath}_1 - \tilde{\imath}_2 + 2) \operatorname{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) \\
&= \Big( \Big( (\frac{r}{2} - \tilde{\imath}_1 + 2)(\frac{r}{2} - \tilde{\imath}_1 - \tilde{\imath}_2 + 2) \Big) k^0 + (-(r+4) + 2\tilde{\imath}_1 + \tilde{\imath}_2)\, k^1 + k^2 \Big)\; A_{\underline{\tilde{\mathbf{I}}}}(\tilde{\imath}_1, \tilde{\imath}_2) \\
&= \sum_{j=0}^{2} C_{\underline{\boldsymbol{\mu}}}^{2-j}(\tilde{\imath}_1, \tilde{\imath}_2) k^j A_{\underline{\tilde{\mathbf{I}}}}(\tilde{\imath}_1, \tilde{\imath}_2)
\end{aligned}
$$

By setting

$$B_{\underline{\boldsymbol{\mu}}}(X) := \sum_{j=0}^{2} C_{\underline{\boldsymbol{\mu}}}^{2-j}(\tilde{\imath}_1, \tilde{\imath}_2) X^j,$$

we get that $B_{\underline{\boldsymbol{\mu}}}$ is a polynomial of degree 2 with leading term $X^2$ and the coefficient of $B_{\underline{\boldsymbol{\mu}}}(X)$ in $X^j$, $j \leqslant 2$, is a polynomial in the variable $\tilde{\imath}_1, \tilde{\imath}_2$ of total degree $\leqslant 2 - j$. Thus,

$$\sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{(j)}}} \operatorname{wt}\Big( (\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}} \Big) = B_{\underline{\boldsymbol{\mu}}}(k) A_{\underline{\tilde{\mathbf{I}}}}(\tilde{\imath}_1, \tilde{\imath}_2).$$

**Corollary 3.18.** *Let $m \in \mathbb{Z}_{>0}$ and $n \in \{0, \dots, m\}$. Let $\underline{\mathbf{I}} \in \mathscr{E}_m$ with $n \leqslant m$ zeroes in positions $j_1, \dots, j_n$, and $\underline{\tilde{\mathbf{I}}}$ as in Lemma 3.16. Then for some polynomial $T_{\underline{\mathbf{I}}} \in \mathbb{C}[X]$ of degree at most $m$, for all $k \in \{1, \dots, r+1\}$,*

$$\sum_{\underline{\tilde{i}} \in \underline{\tilde{\mathbf{I}}}} \sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{\underline{j}}}} \sum_{\substack{(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) \in \hat{\mathscr{P}}_{m-n}(\delta_k), \\ \underline{\tilde{\mu}} \in [\![\delta_{r+1}, \delta_k]\!],\, \operatorname{ht}(\underline{\tilde{\mu}}) = \underline{\tilde{i}}}} \operatorname{wt}\Big( (\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}} \Big) = T_{\underline{\mathbf{I}}}(k).$$

If $n = 0$ or if $\underline{\mathbf{I}}$ has no zero, then $T_{\underline{\mathbf{I}}}$ is the polynomial provided by Lemma 3.14. If $n = m$, then $\underline{\mathbf{I}} = [\underline{0}]$ and $T_{\underline{\mathbf{I}}} = T_m$ is the polynomial provided by Lemma 2.2. So, in these two cases, the statement is known. Also, our notations is compatible with the notation of what follows Lemma 3.14.

*Proof.* Let $k \in \{1, \dots, r+1\}$. By Lemma 3.14 and Lemma 3.16, we have,

$$
\begin{aligned}
\sum_{\underline{\tilde{i}} \in \underline{\tilde{\mathbf{I}}}} \sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{\underline{j}}}} \sum_{\substack{(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) \in \hat{\mathscr{P}}_{m-n}(\delta_k), \\ \underline{\tilde{\mu}} \in [\![\delta_{r+1}, \delta_k]\!],\, \operatorname{ht}(\underline{\tilde{\mu}}) = \underline{\tilde{i}}}} \operatorname{wt}\Big( (\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}} \Big) &= \sum_{\substack{(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) \in \hat{\mathscr{P}}_{m-n}(\delta_k), \\ \underline{\tilde{\mu}} \in [\![\delta_{r+1}, \delta_k]\!],\, \operatorname{ht}(\underline{\tilde{\mu}}) \in \underline{\tilde{\mathbf{I}}}}} B_{\underline{\mathbf{I}}}(k) A_{\underline{\tilde{\mathbf{I}}}}(\tilde{\imath}_1, \dots, \tilde{\imath}_{m-n-1}) \\
&= \sum_{\substack{(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) \in \hat{\mathscr{P}}_{m-n}(\delta_k), \\ \underline{\tilde{\mu}} \in [\![\delta_{r+1}, \delta_k]\!],\, \operatorname{ht}(\underline{\tilde{\mu}}) \in \underline{\tilde{\mathbf{I}}}}} \sum_{j=0}^{n} C_{\underline{\mathbf{I}}}^{(n-j)}(\tilde{\imath}_1, \dots, \tilde{\imath}_{m-n-1}) k^j A_{\underline{\tilde{\mathbf{I}}}}(\tilde{\imath}_1, \dots, \tilde{\imath}_{m-n-1})
\end{aligned}
$$

$$= \sum_{\substack{(\tilde{\underline{\mu}},\tilde{\underline{\alpha}})\in\hat{\mathscr{P}}_{m-n}(\delta_k),\\ \tilde{\underline{\mu}}\in[\![\delta_{r+1},\delta_k]\!],\,\mathrm{ht}(\tilde{\underline{\mu}})\in\tilde{\mathbf{I}}}} \sum_{j=0}^{n} \sum_{\substack{\underline{d}_j=(d_1,\dots,d_{m-n-1}),\\ d_1+\dots+d_{m-n-1}\le n-j}} C_{\underline{d},j}\tilde{\imath}_1^{d_1}\cdots\tilde{\imath}_{m-n-1}^{d_{m-n-1}}k^jA_{\tilde{\mathbf{I}}}(\tilde{\imath}_1,\dots,\tilde{\imath}_{m-n-1}).$$

Set

$$\tilde{T}_{\underline{d}_j,\tilde{\mathbf{I}}} = \sum_{\substack{(\tilde{\underline{\mu}},\tilde{\underline{\alpha}})\in\hat{\mathscr{P}}_{m-n}(\delta_k),\\ \tilde{\underline{\mu}}\in[\![\delta_{r+1},\delta_k]\!],\,\mathrm{ht}(\tilde{\underline{\mu}})\in\tilde{\mathbf{I}}}} \tilde{\imath}_1^{d_1}\cdots\tilde{\imath}_{m-n-1}^{d_{m-n-1}}A_{\tilde{\mathbf{I}}}(\tilde{\imath}_1,\dots,\tilde{\imath}_{m-n-1}).$$

Then by Lemma 3.14, there are some polynomials $T_{\underline{d}_j,\tilde{\mathbf{I}}}$ of degree at most $(n-j)+(m-n)=m-j$, such that

$$\sum_{\tilde{\underline{i}}\in\tilde{\mathbf{I}}}\sum_{\underline{\beta}\in\Pi_{\underline{\mu}^j}} \sum_{\substack{(\tilde{\underline{\mu}},\tilde{\underline{\alpha}})\in\hat{\mathscr{P}}_{m-n}(\delta_k),\\ \tilde{\underline{\mu}}\in[\![\delta_{r+1},\delta_k]\!],\,\mathrm{ht}(\tilde{\underline{\mu}})=\tilde{\underline{i}}}} \mathrm{wt}\left((\tilde{\underline{\mu}},\tilde{\underline{\alpha}})_{j;\underline{\beta}}\right) = \sum_{j=0}^{n} \sum_{\substack{\underline{d}_j=(d_1,\dots,d_{m-n-1}),\\ d_1+\dots+d_{m-n-1}\le n-j}} C_{\underline{d},j}k^jT_{\underline{d}_j,\tilde{\mathbf{I}}}(k).$$

Moreover, if $j<n$, then $T_{\underline{d}_j,\tilde{\mathbf{I}}}$ has degree $<m-j$. By setting

$$T_{\mathbf{I}}(X) := \sum_{j=0}^{n} \sum_{\substack{\underline{d}_j=(d_1,\dots,d_{m-n-1}),\\ d_1+\dots+d_{m-n-1}\le n-j}} C_{\underline{d},j}X^jT_{\underline{d}_j,\tilde{\mathbf{I}}}(X),$$

we have that $T_{\mathbf{I}}$ is a polynomial of degree at most $m-j+j=m$. □

## 3.3 Proof of Theorem 7

The following result is a consequence of Corollary 3.18.

**Lemma 3.19.** *Let $m\in\mathbb{Z}_{>0}$, There is a polynomial $\hat{T}_m$ in $\mathbb{C}[X]$ of degree at most $m$ such that for all $k\in\{1,\dots,r+1\}$,*

$$\sum_{\substack{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\delta_k)\\ \underline{\mu}\in[\![\delta_{r+1},\delta_k]\!]}} \mathrm{wt}(\underline{\mu},\underline{\alpha}) = \hat{T}_m(k).$$

*Proof.* By Corollary 3.18, we have for all $k\in\{1,\dots,r+1\}$,

$$\sum_{\substack{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\delta_k)\\ \underline{\mu}\in[\![\delta_{r+1},\delta_k]\!]}} \mathrm{wt}(\underline{\mu},\underline{\alpha}) = \sum_{\mathbf{I}\in\mathscr{E}_m} \sum_{\substack{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\delta_k),\\ \underline{\mu}\in[\![\delta_{r+1},\delta_k]\!],\,\mathrm{ht}(\underline{i})\in\mathbf{I}}} \mathrm{wt}(\underline{\mu},\underline{\alpha}) = \sum_{\mathbf{I}\in\mathscr{E}_m} T_{\mathbf{I}}(k).$$

Set

$$\hat{T}_m := \sum_{\underline{\mathbf{I}} \in \mathscr{E}_m} T_{\underline{\mathbf{I}}} \in \mathbb{C}[X].$$

By Corollary 3.18, $T_{\underline{\mathbf{I}}}$ has degree at most $m$. Therefore $\hat{T}_m$ has degree at most $m$ and satisfies the condition of the lemma. □

We are now in a position to prove Theorem 7

*Proof of Theorem 7.* By Lemma 1.19, we have

$$\mathrm{ev}_\rho(\widehat{\mathrm{d}p}_{m,k}) = \sum_{\mu \in P(\delta)_k} \sum_{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)} \mathrm{wt}(\underline{\mu},\underline{\alpha}) \langle \mu, \check{\beta}_k \rangle.$$

Remember from Section 2.2 that for $k = 1, \ldots, r$,

$$P(\delta)_k = \{\delta_k, \delta_{k+1}\}.$$

Hence,

$$\mathrm{ev}_\rho(\widehat{\mathrm{d}p}_{m,k}) = \sum_{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k)} \mathrm{wt}(\underline{\mu},\underline{\alpha}) - \sum_{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_{k+1})} \mathrm{wt}(\underline{\mu},\underline{\alpha}),$$

where $\underline{\mu}$ is entirely contained in $[\![\delta_{r+1}, \delta_k]\!]$. Let $\hat{T}_m$ be as in Lemma 3.19 and set

$$\hat{Q}_m := \hat{T}_m(X) - \hat{T}_m(X+1). \tag{3.5}$$

Then $\hat{Q}_m$ is a polynomial of degree at most $m-1$, and we have

$$\mathrm{ev}_\rho(\widehat{\mathrm{d}p}_m) = \mathrm{ev}_\rho\left(\frac{1}{m!}\sum_{k=1}^{r} \widehat{\mathrm{d}p}_{m,k} \otimes \varpi_k^\sharp\right) = \frac{1}{m!}\sum_{k=1}^{r} \mathrm{ev}_\rho\left(\widehat{\mathrm{d}p}_{m,k}\right)\varpi_k^\sharp = \frac{1}{m!}\sum_{k=1}^{r} \hat{Q}_m \varpi_k^\sharp.$$

Moreover, $\hat{Q}_1 = 1$ □

# Chapter 4

## Proof of Theorem 7 for $\mathfrak{sp}_{2r}$

The purpose of this chapter is to prove Theorem 7 for $\mathfrak{sp}_{2r}, r \geqslant 2$. We follow the general strategy of the $\mathfrak{sl}_{r+1}$ case. However, since the simple Lie algebra $\mathfrak{sp}_{2r}$ is non simply-laced, it induces new phenomenon and the proof is much more technical and new tools are needed. Consequently, the results are highly non-trivial generalizations of what we proved in the previous chapter.

Throughout this chapter, it is assumed that $\mathfrak{g} = \mathfrak{sp}_{2r}, r \geqslant 2$, and $\delta = \varpi_1$. We retain all relative notations from previous chapters.

### 4.1 A preliminary result

The goal of this section is to prove the following result for $\mathfrak{sp}_{2r}$ and $\delta = \varpi_1$.

**Theorem 4.1.** *Let $m \in \mathbb{Z}_{>0}$, $\mu \in P(\delta)$ and $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)$. Assume that for some $i \in \{1, \dots, m\}$, $\mu^{(i)} \succ \mu$. Then* $\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = 0$.

According to Theorem 4.1, it will be enough in many situations to consider weighted paths $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)$ such that $\mu^{(i)} \preccurlyeq \mu$ for any $i \in \{1, \dots, m\}$. We will prove the statement, in Section 4.1.4, by induction on the length $m$ of the path.

Let $\mu \in P(\delta)$, $m \in \mathbb{Z}_{>1}$, $\underline{i} \in \mathbb{Z}^m_{\succ \underline{0}}$ and $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)_{\underline{i}}$. Set $p := p(\underline{i})$, $q := q(\underline{i})$ and let $s \in \{1, \dots, p-2\}$ be a positive integer.

In $\mathfrak{sl}_{r+1}$, according to Lemma 3.2, for all positive roots $\alpha^{(s)} \in \underline{\alpha}$, we have that either $\alpha^{(s)} + \alpha^{(p)}$ is not a root or a negative root. Hence for all $u \in U(\mathfrak{g})$, the

Harish-Chandra projection on $(b^*_{\underline{\mu},\underline{\alpha}} e_{\alpha^{(p)}} u)$ is zero. Nonetheless, this is not the case for $\mathfrak{sp}_{2r}$.

Lemma 2.7 shows that, under some conditions, there exists a positive root $\alpha^{(s)} \in \underline{\alpha}$, such that either $\alpha^{(s)} + \alpha^{(p)} \in \Delta_+$ or $\alpha^{(s)} = -\alpha^{(p)}$. In this situation, the cutting vertex operation induces several new paths. The details of this phenomenon will appear in Lemma 4.3 and Lemma 4.4.

However, if for all $s$ either $\alpha^{(s)} + \alpha^{(p)} \notin \Delta \cup \{0\}$ or $\alpha^{(s)} + \alpha^{(p)} \in -\Delta_+$, then Lemma 3.2 can be applied and one can argue as in $\mathfrak{sl}_{r+1}$. Hence, there is a scalar $\bar{K}^{\#p}$ such that:

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \bar{K}^{\#p} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p}.$$

This scalar is described in Lemma 4.2.

For the sake of simplicity, we first consider the case when $p = q$ and assume that the weighted path $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)_{\underline{i}}$ has no loop.

### 4.1.1 Case $p = q$ without loop

In this paragraph, let $\mu \in P(\delta)$, $m \in \mathbb{Z}_{>1}$, $\underline{i} \in \mathbb{Z}^m_{\succ 0}$ and $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)_{\underline{i}}$. Set $p := p(\underline{i})$, $q := q(\underline{i})$.

**Lemma 4.2.** *If for all $s \in \{1, \dots, p-2\}$ either $\alpha^{(s)} + \alpha^{(p)} \notin \Delta \cup \{0\}$ or $\alpha^{(s)} + \alpha^{(p)} \in -\Delta_+$, then there is a scalar $\bar{K}^{\#p}$ such that:*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \bar{K}^{\#p} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p},$$

*where $\bar{K}^{\#p}$ is described as follows:*

1. *If $\alpha^{(p-1)} + \alpha^{(p)} \neq 0$, then*
$$\bar{K}^{\#p} = K^{\#p}_{(\underline{\mu},\underline{\alpha})},$$
*where $K^{\#p}_{(\underline{\mu},\underline{\alpha})}$ is a constant which only depends on constant structures (§2.1.2).*

2. *Assume $\alpha^{(p-1)} + \alpha^{(p)} = 0$*

   *(a) If $p = 2$, then*
$$\bar{K}^{\#p} = (c_{\alpha^{(1)}})^2 \ \mathrm{ht}(\check{\alpha}^{(1)}).$$

   *(b) Otherwise,*
$$\bar{K}^{\#p} = (c_{\alpha^{(p-1)}})^2 \ (\mathrm{ht}(\check{\alpha}^{(p-1)}) - c_{p-1}),$$

*where*

$$c_{p-1} := \sum_{\alpha^{(j_k)} \in \underline{\alpha}_{p-1}} \langle \alpha^{(j_k)}, \check{\alpha}^{(p-1)} \rangle, \tag{4.1}$$

*with*

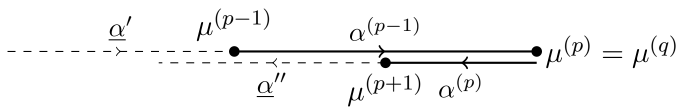
$$\underline{\alpha}_{p-1} := \{\alpha^{(j_k)} \in \underline{\alpha} \mid j_k < p-1, k \in (1, \dots, n), [e_{\alpha^{(j_k)}}, \check{\alpha}^{(p-1)}] \neq 0\},$$

*and* $c_{\alpha^{(p-1)}}$ *be as in Definition 1.14.*

*Proof.* 1. Assume $\alpha^{(p-1)} + \alpha^{(p)} \neq 0$. Write

$$(\underline{\mu}, \underline{\alpha}) = (\underline{\mu}', \underline{\alpha}') \star ((\mu^{(p-1)}, \mu^{(p)}), \alpha^{(p-1)}) \star ((\mu^{(p)}, \mu^{(p+1)}), \alpha^{(p)}) \star (\underline{\mu}'', \underline{\alpha}''),$$

where $(\underline{\mu}', \underline{\alpha}')$ and $(\underline{\mu}'', \underline{\alpha}'')$ have length $p-2$ and $m-p$ respectively.




Set $a = a^{(c_{\alpha^{(p)}} e_{-\alpha^{(p)}})}_{\mu^{(p+1)}, \mu^{(p)}} a^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})}_{\mu^{(p)}, \mu^{(p-1)}}$. We have,

$$\begin{aligned} b^*_{(\underline{\mu}, \underline{\alpha})} &= a b^*_{(\underline{\mu}', \underline{\alpha}')} e_{\alpha^{(p-1)}} e_{\alpha^{(p)}} b^*_{(\underline{\mu}'', \underline{\alpha}'')} \\ &= a b^*_{(\underline{\mu}', \underline{\alpha}')} e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'', \underline{\alpha}'')} + a n_{\alpha^{(p-1)}, \alpha^{(p)}} b^*_{(\underline{\mu}', \underline{\alpha}')} e_{\alpha^{(p-1)} + \alpha^{(p)}} b^*_{(\underline{\mu}'', \underline{\alpha}'')} \\ &= a b^*_{(\underline{\mu}', \underline{\alpha}')} e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'', \underline{\alpha}'')} + n_{\alpha^{(p-1)}, \alpha^{(p)}} \frac{a^{(c_{\alpha^{(p)}} e_{-\alpha^{(p)}})}_{\mu^{(p+1)}, \mu^{(p)}} a^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})}_{\mu^{(p)}, \mu^{(p-1)}}}{a^{(c_{\alpha^{(p-1)} + \alpha^{(p)}} e_{-\alpha^{(p-1)} - \alpha^{(p)}})}_{\mu^{(p+1)}, \mu^{(p-1)}}} b^*_{(\underline{\mu}, \underline{\alpha})^{\#p}}. \end{aligned}$$

By the assumption, the weighted path $(\underline{\mu}', \underline{\alpha}')$ and the positive root $-\alpha^{(p)}$ verify the conditions of Lemma 3.2. Hence

$$\mathrm{hc}(b^*_{(\underline{\mu}', \underline{\alpha}')} e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'', \underline{\alpha}'')}) = 0.$$

By setting $K^{\#p}_{(\underline{\mu}, \underline{\alpha})} := n_{\alpha^{(p-1)}, \alpha^{(p)}} \frac{a^{(c_{\alpha^{(p)}} e_{-\alpha^{(p)}})}_{\mu^{(p+1)}, \mu^{(p)}} a^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})}_{\mu^{(p)}, \mu^{(p-1)}}}{a^{(c_{\alpha^{(p-1)} + \alpha^{(p)}} e_{-\alpha^{(p-1)} - \alpha^{(p)}})}_{\mu^{(p+1)}, \mu^{(p-1)}}}$, we get

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^{\#p}_{(\underline{\mu}, \underline{\alpha})} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p}.$$

2. Assume $\alpha^{(p-1)} + \alpha^{(p)} = 0$. Thus we have $\alpha^{(p-1)} = -\alpha^{(p)}$.

(a) Assume $p = 2$. Write

$$(\underline{\mu}, \underline{\alpha}) = ((\mu^{(1)}, \mu^{(2)}), \alpha^{(1)}) \star ((\mu^{(2)}, \mu^{(3)}), \alpha^{(2)}) \star (\underline{\mu}'', \underline{\alpha}''),$$

where $(\underline{\mu}'', \underline{\alpha}'')$ has length $m-2$. Set $a := a^{(c_{\alpha^{(2)}} e_{-\alpha^{(2)}})}_{\mu^{(3)},\mu^{(2)}} a^{(c_{\alpha^{(1)}} e_{-\alpha^{(1)}})}_{\mu^{(2)},\mu^{(1)}}$. We have

$$\begin{aligned} b^*_{\underline{\mu},\underline{\alpha}} &= a e_{\alpha^{(1)}} e_{-\alpha^{(1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} = a\big(e_{-\alpha^{(1)}} e_{\alpha^{(1)}} + \check{\alpha}^{(1)}\big) b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\ &= a e_{-\alpha^{(1)}} e_{\alpha^{(1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} + a\ b^*_{(\underline{\mu}',\underline{\alpha}')} \check{\alpha}^{(1)} b^*_{(\underline{\mu}'',\underline{\alpha}'')}, \end{aligned}$$

since $\alpha^{(2)} = -\alpha^{(1)}$. From Lemma 1.24 and Lemma 3.2 we have,

$$\mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}}) = a\ \check{\alpha}^{(1)} \mathrm{hc}(b^*_{(\underline{\mu}'',\underline{\alpha}'')}) = (c_{\alpha^{(1)}})^2\ \check{\alpha}^{(1)} \mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})^{\#2}}).$$

Since

$$\mathrm{ev}_\rho(\check{\alpha}^{(1)}) = \langle \rho, \check{\alpha}^{(1)} \rangle = \mathrm{ht}(\check{\alpha}^{(1)}),$$

we get:

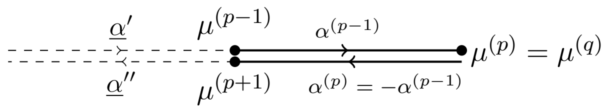
$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (c_{\alpha^{(1)}})^2\ \mathrm{ht}(\check{\alpha}^{(1)})\, \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p},$$

where $(\underline{\mu}'', \underline{\alpha}'')$ is a weighted path of length $m-2$.

(b) Assume $p > 2$. Write

$$(\underline{\mu}, \underline{\alpha}) = (\underline{\mu}', \underline{\alpha}') \star ((\mu^{(p-1)}, \mu^{(p)}), \alpha^{(p-1)}) \star ((\mu^{(p)}, \mu^{(p+1)}), \alpha^{(p)}) \star (\underline{\mu}'', \underline{\alpha}'')$$

as in (1).




Set $a = a^{(c_{\alpha^{(p)}} e_{-\alpha^{(p)}})}_{\mu^{(p+1)},\mu^{(p)}} a^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})}_{\mu^{(p)},\mu^{(p-1)}}$. Hence

$$\begin{aligned} b^*_{(\underline{\mu},\underline{\alpha})} &= a b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p-1)}} e_{\alpha^{(p)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\ &= a b^*_{(\underline{\mu}',\underline{\alpha}')} e_{-\alpha^{(p-1)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} + a b^*_{(\underline{\mu}',\underline{\alpha}')} \check{\alpha}^{(p-1)} b^*_{(\underline{\mu}'',\underline{\alpha}'')}, \end{aligned}$$

since $\alpha^{(p)} = -\alpha^{(p-1)}$. By assumption, the weighted path $(\underline{\mu}', \underline{\alpha}')$ and the positive root $\alpha^{(p-1)}$ satisfy the conditions of Lemma 3.2. Hence,

$$\mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})}) = a\mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')}\check{\alpha}^{(p-1)}b^*_{(\underline{\mu}'',\underline{\alpha}'')}). \tag{4.2}$$

For all other roots $\alpha^{(t)} \in \underline{\alpha}'$, where $t \notin \underline{j}$, $\check{\alpha}^{(p-1)}$ commutes with $\alpha^{(t)}$.

Set $a' := a^{(e_{-\alpha^{(p-2)}})}_{\mu^{(p-1)},\mu^{(p-2)}} \cdots a^{(e_{-\alpha^{(1)}})}_{\mu^{(2)},\mu^{(1)}}$. We get

$$\begin{aligned}
b^*_{(\underline{\mu}',\underline{\alpha}')}\check{\alpha}^{(p-1)}b^*_{(\underline{\mu}'',\underline{\alpha}'')} &= a'e_{\alpha^{(1)}}\cdots e_{\alpha^{(j_1)}}\cdots e_{\alpha^{(j_n)}}\cdots e_{\alpha^{(p-2)}}\check{\alpha}^{(p-1)}b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= a'e_{\alpha^{(1)}}\cdots e_{\alpha^{(j_1)}}\cdots \left(\check{\alpha}^{(p-1)}e_{\alpha^{(j_n)}} - \langle \alpha^{(j_n)}, \check{\alpha}^{(p-1)}\rangle e_{\alpha^{(j_n)}}\right)\cdots e_{\alpha^{(p-2)}}\ b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= a'e_{\alpha^{(1)}}\cdots e_{\alpha^{(j_1)}}\cdots \check{\alpha}^{(p-1)}e_{\alpha^{(j_n)}}\cdots e_{\alpha^{(p-2)}}\ b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\quad - a'\langle \alpha^{(j_n)}, \check{\alpha}^{(p-1)}\rangle e_{\alpha^{(1)}}\cdots e_{\alpha^{(j_1)}}\cdots e_{\alpha^{(j_n)}}\cdots e_{\alpha^{(p-2)}}\ b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= a'e_{\alpha^{(1)}}\cdots e_{\alpha^{(j_1)}}\cdots \left(\check{\alpha}^{(p-1)}e_{\alpha^{(j_{n-1})}} - \langle \alpha^{(j_{n-1})}, \check{\alpha}^{(p-1)}\rangle e_{\alpha^{(j_{n-1})}}\right) \\
&\quad \times \cdots e_{\alpha^{(j_n)}}\cdots e_{\alpha^{(p-2)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\quad - a'\langle \alpha^{(j_n)}, \check{\alpha}^{(p-1)}\rangle e_{\alpha^{(1)}}\cdots e_{\alpha^{(j_1)}}\cdots e_{\alpha^{(j_n)}}\cdots e_{\alpha^{(p-2)}}\ b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= a'e_{\alpha^{(1)}}\cdots e_{\alpha^{(j_1)}}\cdots \check{\alpha}^{(p-1)}e_{\alpha^{(j_{n-1})}}\cdots e_{\alpha^{(j_n)}}\cdots e_{\alpha^{(p-2)}}\ b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\quad - a'\langle \alpha^{(j_{n-1})}, \check{\alpha}^{(p-1)}\rangle e_{\alpha^{(1)}}\cdots e_{\alpha^{(j_1)}}\cdots e_{\alpha^{(j_n)}}\cdots e_{\alpha^{(p-2)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\quad - a'\langle \alpha^{(j_n)}, \check{\alpha}^{(p-1)}\rangle e_{\alpha^{(1)}}\cdots e_{\alpha^{(j_1)}}\cdots e_{\alpha^{(j_n)}}\cdots e_{\alpha^{(p-2)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')}.
\end{aligned}$$

We continue the process, and we get

$$\begin{aligned}
b^*_{(\underline{\mu}',\underline{\alpha}')}\check{\alpha}^{(p-1)}b^*_{(\underline{\mu}'',\underline{\alpha}'')} &= a'\check{\alpha}^{(p-1)}e_{\alpha^{(1)}}\cdots e_{\alpha^{(j_1)}}\cdots e_{\alpha^{(j_n)}}\cdots e_{\alpha^{(p-2)}}\ b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\quad - a'\langle \alpha^{(j_1)}, \check{\alpha}^{(p-1)}\rangle e_{\alpha^{(1)}}\cdots e_{\alpha^{(j_1)}}\cdots e_{\alpha^{(j_n)}}\cdots e_{\alpha^{(p-2)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\quad - \cdots - a'\langle \alpha^{(j_{n-1})}, \check{\alpha}^{(p-1)}\rangle e_{\alpha^{(1)}}\cdots e_{\alpha^{(j_1)}}\cdots e_{\alpha^{(j_n)}}\cdots e_{\alpha^{(p-2)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\quad - a'\langle \alpha^{(j_n)}, \check{\alpha}^{(p-1)}\rangle e_{\alpha^{(1)}}\cdots e_{\alpha^{(j_1)}}\cdots e_{\alpha^{(j_n)}}\cdots e_{\alpha^{(p-2)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= (\check{\alpha}^{(p-1)} - \langle \alpha^{(j_1)}, \check{\alpha}^{(p-1)}\rangle - \cdots - \langle \alpha^{(j_n)}, \check{\alpha}^{(p-1)}\rangle) \qquad (4.3) \\
&\quad \times a'\ e_{\alpha^{(1)}}\cdots e_{\alpha^{(j_1)}}\cdots e_{\alpha^{(j_n)}}\cdots e_{\alpha^{(p-2)}}\ b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= (\check{\alpha}^{(p-1)} - \langle \alpha^{(j_1)}, \check{\alpha}^{(p-1)}\rangle - \cdots - \langle \alpha^{(j_n)}, \check{\alpha}^{(p-1)}\rangle)\ b^*_{(\underline{\mu}',\underline{\alpha}')}b^*_{(\underline{\mu}'',\underline{\alpha}'')},
\end{aligned}$$

since $\check{\alpha}^{(p-1)}$ commutes with all roots $\alpha^{(t)} \in \underline{\alpha}'$ and $t \notin \underline{j}$.

By Lemma 1.24 and (4.2) we have

$$\mathrm{hc}(\underline{\mu}, \underline{\alpha}) = (c_{\alpha^{(p-1)}})^2\ (\check{\alpha}^{(p-1)} - c_{p-1})\mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}).$$

Hence,

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (c_{\alpha^{(p-1)}})^2(\mathrm{ht}(\check{\alpha}^{(p-1)}) - c_{p-1})\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p}.$$

□

**Lemma 4.3** ($\alpha^{(s)}+\alpha^{(p)} \in \Delta_+$)**.** *Assume that for some $s \in \{1, \dots, p-2\}$, $\alpha^{(s)}+\alpha^{(p)} \in \Delta_+$. In this case,*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \bar{K}^{\#p}\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} + \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})}\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\star a},$$

*where $\bar{K}^{\#p}$ is a scalar as in Lemma 4.2, $K^{\star a}$ are some constants which only depend on constant structures, and $(\underline{\mu}, \underline{\alpha})^{\star a} := (\underline{\mu}^{\star a}, \underline{\alpha}^{\star a})$ is a concatenation of paths defined as follows.*

1. *If $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_k - \overline{\delta}_i$, with $k < j < i$, then*

$$(\underline{\mu}, \underline{\alpha})^{\star a} = (\underline{\mu}', \underline{\alpha}') \star \big((\delta_k, \delta_j), (\alpha^{(s)} + \alpha^{(p)})\big) \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}''),$$

*where $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$ is a path of length $< p - s$ between $\delta_j$ and $\delta_i$ whose roots $\underline{\tilde{\alpha}}^{\star a}$ are sums among $(\alpha^{(p-1)}, \dots, \alpha^{(s+1)})$.*

2. *If $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_j - \overline{\delta}_l$, with $j < l < i$, then*

$$(\underline{\mu}, \underline{\alpha})^{\star a} = (\underline{\mu}', \underline{\alpha}') \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}''),$$

*where $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$ is a path of length $< p - s - 1$ between $\delta_j$ and $\delta_i$ whose roots $\underline{\tilde{\alpha}}^{\star a}$ are sums among $(\alpha^{(p-1)}, \dots, \alpha^{(s+1)}, \alpha^{(s)} + \alpha^{(p)})$.*

3. *If $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_i - \overline{\delta}_l$, with $i < j$and $i < l$, then*

$$(\underline{\mu}, \underline{\alpha})^{\star a} = (\underline{\mu}', \underline{\alpha}') \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}''),$$

*where $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$ is a path of length $< p - s - 1$ between $\delta_i$ and $\overline{\delta}_j$ whose roots $\underline{\tilde{\alpha}}^{\star a}$ are sums among $(\alpha^{(p-1)}, \dots, \alpha^{(s+1)}, \alpha^{(s)} + \alpha^{(p)})$.*

4. *If $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_i - \delta_l$, with $i < j < l$, then*

$$(\underline{\mu}, \underline{\alpha})^{\star a} = (\underline{\mu}', \underline{\alpha}') \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}''),$$

*where* $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$ *is a path of length* $< p - s - 1$ *between* $\delta_i$ *and* $\overline{\delta}_j$ *whose roots* $\underline{\tilde{\alpha}}^{\star a}$ *are sums among* $(\alpha^{(p-1)}, \ldots, \alpha^{(s+1)}, \alpha^{(s)} + \alpha^{(p)})$.

5. *If* $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ *and* $\alpha^{(s)} = \delta_k - \delta_j$*, with* $k < i < j$*, then*

$$(\underline{\mu}, \underline{\alpha})^{\star a} = (\underline{\mu}', \underline{\alpha}') \star \big((\delta_k, \delta_i), (\alpha^{(s)} + \alpha^{(p)})\big) \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}''),$$

*where* $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$ *is a path of length* $< p - s$ *between* $\delta_i$ *and* $\overline{\delta}_j$ *whose roots whose roots* $\underline{\tilde{\alpha}}^{\star a}$ *are sums among* $(\alpha^{(p-1)}, \ldots, \alpha^{(s+1)})$.

*In all those cases,*

$$(\underline{\mu}', \underline{\alpha}') = \big((\mu^{(1)}, \ldots, \mu^{(s-1)}, \mu^{(s)}), (\alpha^{(1)}, \ldots, \alpha^{(s-1)})\big),$$

$$(\underline{\mu}'', \underline{\alpha}'') = \big((\mu^{(p+1)}, \ldots, \mu^{(m+1)}), (\alpha^{(p+1)}, \ldots, \alpha^{(m)})\big),$$

*and* $N$ *is the number of possible paths of* $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$.

*Proof.* Let $-\alpha^{(p)}, \alpha^{(s)} \in \Delta_+$ such that $\alpha^{(s)} + \alpha^{(p)} \in \Delta_+$. By Lemma 2.7 the only possibilities for $\alpha^{(p)}$ and $\alpha^{(s)}$ are:

- $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_k - \overline{\delta}_i$, with $k < j < i$,
- $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_j - \overline{\delta}_l$, with $j < l < i$,
- $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_i - \overline{\delta}_l$, with $i < l$,
- $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_i - \delta_l$, with $i < j < l$,
- $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_k - \delta_j$, with $k < i < j$.

1. Assume that $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_k - \overline{\delta}_i$, with $k < j < i$.
   Write

$$\begin{aligned}(\underline{\mu}, \underline{\alpha}) =& (\underline{\mu}', \underline{\alpha}') \star \big((\mu^{(s)} = \delta_k, \mu^{(s+1)} = \overline{\delta}_i, \ldots, \mu^{(p-1)}, \mu^{(p)} = \overline{\delta}_j, \mu^{(p+1)} = \delta_i), \\ & (\alpha^{(s)}, \alpha^{(s+1)}, \ldots, \alpha^{(p-1)}, \alpha^{(p)})\big) \star (\underline{\mu}'', \underline{\alpha}''),\end{aligned}$$

where

$$(\underline{\mu}', \underline{\alpha}') = \big((\mu^{(1)}, \ldots, \mu^{(s-1)}, \mu^{(s)} = \delta_k), (\alpha^{(1)}, \ldots, \alpha^{(s-1)})\big),$$

$$(\underline{\mu}'', \underline{\alpha}'') = \big((\mu^{(p+1)} = \delta_i, \ldots, \mu^{(m+1)}), (\alpha^{(p+1)}, \ldots, \alpha^{(m)})\big),$$

have length $s-1$ and $m-p$ respectively.

In this case, $\mu^{(p+1)} \succ \mu^{(s+1)} \succcurlyeq \mu^{(p-1)}$ and so $\alpha^{(p-1)} + \alpha^{(p)} < 0$. Note that $\alpha^{(p)}$ commutes with all roots $\alpha^{(t)}$ for $t < p$, except with $\alpha^{(p-1)}$ and $\alpha^{(s)}$.

Set

$$a := a_{\mu^{(p+1)},\mu^{(p)}}^{(c_{\alpha^{(p)}} e_{-\alpha^{(p)}})} a_{\mu^{(p)},\mu^{(p-1)}}^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})} \cdots a_{\mu^{(s+2)},\mu^{(s+1)}}^{(c_{\alpha^{(s+1)}} e_{-\alpha^{(s+1)}})} a_{\mu^{(s+1)},\mu^{(s)}}^{(c_{\alpha^{(s)}} e_{-\alpha^{(s)}})}.$$

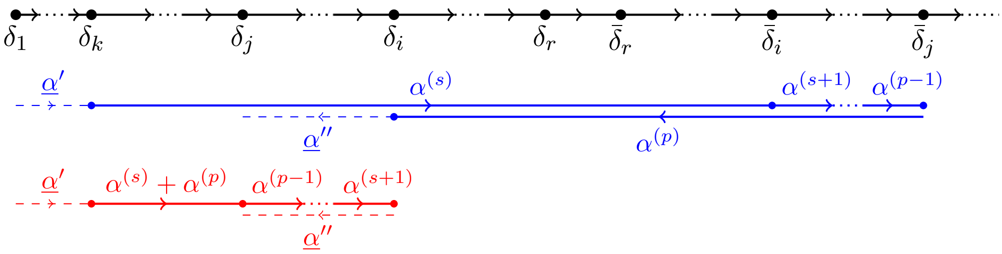



**FIGURE 4.1** – The case for $\alpha^{(p)} = \bar\delta_j - \delta_i$ and $\alpha^{(s)} = \delta_k - \bar\delta_i$.

We have,

$$\begin{aligned}
b^*_{(\underline{\mu},\underline{\alpha})} &= ab^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-2)}} e_{\alpha^{(p-1)}} e_{\alpha^{(p)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= an_{\alpha^{(p-1)},\alpha^{(p)}} b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-2)}} e_{\alpha^{(p-1)}+\alpha^{(p)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\quad + ab^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-2)}} e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= K^{\#p}_{(\underline{\mu},\underline{\alpha})} b^*_{(\underline{\mu},\underline{\alpha})^{\#p}} + ab^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-2)}} e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= K^{\#p}_{(\underline{\mu},\underline{\alpha})} b^*_{(\underline{\mu},\underline{\alpha})^{\#p}} + ab^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} e_{\alpha^{(p)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-2)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= K^{\#p}_{(\underline{\mu},\underline{\alpha})} b^*_{(\underline{\mu},\underline{\alpha})^{\#p}} + a\big(n_{\alpha^{(s)},\alpha^{(p)}} b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-2)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\quad + b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p)}} e_{\alpha^{(s)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-2)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}\big) \\
&= K^{\#p}_{(\underline{\mu},\underline{\alpha})} b^*_{(\underline{\mu},\underline{\alpha})^{\#p}} + an_{\alpha^{(s)},\alpha^{(p)}} b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-2)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\quad + ab^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p)}} e_{\alpha^{(s)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-2)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')},
\end{aligned}$$

where $K^{\#p}_{(\underline{\mu},\underline{\alpha})}$ is as in Lemma 4.2(1).

The weighted path $(\underline{\mu}', \underline{\alpha}')$ and the positive root $-\alpha^{(p)}$ verify the conditions of Lemma 3.2. Hence,

$$\mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p)}} e_{\alpha^{(s)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-2)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}) = 0,$$

and so

$$\begin{aligned}\mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})}) &= K^{\#p}_{(\underline{\mu},\underline{\alpha})}\mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}) \\ &\quad + a n_{\alpha^{(s)},\alpha^{(p)}}\mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(s+1)}}\cdots e_{\alpha^{(p-2)}}e_{\alpha^{(p-1)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')}). \end{aligned} \tag{4.4}$$

Note that $\alpha^{(s)}+\alpha^{(p)} = \varepsilon_k - \varepsilon_j = \delta_k - \delta_j$. Observe also that the target of $\alpha^{(p-1)}$ is $\overline{\delta}_j = -\varepsilon_j$, and the source of $\alpha^{(s+1)}$ is $\overline{\delta}_i = -\varepsilon_i$. We have $\alpha^{(s+1)}+\cdots+\alpha^{(p-1)} = -\varepsilon_i + \varepsilon_j = \delta_j - \delta_i$, while the source of $\alpha^{(p+1)}$ is $\delta_i = \varepsilon_i$ so that $\alpha^{(p+1)}+\cdots+\alpha^{(m)} = \varepsilon_i - \varepsilon_{j_1}$, with $j_1 < k < i$.

Because of the configuration, for all $t \in \{1,\dots,s-1\}$, $\alpha^{(t)} = \delta_{j_t} - \delta_{j_{t+1}} = \varepsilon_{j_t} - \varepsilon_{j_{t+1}}$, with $j_t < j_{t+1} \leqslant k < i$, and for all $t \in \{s+1,\dots,p-1\}$, $\alpha^{(t)} = \overline{\delta}_{j_t} - \overline{\delta}_{j_{t+1}} = \varepsilon_{j_{t+1}} - \varepsilon_{j_t}$, with $j \leqslant j_{t+1} < j_t \leqslant i$. Hence, $\big((\delta_j,\delta_{p-1},\delta_{p-2},\dots,\delta_{s+2},\delta_i),\ (\alpha^{(p-1)},\dots,\alpha^{(s+1)})\big)$ is a path from $\delta_j$ to $\delta_i$ (see Figure 4.1 for an illustration).

So we have to "reverse the order" of $e_{\alpha^{(s+1)}}\cdots e_{\alpha^{(p-1)}}$ in (4.4), in order to get $e_{\alpha^{(p-1)}}\cdots e_{\alpha^{(s+1)}}$. Doing this, it induces several new paths and we denote by $N$ the number of those paths. Let $(\tilde{\underline{\mu}}^{\star a},\tilde{\underline{\alpha}}^{\star a})$, $a = 1,\dots,N$, denote the paths from $\delta_j$ to $\delta_i$ whose roots are sums among the roots $(\alpha^{(p-1)},\dots,\alpha^{(s+1)})$. More precisely, for each $a \in \{1,\dots,N\}$ there exists a partition $(P_1,\dots,P_{n_a})$ of the set $\{p-1,\dots,s+1\}$ such that

$$(\tilde{\alpha}^{\star a})^{(s+j)} = \sum_{t\in P_j}\alpha^{(t)},$$

for $j = 1,\dots,n_a$.

Furthermore, set

$$(\underline{\mu},\underline{\alpha})^{\star a} := (\underline{\mu}',\underline{\alpha}') \star \big((\delta_k,\delta_j),(\alpha^{(s)}+\alpha^{(p)})\big) \star (\tilde{\underline{\mu}}^{\star a},\tilde{\underline{\alpha}}^{\star a}) \star (\underline{\mu}'',\underline{\alpha}'').$$

We have

$$\mathrm{wt}(\underline{\mu},\underline{\alpha}) = K^{\#p}_{(\underline{\mu},\underline{\alpha})}\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p} + \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})}\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\star a},$$

where $K^{\star a}_{(\underline{\mu},\underline{\alpha})}$ are some contants.

2. Assume that $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_j - \overline{\delta}_l$, with $j < l < i$.

   Write

$$\begin{aligned}(\underline{\mu},\underline{\alpha}) = &(\underline{\mu}',\underline{\alpha}') \star \big((\mu^{(s)} = \delta_j, \mu^{(s+1)} = \overline{\delta}_l,\dots,\mu^{(p-1)},\mu^{(p)} = \overline{\delta}_j,\mu^{(p+1)} = \delta_i), \\ &(\alpha^{(s)},\alpha^{(s+1)},\dots,\alpha^{(p-1)},\alpha^{(p)})\big) \star (\underline{\mu}'',\underline{\alpha}''),\end{aligned}$$

where

$$(\underline{\mu}', \underline{\alpha}') = \big((\mu^{(1)}, \dots, \mu^{(s-1)}, \mu^{(s)} = \delta_j),\ (\alpha^{(1)}, \dots, \alpha^{(s-1)})\big),$$
$$(\underline{\mu}'', \underline{\alpha}'') = \big((\mu^{(p+1)} = \delta_i, \dots, \mu^{(m+1)}),\ (\alpha^{(p+1)}, \dots, \alpha^{(m)})\big),$$

have length $s-1$ and $m-p$, respectively.

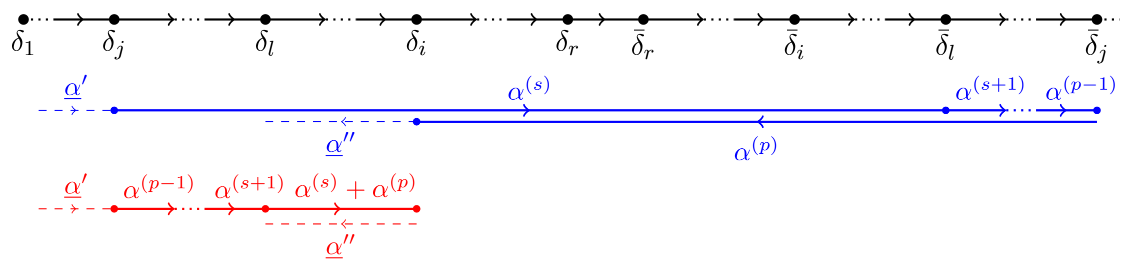


**FIGURE 4.2** – The case for $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_j - \overline{\delta}_l$.

Observe that in this case $\alpha^{(p-1)} + \alpha^{(p)} < 0$ and $\alpha^{(p)}$ commutes with all roots $\alpha^{(t)}, t < p$, except with $\alpha^{(p-1)}$ and $\alpha^{(s)}$.

Set

$$a := a_{\mu^{(p+1)},\mu^{(p)}}^{(c_{\alpha^{(p)}} e_{-\alpha^{(p)}})} a_{\mu^{(p)},\mu^{(p-1)}}^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})} \cdots a_{\mu^{(s+2)},\mu^{(s+1)}}^{(c_{\alpha^{(s+1)}} e_{-\alpha^{(s+1)}})} a_{\mu^{(s+1)},\mu^{(s)}}^{(c_{\alpha^{(s)}} e_{-\alpha^{(s)}})}.$$

Doing the similar calculation as in case (1), we get

$$\begin{aligned} \mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})}) &= K^{\#p}_{(\underline{\mu},\underline{\alpha})} \mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}) \\ &\quad + a n_{\alpha^{(s)},\alpha^{(p)}} \mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-2)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}), \end{aligned} \tag{4.5}$$

where $K^{\#p}_{(\underline{\mu},\underline{\alpha})}$ is as in Lemma 4.2(1).

Note that $\alpha^{(s)} + \alpha^{(p)} = \varepsilon_l - \varepsilon_i = \overline{\delta}_i - \overline{\delta}_l$. Observe that the target of $\alpha^{(p-1)}$ is $\overline{\delta}_j = -\varepsilon_j$ and the source of $\alpha^{(s+1)}$ is $\overline{\delta}_l = -\varepsilon_l$ so that $\alpha^{(s)} + \alpha^{(p)} + \alpha^{(s+1)} + \cdots + \alpha^{(p-1)} = -\varepsilon_i + \varepsilon_j = \delta_j - \delta_i$, while the source of $\alpha^{(p+1)}$ is $\delta_i = \varepsilon_i$ so that $\alpha^{(p+1)} + \cdots + \alpha^{(m)} = \varepsilon_i - \varepsilon_{j_1}$, with $j_1 < j < l$. Because of the configuration, for all $t \in \{1, \dots, s-1\}$, $\alpha^{(t)} = \delta_{j_t} - \delta_{j_{t+1}} = \varepsilon_{j_t} - \varepsilon_{j_{t+1}}$, with $j_t < j_{t+1} \leqslant j < l$, and for all $t \in \{s+1, \dots, p-1\}$, $\alpha^{(t)} = \overline{\delta}_{j_t} - \overline{\delta}_{j_{t+1}} = \varepsilon_{j_{t+1}} - \varepsilon_{j_t}$, with $j \leqslant j_{t+1} < j_t \leqslant l$. Hence, $\big((\delta_j, \delta_{p-1}, \delta_{p-2}, \dots, \delta_{s+2}, \delta_l, \delta_i), (\alpha^{(p-1)}, \dots, \alpha^{(s+1)}, \alpha^{(s)} + \alpha^{(p)})\big)$ is a path from $\delta_j$ to $\delta_i$ (see Figure 4.2 for an illustration).

So we have to "reverse the order" of $e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-1)}}$ in (4.5), in order to get $e_{\alpha^{(p-1)}} \cdots e_{\alpha^{(s+1)}} e_{\alpha^{(s)}+\alpha^{(p)}}$. Doing this, it induces several new paths and we denote by $N$ the number of those paths. Let $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$, $a = 1, \ldots, N$, denote the paths from $\delta_j$ to $\delta_i$ whose roots are sums among the roots $(\alpha^{(p-1)}, \ldots, \alpha^{(s+1)}, \alpha^{(s)} + \alpha^{(p)})$. More precisely, for each $a \in \{1, \ldots, N\}$ there exists a partition $(P_1, \ldots, P_{n_a})$ of the set $\{p-1, \ldots, s+1, s, p\}$, where $s$ and $p$ are always in the same partition, such that

$$(\tilde{\alpha}^{\star a})^{(s+j-1)} = \sum_{t \in P_j} \alpha^{(t)},$$

for $j = 1, \ldots, n_a$.

Furthermore, set

$$(\underline{\mu}, \underline{\alpha})^{\star a} := (\underline{\mu}', \underline{\alpha}') \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}'').$$

Hence,

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^{\#p}_{(\underline{\mu},\underline{\alpha})} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} + \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\star a},$$

where $K^{\star a}_{(\underline{\mu},\underline{\alpha})}$ are some constants.

3. Assume that $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_i - \overline{\delta}_l$, with $i < l$.

Write

$$\begin{aligned}(\underline{\mu}, \underline{\alpha}) =& (\underline{\mu}', \underline{\alpha}') \star \big( (\mu^{(s)} = \delta_i, \mu^{(s+1)} = \overline{\delta}_l, \ldots, \mu^{(p-1)}, \mu^{(p)} = \overline{\delta}_i, \mu^{(p+1)} = \overline{\delta}_j; \\ & (\alpha^{(s)}, \alpha^{(s+1)}, \ldots, \alpha^{(p-1)}, \alpha^{(p)}) \big) \star (\underline{\mu}'', \underline{\alpha}''),\end{aligned}$$

where

$$\begin{aligned}(\underline{\mu}', \underline{\alpha}') &= \big( (\mu^{(1)}, \ldots, \mu^{(s-1)}, \mu^{(s)} = \delta_i); (\alpha^{(1)}, \ldots, \alpha^{(s-1)}) \big), \\ (\underline{\mu}'', \underline{\alpha}'') &= \big( (\mu^{(p+1)} = \overline{\delta}_j, \ldots, \mu^{(m+1)}); (\alpha^{(p+1)}, \ldots, \alpha^{(m)}) \big),\end{aligned}$$

have length $s-1$ and $m-p$, respectively.

Set

$$a := a^{(c_{\alpha^{(p)}} e_{-\alpha^{(p)}})}_{\mu^{(p+1)}, \mu^{(p)}} a^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})}_{\mu^{(p)}, \mu^{(p-1)}} \cdots a^{(c_{\alpha^{(s+1)}} e_{-\alpha^{(s+1)}})}_{\mu^{(s+2)}, \mu^{(s+1)}} a^{(c_{\alpha^{(s)}} e_{-\alpha^{(s)}})}_{\mu^{(s+1)}, \mu^{(s)}}.$$

We have

$$\begin{aligned} b^*_{(\underline{\mu},\underline{\alpha})} &= a b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p-1)}} e_{\alpha^{(p)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\ &= \bar{K}^{\#p} b^*_{(\underline{\mu},\underline{\alpha})^{\#p}} + a\ b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}, \end{aligned}$$

where $\bar{K}^{\#p}$ is as in Lemma 4.2 depending on the different cases for $\alpha^{(p-1)}$ and $\alpha^{(p)}$.

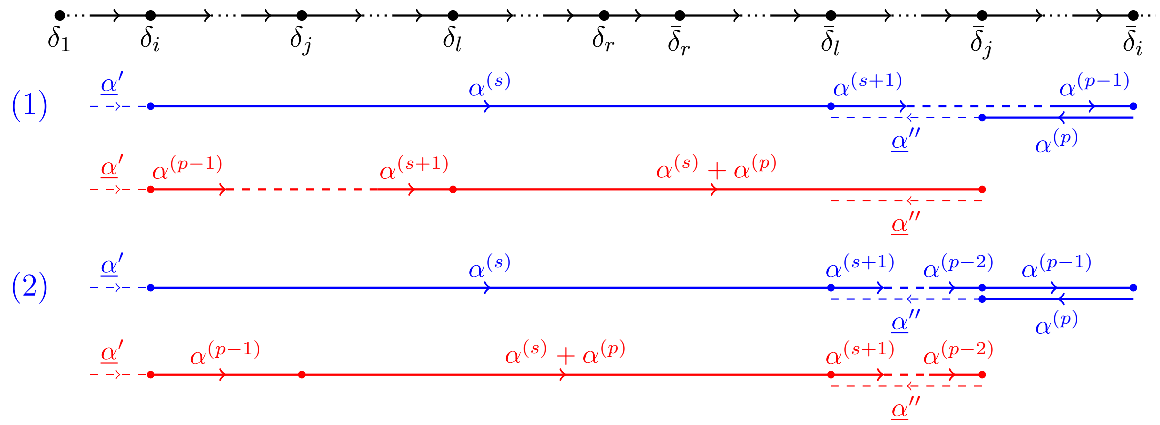



**FIGURE 4.3** – The case for $\alpha^{(p)} = \bar{\delta}_i - \bar{\delta}_j$ and $\alpha^{(s)} = \delta_i - \bar{\delta}_l$.

Assume that there exists a positive root $\alpha^{(t)}$, with $t < p-1$ and $\alpha^{(t)} \neq \alpha^{(s)}$, such that $\alpha^{(t)} + \alpha^{(p)} \in \Delta$. By Lemma 2.7 we observe that there is at most one root $\alpha^{(t)}$ that satisfies such condition (see Figure 4.3). In this case, $\alpha^{(t)} + \alpha^{(p)} \in -\Delta_+$ and $s < t < p-1$.

If the root $\alpha^{(t)}$ exists, then $\alpha^{(p)}$ commutes with all roots $\alpha^{(u)}$ for $u < p$, except with $\alpha^{(p-1)}$, $\alpha^{(t)}$ and $\alpha^{(s)}$. We have

$$
\begin{aligned}
& b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\quad = b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots (e_{\alpha^{(p)}} e_{\alpha^{(t)}} + n_{\alpha^{(t)},\alpha^{(p)}} e_{\alpha^{(t)}+\alpha^{(p)}}) \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\quad = b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p)}} e_{\alpha^{(s)}} \cdots e_{\alpha^{(t)}} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\qquad + n_{\alpha^{(s)},\alpha^{(p)}} b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\qquad + n_{\alpha^{(t)},\alpha^{(p)}} b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(t-1)}} e_{\alpha^{(t)}+\alpha^{(p)}} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}.
\end{aligned}
$$

The weighted path $(\underline{\mu}', \underline{\alpha}')$ and the positive root $\gamma = -\alpha^{(p)}$ verify the conditions of Lemma 3.2, and also the weighted path $(\underline{\mu}', \underline{\alpha}') \star \big((\mu^{(s)}, \ldots, \mu^{(t)}); (\alpha^{(s)}, \ldots, \alpha^{(t-1)})\big)$ and the positive root $-(\alpha^{(t)} + \alpha^{(p)})$. Hence,

$$
\mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}) = n_{\alpha^{(s)},\alpha^{(p)}} \mathrm{hc}\big(b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}\big).
$$

Otherwise, this means that $\alpha^{(p)}$ commutes with all roots $\alpha^{(u)}$ for $u < p$, except with $\alpha^{(p-1)}$ and $\alpha^{(s)}$. We have

$$\begin{aligned} &b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\ &\quad = b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p)}} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} + n_{\alpha^{(s)},\alpha^{(p)}} b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}. \end{aligned}$$

Observe that the weighted path $(\underline{\mu}',\underline{\alpha}')$ and the positive root $\gamma = -\alpha^{(p)}$ satisfy the conditions of Lemma 3.2. Thus,

$$\mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}) = n_{\alpha^{(s)},\alpha^{(p)}} \mathrm{hc}\big(b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}\big).$$

According to the above computation, we have

$$\begin{aligned} \mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})}) &= \bar{K}^{\#p} \mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}) \\ &\quad + a n_{\alpha^{(s)},\alpha^{(p)}} \mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}). \end{aligned} \tag{4.6}$$

Note that $\alpha^{(s)} + \alpha^{(p)} = \varepsilon_l + \varepsilon_j = \delta_j - \overline{\delta}_l$. Observe that the target of $\alpha^{(p-1)}$ is $\overline{\delta}_i = -\varepsilon_i$ and the source of $\alpha^{(s+1)}$ is $\overline{\delta}_l = -\varepsilon_l$ so that $\alpha^{(s)} + \alpha^{(p)} + \alpha^{(s+1)} + \cdots + \alpha^{(p-1)} = \varepsilon_j + \varepsilon_i = \delta_j - \overline{\delta}_i = \delta_i - \overline{\delta}_j$. Note also that the target of $\alpha^{(s-1)}$ is $\delta_i$ and the source of $\alpha^{(p+1)}$ is $\overline{\delta}_j$, while $\alpha^{(p-1)} = \mu^{(p-1)} - \overline{\delta}_i = \delta_i - \overline{\mu}^{(p-1)}$, $\alpha^{(s+1)} = \overline{\delta}_l - \mu^{(s+2)} = \overline{\mu}^{(s+2)} - \delta_l$ and $\alpha^{(s)} + \alpha^{(p)} = \delta_j - \overline{\delta}_l = \delta_l - \overline{\delta}_j$. Hence, we see that $\big((\delta_i, \overline{\mu}^{(p-1)}, \ldots, \overline{\mu}^{(s+2)}, \delta_l, \overline{\delta}_j); (\alpha^{(p-1)}, \ldots, \alpha^{(s+1)}, \alpha^{(s)} + \alpha^{(p)})\big)$ is a path from $\delta_i$ to $\overline{\delta}_j$.

An important remark is that different situation of $\alpha^{(p-1)}$ and $\alpha^{(p)}$ will give different sequences of roots in a path from $\delta_i$ to $\overline{\delta}_j$. If $\alpha^{(p-1)} = -\alpha^{(p)}$ (which is possible in this case) then we have $\alpha^{(p-1)} = \overline{\delta}_j - \overline{\delta}_i = \delta_i - \delta_j$. Note that $\alpha^{(s)} + \alpha^{(p)} = \varepsilon_l + \varepsilon_j = \delta_j - \overline{\delta}_l$. Observe that the source of $\alpha^{(s+1)}$ is $\overline{\delta}_l = -\varepsilon_l$ and the target of $\alpha^{(p-2)}$ is $\overline{\delta}_j = -\varepsilon_j$ so that $\alpha^{(p-1)} + \alpha^{(s)} + \alpha^{(p)} + \alpha^{(s+1)} + \cdots + \alpha^{(p-2)} = \varepsilon_i + \varepsilon_j = \delta_i + \overline{\delta}_j$. Note also that the target of $\alpha^{(s-1)}$ is $\delta_i$, while the source of $\alpha^{(p+1)}$ is $\overline{\delta}_j$. Hence, we see that $\big((\delta_i, \overline{\mu}^{(p-1)}, \ldots, \overline{\delta}_j); (\alpha^{(p-1)}, \alpha^{(s)} + \alpha^{(p)}, \ldots, \alpha^{(p-2)})\big)$ is a path from $\delta_i$ to $\overline{\delta}_j$ as well (see Figure 4.3 for an illustration).

So we have to "reverse the order" of $e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-1)}}$ in (4.6), in order to get $e_{\alpha^{(p-1)}} \cdots e_{\alpha^{(s+1)}} e_{\alpha^{(s)}+\alpha^{(p)}}$ (or to get another order as mention above). Doing this, it induces several new paths and we denote by $N$ the number of those paths. Let $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$, $a = 1, \ldots, N$, denote the paths from $\delta_i$ to $\overline{\delta}_j$ whose roots are sums among the roots $(\alpha^{(p-1)}, \ldots, \alpha^{(s+1)}, \alpha^{(s)} + \alpha^{(p)})$. More precisely, for each $a \in \{1, \ldots, N\}$ there exists a partition $(P_1, \ldots, P_{n_a})$ of the set $\{p-1, \ldots, s+1, s, p\}$, where $s$ and $p$ are always in the same partition, such that

$$(\tilde{\alpha}^{\star a})^{s+(j-1)} = \sum_{t \in P_j} \alpha^{(t)},$$

for $j = 1, \ldots, n_a$. Note also that this partition is not necessary in the sequential order. Furthermore, we set

$$(\underline{\mu}, \underline{\alpha})^{\star a} := (\underline{\mu}', \underline{\alpha}') \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}'').$$

Hence,

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \bar{K}^{\#p} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} + \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu}, \underline{\alpha})} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\star a},$$

where $\bar{K}^{\#p}$ as in Lemma 4.2 depending on the different cases for $\alpha^{(p-1)}$ and $\alpha^{(p)}$, and $K^{\star a}_{(\underline{\mu}, \underline{\alpha})}$ are some constants.

4. Assume that $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_i - \delta_l$, with $i < j < l$.

Write

$$\begin{aligned}(\underline{\mu}, \underline{\alpha}) =& (\underline{\mu}', \underline{\alpha}') \star \big((\mu^{(s)} = \delta_i, \mu^{(s+1)} = \delta_l, \ldots, \mu^{(p-1)}, \mu^{(p)} = \overline{\delta}_i, \mu^{(p+1)} = \overline{\delta}_j; \\ & (\alpha^{(s)}, \alpha^{(s+1)}, \ldots, \alpha^{(p-1)}, \alpha^{(p)})\big) \star (\underline{\mu}'', \underline{\alpha}''),\end{aligned}$$

where

$$(\underline{\mu}', \underline{\alpha}') = \big((\mu^{(1)}, \ldots, \mu^{(s-1)}, \mu^{(s)} = \delta_i); (\alpha^{(1)}, \ldots, \alpha^{(s-1)})\big),$$

$$(\underline{\mu}'', \underline{\alpha}'') = \big((\mu^{(p+1)} = \overline{\delta}_j, \ldots, \mu^{(m+1)}); (\alpha^{(p+1)}, \ldots, \alpha^{(m)})\big),$$

have length $s-1$ and $m-p$ respectively.

Set

$$a := a^{(c_{\alpha^{(p)}} e_{-\alpha^{(p)}})}_{\mu^{(p+1)}, \mu^{(p)}} a^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})}_{\mu^{(p)}, \mu^{(p-1)}} \cdots a^{(c_{\alpha^{(s+1)}} e_{-\alpha^{(s+1)}})}_{\mu^{(s+2)}, \mu^{(s+1)}} a^{(c_{\alpha^{(s)}} e_{-\alpha^{(s)}})}_{\mu^{(s+1)}, \mu^{(s)}}.$$

We have

$$\begin{aligned}b^*_{(\underline{\mu}, \underline{\alpha})} &= a b^*_{(\underline{\mu}', \underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p-1)}} e_{\alpha^{(p)}} b^*_{(\underline{\mu}'', \underline{\alpha}'')} \\ &= \bar{K}^{\#p} b^*_{(\underline{\mu}, \underline{\alpha})^{\#p}} + a\ b^*_{(\underline{\mu}', \underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'', \underline{\alpha}'')},\end{aligned}$$

where $\bar{K}^{\#p}$ is as in Lemma 4.2 depending on the different cases for $\alpha^{(p-1)}$ and $\alpha^{(p)}$.

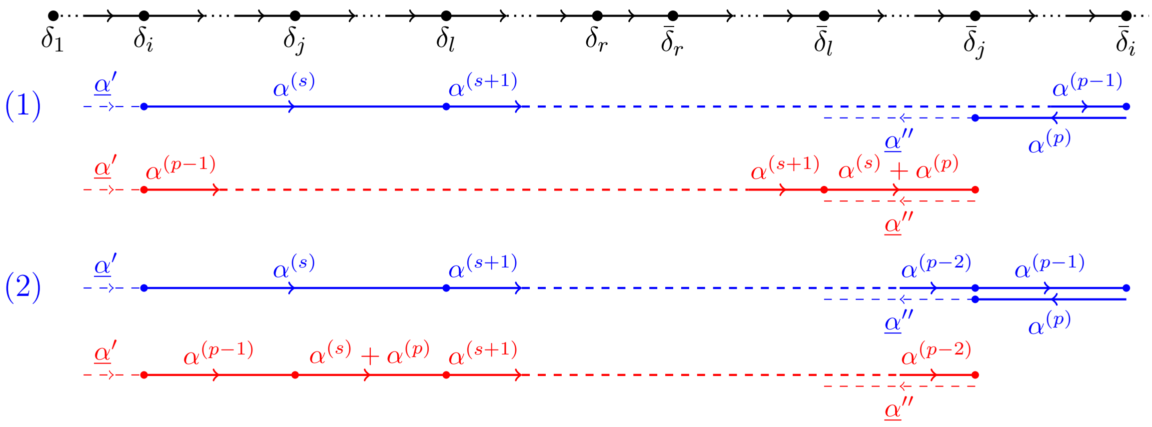



**FIGURE 4.4** – The case for $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_j - \delta_l$.

Assume that there exists a positive root $\alpha^{(t)}$, with $t < p-1$ and $\alpha^{(t)} \neq \alpha^{(s)}$, such that $\alpha^{(t)} + \alpha^{(p)} \in \Delta$. By Lemma 2.7 we observe that there is at most one root $\alpha^{(t)}$ that satisfies such condition (see Figure 4.3). In this case, $\alpha^{(t)} + \alpha^{(p)} \in -\Delta_+$ and $s < t < p-1$.

In the same manner as in case (3), we get

$$\begin{aligned}\mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})}) &= \bar{K}^{\#p}\mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}) \\ &\quad + a n_{\alpha^{(s)},\alpha^{(p)}}\mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(s+1)}}\cdots e_{\alpha^{(p-1)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')}).\end{aligned} \tag{4.7}$$

Note that $\alpha^{(s)} + \alpha^{(p)} = \varepsilon_j - \varepsilon_l = \delta_j - \delta_l$. Observe that the target of $\alpha^{(p-1)}$ is $\overline{\delta}_i = -\varepsilon_i$ and the source of $\alpha^{(s+1)}$ is $\delta_l = \varepsilon_l$ so that $\alpha^{(s)} + \alpha^{(p)} + \alpha^{(s+1)} + \cdots + \alpha^{(p-1)} = \varepsilon_j + \varepsilon_i = \delta_j - \overline{\delta}_i = \delta_i - \overline{\delta}_j$. Note also that the target of $\alpha^{(s-1)}$ is $\delta_i$ and the source of $\alpha^{(p+1)}$ is $\overline{\delta}_j$, while $\alpha^{(p-1)} = \mu^{(p-1)} - \overline{\delta}_i = \delta_i - \overline{\mu}^{(p-1)}$, $\alpha^{(s+1)} = \delta_l - \mu^{(s+2)} = \overline{\mu}^{(s+2)} - \overline{\delta}_l$ and $\alpha^{(s)} + \alpha^{(p)} = \delta_j - \delta_l = \overline{\delta}_l - \overline{\delta}_j$. Hence, we see that $\big((\delta_i, \overline{\mu}^{(p-1)}, \overline{\mu}^{(p-2)}, \ldots, \overline{\mu}^{(s+2)}, \overline{\delta}_l, \overline{\delta}_j); (\alpha^{(p-1)}, \ldots, \alpha^{(s+1)}, \alpha^{(s)} + \alpha^{(p)})\big)$ is a path from $\delta_i$ to $\overline{\delta}_j$. Observe that this case is similar as the case (3) and so there exist different sequences of roots in a path from $\delta_i$ to $\overline{\delta}_j$ as well (see Figure 4.4 for an illustration).

So we have to "reverse the order" of $e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(s+1)}}\cdots e_{\alpha^{(p-1)}}$ in (4.7), in order to get $e_{\alpha^{(p-1)}}\cdots e_{\alpha^{(s+1)}}e_{\alpha^{(s)}+\alpha^{(p)}}$ (or to get another order as mention above). Doing this, it induces several new paths and we denote by $N$ the number of those paths. Let $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$, $a = 1, \ldots, N$, denote the paths from $\delta_i$ to $\overline{\delta}_j$ whose roots are sums among the roots $(\alpha^{(p-1)}, \ldots, \alpha^{(s+1)}, \alpha^{(s)} + \alpha^{(p)})$. More precisely, for each $a \in \{1, \ldots, N\}$ there exists a partition $(P_1, \dots, P_{n_a})$ of the set $\{p-1, \dots, s+1, s, p\}$, where $s$ and $p$ are always in the same partition, such that

$$(\tilde{\alpha}^{\star a})^{(s+j-1)} = \sum_{t \in P_j} \alpha^{(t)},$$

for $j = 1, \dots, n_a$. Note also that this partition is not necessary in the sequential order. Furthermore, we set

$$(\underline{\mu}, \underline{\alpha})^{\star a} := (\underline{\mu}', \underline{\alpha}') \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}'').$$

Hence,

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \bar{K}^{\#p} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} + \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu}, \underline{\alpha})} \mathrm{wt}((\underline{\mu}, \underline{\alpha})^{\star a}),$$

where $\bar{K}^{\#p}$ is as in Lemma 4.2 depending on the different cases for $\alpha^{(p-1)}$ and $\alpha^{(p)}$, and $K^{\star a}_{(\underline{\mu}, \underline{\alpha})}$ are some constants.

5. Assume that $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_k - \delta_j$, with $k < i < j$.
   Write

$$\begin{aligned}(\underline{\mu}, \underline{\alpha}) = &(\underline{\mu}', \underline{\alpha}') \star \big( (\mu^{(s)} = \delta_k, \mu^{(s+1)} = \delta_j, \dots, \mu^{(p-1)}, \mu^{(p)} = \overline{\delta}_i, \mu^{(p+1)} = \overline{\delta}_j; \\ &(\alpha^{(s)}, \alpha^{(s+1)}, \dots, \alpha^{(p-1)}, \alpha^{(p)}) \big) \star (\underline{\mu}'', \underline{\alpha}''),\end{aligned}$$

where

$$(\underline{\mu}', \underline{\alpha}') = \big( (\mu^{(1)}, \dots, \mu^{(s-1)}, \mu^{(s)} = \delta_k); (\alpha^{(1)}, \dots, \alpha^{(s-1)}) \big),$$

$$(\underline{\mu}'', \underline{\alpha}'') = \big( (\mu^{(p+1)} = \overline{\delta}_j, \dots, \mu^{(m+1)}); (\alpha^{(p+1)}, \dots, \alpha^{(m)}) \big),$$

have length $s-1$ and $m-p$ respectively. Set

$$a := a^{(c_{\alpha^{(p)}} e_{-\alpha^{(p)}})}_{\mu^{(p+1)}, \mu^{(p)}} a^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})}_{\mu^{(p)}, \mu^{(p-1)}} \cdots a^{(c_{\alpha^{(s+1)}} e_{-\alpha^{(s+1)}})}_{\mu^{(s+2)}, \mu^{(s+1)}} a^{(c_{\alpha^{(s)}} e_{-\alpha^{(s)}})}_{\mu^{(s+1)}, \mu^{(s)}}.$$

We have

$$\begin{aligned} b^*_{(\underline{\mu}, \underline{\alpha})} &= a b^*_{(\underline{\mu}', \underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p-1)}} e_{\alpha^{(p)}} b^*_{(\underline{\mu}'', \underline{\alpha}'')} \\ &= \bar{K}^{\#p} b^*_{(\underline{\mu}, \underline{\alpha})^{\#p}} + a \; b^*_{(\underline{\mu}', \underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'', \underline{\alpha}'')}, \end{aligned}$$

where $\bar{K}^{\#p}$ is as in Lemma 4.2 depending on the different cases for $\alpha^{(p-1)}$ and $\alpha^{(p)}$.

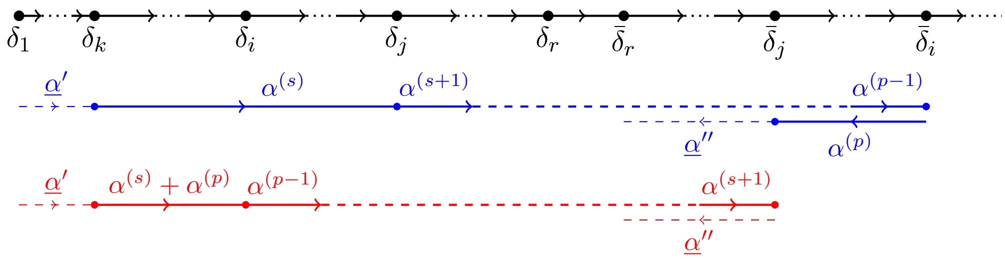



**FIGURE 4.5** – The case for $\alpha^{(p)} = \bar\delta_i - \bar\delta_j$ and $\alpha^{(s)} = \delta_k - \delta_j$.

Assume that there exists a positive root $\alpha^{(t)}$, with $t < p-1$ and $\alpha^{(t)} \neq \alpha^{(s)}$, such that $\alpha^{(t)} + \alpha^{(p)} \in \Delta$. By Lemma 2.7 we observe that there is at most one root $\alpha^{(t)}$ that satisfies such condition (see Figure 4.3). In this case, $\alpha^{(t)} + \alpha^{(p)} \in -\Delta_+$ and $s < t < p-1$.

By doing the same reasoning as in case (3), we get

$$\begin{aligned} \mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})}) &= \bar K^{\#p}\mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}) \\ &\quad + a n_{\alpha^{(s)},\alpha^{(p)}}\mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-2)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}). \end{aligned} \tag{4.8}$$

Note that $\alpha^{(s)} + \alpha^{(p)} = \varepsilon_k - \varepsilon_i = \delta_k - \delta_i$. Observe that the target of $\alpha^{(p-1)}$ is $\bar\delta_i = -\varepsilon_i$ and the source of $\alpha^{(s+1)}$ is $\delta_j = \varepsilon_j$ so that $\alpha^{(s+1)} + \cdots + \alpha^{(p-1)} = \varepsilon_j + \varepsilon_i = \delta_j - \bar\delta_i = \delta_i - \bar\delta_j$. Note also that the target of $\alpha^{(s)} + \alpha^{(p)}$ is $\delta_i$, while $\alpha^{(p-1)} = \mu^{(p-1)} - \bar\delta_i = \delta_i - \overline{\mu}^{(p-1)}$ and $\alpha^{(s+1)} = \delta_j - \mu^{(s+2)} = \overline{\mu}^{(s+2)} - \bar\delta_j$. Hence, $\big((\delta_i, \overline{\mu}^{(p-1)}, \overline{\mu}^{(p-2)}, \ldots, \overline{\mu}^{(s+2)}, \bar\delta_j); (\alpha^{(p-1)}, \ldots, \alpha^{(s+1)})\big)$ is a path from $\delta_i$ to $\bar\delta_j$ (see Figure 4.5 for an illustration).

So we have to "reverse the order" of $e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-1)}}$ in (4.8), in order to get $e_{\alpha^{(p-1)}} \cdots e_{\alpha^{(s+1)}}$. Doing this, it induces several new paths and we denote by $N$ the number of those paths. Let $(\underline{\tilde\mu}^{\star a}, \underline{\tilde\alpha}^{\star a})$, $a = 1, \ldots, N$, denote the paths from $\delta_i$ to $\bar\delta_j$ whose roots are sums among the roots $(\alpha^{(p-1)}, \ldots, \alpha^{(s+1)})$. More precisely, for each $a \in \{1, \ldots, N\}$ there exists a partition $(P_1, \ldots, P_{n_a})$ of the set $\{p-1, \ldots, s+1\}$ such that

$$(\tilde\alpha^{\star a})^{(s+j)} = \sum_{t \in P_j} \alpha^{(t)},$$

for $j = 1, \dots, n_a$. Fix

$$(\underline{\mu}, \underline{\alpha})^{\star a} := (\underline{\mu}', \underline{\alpha}') \star \big((\delta_k, \delta_i), (\alpha^{(s)} + \alpha^{(p)})\big) \star (\underline{\mu}^{\star a}, \underline{\alpha}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}'').$$

We thus get

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \bar{K}^{\#p} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} + \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu}, \underline{\alpha})} \mathrm{wt}((\underline{\mu}, \underline{\alpha})^{\star a}),$$

where $\bar{K}^{\#p}$ is as in Lemma 4.2 depending on the different cases for $\alpha^{(p-1)}$ and $\alpha^{(p)}$, and $K^{\star a}_{(\underline{\mu}, \underline{\alpha})}$ are some constants. □

**Lemma 4.4** ($\alpha^{(s)} + \alpha^{(p)} = 0$)**.** *Assume that for some $s \in \{1, \dots, p-2\}$, $\alpha^{(s)} + \alpha^{(p)} = 0$. In this case,*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \bar{K}^{\#p} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} + (\mathrm{ht}(\check{\alpha}^{(s)}) - \langle \alpha^{(s-1)}, \check{\alpha}^{(s)} \rangle) \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu}, \underline{\alpha})} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\star a},$$

*where $\bar{K}^{\#p}$ is a scalar as in Lemma 4.2, $K^{\star a}_{(\underline{\mu}, \underline{\alpha})}$ are some constants which only depend on constant structures, and $(\underline{\mu}, \underline{\alpha})^{\star a} := (\underline{\mu}^{\star a}, \underline{\alpha}^{\star a})$ is a concatenation of paths defined as follows.*

1. *If $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_j - \overline{\delta}_i$, with $i > j$, then*

$$(\underline{\mu}, \underline{\alpha})^{\star a} = (\underline{\mu}', \underline{\alpha}') \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}''),$$

   *where $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$ is a path of length $< p - s$ between $\delta_j$ and $\delta_i$ whose roots $\underline{\tilde{\alpha}}^{\star a}$ are sums among $(\alpha^{(p-1)}, \dots, \alpha^{(s+1)})$.*

2. *If $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_i - \delta_j = \gamma$, with $i < j$, then*

$$(\underline{\mu}, \underline{\alpha})^{\star a} = (\underline{\mu}', \underline{\alpha}') \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}''),$$

   *where $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$ is a path of length $< p - s$ between $\delta_i$ and $\overline{\delta}_j$ whose roots $\underline{\tilde{\alpha}}^{\star a}$ are sums among $(\alpha^{(p-1)}, \dots, \alpha^{(s+1)})$.*

*In all those cases,*

$$(\underline{\mu}', \underline{\alpha}') = \big((\mu^{(1)}, \dots, \mu^{(s-1)}, \mu^{(s)}), (\alpha^{(1)}, \dots, \alpha^{(s-1)})\big),$$

$$(\underline{\mu}'', \underline{\alpha}'') = \big((\mu^{(p+1)}, \ldots, \mu^{(m+1)}), (\alpha^{(p+1)}, \ldots, \alpha^{(m)})\big),$$

*and $N$ is the number of possible paths of $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$.*

*Proof.* Let $-\alpha^{(p)}, \alpha^{(s)} \in \Delta_+$ such that $\alpha^{(s)} + \alpha^{(p)} = 0$. By Lemma 2.7 the only possibilities for $\alpha^{(p)}$ and $\alpha^{(s)}$ are:

- $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_j - \overline{\delta}_i$, with $j < i$.
- $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_i - \delta_j$, with $i < j$.

1. Assume that $\alpha^{(s)} = \delta_j - \overline{\delta_i} = -\alpha^{(p)}$, with $j < i$.
   Write

$$\begin{aligned}(\underline{\mu}, \underline{\alpha}) =& (\underline{\mu}', \underline{\alpha}') \star \big((\mu^{(s)} = \delta_j, \mu^{(s+1)} = \overline{\delta}_i, \ldots, \mu^{(p-1)}, \mu^{(p)} = \overline{\delta}_j, \mu^{(p+1)} = \delta_i),\\ &(\alpha^{(s)}, \alpha^{(s+1)}, \ldots, \alpha^{(p-1)}, \alpha^{(p)})\big) \star (\underline{\mu}'', \underline{\alpha}''),\end{aligned}$$

where

$$(\underline{\mu}', \underline{\alpha}') = \big((\mu^{(1)}, \ldots, \mu^{(s-1)}, \mu^{(s)} = \delta_j); (\alpha^{(1)}, \ldots, \alpha^{(s-1)})\big),$$

$$(\underline{\mu}'', \underline{\alpha}'') = \big((\mu^{(p+1)} = \delta_i, \ldots, \mu^{(m+1)}); (\alpha^{(p+1)}, \ldots, \alpha^{(m)})\big),$$

have length $s-1$ and $m-p$ respectively.

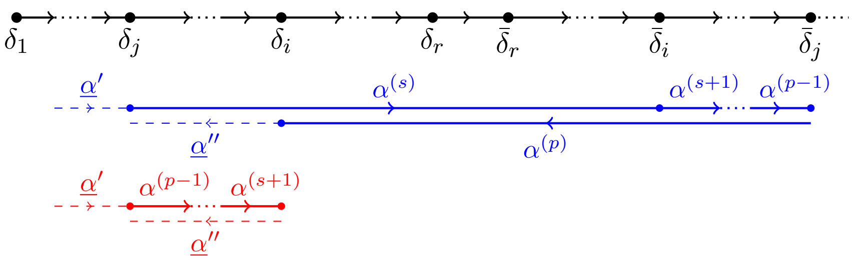



**FIGURE 4.6** – The case for $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_j - \overline{\delta}_i$.

Observe that in this case $\alpha^{(p-1)} + \alpha^{(p)} < 0$ and $\alpha^{(p)}$ commutes with all roots $\alpha^{(t)}$ for $t < p$, except with $\alpha^{(p-1)}$ and $\alpha^{(s)}$.

Set

$$a := a_{\mu^{(p+1)},\mu^{(p)}}^{(c_{\alpha^{(p)}} e_{-\alpha^{(p)}})} a_{\mu^{(p)},\mu^{(p-1)}}^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})} \cdots a_{\mu^{(s+2)},\mu^{(s+1)}}^{(c_{\alpha^{(s+1)}} e_{-\alpha^{(s+1)}})} a_{\mu^{(s+1)},\mu^{(s)}}^{(c_{\alpha^{(s)}} e_{-\alpha^{(s)}})}.$$

We have

$$\begin{aligned}
b^*_{(\underline{\mu},\underline{\alpha})} &= ab^*_{(\underline{\mu}',\underline{\alpha}')}e_{\alpha^{(s)}}e_{\alpha^{(s+1)}}\cdots e_{\alpha^{(p-2)}}e_{\alpha^{(p-1)}}e_{\alpha^{(p)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')}\\
&= a\big(n_{\alpha^{(p-1)},\alpha^{(p)}}b^*_{(\underline{\mu}',\underline{\alpha}')}e_{\alpha^{(s)}}e_{\alpha^{(s+1)}}\cdots e_{\alpha^{(p-2)}}e_{\alpha^{(p-1)}+\alpha^{(p)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')}\\
&\quad + b^*_{(\underline{\mu}',\underline{\alpha}')}e_{\alpha^{(s)}}e_{\alpha^{(s+1)}}\cdots e_{\alpha^{(p-2)}}e_{\alpha^{(p)}}e_{\alpha^{(p-1)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')}\big)\\
&= K^{\#p}_{(\underline{\mu},\underline{\alpha})}b^*_{(\underline{\mu},\underline{\alpha})^{\#p}} + ab^*_{(\underline{\mu}',\underline{\alpha}')}e_{\alpha^{(s)}}e_{\alpha^{(p)}}e_{\alpha^{(s+1)}}\cdots e_{\alpha^{(p-2)}}e_{\alpha^{(p-1)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')}\\
&= K^{\#p}_{(\underline{\mu},\underline{\alpha})}b^*_{(\underline{\mu},\underline{\alpha})^{\#p}} + ab^*_{(\underline{\mu}',\underline{\alpha}')}e_{\alpha^{(p)}}e_{\alpha^{(s)}}e_{\alpha^{(s+1)}}\cdots e_{\alpha^{(p-2)}}e_{\alpha^{(p-1)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')}\\
&\quad + ab^*_{(\underline{\mu}',\underline{\alpha}')}\check{\alpha}^{(s)}e_{\alpha^{(s+1)}}\cdots e_{\alpha^{(p-2)}}e_{\alpha^{(p-1)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')},
\end{aligned}$$

since $\alpha^{(p)} = -\alpha^{(s)}$ and $K^{\#p}_{(\underline{\mu},\underline{\alpha})}$ is as in Lemma 4.2 (1).

Note that $\check{\alpha}^{(s)}$ commutes with all roots in $\underline{\alpha}'$, except with $\alpha^{(s-1)}$. Write

$$(\underline{\mu}',\underline{\alpha}') = (\underline{\mu}'_1,\underline{\alpha}'_1)\star((\mu^{(s-1)},\mu^{(s)}),\alpha^{(s-1)}).$$

With the same arguments as (4.3), we get

$$b^*_{(\underline{\mu}',\underline{\alpha}')}\check{\alpha}^{(s)} = (\check{\alpha}^{(s)} - \langle\alpha^{(s-1)},\check{\alpha}^{(s)}\rangle)\; b^*_{(\underline{\mu}',\underline{\alpha}')},$$

since $\check{\alpha}^{(s)}$ commutes with all roots in $\underline{\alpha}'$, except with $\alpha^{(s-1)}$.

Thus,

$$\begin{aligned}
b^*_{(\underline{\mu},\underline{\alpha})} &= K^{\#p}_{(\underline{\mu},\underline{\alpha})}b^*_{(\underline{\mu},\underline{\alpha})^{\#p}} + a\; b^*_{(\underline{\mu}',\underline{\alpha}')}e_{\alpha^{(p)}}e_{\alpha^{(s)}}e_{\alpha^{(s+1)}}\cdots e_{\alpha^{(p-2)}}e_{\alpha^{(p-1)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')}\\
&\quad + a\;(\check{\alpha}^{(s)} - \langle\alpha^{(s-1)},\check{\alpha}^{(s)}\rangle)b^*_{(\underline{\mu}',\underline{\alpha}')}e_{\alpha^{(s+1)}}\cdots e_{\alpha^{(p-2)}}e_{\alpha^{(p-1)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')}.
\end{aligned}$$

The weighted path $(\underline{\mu}',\underline{\alpha}')$ and the positive root $\gamma = -\alpha^{(p)}$ verify the conditions of Lemma 3.2. Hence we have

$$\begin{aligned}
\mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})}) &= K^{\#p}_{(\underline{\mu},\underline{\alpha})}\mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}})\\
&\quad + a\;(\check{\alpha}^{(s)} - \langle\alpha^{(s-1)},\check{\alpha}^{(s)}\rangle)\mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')}e_{\alpha^{(s+1)}}\cdots e_{\alpha^{(p-1)}}b^*_{(\underline{\mu}'',\underline{\alpha}'')}).
\end{aligned}\tag{4.9}$$

Observe that the target of $\alpha^{(p-1)}$ is $\delta_j$ and the source of $\alpha^{(s+1)}$ is $\overline{\delta}_i = -\varepsilon_i$, so that $\alpha^{(s+1)} + \cdots + \alpha^{(p-1)} = -\varepsilon_i + \varepsilon_j = \delta_j - \delta_i$. Meanwhile the target of $\alpha^{(s-1)}$ is $\delta_j$ and the source of $\alpha^{(p+1)}$ is $\delta_i = \varepsilon_i$ so that $\alpha^{(p+1)} + \cdots + \alpha^{(m)} = \varepsilon_i - \varepsilon_{j_1}$, with $j_1 < j < i$.

Because of the configuration, for all $t \in \{1,\ldots,s-1\}$, $\alpha^{(t)} = \delta_{j_t} - \delta_{j_{t+1}} = \varepsilon_{j_t} - \varepsilon_{j_{t+1}}$, with $j_t < j_{t+1} \leqslant j < i$, and for all $t \in \{s+1,\ldots,p-1\}$, $\alpha^{(t)} = \overline{\delta}_{j_t} - \overline{\delta}_{j_{t+1}} = \varepsilon_{j_{t+1}} - \varepsilon_{j_t}$, with $j \leqslant j_{t+1} < j_t \leqslant i$. We see that $\big((\delta_j, \delta_{p-1}, \delta_{p-2}, \ldots, \delta_{s+2}, \delta_i), (\alpha^{(p-1)}, \ldots, \alpha^{(s+1)})\big)$ is a path from $\delta_j$ to $\delta_i$ (see Figure 4.6 for an illustration).

By reversing the order in the roots, $\alpha^{(s+1)}, \ldots, \alpha^{(p-1)}$, we obtain a path from $\delta_j$ to $\delta_i$. The operations to permute all roots induce several new paths and we denote by $N$ the number of those paths. Let $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$, $a = 1, \ldots, N$, denote the paths from $\delta_j$ to $\delta_i$ whose roots are sums among the roots $(\alpha^{(p-1)}, \ldots, \alpha^{(s+1)})$. More precisely, for each $a \in \{1, \ldots, N\}$ there exists a partition $(P_1, \ldots, P_{n_a})$ of the set $\{p-1, \ldots, s+1\}$ such that

$$(\tilde{\alpha}^{\star a})^{(s+j-1)} = \sum_{t \in P_j} \alpha^{(t)},$$

for $j = 1, \ldots, n_a$. Furthermore, we set

$$(\underline{\mu}, \underline{\alpha})^{\star a} := (\underline{\mu}', \underline{\alpha}') \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}'').$$

Hence,

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^{\#p}_{(\underline{\mu},\underline{\alpha})} \mathrm{wt}((\underline{\mu}, \underline{\alpha})^{\#p}) + (\mathrm{ht}(\check{\alpha}^{(s)}) - \langle \alpha^{(s-1)}, \check{\alpha}^{(s)} \rangle) \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\star a},$$

where $K^{\star a}_{(\underline{\mu},\underline{\alpha})}$ are some constants.

2. Assume that $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_i - \delta_j$, with $i < j$.

Write

$$\begin{aligned}(\underline{\mu}, \underline{\alpha}) =& (\underline{\mu}', \underline{\alpha}') \star \big((\mu^{(s)} = \delta_i, \mu^{(s+1)} = \delta_j, \ldots, \mu^{(p-1)}, \mu^{(p)} = \overline{\delta}_i, \mu^{(p+1)} = \overline{\delta}_j); \\ & (\alpha^{(s)}, \alpha^{(s+1)}, \ldots, \alpha^{(p-1)}, \alpha^{(p)})\big) \star (\underline{\mu}'', \underline{\alpha}''),\end{aligned}$$

where

$$(\underline{\mu}', \underline{\alpha}') = \big((\mu^{(1)}, \ldots, \mu^{(s-1)}, \mu^{(s)} = \delta_i); (\alpha^{(1)}, \ldots, \alpha^{(s-1)})\big),$$

$$(\underline{\mu}'', \underline{\alpha}'') = \big((\mu^{(p+1)} = \overline{\delta}_j, \ldots, \mu^{(m+1)}); (\alpha^{(p+1)}, \ldots, \alpha^{(m)})\big),$$

have length $s-1$ and $m-p$ respectively.

Set

$$a := a^{(c_{\alpha^{(p)}} e_{-\alpha^{(p)}})}_{\mu^{(p+1)}, \mu^{(p)}} a^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})}_{\mu^{(p)}, \mu^{(p-1)}} \cdots a^{(c_{\alpha^{(s+1)}} e_{-\alpha^{(s+1)}})}_{\mu^{(s+2)}, \mu^{(s+1)}} a^{(c_{\alpha^{(s)}} e_{-\alpha^{(s)}})}_{\mu^{(s+1)}, \mu^{(s)}}.$$

We have

$$b^*_{(\underline{\mu},\underline{\alpha})} = a b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p-1)}} e_{\alpha^{(p)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}$$

$$= \bar{K}^{\#p} b^*_{(\underline{\mu},\underline{\alpha})^{\#p}} + a\ b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')},$$

where $\bar{K}^{\#p}$ as in Lemma 4.2 depending on the different cases for $\alpha^{(p-1)}$ and $\alpha^{(p)}$.

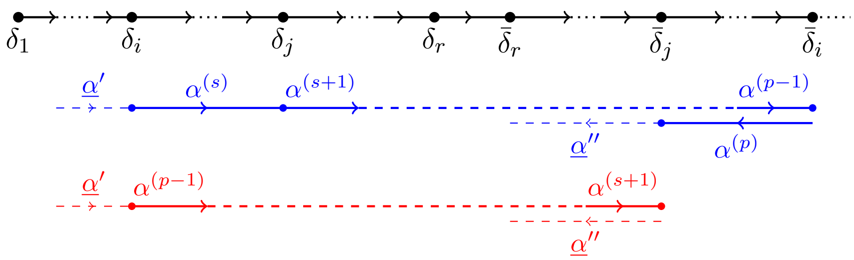



**FIGURE 4.7** – The case for $\alpha^{(p)} = \bar{\delta}_i - \bar{\delta}_j$ and $\alpha^{(s)} = \delta_i - \delta_j$.

Assume that there exists a positive root $\alpha^{(t)}$, with $t < p-1$ and $\alpha^{(t)} \neq \alpha^{(s)}$, such that $\alpha^{(t)} + \alpha^{(p)} \in \Delta$. By Lemma 2.7 we observe that there is at most one root $\alpha^{(t)}$ that satisfies such condition (see Figure 4.7). In this case, $\alpha^{(t)} + \alpha^{(p)} \in -\Delta_+$ and $s < t < p-1$.

If the root $\alpha^{(t)}$ exists, then $\alpha^{(p)}$ commutes with all roots $\alpha^{(u)}$ for $u < p$, except with $\alpha^{(p-1)}$, $\alpha^{(t)}$ and $\alpha^{(s)}$. We have

$$\begin{aligned}
&b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\quad = b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots (e_{\alpha^{(p)}} e_{\alpha^{(t)}} + n_{\alpha^{(t)},\alpha^{(p)}} e_{\alpha^{(t)}+\alpha^{(p)}}) \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\quad = b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p)}} e_{\alpha^{(s)}} \cdots e_{\alpha^{(t)}} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} + b^*_{(\underline{\mu}',\underline{\alpha}')} \check{\alpha}^{(s)} \cdots e_{\alpha^{(t)}} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&\qquad + n_{\alpha^{(t)},\alpha^{(p)}} b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(t-1)}} e_{\alpha^{(t)}+\alpha^{(p)}} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')},
\end{aligned}$$

since $\alpha^{(p)} = -\alpha^{(s)}$.

The weighted path $(\underline{\mu}', \underline{\alpha}')$ and the positive root $\gamma = -\alpha^{(p)}$ verify the conditions of Lemma 3.2, and also the weighted path $(\underline{\mu}', \underline{\alpha}') \star \big((\mu^{(s)}, \ldots, \mu^{(t)}); (\alpha^{(s)}, \ldots, \alpha^{(t-1)})\big)$ and the positive root $-(\alpha^{(t)} + \alpha^{(p)})$. The arguments are similar as (4.3) and so we omit the details. We get

$$\begin{aligned}
&\mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}) \\
&\quad = (\check{\alpha}^{(s)} - \langle \alpha^{(s-1)}, \check{\alpha}^{(s)} \rangle)\ \mathrm{hc}\big(b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}\big).
\end{aligned}$$

If the root $\alpha^{(t)}$ does not exist, then $\alpha^{(p)}$ commutes with all roots $\alpha^{(u)}$ for $u < p$, except with $\alpha^{(p-1)}$ and $\alpha^{(s)}$. We have

$$\begin{aligned} & b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\ & \qquad = b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p)}} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p-2)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} + b^*_{(\underline{\mu}',\underline{\alpha}')} \check{\alpha}^{(s)} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}, \end{aligned}$$

since $\alpha^{(p)} = -\alpha^{(s)}$. Observe that the weighted path $(\underline{\mu}', \underline{\alpha}')$ and the positive root $\gamma = -\alpha^{(p)}$ satisfy the conditions of Lemma 3.2. The arguments are the same as (4.3), and we get

$$\begin{aligned} & \mathrm{hc}(b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s)}} \cdots e_{\alpha^{(p)}} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}) \\ & \qquad = (\check{\alpha}^{(s)} - \langle \alpha^{(s-1)}, \check{\alpha}^{(s)} \rangle)\ \mathrm{hc}\big(b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}\big). \end{aligned}$$

On account of the above computation, we get

$$\begin{aligned} \mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})}) &= \bar{K}^{\#p} \mathrm{hc}(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}) \\ & \quad + a\ (\check{\alpha}^{(s)} - \langle \alpha^{(s-1)}, \check{\alpha}^{(s)} \rangle) \mathrm{hc}\big(b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}\big). \end{aligned} \tag{4.10}$$

Observe that the target of $\alpha^{(p-1)}$ is $\overline{\delta}_i = -\varepsilon_i$ and the source of $\alpha^{(s+1)}$ is $\delta_j = \varepsilon_j$, so that $\alpha^{(s+1)} + \cdots + \alpha^{(p-1)} = \varepsilon_j + \varepsilon_i = \delta_j - \overline{\delta}_i = \delta_i - \overline{\delta}_j$. Note also that the target of $\alpha^{(s-1)}$ is $\delta_i$, while $\alpha^{(p-1)} = \mu^{(p-1)} - \overline{\delta}_i = \delta_i - \overline{\mu}^{(p-1)}$ and $\alpha^{(s+1)} = \delta_j - \mu^{(s+2)} = \overline{\mu}^{(s+2)} - \overline{\delta}_j$. Hence we see that $\big((\delta_i, \overline{\mu}^{(p-1)}, \overline{\mu}^{(p-2)}, \ldots, \overline{\mu}^{(s+2)}, \overline{\delta}_j); (\alpha^{(p-1)}, \ldots, \alpha^{(s+1)})\big)$ is a path from $\delta_i$ to $\overline{\delta}_j$ (see Figure 4.7 for an illustration).

So we have to "reverse the order" of $e_{\alpha^{(s+1)}} \cdots e_{\alpha^{(p-1)}}$ in (4.10), in order to get $e_{\alpha^{(p-1)}} \cdots e_{\alpha^{(s+1)}}$. Doing this, it induces several new paths and we denote by $N$ the number of those paths. Let $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$, $a = 1, \ldots, N$, denote the paths from $\delta_j$ to $\delta_i$ whose roots are sums among the roots $(\alpha^{(p-1)}, \ldots, \alpha^{(s+1)})$. More precisely, for each $a \in \{1, \ldots, N\}$ there exists a partition $(P_1, \ldots, P_{n_a})$ of the set $\{p-1, \ldots, s+1\}$ such that

$$(\tilde{\alpha}^{\star a})^{(s+j-1)} = \sum_{t \in P_j} \alpha^{(t)},$$

for $j = 1, \ldots, n_a$. Furthermore, we set

$$(\underline{\mu}, \underline{\alpha})^{\star a} := (\underline{\mu}', \underline{\alpha}') \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}'').$$

Hence,

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \bar{K}^{\#p}\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} + (\mathrm{ht}(\check{\alpha}^{(s)}) - \langle \alpha^{(s-1)}, \check{\alpha}^{(s)} \rangle) \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\star a},$$

where $\bar{K}^{\#p}$ is as in Lemma 4.2 depending on the different cases for $\alpha^{(p-1)}$ and $\alpha^{(p)}$ and $K^{\star a}_{(\underline{\mu},\underline{\alpha})}$ are some constants. □

*Example* 4.5. Assume that $\mathfrak{g} = \mathfrak{sp}_{12}$, $\delta = \varpi_1$, $\underline{i} \in \mathbb{Z}^m_{\succ \underline{0}}$.

1. Let $\alpha^{(p)} = \overline{\delta}_j - \delta_i = -\varepsilon_j - \varepsilon_i$, and $\alpha^{(s)} = \delta_k - \overline{\delta}_i = \varepsilon_k + \varepsilon_i$, with $k < j < i$. Set $k = 2, j = 3, i = 6$, we have

$$\alpha^{(p)} = -\varepsilon_3 - \varepsilon_6, \quad \alpha^{(s)} = \varepsilon_2 + \varepsilon_6, \quad \alpha^{(s)} + \alpha^{(p)} = \varepsilon_2 - \varepsilon_3.$$

Let $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_8(\delta_1)_{\underline{i}}$ with

$$\begin{aligned}
\underline{\mu} &= (\delta_1, \delta_2, \overline{\delta}_6, \overline{\delta}_5, \overline{\delta}_4, \overline{\delta}_3, \delta_6, \delta_4, \delta_1) = (\varepsilon_1, \varepsilon_2, -\varepsilon_6, -\varepsilon_5, -\varepsilon_4, -\varepsilon_3, \varepsilon_6, \varepsilon_4, \varepsilon_1),\\
\underline{\alpha} &= (\alpha^{(1)}, \alpha^{(2)} = \alpha^{(s)}, \alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)} = \alpha^{(p-1)}, \alpha^{(6)} = \alpha^{(p)}, \alpha^{(7)}, \alpha^{(8)})\\
&= (\varepsilon_1 - \varepsilon_2, \varepsilon_2 + \varepsilon_6, \varepsilon_5 - \varepsilon_6, \varepsilon_4 - \varepsilon_5, \varepsilon_3 - \varepsilon_4, -\varepsilon_3 - \varepsilon_6, \varepsilon_6 - \varepsilon_4, \varepsilon_4 - \varepsilon_1)\\
\underline{i} &= (1, 5, 1, 1, 1, -4, -2, -3).
\end{aligned}$$

We get

$$a^{(c_{\alpha^{(m)}} e_{-\alpha^{(m)}})}_{\mu^{(m+1)},\mu^{(m)}} = a^{(e_{\varepsilon_1 - \varepsilon_4})}_{\varepsilon_1,\varepsilon_4} = 1, \quad a^{(e_{\varepsilon_4 - \varepsilon_6})}_{\varepsilon_4,\varepsilon_6} = 1, \quad a^{(e_{\varepsilon_3 + \varepsilon_6})}_{\varepsilon_6,-\varepsilon_3} = 1, \quad a^{(e_{\varepsilon_4 - \varepsilon_3})}_{-\varepsilon_3,-\varepsilon_4} = -1,$$

$$a^{(e_{\varepsilon_5 - \varepsilon_4})}_{-\varepsilon_4,-\varepsilon_5} = -1, \quad a^{(e_{\varepsilon_6 - \varepsilon_5})}_{-\varepsilon_5,-\varepsilon_6} = -1, \quad a^{(e_{-\varepsilon_2 - \varepsilon_6})}_{-\varepsilon_6,\varepsilon_2} = 1, \quad a^{(e_{\varepsilon_2 - \varepsilon_1})}_{\varepsilon_2,\varepsilon_1} = 1,$$

$$a^{(c_{\alpha^{(p-1)}+\alpha^{(p)}} e_{-\alpha^{(p-1)}-\alpha^{(p)}})}_{\mu^{(p+1)},\mu^{(p-1)}} = a^{(e_{\varepsilon_6 + \varepsilon_4})}_{\varepsilon_6,-\varepsilon_4} = 1, \quad n_{\alpha^{(p-1)},\alpha^{(p)}} = n_{\alpha^{(5)},\alpha^{(6)}} = -1,$$

$$n_{\alpha^{(s)},\alpha^{(p)}} = 1, \quad n_{\alpha^{(3)},\alpha^{(4)}} = n_{\alpha^{(4)},\alpha^{(5)}} = n_{\alpha^{(3)}+\alpha^{(4)},\alpha^{(5)}} = -1.$$

Thus,

$$a_{\underline{\mu},\underline{\alpha}} = 1, \quad \text{and } K^{\#p}_{(\underline{\mu},\underline{\alpha})} = n_{\alpha^{(p-1)},\alpha^{(p)}} \frac{a^{(c_{\alpha^{(p)}} e_{-\alpha^{(p)}})}_{\mu^{(p+1)},\mu^{(p)}} a^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})}_{\mu^{(p)},\mu^{(p-1)}}}{a^{(c_{\alpha^{(p-1)}+\alpha^{(p)}} e_{-\alpha^{(p-1)}-\alpha^{(p)}})}_{\mu^{(p+1)},\mu^{(p-1)}}} = 1.$$

We have by (4.4)

$$\begin{aligned}
\mathrm{hc}\big(b^*_{(\underline{\mu},\underline{\alpha})}\big) &= K^{\#p}_{(\underline{\mu},\underline{\alpha})}\mathrm{hc}\big(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}\big) + a_{\underline{\mu},\underline{\alpha}} n_{\alpha^{(s)},\alpha^{(p)}}\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(3)}}e_{\alpha^{(4)}}e_{\alpha^{(5)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}\big)\\
&= K^{\#p}_{(\underline{\mu},\underline{\alpha})}\mathrm{hc}\big(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}\big) + a_{\underline{\mu},\underline{\alpha}} n_{\alpha^{(s)},\alpha^{(p)}}\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(4)}}e_{\alpha^{(3)}}e_{\alpha^{(5)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}\big)\\
&\quad + a_{\underline{\mu},\underline{\alpha}} n_{\alpha^{(s)},\alpha^{(p)}} n_{\alpha^{(3)},\alpha^{(4)}}\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(3)}+\alpha^{(4)}}e_{\alpha^{(5)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}\big)\\
&= K^{\#p}_{(\underline{\mu},\underline{\alpha})}\mathrm{hc}\big(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}\big) + a_{\underline{\mu},\underline{\alpha}} n_{\alpha^{(s)},\alpha^{(p)}}\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(4)}}e_{\alpha^{(5)}}e_{\alpha^{(3)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}\big)\\
&\quad + n_{\alpha^{(3)},\alpha^{(4)}}\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(5)}}e_{\alpha^{(3)}+\alpha^{(4)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}\big)\\
&\quad + a_{\underline{\mu},\underline{\alpha}} n_{\alpha^{(s)},\alpha^{(p)}} n_{\alpha^{(3)},\alpha^{(4)}} n_{\alpha^{(3)}+\alpha^{(4)},\alpha^{(5)}}\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(3)}+\alpha^{(4)}+\alpha^{(5)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}\big)\\
&= K^{\#p}_{(\underline{\mu},\underline{\alpha})}\mathrm{hc}\big(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}\big) + a_{\underline{\mu},\underline{\alpha}} n_{\alpha^{(s)},\alpha^{(p)}}\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(5)}}e_{\alpha^{(4)}}e_{\alpha^{(3)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}\big)\\
&\quad + a_{\underline{\mu},\underline{\alpha}} n_{\alpha^{(s)},\alpha^{(p)}} n_{\alpha^{(4)},\alpha^{(5)}}\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(4)}+\alpha^{(5)}}e_{\alpha^{(3)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}\big)\\
&\quad + a_{\underline{\mu},\underline{\alpha}} n_{\alpha^{(s)},\alpha^{(p)}} n_{\alpha^{(3)},\alpha^{(4)}}\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(5)}}e_{\alpha^{(3)}+\alpha^{(4)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}\big)\\
&\quad + a_{\underline{\mu},\underline{\alpha}} n_{\alpha^{(s)},\alpha^{(p)}} n_{\alpha^{(3)},\alpha^{(4)}} n_{\alpha^{(3)}+\alpha^{(4)},\alpha^{(5)}}\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(3)}+\alpha^{(4)}+\alpha^{(5)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}\big),
\end{aligned}$$

since $\alpha^{(5)}$ commutes with $\alpha^{(3)}$. Hence,

$$\begin{aligned}
\mathrm{wt}(\underline{\mu},\underline{\alpha}) = \mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p} &- \mathrm{wt}\big(e_{\alpha^{(1)}}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(5)}}e_{\alpha^{(4)}}e_{\alpha^{(3)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}\big)\\
&+ \mathrm{wt}\big(e_{\alpha^{(1)}}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(4)}+\alpha^{(5)}}e_{\alpha^{(3)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}\big)\\
&+ \mathrm{wt}\big(e_{\alpha^{(1)}}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(5)}}e_{\alpha^{(3)}+\alpha^{(4)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}\big)\\
&- \mathrm{wt}\big(e_{\alpha^{(1)}}e_{\alpha^{(s)}+\alpha^{(p)}}e_{\alpha^{(3)}+\alpha^{(4)}+\alpha^{(5)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}\big).
\end{aligned}$$

2. Let $\alpha^{(p)} = \overline{\delta}_j - \delta_i = -\varepsilon_j - \varepsilon_i$, and $\alpha^{(s)} = \delta_j - \overline{\delta}_i = \varepsilon_j + \varepsilon_i$, with $j < i$. Set $j = 2, i = 5$, we have

$$\alpha^{(p)} = -\varepsilon_2 - \varepsilon_5, \quad \alpha^{(s)} = \varepsilon_2 + \varepsilon_5, \quad \alpha^{(s)} = -\alpha^{(p)}.$$

Let $(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_7(\delta_1)_{\underline{i}}$ with

$$\begin{aligned}
\underline{\mu} &= (\delta_1, \delta_2, \overline{\delta}_5, \overline{\delta}_4, \overline{\delta}_3, \overline{\delta}_2, \delta_5, \delta_1) = (\varepsilon_1, \varepsilon_2, -\varepsilon_5, -\varepsilon_4, -\varepsilon_3, -\varepsilon_2, \varepsilon_5, \varepsilon_1),\\
\underline{\alpha} &= (\alpha^{(1)}, \alpha^{(2)} = \alpha^{(s)}, \alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)} = \alpha^{(p-1)}, \alpha^{(6)} = \alpha^{(p)}, \alpha^{(7)})\\
&= (\varepsilon_1 - \varepsilon_2, \varepsilon_2 + \varepsilon_5, \varepsilon_4 - \varepsilon_5, \varepsilon_3 - \varepsilon_4, \varepsilon_2 - \varepsilon_3, -\varepsilon_2 - \varepsilon_5, \varepsilon_5 - \varepsilon_1),\\
\underline{i} &= (1, 6, 1, 1, 1, -6, -4).
\end{aligned}$$

We get,

$$a_{\varepsilon_1,\varepsilon_5}^{(e_{\varepsilon_1-\varepsilon_5})} = 1, \quad a_{\varepsilon_5,-\varepsilon_2}^{(e_{\varepsilon_2+\varepsilon_5})} = 1, \quad a_{-\varepsilon_2,-\varepsilon_3}^{(e_{\varepsilon_3-\varepsilon_2})} = -1, \quad a_{-\varepsilon_3,-\varepsilon_4}^{(e_{\varepsilon_4-\varepsilon_3})} = -1, \quad a_{-\varepsilon_4,-\varepsilon_5}^{(e_{\varepsilon_5-\varepsilon_4})} = -1,$$

$$a_{-\varepsilon_5,\varepsilon_2}^{(e_{-\varepsilon_2-\varepsilon_5})} = 1, \quad a_{\varepsilon_2,\varepsilon_1}^{(e_{\varepsilon_2-\varepsilon_1})} = 1, \quad a_{\mu^{(p+1)},\mu^{(p-1)}}^{(c_{\alpha^{(p-1)}+\alpha^{(p)}}e_{-\alpha^{(p-1)}-\alpha^{(p)}})} = a_{\varepsilon_5,-\varepsilon_3}^{(e_{\varepsilon_5+\varepsilon_3})} = 1,$$

$$n_{\alpha^{(5)},\alpha^{(6)}} = n_{\alpha^{(3)},\alpha^{(4)}} = n_{\alpha^{(4)},\alpha^{(5)}} = n_{\alpha^{(3)}+\alpha^{(4)},\alpha^{(5)}} = -1.$$

Thus,

$$a_{\underline{\mu},\underline{\alpha}} = -1 \quad \text{and} \quad K_{(\underline{\mu},\underline{\alpha})}^{\#p} = n_{\alpha^{(p-1)},\alpha^{(p)}} \frac{a_{\mu^{(p+1)},\mu^{(p)}}^{(c_{\alpha^{(p)}}e_{-\alpha^{(p)}})} a_{\mu^{(p)},\mu^{(p-1)}}^{(c_{\alpha^{(p-1)}}e_{-\alpha^{(p-1)}})}}{a_{\mu^{(p+1)},\mu^{(p-1)}}^{(c_{\alpha^{(p-1)}+\alpha^{(p)}}e_{-\alpha^{(p-1)}-\alpha^{(p)}})}} = 1.$$

We have by (4.9),

$$\begin{aligned}
\mathrm{hc}\big(b^*_{(\underline{\mu},\underline{\alpha})}\big) &= K_{(\underline{\mu},\underline{\alpha})}^{\#p}\mathrm{hc}\big(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}\big) + a_{\underline{\mu},\underline{\alpha}}(\check{\alpha}^{(s)}+1)\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(3)}}e_{\alpha^{(4)}}e_{\alpha^{(5)}}e_{\alpha^{(7)}}\big) \\
&= K_{(\underline{\mu},\underline{\alpha})}^{\#p}\mathrm{hc}\big(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}\big) + a_{\underline{\mu},\underline{\alpha}}(\check{\alpha}^{(s)}+1)\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(4)}}e_{\alpha^{(3)}}e_{\alpha^{(5)}}e_{\alpha^{(7)}}\big) \\
&\quad + a_{\underline{\mu},\underline{\alpha}}n_{\alpha^{(3)},\alpha^{(4)}}(\check{\alpha}^{(s)}+1)\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(3)}+\alpha^{(4)}}e_{\alpha^{(5)}}e_{\alpha^{(7)}}\big) \\
&= K_{(\underline{\mu},\underline{\alpha})}^{\#p}\mathrm{hc}\big(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}\big) + a_{\underline{\mu},\underline{\alpha}}(\check{\alpha}^{(s)}+1)\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(4)}}e_{\alpha^{(5)}}e_{\alpha^{(3)}}e_{\alpha^{(7)}}\big) \\
&\quad + a_{\underline{\mu},\underline{\alpha}}n_{\alpha^{(3)},\alpha^{(4)}}(\check{\alpha}^{(s)}+1)\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(5)}}e_{\alpha^{(3)}+\alpha^{(4)}}e_{\alpha^{(7)}}\big) \\
&\quad + a_{\underline{\mu},\underline{\alpha}}n_{\alpha^{(3)},\alpha^{(4)}}n_{\alpha^{(3)}+\alpha^{(4)},\alpha^{(5)}}(\check{\alpha}^{(s)}+1)\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(3)}+\alpha^{(4)}+\alpha^{(5)}}e_{\alpha^{(7)}}\big) \\
&= K_{(\underline{\mu},\underline{\alpha})}^{\#p}\mathrm{hc}\big(b^*_{(\underline{\mu},\underline{\alpha})^{\#p}}\big) + a_{\underline{\mu},\underline{\alpha}}(\check{\alpha}^{(s)}+1)\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(5)}}e_{\alpha^{(4)}}e_{\alpha^{(3)}}e_{\alpha^{(7)}}\big) \\
&\quad + a_{\underline{\mu},\underline{\alpha}}n_{\alpha^{(4)},\alpha^{(5)}}(\check{\alpha}^{(s)}+1)\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(4)}+\alpha^{(5)}}e_{\alpha^{(3)}}e_{\alpha^{(7)}}\big) \\
&\quad + a_{\underline{\mu},\underline{\alpha}}n_{\alpha^{(3)},\alpha^{(4)}}(\check{\alpha}^{(s)}+1)\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(5)}}e_{\alpha^{(3)}+\alpha^{(4)}}e_{\alpha^{(7)}}\big) \\
&\quad + a_{\underline{\mu},\underline{\alpha}}n_{\alpha^{(3)},\alpha^{(4)}}n_{\alpha^{(3)}+\alpha^{(4)},\alpha^{(5)}}(\check{\alpha}^{(s)}+1)\mathrm{hc}\big(e_{\alpha^{(1)}}e_{\alpha^{(3)}+\alpha^{(4)}+\alpha^{(5)}}e_{\alpha^{(7)}}\big),
\end{aligned}$$

since $\alpha^{(5)}$ commutes with $\alpha^{(3)}$. Hence,

$$\begin{aligned}
\mathrm{wt}(\underline{\mu},\underline{\alpha}) = \mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p} &- (\mathrm{ht}(\check{\alpha}^{(s)})+1)\mathrm{wt}(e_{\alpha^{(1)}}e_{\alpha^{(5)}}e_{\alpha^{(4)}}e_{\alpha^{(3)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}) \\
&+ (\mathrm{ht}(\check{\alpha}^{(s)})+1)\mathrm{wt}(e_{\alpha^{(1)}}e_{\alpha^{(4)}+\alpha^{(5)}}e_{\alpha^{(3)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}) \\
&+ (\mathrm{ht}(\check{\alpha}^{(s)})+1)\mathrm{wt}(e_{\alpha^{(1)}}e_{\alpha^{(5)}}e_{\alpha^{(3)}+\alpha^{(4)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}) \\
&- (\mathrm{ht}(\check{\alpha}^{(s)})+1)\mathrm{wt}(e_{\alpha^{(1)}}e_{\alpha^{(3)}+\alpha^{(4)}+\alpha^{(5)}}e_{\alpha^{(7)}}e_{\alpha^{(8)}}).
\end{aligned}$$

3. Let $\alpha^{(p-1)} = \overline{\delta}_j - \overline{\delta}_i = -\alpha^{(p)}$ and $\alpha^{(s)} = \delta_i - \delta_l = \varepsilon_i - \varepsilon_l$, with $i < j < l$.
   Set $i = 2,\ j = 3,\ l = 4$, we have

$$\alpha^{(p-1)} = \varepsilon_2 - \varepsilon_3, \quad -\alpha^{(p)} = \varepsilon_3 - \varepsilon_2, \quad \alpha^{(s)} = \varepsilon_2 - \varepsilon_4, \quad \alpha^{(s)} + \alpha^{(p)} = \varepsilon_3 - \varepsilon_4.$$

   Let $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_7(\delta_2)_{\underline{i}}$ with

$$\begin{aligned}
\underline{\mu} &= (\delta_2, \delta_4, \delta_5, \overline{\delta}_4, \overline{\delta}_3, \overline{\delta}_2, \overline{\delta}_3, \delta_2) = (\varepsilon_2, \varepsilon_4, \varepsilon_5, -\varepsilon_4, -\varepsilon_3, -\varepsilon_2, -\varepsilon_3, \varepsilon_2),\\
\underline{\alpha} &= (\alpha^{(1)} = \alpha^{(s)}, \alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)} = \alpha^{(p-1)}, \alpha^{(6)} = \alpha^{(p)}, \alpha^{(7)})\\
&= (\varepsilon_2 - \varepsilon_4, \varepsilon_4 - \varepsilon_5, \varepsilon_5 + \varepsilon_4, \varepsilon_3 - \varepsilon_4, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_2, -\varepsilon_3 - \varepsilon_2),\\
\underline{i} &= (2, 1, 4, 1, 1, -1, -8).
\end{aligned}$$

   Thus $a_{\underline{\mu},\underline{\alpha}} = -1$ and

$$\begin{aligned}
\bar{K}^{\#p}_{(\underline{\mu},\underline{\alpha})} &= (c_{\alpha^{(p-1)}})^2\,(\mathrm{ht}(\check{\alpha}^{(p-1)}) - c_{p-1})\\
&= \mathrm{ht}(\check{\alpha}^{(5)}) - \langle \varepsilon_2 - \varepsilon_4, \varepsilon_2 - \varepsilon_3 \rangle - \langle \varepsilon_3 - \varepsilon_4, \varepsilon_2 - \varepsilon_3 \rangle\\
&= \mathrm{ht}(\check{\alpha}^{(5)}).
\end{aligned}$$

   Doing the similar calculation as above examples, we get

$$\begin{aligned}
\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \mathrm{ht}(\check{\alpha}^{(5)})\,\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} &- \mathrm{wt}(e_{\alpha^{(5)}} e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(2)}} e_{\alpha^{(3)}} e_{\alpha^{(4)}} e_{\alpha^{(7)}})\\
&+ 2\,\mathrm{wt}(e_{\alpha^{(5)}+\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(2)}} e_{\alpha^{(3)}} e_{\alpha^{(4)}} e_{\alpha^{(7)}})\\
&+ \mathrm{wt}(e_{\alpha^{(4)}+\alpha^{(5)}} e_{\alpha^{(3)}} e_{\alpha^{(2)}+\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(7)}})\\
&- \mathrm{wt}(e_{\alpha^{(3)}+\alpha^{(4)}+\alpha^{(5)}} e_{\alpha^{(2)}+\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(7)}})\\
&- \mathrm{wt}(e_{\alpha^{(3)}+\alpha^{(4)}+\alpha^{(5)}} e_{\alpha^{(2)}} e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(7)}})\\
&+ \mathrm{wt}(e_{\alpha^{(4)}+\alpha^{(5)}} e_{\alpha^{(3)}+\alpha^{(2)}+\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(7)}})\\
&+ \mathrm{wt}(e_{\alpha^{(4)}+\alpha^{(5)}} e_{\alpha^{(2)}} e_{\alpha^{(3)}+\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(7)}})\\
&- \mathrm{wt}(e_{\alpha^{(2)}+\alpha^{(4)}+\alpha^{(5)}} e_{\alpha^{(3)}} e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(7)}})\\
&- \mathrm{wt}(e_{\alpha^{(2)}+\alpha^{(4)}+\alpha^{(5)}} e_{\alpha^{(3)}+\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(7)}})\\
&- \mathrm{wt}(e_{\alpha^{(2)}+\alpha^{(3)}+\alpha^{(4)}+\alpha^{(5)}} e_{\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(7)}})\\
&- \mathrm{wt}(e_{\alpha^{(2)}+\alpha^{(3)}+\alpha^{(4)}+\alpha^{(5)}+\alpha^{(s)}+\alpha^{(p)}} e_{\alpha^{(7)}}).
\end{aligned}$$

### 4.1.2 Case $p = q$ with loops

We now consider in this subsection the weighted path of lengths $m$ with $n$ loops. We assume in this paragraph that $n < m$. If $n = m$, then we can argue as in the symmetric case (see Section 2.3) and the weight of the paths is known by Lemma 2.4.

Let $\underline{i} \in \mathbb{Z}^m_{\succ 0}$ and $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)_{\underline{i}}$ with $n$ loops in the positions $j_1 \leqslant \ldots \leqslant j_n$ with $n < m$. This means that for $l = 1, \ldots, n$, $\mu^{(j_l)} = \mu^{(j_l+1)}$, $\alpha^{(j_l)} \in \Pi$ and $b^*_{(\underline{\mu},\underline{\alpha}),j_l} = \varpi^\sharp_{\alpha^{(j_l)}}$.

**Lemma 4.6.** *Let $m \in \mathbb{Z}_{>0}$, $\mu \in P(\delta)$, $\underline{i} \in \mathbb{Z}^m_{\succ 0}$ and $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)_{\underline{i}}$. Assume that the weighted path $(\underline{\mu}, \underline{\alpha})$ has $n$ loops in the position $j_1, \ldots, j_n$, $0 < n < m$. Let $j'_{l,1}, \ldots, j'_{l,n'}$ be integers of $\{1, \ldots, j_l - 1\}$, for $l = 1, \ldots, n$, such that* $\mathrm{supp}(\alpha^{(j'_{l,t})})$ *contains the simple root $\alpha^{(j_l)}$. Hence,*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \hat{K}\mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}),$$

*where*

$$\hat{K} := \prod_{j_l=1}^{n} \langle \mu^{(j_l)}, \check{\alpha}^{(j_l)} \rangle \Big( \langle \rho, \varpi^\sharp_{\alpha^{(j_l)}} \rangle - \sum_{\substack{j'_{l,t} \in \{1,\ldots,j_l-1\}, \\ \alpha^{(j_l)} \in \mathrm{supp}(\alpha^{(j'_{l,t})})}} \langle \alpha^{(j'_{l,t})}, \varpi^\sharp_{\alpha^{(j_l)}} \rangle \Big)$$

*and $(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})$ is the weighted path of length $m - n$ obtained from $(\underline{\mu}, \underline{\alpha})$ by "removing all loops" from the path. In particular, if $n = 0$, we have $\hat{K} = 1$.*

*Proof.* First of all, if $n = 0$, then the results are known by Lemma 4.2, Lemma 4.3 and Lemma 4.4. Assume that the weighted path $(\underline{\mu}, \underline{\alpha})$ has $n$ loops in the position $j_1, \ldots, j_n$. Write

$$(\underline{\mu}, \underline{\alpha}) = \big((\mu^{(1)}, \ldots, \mu^{(j_1)}, \mu^{(j_1+1)} \ldots, \mu^{(j_n+1)}), (\alpha^{(1)}, \ldots, \alpha^{(j_1)}, \alpha^{(j_1+1)}, \ldots, \alpha^{(j_n)})\big) \star (\underline{\mu}'', \underline{\alpha}''),$$

where

$$(\underline{\mu}'', \underline{\alpha}'') = \big((\mu^{(j_n+1)}, \ldots, \mu^{(m+1)}), (\alpha^{(j_n+1)}, \ldots, \alpha^{(m)})\big)),$$

and $(\underline{\mu}'', \underline{\alpha}'')$ has length $m - j_n$. We have $i_{j_l} = 0$ and $\alpha^{(j_l)} \in \Pi$. For each $j_l, l \in 1, \ldots, n$, denote by $j'_{l,1}, \ldots, j'_{l,n'}$ the integers of $\{1, \ldots, j_l - 1\}$ such that simple root $\alpha^{(j_l)}$ appears in the support of $(\alpha^{(j'_{l,t})})$, thus $[e_{\alpha^{(j'_{l,t})}}, \varpi^\sharp_{\alpha^{(j_l)}}] = -\langle \alpha^{(j'_{l,t})}, \varpi^\sharp_{\alpha^{(j_l)}} \rangle e_{\alpha^{(j'_{l,t})}} \neq 0$. Hence, $e_{\alpha^{(j'_{l,t})}} \varpi^\sharp_{\alpha^{(j_l)}} \neq \varpi^\sharp_{\alpha^{(j_l)}} e_{\alpha^{(j'_{l,t})}}$. In other words, $\varpi^\sharp_{\alpha^{(j_l)}}$ commutes with $e_{\alpha^{(s)}}$, where $s \neq j'_{l,t}$ for $l \in \{1, \ldots, n\}, t \in \{1, \ldots, n'\}$.

Set

$$a := a^{(b_{(\underline{\mu},\underline{\alpha}),j_n})}_{\mu^{(j_n+1)},\mu^{(j_n)}} a^{(b_{(\underline{\mu},\underline{\alpha}),j_n-1})}_{\mu^{(j_n)},\mu^{(j_n-1)}} \cdots a^{(b_{(\underline{\mu},\underline{\alpha}),j_1})}_{\mu^{(j_1+1)},\mu^{(j_1)}} \cdots a^{(b_{(\underline{\mu},\underline{\alpha}),1})}_{\mu^{(2)},\mu^{(1)}}$$

In the same manner as in (4.3), we have

$$\begin{aligned}
b^*_{(\underline{\mu},\underline{\alpha})} &= a e_{\alpha^{(1)}} \cdots \varpi^\sharp_{\alpha^{(j_1)}} \cdots \varpi^\sharp_{\alpha^{(j_2)}} \cdots \varpi^\sharp_{\alpha^{(j_n)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= a\big(\varpi^\sharp_{\alpha^{(j_1)}} - \langle \alpha^{(j'_{1,1})}, \varpi^\sharp_{\alpha^{(j_1)}}\rangle - \cdots \langle \alpha^{(j'_{1,n'_1})}, \varpi^\sharp_{\alpha^{(j_1)}}\rangle\big) e_{\alpha^{(1)}} \cdots \varpi^\sharp_{\alpha^{(j_2)}} \cdots \varpi^\sharp_{\alpha^{(j_n)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= a\big(\varpi^\sharp_{\alpha^{(j_1)}} - \langle \alpha^{(j'_{1,1})}, \varpi^\sharp_{\alpha^{(j_1)}}\rangle - \cdots \langle \alpha^{(j'_{1,n'_1})}, \varpi^\sharp_{\alpha^{(j_1)}}\rangle\big) \\
&\quad \times \big(\varpi^\sharp_{\alpha^{(j_2)}} - \langle \alpha^{(j'_{2,1})}, \varpi^\sharp_{\alpha^{(j_2)}}\rangle - \cdots \langle \alpha^{(j'_{2,n'_2})}, \varpi^\sharp_{\alpha^{(j_2)}}\rangle\big) e_{\alpha^{(1)}} \cdots \varpi^\sharp_{\alpha^{(j_n)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= a\big(\varpi^\sharp_{\alpha^{(j_1)}} - \langle \alpha^{(j'_{1,1})}, \varpi^\sharp_{\alpha^{(j_1)}}\rangle - \cdots \langle \alpha^{(j'_{1,n'_1})}, \varpi^\sharp_{\alpha^{(j_1)}}\rangle\big) \\
&\quad \times \big(\varpi^\sharp_{\alpha^{(j_2)}} - \langle \alpha^{(j'_{2,1})}, \varpi^\sharp_{\alpha^{(j_2)}}\rangle - \cdots \langle \alpha^{(j'_{2,n'_2})}, \varpi^\sharp_{\alpha^{(j_2)}}\rangle\big) \\
&\quad \times \cdots \times \big(\varpi^\sharp_{\alpha^{(j_n)}} - \langle \alpha^{(j'_{n,1})}, \varpi^\sharp_{\alpha^{(j_n)}}\rangle - \cdots \langle \alpha^{(j'_{n,n'_n})}, \varpi^\sharp_{\alpha^{(j_n)}}\rangle\big) e_{\alpha^{(1)}} \cdots b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= \prod_{j_k=1}^{n} \langle \mu^{(j_k)}, \check{\alpha}^{(j_k)}\rangle \Big(\varpi^\sharp_{\alpha^{(j_k)}} - \sum_{\substack{j'_{k,t}\in\{1,\dots,j_k-1\},\\ \alpha^{(j_k)}\in \operatorname{supp}(\alpha^{(j'_{k,t})})}} \langle \alpha^{(j'_{k,t})}, \varpi^\sharp_{\alpha^{(j_k)}}\rangle\Big) b^*_{(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})}.
\end{aligned}$$

Hence we get the statement. □

*Example* 4.7. Let $r = 8$, $\underline{i} \in \mathbb{Z}^m_{\succ 0}$ and $(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_1)_{\underline{i}}$ with

$$(\underline{\mu},\underline{\alpha}) = (\underline{\mu}',\underline{\alpha}') \star (\underline{\mu}'',\underline{\alpha}''),$$

where

$$(\underline{\mu}',\underline{\alpha}') = \big((\delta_1,\delta_2,\overline{\delta}_6,\overline{\delta}_6,\overline{\delta}_5,\overline{\delta}_5,\overline{\delta}_3,\delta_6),(\varepsilon_1-\varepsilon_2,\varepsilon_2+\varepsilon_6,\beta_6,\varepsilon_5-\varepsilon_6,\beta_5,\varepsilon_3-\varepsilon_5,-\varepsilon_3-\varepsilon_6)\big),$$

and $(\underline{\mu}'',\underline{\alpha}'') = \big((\delta_6,\dots,\delta_1),(\alpha^{(9)},\dots,\alpha^{(m)})\big)$ is a weighted path without loop.
We have

$$j_1 = 3,\quad \alpha^{(j_1)} = \beta_6,\quad b^*_{(\underline{\mu},\underline{\alpha}),j_1} = \varpi^\sharp_{\beta_6} = \varpi^\sharp_6,$$

$$j_2 = 5,\quad \alpha^{(j_2)} = \beta_3,\quad b^*_{(\underline{\mu},\underline{\alpha}),j_2} = \varpi^\sharp_{\beta_3} = \varpi^\sharp_3,$$

Let $j'_{l,t}$ be integers of $\{1,\dots,j_l-1\}$, where $l = 1,2$, such that $\operatorname{supp}(\alpha^{(j'_{l,t})})$ contains the simple root $\alpha^{(j_l)}$. Thus,

$$j'_{1,1} = 2,\quad j'_{2,1} = 2 \text{ and } j'_{2,2} = 4.$$

It means that $\varpi^\sharp_6$ does not commute with $\alpha^{(2)}$, and $\varpi^\sharp_3$ does not commute with $\alpha^{(2)}$ and $\alpha^{(3)}$.

Set

$$a' = a^{(e_{\varepsilon_3+\varepsilon_6})}_{\varepsilon_6,-\varepsilon_3} a^{(e_{\varepsilon_5-\varepsilon_3})}_{-\varepsilon_3,-\varepsilon_5} a^{(\check{\beta}_5)}_{-\varepsilon_5,-\varepsilon_5} a^{(e_{\varepsilon_6-\varepsilon_5})}_{-\varepsilon_5,-\varepsilon_6} a^{(\check{\beta}_6)}_{-\varepsilon_5,-\varepsilon_5} a^{(e_{-\varepsilon_2-\varepsilon_6})}_{-\varepsilon_6,\varepsilon_2} a^{(e_{\varepsilon_2-\varepsilon_1})}_{\varepsilon_2,\varepsilon_1}.$$

Hence,

$$\begin{aligned}
b^*_{(\underline{\mu},\underline{\alpha})} &= a' e_{\alpha^{(1)}} e_{\alpha^{(2)}} \check{\varpi}_6 e_{\alpha^{(4)}} \check{\varpi}_5 e_{\alpha^{(6)}} e_{\alpha^{(p)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= a' \left(\check{\varpi}_6 - \langle \alpha^{(2)}, \check{\varpi}_6\rangle\right) e_{\alpha^{(1)}} e_{\alpha^{(2)}} e_{\alpha^{(4)}} \check{\varpi}_5 e_{\alpha^{(6)}} e_{\alpha^{(p)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= a' \left(\check{\varpi}_6 - \langle \alpha^{(2)}, \check{\varpi}_6\rangle\right) \left(\check{\varpi}_5 - \langle \alpha^{(2)}, \check{\varpi}_5\rangle - \langle \alpha^{(4)}, \check{\varpi}_5\rangle\right) e_{\alpha^{(1)}} e_{\alpha^{(2)}} e_{\alpha^{(4)}} e_{\alpha^{(6)}} e_{\alpha^{(p)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')}.
\end{aligned}$$

Then

$$(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) = \left((\delta_1, \delta_2, \overline{\delta}_6, \overline{\delta}_5, \overline{\delta}_3, \delta_6), (\alpha^{(1)}, \alpha^{(2)}, \alpha^{(4)}, \alpha^{(6)}, \alpha^{(7)})\right) \star (\underline{\mu}'', \underline{\alpha}'').$$

Thus

$$\begin{aligned}
b^*_{(\underline{\mu},\underline{\alpha})} &= a^{(\check{\beta}_6)}_{-\varepsilon_6,-\varepsilon_6} a^{(\check{\beta}_5)}_{-\varepsilon_5,-\varepsilon_5} (\check{\varpi}_6 - \langle \alpha^{(2)}, \check{\varpi}_6\rangle)(\check{\varpi}_5 - \langle \alpha^{(2)}, \check{\varpi}_5\rangle - \langle \alpha^{(4)}, \check{\varpi}_5\rangle) b^*_{(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})} \\
&= \langle \mu^{(3)}, \check{\alpha}^{(3)}\rangle \langle \mu^{(4)}, \check{\alpha}^{(4)}\rangle (\check{\varpi}_6 - \langle \alpha^{(2)}, \check{\varpi}_6\rangle)(\check{\varpi}_5 - \langle \alpha^{(2)}, \check{\varpi}_5\rangle - \langle \alpha^{(4)}, \check{\varpi}_5\rangle) b^*_{(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})}.
\end{aligned}$$

Set

$$\hat{K} = \langle \mu^{(3)}, \check{\alpha}^{(3)}\rangle \langle \mu^{(4)}, \check{\alpha}^{(4)}\rangle (\langle \rho, \check{\varpi}_6\rangle - \langle \alpha^{(2)}, \check{\varpi}_6\rangle)(\langle \rho, \check{\varpi}_5\rangle - \langle \alpha^{(2)}, \check{\varpi}_5\rangle - \langle \alpha^{(4)}, \check{\varpi}_5\rangle).$$

Hence

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \hat{K} \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}).$$

### 4.1.3 Case $p < q$ with loops

We continue to assume $m \in \mathbb{Z}_{>1}$, $\mu \in P(\delta)$, $\underline{i} \in \mathbb{Z}^m_{\succ \underline{0}}$ and $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)_{\underline{i}}$. We consider in this subsection the weighted paths with $p < q$, where $p = p(\underline{i})$ and $q := q(\underline{i})$. Thus, $i_p = 0$ and $\alpha^{(p)} \in \Pi_{\mu^{(p)}} = \{\beta \in \Pi \mid \langle \mu^{(p)}, \check{\beta}\rangle \neq 0\}$.

**Lemma 4.8.** *1. Assume $\mu^{(p)} \in \{\delta_1, \ldots, \delta_r\}$.*

*(a) If $\langle \mu^{(p)}, \check{\alpha}^{(p)}\rangle = 1$ , then*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \langle \rho, \varpi^\sharp_{\alpha^{(p)}}\rangle \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p}.$$

(b) *If* $\langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle = -1$ , *then*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (-\langle \rho, \varpi^{\sharp}_{\alpha^{(p)}} \rangle + 1)\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\# p}.$$

2. *Assume* $\mu^{(p)} \in \{\overline{\delta}_1, \dots, \overline{\delta}_r\}$.

(a) *If* $\langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle = 1$ , *then*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (\langle \rho, \check{\varpi}_{\alpha^{(p)}} \rangle - 1)\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\# p}.$$

(b) *If* $\langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle = -1$ , *then*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (2 - \langle \rho, \varpi^{\sharp}_{\alpha^{(p)}} \rangle)\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\# p}.$$

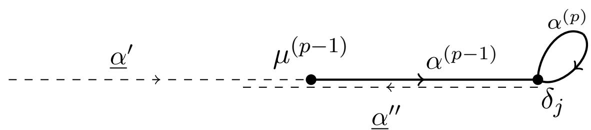
*Proof.* Write

$$(\underline{\mu}, \underline{\alpha}) = (\underline{\mu}', \underline{\alpha}') \star ((\mu^{(p-1)}, \mu^{(p)}), \alpha^{(p-1)}) \star ((\mu^{(p)}, \mu^{(p+1)}), \alpha^{(p)}) \star (\underline{\mu}'', \underline{\alpha}''),$$

where $(\underline{\mu}', \underline{\alpha}')$ and $(\underline{\mu}'', \underline{\alpha}'')$ have length $p-2$ and $m-p$, respectively. Since $p < q$, then $i_p = 0$ and $\alpha^{(p)} \in \Pi_{\mu^{(p)}} = \{\beta \in \Pi \mid \langle \mu^{(p)}, \check{\beta} \rangle \neq 0\}$.

1. Assume $\mu^{(p)} \in \{\delta_1, \dots, \delta_r\}$.

We have $\mu^{(p)} = \delta_j$ for some $j = 1, \dots, r$ and $\alpha^{(p)} = \beta_j$ or $\beta_{j-1}$.




Observe that for $s = 1, \dots, p-2$, the support of $\alpha^{(s)}$ does not contain the simple root $\alpha^{(p)}$, and so $b^*_{(\underline{\mu},\underline{\alpha}),s}\varpi^{\sharp}_{\alpha^{(p)}} = \varpi^{\sharp}_{\alpha^{(p)}} b^*_{(\underline{\mu},\underline{\alpha}),s}$ In other words, $\varpi^{\sharp}_{\alpha^{(p)}}$ commutes with all roots in $\underline{\alpha}'$. This case is similar as the $\mathfrak{sl}_{r+1}$ case (cf. Lemma 3.3 (3)). By doing the same kind of reasoning, we get

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle \left( \langle \rho, \varpi^{\sharp}_{\alpha^{(p)}} \rangle - \langle \alpha^{(p-1)}, \varpi^{\sharp}_{\alpha^{(p)}} \rangle \right) \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\# p}.$$

(a) If $\langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle = 1$ then $\alpha^{(p)} = \beta_j$, for some $j = 1, \dots r$, and so $\langle \alpha^{(p-1)}, \varpi^{\sharp}_{\alpha^{(p)}} \rangle = 0$. Hence,

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \langle \rho, \varpi^{\sharp}_{\alpha^{(p)}} \rangle \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\# p}.$$

(b) If $\langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle = -1$ then $\alpha^{(p)} = \beta_{j-1}$, for some $j = 2, \ldots r$, so $\langle \alpha^{(p-1)}, \varpi^\sharp_{\alpha^{(p)}} \rangle = 1$. Thus,

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (-\langle \rho, \varpi^\sharp_{\alpha^{(p)}} \rangle + 1)\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\# p}.$$

2. Assume that $\mu^{(p)} \in \{\overline{\delta}_1, \ldots, \overline{\delta}_r\}$.

We have $\mu^{(p)} = \overline{\delta}_j$ for some $j = 1, \ldots, r$ and $\alpha^{(p)} = \beta_j$ or $\beta_{j-1}$. Let $s$ be an integer in $\{1, \ldots, p-2\}$ such that the support of $\alpha^{(s)}$ contains the simple root $\alpha^{(p)}$, then either $i_s = 0$ or $i_s \neq 0$.

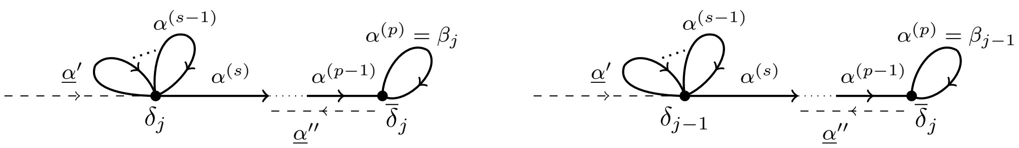



**FIGURE 4.8** – Path in case $p < q$ and $\mu^{(p)} = \overline{\delta}_j$

If $i_s = 0$, then $b^*_{(\underline{\mu},\underline{\alpha}),s} = \varpi^\sharp_{\alpha^{(s)}} \in U(\mathfrak{h})$, and so $\varpi^\sharp_{\alpha^{(p)}}$ commutes with $b^*_{(\underline{\mu},\underline{\alpha}),s}$.

Otherwise, there is at most one root $\alpha^{(s)}$ with $i_s \neq 0$ such that $\alpha^{(p)} \in \mathrm{supp}(\alpha^{(s)})$ (see Figure 4.8), and so $\varpi^\sharp_{\alpha^{(p)}} e_{\alpha^{(s)}} \neq e_{\alpha^{(s)}} \varpi^\sharp_{\alpha^{(p)}}$. Hence $\varpi^\sharp_{\alpha^{(p)}}$ commutes with $b^*_{(\underline{\mu}',\underline{\alpha}')}$ except with $\alpha^{(s)}$.

Write

$$(\underline{\mu}', \underline{\alpha}') = (\underline{\mu}'_1, \underline{\alpha}'_1) \star ((\mu^{(s)}, \mu^{(s+1)}), \alpha^{(s)}) \star (\underline{\mu}'_2, \underline{\alpha}'_2).$$

Doing similar calculation as (4.3), we get

$$\begin{aligned}
b^*_{(\underline{\mu},\underline{\alpha})} &= \langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle a^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})}_{\mu^{(p)},\mu^{(p-1)}} b^*_{(\underline{\mu}',\underline{\alpha}')} e_{\alpha^{(p-1)}} \varpi^\sharp_{\alpha^{(p)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= \langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle a^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})}_{\mu^{(p)},\mu^{(p-1)}} a^{(c_{\alpha^{(s)}} e_{-\alpha^{(s)}})}_{\mu^{(s+1)},\mu^{(s)}} b^*_{(\underline{\mu}'_1,\underline{\alpha}'_1)} e_{\alpha^{(s)}} b^*_{(\underline{\mu}'_2,\underline{\alpha}'_2)} e_{\alpha^{(p-1)}} \varpi^\sharp_{\alpha^{(p)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= \langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle a^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})}_{\mu^{(p)},\mu^{(p-1)}} a^{(c_{\alpha^{(s)}} e_{-\alpha^{(s)}})}_{\mu^{(s+1)},\mu^{(s)}} \left( \varpi^\sharp_{\alpha^{(p)}} - \langle \alpha^{(s)}, \varpi^\sharp_{\alpha^{(p)}} \rangle - \langle \alpha^{(p-1)}, \varpi^\sharp_{\alpha^{(p)}} \rangle \right) \\
&\quad \times b^*_{(\underline{\mu}'_1,\underline{\alpha}'_1)} e_{\alpha^{(s)}} b^*_{(\underline{\mu}'_2,\underline{\alpha}'_2)} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} \\
&= \langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle \left( \varpi^\sharp_{\alpha^{(p)}} - \langle \alpha^{(s)}, \varpi^\sharp_{\alpha^{(p)}} \rangle - \langle \alpha^{(p-1)}, \varpi^\sharp_{\alpha^{(p)}} \rangle \right) b^*_{(\underline{\mu},\underline{\alpha})^{\#}},
\end{aligned}$$

since $\varpi^\sharp_{\alpha^{(p)}}$ commutes with all roots of $\underline{\alpha}'_1$ and $\underline{\alpha}'_2$ and since

$$a^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})}_{\mu^{(p)},\mu^{(p-1)}} a^{(c_{\alpha^{(s)}} e_{-\alpha^{(s)}})}_{\mu^{(s+1)},\mu^{(s)}} b^*_{(\underline{\mu}'_1,\underline{\alpha}'_1)} e_{\alpha^{(s)}} b^*_{(\underline{\mu}'_2,\underline{\alpha}'_2)} e_{\alpha^{(p-1)}} b^*_{(\underline{\mu}'',\underline{\alpha}'')} = b^*_{(\underline{\mu},\underline{\alpha})^{\#}}.$$

Hence,

$$\mathrm{wt}(\underline{\mu},\underline{\alpha}) = \langle \mu^{(p)}, \check{\alpha}^{(p)}\rangle \left(\langle \rho, \varpi^{\sharp}_{\alpha^{(p)}}\rangle - \langle \alpha^{(s)}, \varpi^{\sharp}_{\alpha^{(p)}}\rangle - \langle \alpha^{(p-1)}, \varpi^{\sharp}_{\alpha^{(p)}}\rangle\right) \mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p}.$$

(a) If $\langle \mu^{(p)}, \check{\alpha}^{(p)}\rangle = 1$ then $\alpha^{(p)} = \beta_{j-1}$, for some $j = 2,\dots,r$. Observe that if $\langle \alpha^{(p-1)}, \varpi^{\sharp}_{\alpha^{(p)}}\rangle = 1$ then $\langle \alpha^{(s)}, \varpi^{\sharp}_{\alpha^{(p)}}\rangle = 0$, and vice versa. Thus,

$$\mathrm{wt}(\underline{\mu},\underline{\alpha}) = (\langle \rho, \varpi^{\sharp}_{\alpha^{(p)}}\rangle - 1)\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p}.$$

(b) Assume that $\langle \mu^{(p)}, \check{\alpha}^{(p)}\rangle = -1$ then $\alpha^{(p)} = \beta_j$, for some $i = 1,\dots,r$. Observe that $\langle \alpha^{(p-1)}, \varpi^{\sharp}_{\alpha^{(p)}}\rangle = 1$ or 2. If $\langle \alpha^{(p-1)}, \varpi^{\sharp}_{\alpha^{(p)}}\rangle = 2$, it means that $\alpha^{(p-1)} = \varepsilon_k + \varepsilon_j, k = 1,\dots,j$, whence $\langle \alpha^{(s)}, \varpi^{\sharp}_{\alpha^{(p)}}\rangle = 0$.
Otherwise $\langle \alpha^{(p-1)}, \varpi^{\sharp}_{\alpha^{(p)}}\rangle = 1$, and so $\alpha^{(p-1)} = \varepsilon_k + \varepsilon_j, k > j$, whence $\langle \alpha^{(s)}, \varpi^{\sharp}_{\alpha^{(p)}}\rangle = 1$. Hence,

$$\mathrm{wt}(\underline{\mu},\underline{\alpha}) = (2 - \langle \rho, \varpi^{\sharp}_{\alpha^{(p)}}\rangle)\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p}.$$

□

We summarize the results that we obtained in Lemma 4.2, Lemma 4.3, Lemma 4.4, Lemma 4.8 and Lemma 4.6, in the following proposition:

**Proposition 4.9.** *Let $m \in \mathbb{Z}_{>1}$, $\mu \in P(\delta)$, $\underline{i} \in \mathbb{Z}^m_{\succ \underline{0}}$ and $(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\mu)_{\underline{i}}$. Set $p := p(\underline{i})$ and $q := q(\underline{i})$.*

1. *Assume $\alpha^{(p-1)} + \alpha^{(p)} \neq 0$.*

   (a) *If for all $s \in \{1,\dots,p-2\}$, either $\alpha^{(s)}+\alpha^{(p)} \in -\Delta_+$ or $\alpha^{(s)}+\alpha^{(p)} \notin \Delta \cup \{0\}$ then there is a constant $K^{\#p}_{(\underline{\mu},\underline{\alpha})}$ and a scalar $\hat{K}$ such that*

   $$\mathrm{wt}(\underline{\mu},\underline{\alpha}) = \hat{K} K^{\#p}_{(\underline{\mu},\underline{\alpha})} \mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p}.$$

   (b) *If for some $s \in \{1,\dots,p-2\}$, $\alpha^{(s)} + \alpha^{(p)} \in \Delta_+$ then there are some constants $K^{\#p}_{(\underline{\mu},\underline{\alpha})}$, $K^{\star a}_{(\underline{\mu},\underline{\alpha})}$ and scalar $\hat{K}$ such that*

   $$\mathrm{wt}(\underline{\mu},\underline{\alpha}) = \hat{K}\left(K^{\#p}_{(\underline{\mu},\underline{\alpha})}\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p} + \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})}\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\star a}\right),$$

*with* $(\underline{\mu}, \underline{\alpha})^{\star a}$ *is a weighted path of length strictly smaller than* $m$.

(c) *If for some* $s \in \{1, \dots, p-2\}$, $\alpha^{(s)} = -\alpha^{(p)}$ *then there are some constants* $K^{\#p}_{(\underline{\mu},\underline{\alpha})}$, $K^{\star a}_{(\underline{\mu},\underline{\alpha})}$ *and a scalar* $\hat{K}$ *such that*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \hat{K}\left(K^{\#p}_{(\underline{\mu},\underline{\alpha})}\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} + (\mathrm{ht}(\check{\alpha}^{(s)}) - c_s)\sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})}\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\star a}\right),$$

*with* $c_s := \langle \alpha^{(s-1)}, \check{\alpha}^{(s)}\rangle$.

2. *Assume* $\alpha^{(p-1)} + \alpha^{(p)} = 0$.

   (a) *If* $i_1 = \dots = i_{p-2} = 0$, *or if* $p = 2$, *then*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (c_{\alpha^{(p-1)}})^2 \ \mathrm{ht}(\check{\alpha}^{(p-1)})\, \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p},$$

   *with* $c_{\alpha^{(p-1)}}$ *as in Definition 1.14.*

   (b) *Otherwise,*

   i. *If for all* $s \in \{1, \dots, p-2\}$, *either* $\alpha^{(s)} + \alpha^{(p)} \in -\Delta_+$ *or* $\alpha^{(s)} + \alpha^{(p)} \notin \Delta \cup \{0\}$ *then there is a scalar* $\hat{K}$ *such that*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \hat{K}(c_{\alpha^{(p-1)}})^2 \ (\mathrm{ht}(\check{\alpha}^{(p-1)}) - c_{p-1})\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p},$$

   *with* $c_{\alpha^{(p-1)}}$ *as in Definition 1.14 and* $c_{p-1}$ *is an integer as in* (4.1).

   ii. *If for some* $s \in \{1, \dots, p-2\}$, $\alpha^{(s)} + \alpha^{(p)} \in \Delta_+$ *then there are some constants* $K^{\star a}_{(\underline{\mu},\underline{\alpha})}$ *and a scalar* $\hat{K}$ *such that*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \hat{K}\left((\mathrm{ht}(\check{\alpha}^{(p-1)}) - c_{p-1})\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} + \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})}\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\star a}\right),$$

   *with* $(\underline{\mu}, \underline{\alpha})^{\star a}$ *is a weighted path of length strictly smaller than* $m$ *and* $c_{p-1}$ *is an integer as in* (4.1).

   iii. *If for some* $s \in \{1, \dots, p-2\}$, $\alpha^{(s)} = -\alpha^{(p)}$ *then there are some constants* $K^{\star a}_{(\underline{\mu},\underline{\alpha})}$ *and a scalar* $\hat{K}$ *such that*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \hat{K}(\mathrm{ht}(\check{\alpha}^{(p-1)}) - c_{p-1})\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} + \hat{K}(\mathrm{ht}(\check{\alpha}^{(s)}) - c_s)\sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})}\mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\star a},$$

*with* $(\underline{\mu}, \underline{\alpha})^{\star a}$ *is a weighted path of length strictly smaller than* $m$*,* $c_{p-1}$ *is an integer as in* (4.1)*, and* $c_s := \langle \alpha^{(s-1)}, \check{\alpha}^{(s)} \rangle$.

3. *Assume* $p < q$*, then*

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \langle \mu^{(p)}, \check{\alpha}^{(p)} \rangle (\langle \rho, \varpi^{\sharp}_{\alpha^{(p)}} \rangle - c_{p-1}) \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\# p},$$

*with* $c_{p-1}$ *is an integer as in* (4.1).

*Note that* $(\underline{\mu}, \underline{\alpha})^{\star a}$ *is a weighted path as in Lemma 4.3 or Lemma 4.4, depending on different cases, and* $N$ *the number of possible paths of* $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$ *as in Lemma 4.3 or Lemma 4.4. Note also that what we mean by "constant" is a complex number which only depends on constant structures* $(n_{\alpha,\beta}, a^{(b)}_{\lambda,\mu})$*, but not on the height of* $\alpha$*. Moreover, the scalar* $\hat{K}$ *is described in Lemma 4.6.*

### 4.1.4 Proof of Theorem 4.1

We are now in a position to prove Theorem 4.1 for $\mathfrak{g} = \mathfrak{sp}_{2r}$ and $\delta = \varpi_1$.

*Proof of Theorem 4.1.* Let $(\underline{\mu}, \underline{\alpha})$ as in the theorem and set $\underline{i} := \mathrm{ht}(\underline{\mu})$. First of all, we observe that for all $\underline{i} \in \mathbb{Z}^m_{\prec 0}$ there exists $1 \leq p \leq m$ such that $i_1 = i_2 = \ldots = i_{p-1} = 0, i_p < 0$. So

$$b^*_{(\underline{\mu},\underline{\alpha})} = a_{\underline{\mu},\underline{\alpha}} b^*_{(\underline{\mu},\underline{\alpha}),1} \ldots b^*_{(\underline{\mu},\underline{\alpha}),p-1} b^*_{(\underline{\mu},\underline{\alpha}),p} \ldots b^*_{(\underline{\mu},\underline{\alpha}),m} \in n_{-}U(\mathfrak{g}).$$

Hence $\mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}}) = 0$ and so the theorem is clear for $\underline{i} \in \mathbb{Z}^m_{\prec 0}$.

We prove the statement by induction on $m$. Necessarily, $m \geqslant 2$.

∗ If $m = 2$, then the hypothesis implies that $\underline{i} \in \mathbb{Z}^m_{\prec 0}$ and so the statement is true.

∗ Assume $m \geqslant 3$ and that for all weighted paths $(\underline{\mu'}, \underline{\alpha'}) \in \hat{\mathscr{P}}_{m'}(\mu)$, with $m' < m$, such that for some $i' \in \{1, \ldots, m'\}$, $\mu'^{(i')} \geqslant \mu$, we have $\mathrm{wt}(\underline{\mu'}, \underline{\alpha'}) = 0$.

If $\underline{i} \in \mathbb{Z}^m_{\prec 0}$ the statement is true. So we can assume that $\underline{i} \in \mathbb{Z}^m_{\not\prec 0}$, and by the assumption, necessarily, $\underline{i} \in \mathbb{Z}^m_{\succ 0}$. Set $p := p(\underline{i})$ and $q := q(\underline{i})$. By Proposition 4.9 there are some scalars $K^{\# p}$ and $K^{\star a}$ such that

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^{\# p} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\# p}$$

or

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^{\# p} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\# p} + \sum_{a=1}^{N} K^{\star a} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\star a}.$$

Assume $\mathrm{wt}(\underline{\mu},\underline{\alpha}) = K^{\#p}\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p}$. We observe that the weighted path $(\underline{\mu},\underline{\alpha})^{\#p}$ satisfies the hypothesis of the theorem and it is not empty. Hence by our induction hypothesis and Proposition 4.9 we get the statement.

Assume that

$$\mathrm{wt}(\underline{\mu},\underline{\alpha}) = K^{\#p}\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p} + \sum_{a=1}^{N} K^{\star a}\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\star a}.$$

Let us take as an example the path $(\underline{\mu},\underline{\alpha})^{\star a}$ as in Lemma 4.3 (1) as follows :

$$(\underline{\mu},\underline{\alpha})^{\star a} := (\underline{\mu}',\underline{\alpha}') \star \big((\delta_k,\delta_j);(\alpha^{(s)}+\alpha^{(p)})\big) \star (\underline{\tilde{\mu}}^{\star a},\underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'',\underline{\alpha}''),$$

where,

$$(\underline{\mu}',\underline{\alpha}') = \big((\mu^{(1)},\dots,\mu^{(s-1)},\mu^{(s)}=\delta_k);(\alpha^{(1)},\dots,\alpha^{(s-1)})\big),$$
$$(\underline{\mu}'',\underline{\alpha}'') = \big((\delta_i=\mu^{(p+1)},\dots,\mu^{(m+1)});(\alpha^{(p+1)},\dots,\alpha^{(m)})\big),$$

and $(\underline{\tilde{\mu}}^{\star a},\underline{\tilde{\alpha}}^{\star a})$ is a path of length $< p-s$ between $\delta_j$ and $\delta_i$ whose roots $(\tilde{\alpha}^{\star a})^{(l)}$ have height 0 or strictly positive height. We can similarly argue for the other cases.

Let $t$ be the smallest integer such that $\mu^{(t)} > \mu$. Since the root $\alpha^{(s)}+\alpha^{(p)}$ and all roots in path $(\underline{\mu}',\underline{\alpha}'),(\underline{\tilde{\mu}}^{\star a},\underline{\tilde{\alpha}}^{\star a})$ have height 0 or strictly positive, then $t > p$ and $\alpha^{(t)}$ is belongs to $\underline{\alpha}''$. Observe that the weighted paths $(\underline{\mu},\underline{\alpha})^{\#p}$ and $(\underline{\mu}'',\underline{\alpha}'')$ satisfy the hypothesis of the theorem and it is not empty. Note that the path $(\underline{\mu}'',\underline{\alpha}'')$ have length $m-p$ for each case. Hence by our induction hypothesis we have

$$\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p} = 0 \quad \text{and} \quad \mathrm{wt}(\underline{\mu}'',\underline{\alpha}'') = 0.$$

By Proposition 4.9 we get the statement. □

## 4.2 An equivalence relation on the set of weighted paths

Let $\alpha \in \Delta_+$ and $\alpha = \mu - \nu$. We say that $\alpha$ has type I if the admissible triple $(\alpha,\mu,\nu)$ (see §2.4) has type I, $\alpha$ has type II if the admissible triple $(\alpha,\mu,\nu)$ has type II, and $\alpha$ has type III if the admissible triple $(\alpha,\mu,\nu)$ has type III (a) or III (b). The type of $\alpha$ depends only on $\alpha$. We will use this notion in the whole section.

Recall that for $\lambda,\mu \in P(\delta)$, $[\![\lambda,\mu]\!]$ denotes the set of $\nu \in P(\delta)$ such that $\lambda \leqslant \nu \leqslant \mu$.

### 4.2.1 Definitions and first properties

This paragraph is devoted to the proof of the following result.

**Lemma 4.10.** *Let $m \in \mathbb{Z}_{>0}$. There are polynomials $\hat{T}_{1,m}$ and $\hat{T}_{2,m}$ in $\mathbb{C}[X]$ of degree at most $m$ such that for all $k \in \{1, \dots, r\}$,*

$$\sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \\ \underline{\mu} \in [\![\overline{\delta_1}, \delta_k]\!]}} \mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \hat{T}_{1,m}(k),$$

$$\sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\overline{\delta}_k) \\ \underline{\mu} \in [\![\overline{\delta}_1, \overline{\delta}_k]\!]}} \mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \hat{T}_{2,m}(k).$$

According to Theorem 4.1, the weighted paths which starting from $\overline{\delta}_k$ have weights entirely contained in $[\![\overline{\delta}_1, \overline{\delta}_k]\!]$. So the sum

$$\sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\overline{\delta}_k) \\ \underline{\mu} \in [\![\overline{\delta}_1, \overline{\delta}_k]\!]}} \mathrm{wt}(\underline{\mu}, \underline{\alpha})$$

can be computed exactly as in the $\mathfrak{sl}_{r+1}$ case.

So it remains to consider the paths starting from $\delta_k$ and contained in $[\![\overline{\delta}_1, \delta_k]\!]$. This is our purpose. Thus we have to show that the corresponding sum is a polynomial in $k$ of degree $\leqslant m-1$.

Next, we introduce an equivalence relation on paths.

Since we cannot here argue only on the heights of roots, as for $\mathfrak{sl}_{r+1}$, we introduce an equivalence relation directly on such paths as follows. This definition is a generalization of Definition 3.6.

**Definition 4.11** (equivalence relation on the paths for $\mathfrak{sp}_{2r}$)**.** *We define an equivalence relation $\sim$ on $\hat{\mathscr{P}}_m$ by induction on $m$ as follows.*

1. *If $m = 1$, there is only one equivalence class represented by the trivial path of length 0.*

2. *If $m = 2$, then two paths $(\underline{\mu}, \underline{\alpha}), (\underline{\mu}', \underline{\alpha}')$ in $\hat{\mathscr{P}}_m$ are* equivalent *if the following condition holds:*

   *(a) there is $(\varepsilon_1, \varepsilon_2) \in \{0,1\}^2$ such that $\mathrm{ht}(\underline{\mu}) \in \varepsilon_1 \mathbb{Z}_{\succcurlyeq \underline{0}} \times \varepsilon_2 \mathbb{Z}^m_{\succcurlyeq \underline{0}}$ and $\mathrm{ht}(\underline{\mu}') \in \varepsilon_1 \mathbb{Z}^m_{\succcurlyeq \underline{0}} \times \varepsilon_2 \mathbb{Z}^m_{\succcurlyeq \underline{0}}$*

(b) *the roots* $\alpha^{(1)} \in \underline{\alpha}$ *and* $\alpha'^{(1)} \in \underline{\alpha}'$ *have same types.*

3. *If* $m > 2$, *then two paths* $(\underline{\mu}, \underline{\alpha}), (\underline{\mu}', \underline{\alpha}')$ *in* $\hat{\mathscr{P}}_m$ *are equivalent if the following conditions hold, in the notations of Proposition 4.9:*

   (a) *for all* $i \in \{1, \ldots, m\}$, *there is* $(\varepsilon_1, \ldots, \varepsilon_m) \in \{-1, 0, 1\}^m$ *such that* $\mathrm{ht}(\underline{\mu})_i \in \prod_{i=1}^m \varepsilon_i \mathbb{Z}^m_{\geqslant 0}$ *and* $\mathrm{ht}(\underline{\mu}')_i \in \prod_{i=1}^m \varepsilon_i \mathbb{Z}^m_{\geqslant 0}$;

   (b) *we have* $p(\underline{i}) = p(\underline{i}') =: p$, *with* $\underline{i} = \mathrm{ht}(\underline{\mu})$, $\underline{i}' = \mathrm{ht}(\underline{\mu}')$, *and the weighted paths* $(\underline{\mu}, \underline{\alpha})^{\# p}$ *and* $(\underline{\mu}', \underline{\alpha}')^{\# p}$ *are equivalent,*

   (c) *if there is an* $s \in \{1, \ldots, p-2\}$ *such that* $\alpha^{(s)} + \alpha^{(q)} \in \Delta_+ \cup \{0\}$, *with* $q := q(\underline{i}) = q(\underline{i}')$, *then* $s$ *is the unique integer of* $\{1, \ldots, p-2\}$ *such that* $\alpha'^{(s)} + \alpha'^{(q)} \in \Delta_+ \cup \{0\}$. *Moreover, all paths* $(\underline{\mu}, \underline{\alpha})^{\star a}$ *and* $(\underline{\mu}', \underline{\alpha}')^{\star a}$ *are equivalent,*

   (d) *we have* $K^{\# p}_{(\underline{\mu}, \underline{\alpha})} = K^{\# p}_{(\underline{\mu}', \underline{\alpha}')}$, *and if an* $s$ *as in (c) exists, then for all the* $N$ *possible paths* $(\underline{\mu}, \underline{\alpha})^{\star a}$ , $K^{\star a}_{(\underline{\mu}, \underline{\alpha})} = K^{\star a}_{(\underline{\mu}', \underline{\alpha}')}$.

*Remark* 4.12. If $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m$ is a weighted path starting at $\delta_k$ and contained in $[\![\delta_r, \delta_k]\!]$ or starting at $\overline{\delta}_k$ and contained in $[\![\overline{\delta}_1, \overline{\delta}_k]\!]$, then Definition 3.6 and Definition 4.11 are equivalent.

By the above remark, the paths as above can be dealt as in $\mathfrak{sl}_{r+1}$. Thus, one can use the results for the weights of the paths as for $\mathfrak{sl}_{r+1}$. We denote by $[(\underline{\mu}, \underline{\alpha})]_{[\![\overline{\delta}_1, \delta_k]\!]}$ the class of a weighted paths of length $m$ which is contained in $[\![\overline{\delta}_1, \delta_k]\!]$, $\mathscr{E}_m$ the set of equivalence classes $[(\underline{\mu}, \underline{\alpha})]_{[\![\overline{\delta}_1, \delta_k]\!]}$, and $\bar{\mathscr{E}}_m$ the set of elements of $\mathscr{E}_m$ whose representative are not contained in $[\![\delta_r, \delta_k]\!]$.

We observe that an equivalent class in $\mathscr{E}_m$ can simply be described by the sequence $\underline{\mu}$. Hence, we will often write $[\mu^{(1)}, \ldots, \mu^{(m)}]$ or simply by $\boldsymbol{\mu}$ for the class $[(\underline{\mu}, \underline{\alpha})]_{[\![\overline{\delta}_1, \delta_k]\!]}$.

**Lemma 4.13.** *Let* $(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k)$ *and* $\alpha, \beta \in \underline{\alpha}$ *such that* $\alpha + \beta \in \Delta$. *We have* $(n_{\alpha,\beta})^2 = 4$ *if and only if the roots* $\alpha$ *and* $\beta$ *are of the following form:*

$$\alpha = \delta_i - \overline{\delta}_j \text{ and } \beta = \overline{\delta}_j - \overline{\delta}_i, \ i \neq j,$$
$$\alpha = \overline{\delta}_i - \overline{\delta}_j \text{ and } \beta = \overline{\delta}_j - \delta_i, \ i \neq j.$$

*Moreover, in those cases* $n_{\alpha,\beta} = -2$.

*Proof.* Clear by formula (2.1) (see Section 2.1.2). □

*Example* 4.14. 1. Assume $m = 2$, then there are four equivalence classes :

$$[\delta_k, \delta_k]; \quad [\delta_k, \overline{\delta}_k]; \quad [\delta_k, \delta_j], k < j; \quad [\delta_k, \overline{\delta}_j], k \neq j.$$

There are two elements of $\bar{\mathscr{E}}_m$.

2. Assume $m = 3$, we will seek the elements of $\bar{\mathscr{E}}_m$.

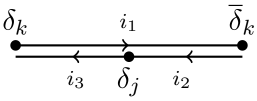
Recall that for $\alpha^{(p-1)} + \alpha^{(p)} \neq 0$

$$K^{\#p}_{(\underline{\mu},\underline{\alpha})} := n_{\alpha^{(p-1)},\alpha^{(p)}} \frac{a^{(c_{\alpha^{(p)}} e_{-\alpha^{(p)}})}_{\mu^{(p+1)},\mu^{(p)}} \, a^{(c_{\alpha^{(p-1)}} e_{-\alpha^{(p-1)}})}_{\mu^{(p)},\mu^{(p-1)}}}{a^{(c_{\alpha^{(p-1)}+\alpha^{(p)}} e_{-\alpha^{(p-1)}-\alpha^{(p)}})}_{\mu^{(p+1)},\mu^{(p-1)}}}.$$

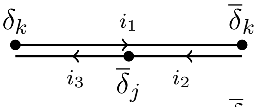
Thus there are 16 elements of $\bar{\mathscr{E}}_m$, with 6 elements with loops as follows:

$$[\delta_k, \delta_k, \overline{\delta}_k]; \quad [\delta_k, \delta_k, \overline{\delta}_j], k \neq j; \quad [\delta_k, \overline{\delta}_k, \delta_k];$$
$$[\delta_k, \overline{\delta}_j, \delta_k], k \neq j; \quad [\delta_k, \overline{\delta}_k, \overline{\delta}_k]; \quad [\delta_k, \overline{\delta}_j, \overline{\delta}_j], k \neq j;$$

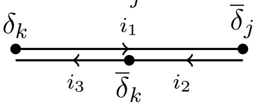
and 10 elements without loops as follows:

$[\delta_k, \overline{\delta}_k, \delta_j], k < j$, represented by 

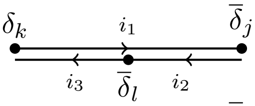
;

$[\delta_k, \overline{\delta}_k, \overline{\delta}_j], k < j$, represented by 

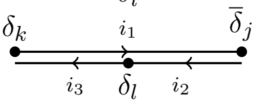
;

$[\delta_k, \overline{\delta}_j, \overline{\delta}_k], j < k$, represented by 

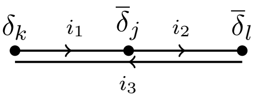
;

$[\delta_k, \overline{\delta}_j, \overline{\delta}_l], j \neq k, l \neq k, j < l$, represented by 

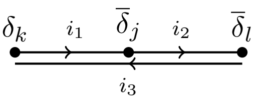
;

$[\delta_k, \overline{\delta}_j, \delta_l], j \neq k, l \neq j, k < l$, represented by 

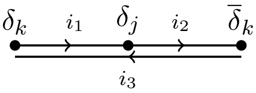
;

$[\delta_k, \overline{\delta}_j, \overline{\delta}_l], j \neq k, l \neq k, l < j$, represented by 

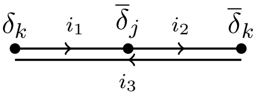
;

$[\delta_k, \overline{\delta}_j, \overline{\delta}_l], l \neq k, l \neq k, k < j$, represented by 

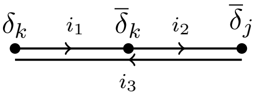
;

$[\delta_k, \delta_j, \overline{\delta}_k], k < j$, represented by 

;

$[\delta_k, \overline{\delta}_j, \overline{\delta}_k], k < j$, represented by 

;

$[\delta_k, \overline{\delta}_5, \overline{\delta}_j], k < j$, represented by 


One easily verifies that the above paths are pairwise non-equivalent. For example, let $(\underline{\mu}, \underline{\alpha})$ and $(\underline{\mu}', \underline{\alpha}')$ be the weighted paths with $\underline{\mu} = (\delta_k, \overline{\delta}_k, \delta_j, \delta_k), k < j$ and $\underline{\mu}' = (\delta_k, \overline{\delta}_j, \delta_l, \delta_k), j \neq k, l \neq j, k < l$. Condition 3(a) and 3(b) hold, but $K^{\#p}_{(\underline{\mu},\underline{\alpha})} = 1 \times \frac{1\times 2}{1} = 2 \neq K^{\#p}_{(\underline{\mu}',\underline{\alpha}')} = 1 \times \frac{1\times 1}{1} = 1$. So the condition 3(d) does not hold, therefore they are not equivalent.

Meanwhile, the weighted paths $(\underline{\mu}, \underline{\alpha})$ and $(\underline{\mu}', \underline{\alpha}')$ with $\underline{\mu} = (\delta_k, \overline{\delta}_j, \delta_j, \delta_k)$ and $\underline{\mu}' = (\delta_k, \overline{\delta}_k, \delta_j, \delta_k)$,$k < j$, are equivalent, since their heights have same sign, $(\underline{\mu}, \underline{\alpha})^{\#p} \sim (\underline{\mu}', \underline{\alpha}')^{\#p}$ and $K^{\#p}_{(\underline{\mu},\underline{\alpha})} = K^{\#p}_{(\underline{\mu}',\underline{\alpha}')} = 2$.

Let $m \in \mathbb{Z}_{>0}$ and $\underline{\boldsymbol{\mu}} := [(\underline{\mu}, \underline{\alpha})] \in \mathscr{E}_m$. The number $n$ of zero values of $\underline{i} := \mathrm{ht}(\underline{\mu})$ does not depend on $(\underline{\mu}', \underline{\alpha}')$ in $\underline{\boldsymbol{\mu}}$. We adopt the terminology of paths with zeroes and without zero as in Definition 3.8. By definition, the position $p(\underline{i})$ of the first returning back does not depend on $(\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}$. Similarly, the integers $q(\underline{i})$ and $s \in \{1, \dots, p-2\}$ (if such an $s$ exists), such that $\alpha^{(s)} + \alpha^{(q)} \in \Delta_+ \cup \{0\}$, do not depend on $(\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}$. Furthermore, the class of $(\underline{\mu}, \underline{\alpha})^{\#p}$ and the class of $(\underline{\mu}, \underline{\alpha})^{\star a}$ only depend on $\underline{\boldsymbol{\mu}}$. We denote by $\underline{\boldsymbol{\mu}}^{\#}$ and $\underline{\boldsymbol{\mu}}^{\star a}$ these equivalence classes, respectively. Moreover, we denote by $K^{\#}_{\underline{\boldsymbol{\mu}}}$ the scalar $K^{\#p}_{(\underline{\mu},\underline{\alpha})}$ and, if an $s$ as in (c) exists, we denote by $K^{\star a}_{\underline{\boldsymbol{\mu}}}$ the scalar $K^{\star a}_{(\underline{\mu},\underline{\alpha})}$.

We denote by $\ell(\underline{\boldsymbol{\mu}}') := m'$ the length of $\underline{\boldsymbol{\mu}}'$ for some equivalence class $\underline{\boldsymbol{\mu}} \in \mathscr{E}_{m'}$, $m' \in \mathbb{Z}_{>0}$. We have $\ell(\underline{\boldsymbol{\mu}}) = m$, $\ell(\underline{\boldsymbol{\mu}}^{\#}) = m - 1$ if $i_p + i_{p-1} \neq 0$, $\ell(\underline{\boldsymbol{\mu}}^{\#}) = m - 2$ if $i_p + i_{p-1} = 0$, and $\ell(\underline{\boldsymbol{\mu}}^{\star a}) < m$ for all $a \in \{1, \dots, N\}$.

### 4.2.2 Elements of $\mathscr{E}_m$ without zero

This subsection is devoted to the study of the elements of $\mathscr{E}_m$ that has no zero. Note that if $\underline{\boldsymbol{\mu}}$ has no zero then $\underline{\boldsymbol{\mu}}^{\#}$ and $\underline{\boldsymbol{\mu}}^{\star a}$ has no zero, too.

**Lemma 4.15.** *Let $\underline{\boldsymbol{\mu}} \in \mathscr{E}_m$ without zero, and set $p := p(\underline{\boldsymbol{\mu}})$.*

1. *There is a polynomial* $A_{\underline{\boldsymbol{\mu}}} \in \mathbb{C}[X_1, \ldots, X_{m-1}]$ *of total degree* $\leqslant \lfloor \frac{m}{2} \rfloor$ *such that for all weighted paths* $(\underline{\mu}, \underline{\alpha})$ *such that* $[(\underline{\mu}, \underline{\alpha})]_{[\![\overline{\delta}_1, \delta_k]\!]} = \underline{\boldsymbol{\mu}}$,

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = A_{\underline{\boldsymbol{\mu}}}(i_1, \ldots, i_{m-1}).$$

*Here, the integer* $i_j$ *denotes the heights of* $\alpha^{(j)}$. *Moreover,* $A_{\underline{\boldsymbol{\mu}}}$ *is a sum of monomials of the form* $X_{j_1} \cdots X_{j_l}$, *with* $1 \leqslant j_1 < \cdots < j_l < m$.

2. *The polynomial* $A_{\underline{\boldsymbol{\mu}}}$ *is defined by induction as follows.*

   (a) *Assume* $m = 2$

$$\begin{aligned}
A_{[\delta_k, \overline{\delta}_k]}(X_1) &= 2(X_1 + 1);\\
A_{[\delta_k, \delta_j], k<j}(X_1) &= X_1;\\
A_{[\delta_k, \overline{\delta}_j]}(X_1) &= X_1 + 1.
\end{aligned}$$

   (b) *Assume* $m = 3$

$$\begin{aligned}
A_{[\delta_k, \delta_l, \delta_j], k<l<j}(X_1, X_2) &= X_1;\\
A_{[\delta_k, \delta_l, \delta_j], k<j<l}(X_1, X_2) &= X_1 + X_2;\\
A_{[\delta_k, \overline{\delta}_k, \delta_j], k<j}(X_1, X_2) &= 2(X_1 + X_2);\\
A_{[\delta_k, \overline{\delta}_k, \overline{\delta}_j], k<j}(X_1, X_2) &= 2(X_1 + X_2 + 1);\\
A_{[\delta_k, \overline{\delta}_j, \overline{\delta}_k], j<k}(X_1, X_2) &= 2(X_1 + X_2 + 1);\\
A_{[\delta_k, \overline{\delta}_j, \overline{\delta}_l], j\neq k, l\neq k, j<l}(X_1, X_2) &= X_1 + X_2 + 1;\\
A_{[\delta_k, \overline{\delta}_j, \delta_l], j\neq k, l\neq j, k<l}(X_1, X_2) &= X_1 + X_2;\\
A_{[\delta_k, \delta_j, \overline{\delta}_k], k<j}(X_1, X_2) &= 2X_1;\\
A_{[\delta_k, \overline{\delta}_j, \overline{\delta}_k], k<j}(X_1, X_2) &= 2(X_1 + 1);\\
A_{[\delta_k, \overline{\delta}_k, \overline{\delta}_j], j<k}(X_1, X_2) &= 2(X_1 + 1);\\
A_{[\delta_k, \overline{\delta}_l, \overline{\delta}_j], j\neq k, l\neq k, j<l}(X_1, X_2) &= X_1 + 1;\\
A_{[\delta_k, \delta_l, \overline{\delta}_j], j\neq k, l\neq j, k<l}(X_1, X_2) &= X_1.
\end{aligned}$$

(c) *Assume $m \geqslant 4$. Set $P_\alpha(X)$ a polynomial of degree 1 by :*

$$P_\alpha(X) := \begin{cases} \frac{X+1}{2} & \text{if} \quad \alpha \text{ has type I}, \\ X+1 & \text{if} \quad \alpha \text{ has type II}, \\ X & \text{if} \quad \alpha \text{ has type III}, \end{cases} \tag{4.11}$$

*Let $m^{\star a} := \ell(\underline{\boldsymbol{\mu}}^{\star a})$ and $\underline{\alpha}^{\star a}$ be a sequence of roots as in Lemma 4.3 and Lemma 4.4. We will denote by $\underline{X}^{\star a}$ the sequence of variables associated with $(\alpha^{\star a})^{(j)} \in \underline{\alpha}^{\star a}$, $j = 1, \dots, m^{\star a} - 1$, where $(\alpha^{\star a})^{(j)}$ is replaced by $X_j$. Let $N$ denotes the number of possible paths $\underline{\boldsymbol{\mu}}^{\star a}$.*

i. *Assume that $\alpha^{(p)} + \alpha^{(p-1)} \neq 0$.*
∗ *If for all $s \in \{1, \dots, p-2\}$, either $\alpha^{(s)} + \alpha^{(p)} \in -\Delta_+$ or $\alpha^{(s)} + \alpha^{(p)} \notin \Delta \cup \{0\}$, then*

$$A_{\underline{\boldsymbol{\mu}}}(X_1, \dots, X_{m-1}) = K^{\#}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}}(X_1, \dots, X_{p-2}, X_{p-1} + X_p, \dots, X_{m-1}).$$

∗ *If for some $s \in \{1, \dots, p-2\}$, $\alpha^{(s)} + \alpha^{(p)} \in \Delta_+$, then*

$$\begin{aligned} A_{\underline{\boldsymbol{\mu}}}(X_1, \dots, X_{m-1}) &= K^{\#}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1, \dots, X_{p-2}, X_{p-1} + X_p, \dots, X_{m-1}) \\ &\quad + \sum_{a=1}^{N} K^{\star a}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{X}^{\star a}). \end{aligned}$$

∗ *If for some $s \in \{1, \dots, p-2\}$, $\alpha^{(s)} = -\alpha^{(p)} \in \Delta_+$, then*

$$\begin{aligned} A_{\underline{\boldsymbol{\mu}}}(X_1, \dots, X_{m-1}) &= K^{\#}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1, \dots, X_{p-2}, X_{p-1} + X_p, \dots, X_{m-1}) \\ &\quad + (P_{\alpha^{(s)}}(X_s) - c_s) \sum_{a=1}^{N} K^{\star a}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{X}^{\star a}). \end{aligned}$$

ii. *Assume $\alpha^{(p)} + \alpha^{(p-1)} = 0$.*
∗ *If for all $s \in \{1, \dots, p-2\}$, either $\alpha^{(s)} + \alpha^{(p)} \in -\Delta_+$ or $\alpha^{(s)} + \alpha^{(p)} \notin \Delta \cup \{0\}$, then*

$$\begin{aligned} A_{\underline{\boldsymbol{\mu}}}(X_1, \dots, X_{m-1}) &= (c_{\alpha^{(p-1)}})^2 (P_{\alpha^{(p-1)}}(X_{p-1}) - c_{p-1}) \\ &\quad \times A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1, \dots, X_{p-2}, X_{p+1}, \dots, X_{m-1}). \end{aligned}$$

$*$ *If for some* $s \in \{1,\dots,p-2\}$, $\alpha^{(s)} + \alpha^{(p)} \in \Delta_+$, *then*

$$\begin{aligned}
&A_{\underline{\boldsymbol{\mu}}}(X_1,\dots,X_{m-1})\\
&\quad = (P_{\alpha^{(p-1)}}(X_{p-1}) - c_{p-1})A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1,\dots,X_{p-2},X_{p+1},\dots,X_{m-1})\\
&\qquad + \sum_{a=1}^{N} K_{\underline{\boldsymbol{\mu}}}^{\star a} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{X}^{\star a}).
\end{aligned}$$

$*$ *If for some* $s \in \{1,\dots,p-2\}$, $\alpha^{(s)} = -\alpha^{(p)} \in \Delta_+$, *then*

$$\begin{aligned}
&A_{\underline{\boldsymbol{\mu}}}(X_1,\dots,X_{m-1})\\
&\quad = (P_{\alpha^{(p-1)}}(X_{p-1}) - c_{p-1})A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1,\dots,X_{p-2},X_{p+1},\dots,X_{m-1})\\
&\qquad + (P_{\alpha^{(s)}}(X_s) - c_s)\sum_{a=1}^{N} K_{\underline{\boldsymbol{\mu}}}^{\star a} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{X}^{\star a}),
\end{aligned}$$

*where* $K_{\underline{\boldsymbol{\mu}}}^{\#}$, $K_{\underline{\boldsymbol{\mu}}}^{\star a}$ *are some constants,* $c_{\alpha^{(p-1)}}$ *is a constant as in Definition 1.14,* $c_{p-1}$ *is an integer as in equation (4.1) and* $c_s := \langle \alpha^{(s-1)}, \check{\alpha}^{(s)} \rangle$.

*Proof.* Let $\alpha \in \Delta$.

i) If $\alpha = 2\varepsilon_i$ has type I, then $\mathrm{ht}(\alpha) = 2(r+1-i)-1$ and $\mathrm{ht}(\check{\alpha}) = \mathrm{ht}((2\varepsilon_i)^\vee) = \mathrm{ht}(\varepsilon_i) = r+1-i = \frac{\mathrm{ht}(\alpha)+1}{2}$.

ii) If $\alpha = \varepsilon_i + \varepsilon_j$ has type II, then $\mathrm{ht}(\alpha) = 2r-i-j+1$ and $\mathrm{ht}(\check{\alpha}) = 2r-i-j+2 = \mathrm{ht}(\alpha)+1$.

iii) If $\alpha = \varepsilon_i - \varepsilon_j$ has type III, then $\mathrm{ht}(\alpha) = j-i$ and $\mathrm{ht}(\check{\alpha}) = j-i = \mathrm{ht}(\alpha)$.

We prove the statements by induction on $m$.

$*$ Assume $m = 2$. According to Example 4.14, there are three equivalence classes without zero as follows :

1. For $\underline{\boldsymbol{\mu}} = [\delta_k, \overline{\delta}_k]$,

$$\mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}}) = a^{(2e_{2\varepsilon_k})}_{\varepsilon_k,-\varepsilon_k} a^{2(e_{-2\varepsilon_k})}_{-\varepsilon_k,\varepsilon_k} e_{2\varepsilon_k} e_{-2\varepsilon_k} = 4e_{-2\varepsilon_k} e_{2\varepsilon_k} + 4((2\varepsilon_k)^\vee) = 4((2\varepsilon_k)^\vee).$$

Hence $\mathrm{wt}(\underline{\mu},\underline{\alpha}) = 4\mathrm{ht}((2\varepsilon_k)^\vee) = 4(\frac{i_1+1}{2}) = 2(i_1+1)$, and so $A_{[\delta_k,\overline{\delta}_k]}(X_1) = 2(X_1+1)$.

2. For $\underline{\boldsymbol{\mu}} = [\delta_k, \delta_j], k < j$, since $\underline{\boldsymbol{\mu}}$ is entirely contained in $[\![\delta_r, \delta_k]\!]$, then by $\mathfrak{sl}_{r+1}$ case we have $A_{[\delta_k,\delta_j],k<j}(X_1) = X_1$.

3. For $\underline{\boldsymbol{\mu}} = [\delta_k, \overline{\delta}_j]$,

$$\mathrm{hc}(b^*_{\underline{\mu},\underline{\alpha}}) = a^{(e_{\varepsilon_k+\varepsilon_j})}_{\varepsilon_k,-\varepsilon_j} a^{(e_{-\varepsilon_k-\varepsilon_j})}_{-\varepsilon_j,\varepsilon_k} e_{\varepsilon_k+\varepsilon_j} e_{-\varepsilon_k-\varepsilon_j} = ((\varepsilon_k+\varepsilon_j)^\vee).$$

   Hence $\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \mathrm{ht}((\varepsilon_k+\varepsilon_j)^\vee) = \mathrm{ht}(\varepsilon_k+\varepsilon_j)+1 = i_1+1$, and so $A_{[\delta_k,\overline{\delta}_j]}(X_1) = X_1+1$. Thus for all $\underline{\boldsymbol{\mu}} \in \mathscr{E}_2$ without zero, $A_{\underline{\boldsymbol{\mu}}}$ is polynomial of degree 1 in $\mathbb{C}[X_1]$.

∗ Assume $m = 3$. On account of Example 4.14, there are 12 equivalence classes without zero as follows.

1. For $\underline{\boldsymbol{\mu}} = [\delta_k, \delta_l, \delta_j], k < l < j$ and $\mu = [\delta_k, \delta_l, \delta_j], k < j < l$, since those $\underline{\boldsymbol{\mu}}$ are entirely contained in $[\![\delta_r, \delta_k]\!]$, then according to the $\mathfrak{sl}_{r+1}$ case, we have

$$A_{[\delta_k,\delta_l,\delta_j],k<l<j}(X_1, X_2) = X_1 \quad \text{and} \quad A_{[\delta_k,\delta_l,\delta_j],k<j<l}(X_1, X_2) = X_1 + X_2.$$

2. For $\underline{\boldsymbol{\mu}} = [\delta_k, \overline{\delta}_k, \delta_j], k < j$, we have $p = 2$, $\underline{\boldsymbol{\mu}}^\# = [\delta_k, \delta_j]$, $\underline{\alpha}^{\#p} = (\alpha^{(1)} + \alpha^{(2)}, \alpha^{(3)})$ and $\underline{i}^{\#p} = (i_1 + i_2, i_3)$. We get

$$K^\#_{\underline{\boldsymbol{\mu}}} = n_{2\varepsilon_k,-\varepsilon_k-\varepsilon_j} \frac{a^{(e_{\varepsilon_k+\varepsilon_j})}_{\varepsilon_j,-\varepsilon_k} a^{(2e_{-2\varepsilon_k})}_{-\varepsilon_k,-\varepsilon_k}}{a^{(e_{\varepsilon_j-\varepsilon_k})}_{\varepsilon_j,\varepsilon_k}} = 1 \times \frac{2\times 1}{1} = 2.$$

   Hence $\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^\#_{\underline{\boldsymbol{\mu}}} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} = 2(i_1^{\#p}) = 2(i_1 + i_2)$, and so

$$A_{[\delta_k,\overline{\delta}_k,\delta_j],k<j}(X_1, X_2) = 2(X_1 + X_2).$$

3. For $\underline{\boldsymbol{\mu}} = [\delta_k, \overline{\delta}_k, \overline{\delta}_j], k < j$, thus $p = 2$, $\underline{\boldsymbol{\mu}}^\# = [\delta_k, \overline{\delta}_j]$, $\underline{\alpha}^{\#p} = (\alpha^{(1)} + \alpha^{(2)}, \alpha^{(3)})$ and $\underline{i}^{\#p} = (i_1 + i_2, i_3)$. We get

$$K^\#_{\underline{\boldsymbol{\mu}}} = n_{2\varepsilon_k,\varepsilon_j-\varepsilon_k} \frac{a^{(e_{\varepsilon_k-\varepsilon_j})}_{-\varepsilon_j,-\varepsilon_k} a^{(2e_{-2\varepsilon_k})}_{-\varepsilon_k,\varepsilon_k}}{a^{(e_{\varepsilon_j-\varepsilon_k})}_{-\varepsilon_j,\varepsilon_k}} = -1 \times \frac{-1\times 2}{1} = 2.$$

   Hence $\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^\#_{\underline{\boldsymbol{\mu}}} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} = 2(i_1^{\#p} + 1) = 2(i_1 + i_2 + 1)$, and so

$$A_{[\delta_k,\overline{\delta}_k,\overline{\delta}_j],k<j}(X_1, X_2) = 2(X_1 + X_2 + 1).$$

4. For $\underline{\boldsymbol{\mu}} = [\delta_k, \overline{\delta}_j, \overline{\delta}_k], k < j$, thus $p = 2$, $\underline{\boldsymbol{\mu}}^{\#} = [\delta_k, \overline{\delta}_k]$, $\underline{\alpha}^{\#p} = (\alpha^{(1)} + \alpha^{(2)}, \alpha^{(3)})$ and $\underline{i}^{\#p} = (i_1 + i_2, i_3)$. We get

$$K^{\#}_{\underline{\boldsymbol{\mu}}} = n_{\varepsilon_k+\varepsilon_j,\varepsilon_k-\varepsilon_j} \frac{a^{(e_{\varepsilon_j-\varepsilon_k})}_{-\varepsilon_k,-\varepsilon_j} a^{(e_{-\varepsilon_j-\varepsilon_k})}_{-\varepsilon_j,\varepsilon_k}}{a^{(2e_{-\varepsilon_k})}_{-\varepsilon_k,\varepsilon_k}} = -2 \times \frac{-1 \times 1}{2} = 1.$$

Hence $\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^{\#}_{\underline{\boldsymbol{\mu}}} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} = 1 \times 2(i^{\#p}_1 + 1) = 2(i_1 + i_2 + 1)$, and so

$$A_{[\delta_k,\overline{\delta}_k,\overline{\delta}_j],k<j}(X_1, X_2) = 2(X_1 + X_2 + 1).$$

5. For $\underline{\boldsymbol{\mu}} = [\delta_k, \overline{\delta}_j, \overline{\delta}_l], j \neq k, l \neq k, j < l$, thus $p = 2$, $\underline{\boldsymbol{\mu}}^{\#} = [\delta_k, \overline{\delta}_l]$, $\underline{\alpha}^{\#p} = (\alpha^{(1)} + \alpha^{(2)}, \alpha^{(3)})$ and $\underline{i}^{\#p} = (i_1 + i_2, i_3)$. We get

$$K^{\#}_{\underline{\boldsymbol{\mu}}} = n_{\varepsilon_k+\varepsilon_j,\varepsilon_l-\varepsilon_j} \frac{a^{(e_{\varepsilon_j-\varepsilon_l})}_{-\varepsilon_l,-\varepsilon_j} a^{(e_{-\varepsilon_k-\varepsilon_j})}_{-\varepsilon_j,\varepsilon_k}}{a^{(e_{-\varepsilon_l-\varepsilon_k})}_{-\varepsilon_l,\varepsilon_k}} = -1 \times \frac{-1 \times 1}{1} = 1.$$

Hence $\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^{\#}_{\underline{\boldsymbol{\mu}}} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} = i^{\#p} + 1 = i_1 + i_2 + 1$, and so

$$A_{[\delta_k,\overline{\delta}_j,\overline{\delta}_l],j\neq k,l\neq k,j<l}(X_1, X_2) = X_1 + X_2 + 1.$$

6. For $\underline{\boldsymbol{\mu}} = [\delta_k, \overline{\delta}_j, \delta_l], j \neq k, l \neq j, k < l$, thus $p = 2$, $\underline{\boldsymbol{\mu}}^{\#} = [\delta_k, \delta_l]$, $\underline{\alpha}^{\#p} = (\alpha^{(1)} + \alpha^{(2)}, \alpha^{(3)})$ and $\underline{i}^{\#p} = (i_1 + i_2, i_3)$. We get

$$K^{\#}_{\underline{\boldsymbol{\mu}}} = n_{\varepsilon_k+\varepsilon_j,-\varepsilon_l-\varepsilon_j} \frac{a^{(e_{\varepsilon_l+\varepsilon_j})}_{\varepsilon_l,-\varepsilon_j} a^{(e_{-\varepsilon_j-\varepsilon_l})}_{-\varepsilon_j,\varepsilon_l}}{a^{(e_{\varepsilon_l-\varepsilon_k})}_{\varepsilon_l,\varepsilon_k}} = 1 \times \frac{1 \times 1}{1} = 1.$$

Hence $\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^{\#}_{\underline{\boldsymbol{\mu}}} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} = i^{\#p} = i_1 + i_2$, and so

$$A_{[\delta_k,\overline{\delta}_j,\delta_l],j\neq k,l\neq j,k<l}(X_1, X_2) = X_1 + X_2.$$

7. For $\underline{\boldsymbol{\mu}} = [\delta_k, \delta_j, \overline{\delta}_k], k < j$, thus $p = 3$, $\underline{\boldsymbol{\mu}}^{\#} = [\delta_k, \delta_j]$, $\underline{\alpha}^{\#p} = (\alpha^{(1)}, \alpha^{(2)} + \alpha^{(3)})$ and $\underline{i}^{\#p} = (i_1, i_2 + i_3)$. We get

$$K^{\#}_{\underline{\boldsymbol{\mu}}} = n_{\varepsilon_j+\varepsilon_k,-2\varepsilon_k} \frac{a^{(2e_{2\varepsilon_k})}_{\varepsilon_k,-\varepsilon_k} a^{(e_{-\varepsilon_k-\varepsilon_j})}_{-\varepsilon_k,\varepsilon_j}}{a^{(e_{\varepsilon_k-\varepsilon_j})}_{\varepsilon_k,\varepsilon_j}} = 1 \times \frac{2 \times 1}{1} = 2.$$

Hence $\mathrm{wt}(\underline{\mu},\underline{\alpha}) = K^{\#}_{\underline{\boldsymbol{\mu}}}\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p} = 2(i_1^{\#p}) = 2(i_1)$, and so

$$A_{[\delta_k,\delta_j,\overline{\delta}_k],k<j}(X_1,X_2) = 2X_1.$$

8. For $\underline{\boldsymbol{\mu}} = [\delta_k,\overline{\delta}_j,\overline{\delta}_k], k<j$, thus $p=3$, $\underline{\boldsymbol{\mu}}^{\#} = [\delta_k,\overline{\delta}_j]$, $\underline{\alpha}^{\#p} = (\alpha^{(1)},\alpha^{(2)}+\alpha^{(3)})$ and $\underline{i}^{\#p} = (i_1, i_2+i_3)$. We get

$$K^{\#}_{\underline{\boldsymbol{\mu}}} = n_{\varepsilon_k-\varepsilon_j,-2\varepsilon_k}\frac{a^{(2e_{2\varepsilon_k})}_{\varepsilon_k,-\varepsilon_k}a^{(e_{\varepsilon_j-\varepsilon_k})}_{-\varepsilon_k,-\varepsilon_j}}{a^{(e_{\varepsilon_k+\varepsilon_j})}_{\varepsilon_k,-\varepsilon_j}} = -1\times\frac{2\times -1}{1} = 2.$$

Hence $\mathrm{wt}(\underline{\mu},\underline{\alpha}) = K^{\#}_{\underline{\boldsymbol{\mu}}}\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p} = 2(i_1^{\#p}+1) = 2(i_1+1)$, and so

$$A_{[\delta_k,\overline{\delta}_j,\overline{\delta}_k],k<j}(X_1,X_2) = 2X_1+1.$$

9. For $\underline{\boldsymbol{\mu}} = [\delta_k,\overline{\delta}_k,\overline{\delta}_j], j<k$, thus $p=3$, $\underline{\boldsymbol{\mu}}^{\#} = [\delta_k,\overline{\delta}_k]$, $\underline{\alpha}^{\#p} = (\alpha^{(1)},\alpha^{(2)}+\alpha^{(3)})$ and $\underline{i}^{\#p} = (i_1, i_2+i_3)$. We get

$$K^{\#}_{\underline{\boldsymbol{\mu}}} = n_{\varepsilon_j-\varepsilon_k,-\varepsilon_k-\varepsilon_j}\frac{a^{(e_{\varepsilon_k+\varepsilon_j})}_{\varepsilon_k,-\varepsilon_j}a^{(e_{\varepsilon_k-\varepsilon_j})}_{-\varepsilon_j,-\varepsilon_k}}{a^{(2e_{2\varepsilon_k})}_{\varepsilon_k,-\varepsilon_k}} = -2\times\frac{1\times -1}{2} = 1.$$

Hence $\mathrm{wt}(\underline{\mu},\underline{\alpha}) = K^{\#}_{\underline{\boldsymbol{\mu}}}\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p} = 2(\underline{i}_1+1)$, and so

$$A_{[\delta_k,\overline{\delta}_k,\overline{\delta}_j],j<k}(X_1,X_2) = 2(X_1+1).$$

10. For $\underline{\boldsymbol{\mu}} = [\delta_k,\overline{\delta}_l,\overline{\delta}_j], j\neq k, l\neq k, j<l$, thus $p=3$, $\underline{\boldsymbol{\mu}}^{\#} = [\delta_k,\overline{\delta}_l]$, $\underline{\alpha}^{\#p} = (\alpha^{(1)},\alpha^{(2)}+\alpha^{(3)})$ and $\underline{i}^{\#p} = (i_1, i_2+i_3)$. We get

$$K^{\#}_{\underline{\boldsymbol{\mu}}} = n_{\varepsilon_j-\varepsilon_l,-\varepsilon_k-\varepsilon_j}\frac{a^{(e_{\varepsilon_k+\varepsilon_j})}_{\varepsilon_k,-\varepsilon_j}a^{(e_{\varepsilon_l-\varepsilon_j})}_{-\varepsilon_j,-\varepsilon_l}}{a^{(e_{\varepsilon_k+\varepsilon_l})}_{\varepsilon_k,-\varepsilon_l}} = -1\times\frac{1\times -1}{1} = 1.$$

Hence $\mathrm{wt}(\underline{\mu},\underline{\alpha}) = K^{\#}_{\underline{\boldsymbol{\mu}}}\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p} = (i^{\#p}+1) = (i_1+1)$, and so

$$A_{[\delta_k,\overline{\delta}_l,\overline{\delta}_j],j\neq k,l\neq k,j<l}(X_1,X_2) = X_1+1.$$

11. For $\underline{\boldsymbol{\mu}} = [\delta_k,\delta_l,\overline{\delta}_j], j\neq k, l\neq j, k<l$, thus $p=3$, $\underline{\boldsymbol{\mu}}^{\#} = [\delta_k,\delta_l]$, $\underline{\alpha}^{\#p} =$

$(\alpha^{(1)}, \alpha^{(2)} + \alpha^{(3)})$ and $\underline{i}^{\#p} = (i_1, i_2 + i_3)$. We get

$$K^{\#}_{\underline{\boldsymbol{\mu}}} = n_{\varepsilon_l+\varepsilon_j,-\varepsilon_j-\varepsilon_k} \frac{a^{(e_{\varepsilon_k+\varepsilon_j})}_{\varepsilon_k,-\varepsilon_j} a^{(e_{-\varepsilon_j-\varepsilon_l})}_{-\varepsilon_j,\varepsilon_l}}{a^{(e_{\varepsilon_k-\varepsilon_l})}_{\varepsilon_k,\varepsilon_l}} = 1 \times \frac{1 \times 1}{1} = 1.$$

Hence $\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^{\#}_{\underline{\boldsymbol{\mu}}} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} = i^{\#p} = i_1$, and so

$$A_{[\delta_k,\delta_l,\bar{\delta}_j],j\neq k,l\neq j,k<l}(X_1, X_2) = X_1.$$

Thus for all $\underline{\boldsymbol{\mu}} \in \mathscr{E}_3$ without zero, $A_{\underline{\boldsymbol{\mu}}}$ is polynomial of degree 1 in $\mathbb{C}[X_1, X_2]$.

$*$ Let $m \geqslant 4$ and assume the proposition true for any $m' \in \{2, \dots, m-1\}$ and any $\underline{\boldsymbol{\mu}}' \in \mathscr{E}_{m'}$. Let $\underline{\boldsymbol{\mu}} \in \mathscr{E}_m$ and $(\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}$. For $\alpha^{(j)} \in \underline{\alpha}$, let $P_{\alpha^{(j)}}$ be a polynomial as in (4.11). Observe that $P_{\alpha^{(j)}}$ is a polynomial of degree 1.

Let $\hat{\underline{i}}^{\star a}$ denotes the sequence of heights $((\alpha^{\star a})^{(1)}, \dots, (\alpha^{\star a})^{(m^{\star a}-1)})$.
Recall the constants $c_{\alpha^{(p-1)}}$ as in Definition 1.14, $c_{p-1}$ as in equation (4.1) and $c_s := \langle \alpha^{(s-1)}, \check{\alpha}^{(s)} \rangle$.

(i) Assume that $\alpha^{(p-1)} + \alpha^{(p)} \neq 0$.
$*$ If for all $s \in \{1, \dots, p-2\}$, either $\alpha^{(s)} + \alpha^{(p)} \in -\Delta_+$ or $\alpha^{(s)} + \alpha^{(p)} \notin \Delta \cup \{0\}$, then by Proposition 4.9 there is a constant $K^{\#p}_{(\underline{\mu},\underline{\alpha})}$ such that

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^{\#p} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p}.$$

We have $(\underline{\mu}, \underline{\alpha})^{\#p} \in \underline{\boldsymbol{\mu}}^{\#}$ and by our induction hypothesis, there exists a polynomial $A_{\underline{\boldsymbol{\mu}}^{\#}} \in \mathbb{C}[Y_1, \dots, Y_{m-2}]$ of total degree $\leqslant \lfloor \frac{m-1}{2} \rfloor$ such that

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^{\#p}_{(\underline{\mu},\underline{\alpha})} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} = K^{\#p}_{(\underline{\mu},\underline{\alpha})} A_{\underline{\boldsymbol{\mu}}^{\#}}(i_1, \dots, i_{p-1} + i_p, \dots, i_{m-1}).$$

Hence the polynomial

$$A_{\underline{\boldsymbol{\mu}}}(X_1, \dots, X_{m-1}) := K^{\#}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1, \dots, X_{p-2}, X_{p-1} + X_p, \dots, X_{m-1})$$

satisfies the conditions of the lemma.
$*$ If for some $s \in \{1, \dots, p-2\}$, $\alpha^{(s)} + \alpha^{(p)} \in \Delta_+$, then by Proposition 4.9 there are some constants $K^{\#p}_{(\underline{\mu},\underline{\alpha})}$ and $K^{\star a}_{(\underline{\mu},\underline{\alpha})}$ such that

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^{\#p}_{(\underline{\mu},\underline{\alpha})} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} + \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\star a}.$$

We have $(\underline{\mu}, \underline{\alpha})^{\#p} \in \underline{\boldsymbol{\mu}}^{\#}$ and $(\underline{\mu}, \underline{\alpha})^{\star a} \in \underline{\boldsymbol{\mu}}^{\star a}$. By our induction hypothesis, there are polynomials $A_{\underline{\boldsymbol{\mu}}^{\#}} \in \mathbb{C}[Y_1, \ldots, Y_{m-2}]$, and $A_{\underline{\boldsymbol{\mu}}^{\star a}} \in \mathbb{C}[Y_1, \ldots, Y_{m-2}]$ of total degree $\leqslant \lfloor \frac{m-1}{2} \rfloor$ such that

$$\begin{aligned}\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = &K^{\#p}_{(\underline{\mu},\underline{\alpha})} A_{\underline{\boldsymbol{\mu}}^{\#}}(i_1, \ldots, i_{p-1} + i_p, \ldots, i_{m-1}) \\ &+ \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{\hat{\imath}}^{\star a}),\end{aligned}$$

where $K^{\#p}_{(\underline{\mu},\underline{\alpha})}$ and $K^{\star a}_{(\underline{\mu},\underline{\alpha})}$ are some constants.

Set

$$\begin{aligned}A_{\underline{\boldsymbol{\mu}}}(X_1, \ldots, X_{m-1}) := &K^{\#}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1, \ldots, X_{p-2}, X_{p-1} + X_p, \ldots, X_{m-1}) \\ &+ \sum_{a=1}^{N} K^{\star a}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{X}^{\star a}).\end{aligned}$$

By our computation above, the polynomial $A_{\underline{\boldsymbol{\mu}}}$ satisfies the condition of the lemma.

$*$ If for some $s \in \{1, \ldots, p-2\}$, $\alpha^{(s)} = -\alpha^{(p)}$ then

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = K^{\#p}_{(\underline{\mu},\underline{\alpha})} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\#p} + (\mathrm{ht}(\check{\alpha}^{(s)}) - c_s) \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\star a}.$$

We have $(\underline{\mu}, \underline{\alpha})^{\#p} \in \underline{\boldsymbol{\mu}}^{\#}$ and $(\underline{\mu}, \underline{\alpha})^{\star a} \in \underline{\boldsymbol{\mu}}^{\star a}$. By our induction hypothesis, there are polynomials $A_{\underline{\boldsymbol{\mu}}^{\#}} \in \mathbb{C}[Y_1, \ldots, Y_{m-2}]$ of total degree $\leqslant \lfloor \frac{m-1}{2} \rfloor$ and $A_{\underline{\boldsymbol{\mu}}^{\star a}} \in \mathbb{C}[Y_1, \ldots, Y_{m-3}]$ of total degree $\leqslant \lfloor \frac{m-2}{2} \rfloor$ such that

$$\begin{aligned}\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = &K^{\#p}_{(\underline{\mu},\underline{\alpha})} A_{\underline{\boldsymbol{\mu}}^{\#}}(i_1, \ldots, i_{p-1} + i_p, \ldots, i_{m-1}) \\ &+ (P_{\alpha^{(s)}}(i_s) - c_s) \sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{\hat{\imath}}^{\star a}),\end{aligned}$$

where $K^{\#p}_{(\underline{\mu},\underline{\alpha})}$, $K^{\star a}_{(\underline{\mu},\underline{\alpha})}$, and $c_s$ are some constants, whereas $P_{\alpha^{(s)}}$ is a polynomial of degree 1.

Set

$$A_{\underline{\boldsymbol{\mu}}}(X_1, \ldots, X_{m-1}) := K^{\#}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1, \ldots, X_{p-2}, X_{p-1} + X_p, \ldots, X_{m-1})$$

$$+ (P_{\alpha^{(s)}}(X_s) - c_s) \sum_{a=1}^{N} K_{\underline{\boldsymbol{\mu}}}^{\star a} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{X}^{\star a}).$$

Thus $A_{\underline{\boldsymbol{\mu}}}$ has total degree $\leqslant 1 + \lfloor \frac{m-2}{2} \rfloor = \lfloor \frac{m}{2} \rfloor$. Hence this polynomial satisfies the conditions of the lemma.

(ii) Assume that $\alpha^{(p-1)} + \alpha^{(p)} = 0$.
$*$ If for all $s \in \{1, \ldots, p-2\}$, either $\alpha^{(s)} + \alpha^{(p)} \in -\Delta_+$ or $\alpha^{(s)} + \alpha^{(p)} \notin \Delta \cup \{0\}$, By Proposition 4.9 we have

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (c_{\alpha^{(p-1)}})^2 \, (\mathrm{ht}(\check{\alpha}^{(p-1)}) - c_{p-1}) \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\# p}.$$

We have $(\underline{\mu}, \underline{\alpha})^{\# p} \in \underline{\boldsymbol{\mu}}^{\#}$ and by our induction hypothesis, there is a polynomial $A_{\underline{\boldsymbol{\mu}}^{\#}} \in \mathbb{C}[Y_1, \ldots, Y_{m-3}]$ of total degree $\leqslant \lfloor \frac{m-2}{2} \rfloor$ such that

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (c_{\alpha^{(p-1)}})^2 \, (P_{\alpha^{(p-1)}}(i_{p-1}) - c_{p-1}) A_{\underline{\boldsymbol{\mu}}^{\#}}(i_1, \ldots, i_{p-2}, i_{p+1}, \ldots, i_{m-1}),$$

where $c_{p-1}$ is a constant and $P_{\alpha^{(p-1)}}$ is a polynomial of degree 1. Set

$$A_{\underline{\boldsymbol{\mu}}}(X_1, \ldots, X_{m-1}) := (P_{\alpha^{(p-1)}}(X_{p-1}) - c_{p-1}) A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1, \ldots, X_{p-2}, X_{p+1}, \ldots, X_{m-1}).$$

Thus $A_{\underline{\boldsymbol{\mu}}}$ has total degree $\leqslant 1 + \lfloor \frac{m-2}{2} \rfloor = \lfloor \frac{m}{2} \rfloor$. Hence the polynomial $A_{\underline{\boldsymbol{\mu}}}$ satisfies the condition of the lemma.
$*$ If for some $s \in \{1, \ldots, p-2\}$, $\alpha^{(s)} + \alpha^{(p)} \in \Delta_+$, then by Proposition 4.9 there are some constants $K_{(\underline{\mu}, \underline{\alpha})}^{\# p}$ and $K_{(\underline{\mu}, \underline{\alpha})}^{\star a}$ such that

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = (\mathrm{ht}(\check{\alpha}^{(p-1)}) - c_{p-1}) \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\# p} + \sum_{a=1}^{N} K_{(\underline{\mu}, \underline{\alpha})}^{\star a} \mathrm{wt}(\underline{\mu}, \underline{\alpha})^{\star a}.$$

We have $(\underline{\mu}, \underline{\alpha})^{\# p} \in \underline{\boldsymbol{\mu}}^{\#}$ and $(\underline{\mu}, \underline{\alpha})^{\star a} \in \underline{\boldsymbol{\mu}}^{\star a}$. By our induction hypothesis, there are polynomials $A_{\underline{\boldsymbol{\mu}}^{\#}} \in \mathbb{C}[Y_1, \ldots, Y_{m-3}]$ of total degree $\leqslant \lfloor \frac{m-2}{2} \rfloor$ and $A_{\underline{\boldsymbol{\mu}}^{\star a}} \in \mathbb{C}[Y_1, \ldots, Y_{m-2}]$ of total degree $\leqslant \lfloor \frac{m-1}{2} \rfloor$ such that

$$\begin{aligned} \mathrm{wt}(\underline{\mu}, \underline{\alpha}) &= (P_{\alpha^{(p-1)}}(i_{p-1}) - c_{p-1}) A_{\underline{\boldsymbol{\mu}}^{\#}}(i_1, \ldots, i_{p-2}, i_{p+1}, \ldots, i_{m-1}) \\ &\quad + \sum_{a=1}^{N} K_{(\underline{\mu}, \underline{\alpha})}^{\star a} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{\hat{\imath}}^{\star a}), \end{aligned}$$

where $K^{\star a}_{(\underline{\mu},\underline{\alpha})}$, $c_{p-1}$ are some constants and $P_{\alpha^{(p-1)}}$ is a polynomial of degree 1. Set

$$A_{\underline{\boldsymbol{\mu}}}(X_1,\dots,X_{m-1}) = (P_{\alpha^{(p-1)}}(X_{p-1}) - c_{p-1})A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1,\dots,X_{p-2},X_{p+1},\dots,X_{m-1}) + \sum_{a=1}^{N} K^{\star a}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{X}^{\star a}).$$

By our computation above, $A_{\underline{\boldsymbol{\mu}}}$ has total degree $\leqslant 1 + \lfloor \frac{m-2}{2} \rfloor = \lfloor \frac{m}{2} \rfloor$, hence it satisfies the conditions of the lemma.
$*$ If for some $s \in \{1,\dots,p-2\}$, $\alpha^{(s)} = -\alpha^{(p)}$ then

$$\mathrm{wt}(\underline{\mu},\underline{\alpha}) = (\mathrm{ht}(\check{\alpha}^{(p-1)}) - c_{p-1})\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\#p} + (\mathrm{ht}(\check{\alpha}^{(s)}) - c_s)\sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})}\mathrm{wt}(\underline{\mu},\underline{\alpha})^{\star a}.$$

By our induction hypothesis, there are polynomials $A_{\underline{\boldsymbol{\mu}}^{\#}} \in \mathbb{C}[Y_1,\dots,Y_{m-3}]$ and $A_{\underline{\boldsymbol{\mu}}^{\star a}} \in \mathbb{C}[Y_1,\dots,Y_{m-3}]$ of total degree $\leqslant \lfloor \frac{m-2}{2} \rfloor$ such that

$$\mathrm{wt}(\underline{\mu},\underline{\alpha}) = (P_{\alpha^{(p-1)}}(i_{p-1}) - c_{p-1})A_{\underline{\boldsymbol{\mu}}^{\#}}(i_1,\dots,i_{p-1}+i_p,\dots,i_{m-1}) + (P_{\alpha^{(s)}}(i_s) - c_s)\sum_{a=1}^{N} K^{\star a}_{(\underline{\mu},\underline{\alpha})} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{\imath}^{\star a}),$$

where $c_{p-1}$ and $c_s$ are some constants, whereas $P_{\alpha^{(p-1)}}$ and $P_{\alpha^{(s)}}$ are polynomials of degree 1. Hence the polynomial

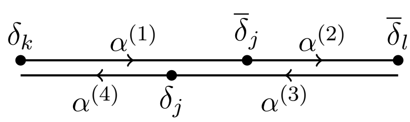
$$A_{\underline{\boldsymbol{\mu}}}(X_1,\dots,X_{m-1}) := (P_{\alpha^{(p-1)}}(X_{p-1}) - c_{p-1})A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1,\dots,X_{p-2},X_{p+1},\dots,X_{m-1}) + (P_{\alpha^{(s)}}(X_s) - c_s)\sum_{a=1}^{N} K^{\star a}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{X}^{\star a})$$

satisfies the conditions of the lemma.

□

*Example* 4.16. Let $\underline{\boldsymbol{\mu}} \in \mathscr{E}_m$ without zero.

1. Assume $m = 4$, and $\underline{\boldsymbol{\mu}} = [\delta_k, \overline{\delta}_j, \overline{\delta}_l, \delta_j]$, with $k < l < j$.



Thus $p = 3$ and there is an integer $s = 1$ such that $\alpha^{(p)} + \alpha^{(s)} \in \Delta_+$. We get

$$\underline{\boldsymbol{\mu}}^{\#} = [\delta_k, \overline{\delta}_j, \delta_j], k < j \sim [\delta_k, \overline{\delta}_k, \delta_j], k < j,$$

and $K^{\#}_{\underline{\boldsymbol{\mu}}} = n_{\varepsilon_l-\varepsilon_j,-\varepsilon_l-\varepsilon_j} \frac{a^{(e_{\varepsilon_j+\varepsilon_l})}_{\varepsilon_j,-\varepsilon_l} a^{(e_{-\varepsilon_l-\varepsilon_j})}_{-\varepsilon_l,-\varepsilon_j}}{a^{(e_{2\varepsilon_j})}_{\varepsilon_j,-\varepsilon_j}} = -2 \times \frac{1\times -1}{2} = 1.$
Since $s + 1 = p - 1$ then $N = 1$ and $\underline{\boldsymbol{\mu}}^{\star a} = [\delta_k, \delta_l, \delta_j], k < l < j$, with

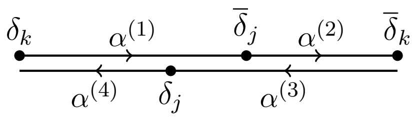
$$K^{\star a}_{\underline{\boldsymbol{\mu}}} = n_{\varepsilon_k+\varepsilon_j,-\varepsilon_l-\varepsilon_j} \frac{a^{(e_{\varepsilon_j+\varepsilon_l})}_{\varepsilon_j,-\varepsilon_l} a^{(e_{-\varepsilon_l-\varepsilon_j})}_{-\varepsilon_l,-\varepsilon_j} a^{(e_{-\varepsilon_j-\varepsilon_k})}_{-\varepsilon_j,\varepsilon_k}}{a^{(e_{\varepsilon_j-\varepsilon_l})}_{\varepsilon_j,\varepsilon_l} a^{(e_{\varepsilon_l-\varepsilon_k})}_{\varepsilon_l,\varepsilon_k}} = 1 \times \frac{1 \times -1 \times 1}{1 \times 1} = -1.$$

Hence by Lemma 4.15 (2c (i)) we get

$$\begin{aligned} A_{\underline{\boldsymbol{\mu}}}(X_1, X_2, X_3) &= K^{\#}_{\underline{\boldsymbol{\mu}}} A_{\mu^{\#}}(X_1, X_2 + X_3) + K^{\star a}_{\underline{\boldsymbol{\mu}}} A_{\mu^{\star a}}(X_1 + X_3, X_2) \\ &= 1 \times 2(X_1 + X_2 + X_3) + (-1) \times (X_1 + X_3) = X_1 + 2X_2 + X_3. \end{aligned}$$

So $A_{[\delta_k,\overline{\delta}_j,\overline{\delta}_l,\delta_j],k<l<j}(X_1, X_2, X_3) = X_1 + 2X_2 + X_3$.

2. Assume $m = 4$, and $\underline{\boldsymbol{\mu}} = [\delta_k, \overline{\delta}_j, \overline{\delta}_k, \delta_j]$, with $k < l < j$.




Thus $p = 3$ and there is an integer $s = 1$ such that $\alpha^{(p)} = -\alpha^{(s)}$. We get $\underline{\boldsymbol{\mu}}^{\#}$ as in (1), $P_{\alpha^{(s)}}(X_s) = X_s + 1$ and

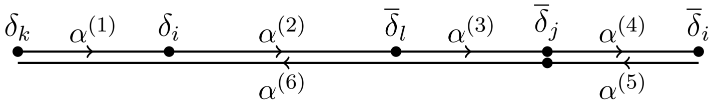
$$\underline{\boldsymbol{\mu}}^{\star a} = [\delta_k, \delta_j], k < j, K^{\star a}_{\underline{\boldsymbol{\mu}}} = a^{(e_{\varepsilon_j+\varepsilon_k})}_{\varepsilon_j,-\varepsilon_k} a^{(e_{\varepsilon_j-\varepsilon_k})}_{-\varepsilon_k,-\varepsilon_j} = -1.$$

Hence by Lemma 4.15 we get

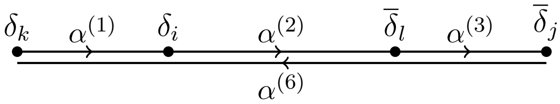
$$\begin{aligned} A_{\underline{\boldsymbol{\mu}}}(X_1, X_2, X_3) &= K^{\#}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1, X_2 + X_3) + (P_{\alpha^{(s)}}(X_s)) K^{\star a}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\star a}}(X_2) \\ &= 1 \times 2(X_1 + X_2 + X_3) + (X_1 + 1) \times (-1) \times (X_2) \\ &= 2X_2 - X_1X_2 - X_2 = X_2 - X_1X_2, \end{aligned}$$

since $X_1 = -X_3$. Hence $A_{[\delta_k,\overline{\delta}_j,\overline{\delta}_k,\delta_j],k<l<j}(X_1, X_2, X_3) = X_2 - X_1X_2$.

3. Assume $m = 6$ and $\underline{\boldsymbol{\mu}} = [\delta_k, \delta_i, \overline{\delta}_l, \overline{\delta}_j, \overline{\delta}_i, \overline{\delta}_j]$, with $k < i < j < l$.




Thus $p = 5$ and there is an integer $s = 2$ such that $\alpha^s + \alpha^p \in \Delta_+$. We have $\underline{\boldsymbol{\mu}}^{\#} = [\delta_k, \delta_i, \overline{\delta}_l, \overline{\delta}_j], k < i < j < l$.



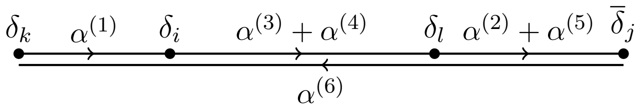

Hence by Lemma 4.15,

$$\begin{aligned} A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1, X_2, X_3) &= (K^{\#}_{\underline{\boldsymbol{\mu}}})^{\#} A_{(\underline{\boldsymbol{\mu}}^{\#})^{\#}}(X_1, X_2) \\ &= n_{\varepsilon_j - \varepsilon_l, -\varepsilon_j - \varepsilon_k} \frac{a^{(e_{\varepsilon_k+\varepsilon_j})}_{\varepsilon_k, -\varepsilon_j} a^{(e_{\varepsilon_l-\varepsilon_j})}_{-\varepsilon_j, -\varepsilon_l}}{a^{(e_{\varepsilon_k+\varepsilon_l})}_{\varepsilon_k, -\varepsilon_l}} A_{[\delta_k, \delta_i, \overline{\delta}_l], k<i<l}(X_1, X_2) \\ &= -1 \times \frac{1 \times -1}{1}(X_1) = X_1. \end{aligned}$$

The root $\alpha^{(p-1)}$ has type III. So $P_{\alpha^{(p-1)}}(X_{p-1}) = X_4$ and $c_{p-1} = \langle \alpha^{(3)}, \check{\alpha}^{(4)} \rangle + \langle \alpha^{(2)}, \check{\alpha}^{(4)} \rangle + \langle \alpha^{(1)}, \check{\alpha}^{(4)} \rangle = -1 + 1 + -1 = -1$.

Note that $\alpha^s + \alpha^p = \varepsilon_l + \varepsilon_j = \delta_l - \overline{\delta}_j$. The several new paths $(\underline{\boldsymbol{\mu}})^{\star a}$ that are generated by "reversing the order" as in Lemma 4.4 are the following:

(a) $\underline{\boldsymbol{\mu}}^{\star 1} = [\delta_k, \delta_i, \delta_l, \overline{\delta}_j]$,




In this path,

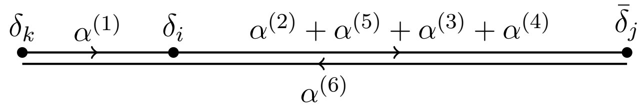
$$\begin{aligned} &A_{\underline{\boldsymbol{\mu}}^{\star 1}}(X_1, X_3 + X_4, X_2 + X_5) = (K^{\star 1}_{\underline{\boldsymbol{\mu}}})^{\#} A_{(\underline{\boldsymbol{\mu}}^{\star 1})^{\#}}(X_1, X_3 + X_4) \\ &= n_{\varepsilon_l + \varepsilon_j, -\varepsilon_j - \varepsilon_k} \frac{a^{(e_{\varepsilon_k+\varepsilon_j})}_{\varepsilon_k, -\varepsilon_j} a^{(e_{-\varepsilon_j-\varepsilon_l})}_{-\varepsilon_j, \varepsilon_l}}{a^{(e_{\varepsilon_k-\varepsilon_l})}_{\varepsilon_k, \varepsilon_l}} A_{[\delta_k, \delta_i, \delta_l]}(X_1, X_3 + X_4) = (-1)(X_1) = -X_1, \end{aligned}$$

and

$$K^{\star 1}_{\underline{\boldsymbol{\mu}}} = n_{\alpha^{(2)}, \alpha^{(5)}} n_{\alpha^{(3)}, \alpha^{(4)}} \frac{a^{(e_{\varepsilon_i-\varepsilon_j})}_{-\varepsilon_j, -\varepsilon_i} a^{(e_{\varepsilon_j-\varepsilon_i})}_{-\varepsilon_i, -\varepsilon_j} a^{(e_{\varepsilon_l-\varepsilon_j})}_{-\varepsilon_j, -\varepsilon_l} a^{(e_{-\varepsilon_l-\varepsilon_j})}_{-\varepsilon_l, \varepsilon_j}}{a^{(e_{-\varepsilon_j-\varepsilon_l})}_{-\varepsilon_j, \varepsilon_l} a^{(e_{\varepsilon_l-\varepsilon_i})}_{\varepsilon_l, \varepsilon_i}}$$

$$= 1 \times -1 \times \frac{-1 \times -1 \times -1 \times 1}{1 \times 1} = 1.$$

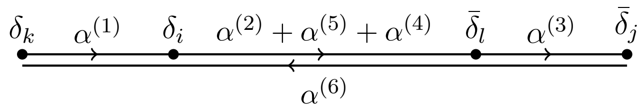
(b) $\underline{\boldsymbol{\mu}}^{\star 2} = [\delta_k, \delta_i, \overline{\delta}_j]$;




In this path, $A_{\underline{\boldsymbol{\mu}}^{\star 2}}(X_1, X_2 + X_5 + X_3 + X_4) = A_{[\delta_k, \delta_l, \overline{\delta}_j]}(X_1, X_2 + X_5 + X_3 + X_4) = X_1$, and

$$K_{\underline{\boldsymbol{\mu}}}^{\star 2} = n_{\alpha^{(2)},\alpha^{(5)}} n_{\alpha^{(2)}+\alpha^{(5)},\alpha^{(3)}+\alpha^{(4)}} \frac{a^{(e_{\varepsilon_i - \varepsilon_j})}_{-\varepsilon_j,-\varepsilon_i} a^{(e_{\varepsilon_j - \varepsilon_i})}_{-\varepsilon_i,-\varepsilon_j} a^{(e_{\varepsilon_l - \varepsilon_j})}_{-\varepsilon_j,-\varepsilon_l} a^{(e_{-\varepsilon_l - \varepsilon_j})}_{-\varepsilon_l,\varepsilon_j}}{a^{(e_{-\varepsilon_j - \varepsilon_i})}_{-\varepsilon_j,\varepsilon_i}}$$
$$= 1 \times 1 \times \frac{-1 \times -1 \times -1 \times 1}{1} = -1.$$

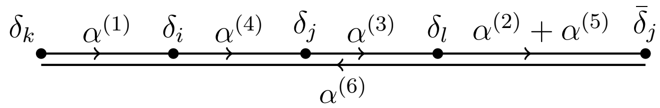
(c) $\underline{\boldsymbol{\mu}}^{\star 3} = [\delta_k, \delta_i, \overline{\delta}_l, \overline{\delta}_j]$




In this path $A_{\underline{\boldsymbol{\mu}}^{\star 3}}(X_1, X_2 + X_5 + X_4, X_3) = A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1, X_2, X_3) = X_1$ and

$$K_{\underline{\boldsymbol{\mu}}}^{\star 3} = n_{\alpha^{(2)},\alpha^{(5)}} n_{\alpha^{(2)}+\alpha^{(5)},\alpha^{(4)}} a^{(e_{\varepsilon_i - \varepsilon_j})}_{-\varepsilon_j,-\varepsilon_i} a^{(e_{\varepsilon_j - \varepsilon_i})}_{-\varepsilon_i,-\varepsilon_j}$$
$$= 1 \times 1 \times -1 \times -1 = 1.$$

(d) $\underline{\boldsymbol{\mu}}^{\star 4} = [\delta_k, \delta_i, \delta_j, \delta_l, \overline{\delta}_j]$.



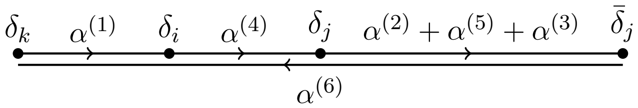

In this path,

$$A_{\underline{\boldsymbol{\mu}}^{\star 4}}(X_1, X_4, X_3, X_2 + X_5) = -1 \times \frac{1 \times 1}{1} A_{(\underline{\boldsymbol{\mu}}^{\star 4})^{\#}}(X_1, X_4, X_3)$$
$$= (-1)(X_1) = -X_1,$$

since $(\underline{\boldsymbol{\mu}}^{\star 4})^{\#} \sim [1,1,1,-3]$ and

$$K_{\underline{\boldsymbol{\mu}}}^{\star 4} = n_{\alpha^{(2)},\alpha^{(5)}} \frac{a_{-\varepsilon_j,-\varepsilon_i}^{(e_{\varepsilon_i-\varepsilon_j})} a_{-\varepsilon_i,-\varepsilon_j}^{(e_{\varepsilon_j-\varepsilon_i})} a_{-\varepsilon_j,-\varepsilon_l}^{(e_{\varepsilon_l-\varepsilon_j})} a_{-\varepsilon_l,\varepsilon_j}^{(e_{-\varepsilon_l-\varepsilon_j})}}{a_{-\varepsilon_j,\varepsilon_l}^{(e_{-\varepsilon_j-\varepsilon_l})} a_{\varepsilon_l,\varepsilon_j}^{(e_{\varepsilon_l-\varepsilon_j})} a_{\varepsilon_j,\varepsilon_i}^{(e_{\varepsilon_j-\varepsilon_i})}} = -1.$$

(e) $\underline{\boldsymbol{\mu}}^{\star 5} = [\delta_k, \delta_i, \delta_j, \overline{\delta}_j]$




In this path,

$$\begin{aligned}
&A_{\underline{\boldsymbol{\mu}}^{\star 5}}(X_1, X_4, X_2+X_5+X_3) = (K_{\underline{\boldsymbol{\mu}}}^{\star 5})^{\#} A_{(\underline{\boldsymbol{\mu}}^{\star 5})^{\#}}(X_1, X_4) \\
&= n_{2\varepsilon_j,-\varepsilon_j-\varepsilon_k} \frac{a_{\varepsilon_k,-\varepsilon_j}^{(e_{\varepsilon_k+\varepsilon_j})} a_{-\varepsilon_j,\varepsilon_j}^{(2e_{-2\varepsilon_j})}}{a_{\varepsilon_k,\varepsilon_j}^{(e_{\varepsilon_k-\varepsilon_j})}} A_{[\delta_k,\delta_i,\delta_j,\delta_k]}(X_1, X_4) = 1 \times \frac{1\times 2}{1}(X_1) = 2X_1,
\end{aligned}$$

and

$$\begin{aligned}
K_{\underline{\boldsymbol{\mu}}}^{\star 5} &= n_{\alpha^{(2)},\alpha^{(5)}} n_{\alpha^{(2)}+\alpha^{(5)},\alpha^{(3)}} \frac{a_{-\varepsilon_j,-\varepsilon_i}^{(e_{\varepsilon_i-\varepsilon_j})} a_{-\varepsilon_i,-\varepsilon_j}^{(e_{\varepsilon_j-\varepsilon_i})} a_{-\varepsilon_j,-\varepsilon_l}^{(e_{\varepsilon_l-\varepsilon_j})} a_{-\varepsilon_l,\varepsilon_j}^{(e_{-\varepsilon_l-\varepsilon_j})}}{a_{-\varepsilon_j,\varepsilon_j}^{(2e_{-2\varepsilon_j})} a_{\varepsilon_j,\varepsilon_i}^{(e_{\varepsilon_j-\varepsilon_i})}} \\
&= 1 \times -2 \times \frac{-1\times -1 \times -1 \times 1}{2\times 1} = 1.
\end{aligned}$$

By Lemma 4.15, we have

$$\begin{aligned}
&A_{\underline{\boldsymbol{\mu}}}(X_1, \ldots, X_5) \\
&= (c_{\alpha^{(p-1)}})^2 (P_{\alpha^{(p-1)}}(X_{p-1}) - c_{p-1}) A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1, X_2, X_3) + \sum_{a=1}^{5} K_{\underline{\boldsymbol{\mu}}}^{\star a} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{X}^{\star a}) \\
&= (X_4+1)(X_1) + (1\times -X_1) + (-1\times X_1) + (1\times X_1) + (-1\times -X_1) + (1\times 2X_1) \\
&= X_4X_1 + X_1 - X_1 - X_1 + X_1 + X_1 + 2X_1 = X_1X_4 + 3X_1.
\end{aligned}$$

*Remark* 4.17. Let $m \in \mathbb{Z}_{>0}$ and $\underline{\boldsymbol{\mu}} \in \mathscr{E}_m$ without zero. There are some classes $\underline{\boldsymbol{\mu}}_1, \ldots, \underline{\boldsymbol{\mu}}_N$ without zero of length $\ell_h := \ell(\underline{\boldsymbol{\mu}}_h) < m$ such that

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = \sum_{h=1}^{N} K_{\underline{\boldsymbol{\mu}}_h} A_{\underline{\boldsymbol{\mu}}_h}(i_1', \ldots, i_{\ell_h-1}').$$

where $\operatorname{ht}(\underline{\mu}_h) = (i'_1, \ldots, i'_{\ell_h})$.

### 4.2.3 A key lemma

We start this paragraph with the following useful result that will be needed to prove the key lemma (Lemma 4.20).

**Lemma 4.18.** *Let $S \in \mathbb{Z}_{\geqslant 1}$, $\ell_1, \ldots, \ell_S$ be a family of strictly positive integers. For each $1 \leqslant j \leqslant \ell_i$, let $d_{i,j} \in \mathbb{Z}_{\geqslant 0}$. Set $\underline{\ell} = (\ell_1, \ldots, \ell_S)$ and $\underline{d} = (d_{ij})$. Then there is a polynomial $Q_{S,\underline{\ell},\underline{d}} \in \mathbb{C}[X_1, \ldots, X_S]$ of degree $\leqslant \sum_{i,j} d_{ij} + \sum_{i=1}^{S}(\ell_i - 1)$ such that*

$$\sum_{\substack{a_{ij} \in \mathbb{Z}>0 \\ 1 \leqslant i \leqslant S,\, 1 \leqslant j \leqslant \ell_i, \\ \sum_{j=1}^{\ell_i} a_{ij} = n_i}} \prod_{i,j} (a_{ij})^{d_{ij}} = Q_{S,\underline{\ell},\underline{d}}(n_1, \ldots, n_S).$$

*Proof.* First of all we consider the case where $S = 1$. We prove the lemma by induction on $\ell$. For $\ell = 1$, the result is clear.
We assume that the lemma is true for some $\ell \geqslant 1$ and we prove the statement for $\ell + 1$. By the induction hypothesis there exists a polynomial $Q_{1,\ell,\underline{d}} \in \mathbb{C}[X]$ of degree $\leqslant d_1 + \cdots + \mathrm{d}_\ell + \ell - 1$ such that

$$Q_{1,\ell,\underline{d}}(n) = \sum_{\substack{a_i \in \mathbb{Z}_{>0} \\ \sum_{i=1}^{\ell} a_i = n}} (a_1)^{d_1} \ldots (a_\ell)^{d_\ell}.$$

Write $Q_{1,\ell,\underline{d}}(X) = \sum_{j=0}^{d_1+\cdots+d_\ell+\ell-1} C_j (X)^j$. Thus,

$$\begin{aligned}
\sum_{\substack{a_i \in \mathbb{Z}_{>0} \\ \sum_{i=1}^{\ell+1} a_i = n}} (a_1)^{d_1} \ldots (a_\ell)^{d_\ell} (a_{\ell+1})^{d_{\ell+1}} &= \sum_{\imath=1}^{n-\ell} \sum_{\substack{a_i \in \mathbb{Z}_{>0} \\ \sum_{i=1}^{l} a_i = n-\imath}} (\imath)^{d_{\ell+1}} (a_1)^{d_1} \ldots (a_\ell)^{d_\ell} \\
&= \sum_{\imath=1}^{n-\ell} (\imath)^{d_{\ell+1}} Q_{1,\ell,\underline{d}}(n-\imath) = \sum_{\imath=1}^{n-\ell} (\imath)^{d_{\ell+1}} \sum_{j=0}^{d_1+\cdots+d_\ell+\ell-1} C_j (n-\imath)^j \\
&= \sum_{\imath=1}^{n-\ell} (\imath)^{d_{\ell+1}} \sum_{j=0}^{d_1+\cdots+d_\ell+\ell-1} C_j \sum_{t=0}^{j} \binom{j}{t} (n)^{j-t} (-1)^t (\imath)^t \\
&= \sum_{j=0}^{d_1+\cdots+d_\ell+\ell-1} C_j \sum_{t=0}^{j} \binom{j}{t} (n)^{j-t} (-1)^t \sum_{\imath=1}^{n-\ell} (\imath)^{d_{\ell+1}+t} \\
&= \sum_{j=0}^{d_1+\cdots+d_\ell+\ell-1} C_j \sum_{t=0}^{j} \binom{j}{t} (n)^{j-t} (-1)^t S_{d_{\ell+1}+t}(n-\ell).
\end{aligned}$$

$S_{d_{\ell+1}+t}$ is a polynomial of degree $d_{\ell+1}+t+1$ with leading term $(d_{\ell+1}+t+1)^{-1}\, n^{d_{\ell+1}+t+1}$. Set $\underline{d}' = (d_1, \ldots, d_{\ell+1})$, thus there exists a polynomial $Q_{1,\ell+1,\underline{d}'} \in \mathbb{C}[X]$ of degree $\leqslant d_1 + \ldots + d_\ell + \ell - 1 + d_{\ell+1} + 1 = d_1 + \ldots + d_\ell + d_{\ell+1} + (\ell+1) - 1$ such that

$$Q_{1,\ell+1,\underline{d}'}(n) = \sum_{\substack{a_i \in \mathbb{Z}_{>0} \\ \sum_{i=1}^{\ell+1} a_i = n}} (a_1)^{d_1} \ldots (a_\ell)^{d_\ell} (a_{\ell+1})^{d_{\ell+1}}.$$

And so the lemma is true for $\ell+1$. By induction, it is true for all $\ell \geqslant 2$

Now we consider the case when $S \geqslant 2$. We get

$$\sum_{\substack{a_{ij} \in \mathbb{Z}>0 \\ 1 \leqslant i \leqslant S,\, 1 \leqslant j \leqslant \ell_i, \\ \sum_{j=1}^{\ell_i} a_{ij} = n_i}} \prod_{i,j} (a_{ij})^{d_{ij}} = \sum_{\substack{a_{1j} \in \mathbb{Z}>0 \\ 1 \leqslant j \leqslant \ell_1, \\ \sum_{j=1}^{\ell_1} a_{1j} = n_1}} (a_{11})^{d_{11}} \cdots (a_{1\ell_1})^{d_{1\ell_1}} \cdots \sum_{\substack{a_{Sj} \in \mathbb{Z}>0 \\ 1 \leqslant j \leqslant \ell_S, \\ \sum_{j=1}^{\ell_S} a_{Sj} = n_S}} (a_{S1})^{d_{S1}} \cdots (a_{S\ell_S})^{d_{S\ell_S}}$$
$$= Q_{1,\ell_1,\underline{d}_1}(n_1)\, Q_{1,\ell_2,\underline{d}_2}(n_2) \ \cdots \ Q_{1,\ell_S,\underline{d}_S}(n_S) = Q_{S,\underline{\ell},\underline{d}}(n_1, n_2, \ldots, n_S),$$

where $\underline{\ell} = (\ell_1, \ldots, \ell_S)$ and $\underline{d} = (d_{ij})$. Hence, $Q_{S,\underline{\ell},\underline{d}}$ is a polynomial of degree $\leqslant \sum_{i,j} d_{ij} + \sum_{i=1}^{S} (\ell_i - 1)$. □

*Remark* 4.19. The polynomial $Q_{S,\underline{\ell},\underline{d}}$ has degree $\sum_{i,j} d_{ij} + \sum_{i=1}^{S}(\ell_i - 1)$, for $1 \leqslant i \leqslant S$, $1 \leqslant j \leqslant \ell_i$, with leading term

$$\frac{\prod_{ij} d_{ij}!}{\prod_{i=1}^{S} \left( \sum_{j=1}^{\ell_i} d_{ij} + \ell_i - 1 \right)!} X^{\sum_{i,j} d_{ij} + \sum_{i=1}^{S}(\ell_i - 1)}.$$

**Lemma 4.20.** *Let $m \in \mathbb{Z}_{>0}$, $d \in \mathbb{Z}_{\geqslant 0}$, and $\boldsymbol{\mu} \in \mathscr{E}_m$ without zero. Let $\underline{d} = (d_1, \ldots, d_{m-1})$ with $d_1 + \cdots + d_{m-1} = d$. Then for some polynomial $T_{\underline{d},\boldsymbol{\mu}} \in \mathbb{C}[X]$ of degree $\leqslant d + m - \deg A_{\boldsymbol{\mu}}$, we have*

$$\forall k \in \{1, \ldots, r\}, \qquad \sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k), \\ \underline{\mu} \in [\![\bar{\delta}_1, \delta_k]\!],\, (\underline{\mu},\underline{\alpha}) \in \boldsymbol{\mu}}} i_1^{d_1} \ldots i_{m-1}^{d_{m-1}} = T_{\underline{d},\boldsymbol{\mu}}(k).$$

*Here, the integer $i_j$, for $j = 1, \ldots, m-1$, denotes the height of $\alpha^{(j)}$. In particular, if for some $k \in \{1, \ldots, r\}$, the set $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\bar{\delta}_1, \delta_k]\!],\, (\underline{\mu}, \underline{\alpha}) \in \boldsymbol{\mu}\}$ is empty, we have $T_{\underline{d},\boldsymbol{\mu}}(k) = 0$.*

The lemma implies that, with the same arguments as in Lemma 3.14, for all $k \in \{1, \ldots, r\}$,

$$\sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k), \\ \underline{\mu} \in [\![\bar{\delta}_1, \delta_k]\!], (\underline{\mu},\underline{\alpha}) \in \boldsymbol{\mu}}} i_1^{d_1} \ldots i_{m-1}^{d_{m-1}} A_{\boldsymbol{\mu}}(i_1, \ldots, i_{m-1})$$

is a polynomial of degree $d+m$.

*Proof.* We prove the lemma by induction on $m$. More precisely, we prove by induction on $m$ the following:

*For all* $\underline{\boldsymbol{\mu}} \in \mathscr{E}_m$ *without zero and all* $\underline{d} = (d_1, \ldots, d_{m-1})$ *with* $d_1 + \cdots + d_{p-1} = d$, $d \in \mathbb{Z}_{\geqslant 0}$,
*Set*

$$\forall k \in \{1, \ldots, r\}, \quad \widetilde{T}_{\underline{d},\underline{\boldsymbol{\mu}}}(k) := \sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k), \\ \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!],\ (\underline{\mu},\underline{\alpha}) \in \underline{\boldsymbol{\mu}}}} i_1^{d_1} \ldots i_{m-1}^{d_{m-1}}.$$

*Then there exists a polynomial* $T_{\underline{d},\underline{\boldsymbol{\mu}}} \in \mathbb{C}[X]$ *of degree* $\leqslant d + m - \deg A_{\underline{\boldsymbol{\mu}}}$ *such that for all* $k \in \{1, \ldots, r\}$,

$$T_{\underline{d},\underline{\boldsymbol{\mu}}}(k) = \widetilde{T}_{\underline{d},\underline{\boldsymbol{\mu}}}(k).$$

*In particular, if for some* $k \in \{1, \ldots, r\}$, *the set* $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!],\ (\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}\}$ *is empty, we have* $T_{\underline{d},\underline{\boldsymbol{\mu}}}(k) = 0$.

The case $m = 1$ is empty.

Assume $m = 2$, and let $\underline{\boldsymbol{\mu}} \in \mathscr{E}_2$ without zero. The equivalence classes in $\mathscr{E}_2$ without zero are $[\delta_k, \overline{\delta}_k], [\delta_k, \overline{\delta}_j], [\delta_k, \delta_j]$ with $j \neq k$. Let $d = d_1 \in \mathbb{Z}_{\geqslant 0}$.

- We first compute for $\underline{\boldsymbol{\mu}} = [\delta_k, \overline{\delta}_k]$.
  Let $k \in \{1, \ldots, r\}$. Then the set $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!],\ (\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}\}$ is nonempty. Observe that there is only one path in this class. Thus, we get

$$\sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k), \\ \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!],\ (\underline{\mu},\underline{\alpha}) \in \underline{\boldsymbol{\mu}}}} i_1^{d_1} = i_1^{d_1} = (2(r-k)+1)^{d_1}.$$

  Hence, the polynomial $T_{\underline{d},\underline{\boldsymbol{\mu}}}(k) := (2r+1-2X)^{d_1}$ has degree $d_1 = d \leqslant d + 2 - \deg A_{\underline{\boldsymbol{\mu}}} = d+1$, since $A_{\underline{\boldsymbol{\mu}}}(X_1) = 2(X_1) + 1$ has degree 1 by Lemma 4.15.

- Assume $\underline{\boldsymbol{\mu}} = [\delta_k, \overline{\delta}_j], j \neq k$.
  For $k \in \{1, \ldots, r\}$, the set $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!],\ (\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}\}$ is nonempty. Observe that the height of $\alpha^{(1)}$ run through $\{r-k+1, \ldots 2r-k\}$. Thus, we get

$$\begin{aligned}\sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k), \\ \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!], \\ (\underline{\mu},\underline{\alpha}) \in \underline{\boldsymbol{\mu}}}} i_1^{d_1} &= \sum_{i=r-k+1}^{2r-k} i_1^{d_1} - (2r-k+1)^{d_1} \\ &= S_{d_1}(2r-k) - S_{d_1}(r-k+1) - (2(r-k)+1)^{d_1},\end{aligned}$$

By Lemma 3.13 (1), the polynomial $T_{\underline{d},\underline{\boldsymbol{\mu}}}(k) := S_{d_1}(2r-k) - S_{d_1}(r-k+1) - (2(r-k)+1)^{d_1}$ has degree $d_1 = d \leqslant d+2-\deg A_{\underline{\boldsymbol{\mu}}} = d+1$, since $A_{\underline{\boldsymbol{\mu}}}(X_1) = X_1+1$ has degree 1 by Lemma 4.15.

- Assume $\underline{\boldsymbol{\mu}} = [\delta_k, \delta_j], k < j$.
  Let $k \in \{1, \dots, r-1\}$. Then the set $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}\}$ is nonempty. It is empty for $k = r$. We get

$$\sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k), \\ \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!],\, (\underline{\mu},\underline{\alpha}) \in \underline{\boldsymbol{\mu}}}} i_1^{d_1} = \sum_{i_1=1}^{r+1-k} i_1^{d_1} = S_{d_1}(r+1-k).$$

  By Lemma 3.13 (1), the polynomial

$$T_{d,\underline{\boldsymbol{\mu}}}(X) := S_{d_1}(r+1-X),$$

  has degree $d_1 + 1 = d + 2 - \deg A_{\underline{\boldsymbol{\mu}}}$ since $A_{\underline{\boldsymbol{\mu}}}(X_1) = X_1$ has degree 1 by Lemma 4.15. For $k = r$, the equality still holds since $T_{d,\underline{\boldsymbol{\mu}}}(r) = S_d(0) = 0$.

These prove the claim for $m = 2$.

Assume $m \geqslant 3$ and the formula holds for any $m' \in \{1, \dots, m-1\}$.

Let $\underline{\boldsymbol{\mu}} \in \mathscr{E}_m$ without zero, set $p := p(\underline{\boldsymbol{\mu}})$, and let $\underline{d} = (d_1, \dots, d_{m-1})$ with $d_1 + \cdots + d_{m-1} = d$. Let $k \in \{1, \dots, r\}$ such that the set $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}\}$ is nonempty. Then the sets $\{(\underline{\mu}', \underline{\alpha}') \in \hat{\mathscr{P}}_{m-1}(\delta_k) \mid \underline{\mu}' \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}', \underline{\alpha})' \in \underline{\boldsymbol{\mu}}^{\#}\}$ and $\{(\underline{\mu}', \underline{\alpha}') \in \hat{\mathscr{P}}_{m-1}(\delta_k) \mid \underline{\mu}' \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}', \underline{\alpha})' \in \underline{\boldsymbol{\mu}}^{\star a}\}$ are nonempty, too.

Let $\underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!]$ such that $(\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}$ and let $\underline{i} := \mathrm{ht}(\underline{\boldsymbol{\mu}})$.

We first assume that $\mu^{(p)} \in [\![\delta_r, \delta_k]\!]$. By Lemma 2.7 the admissible triple of $-\alpha^{(p)}$ has type III (a) and so this case can be dealt similarly as in $\mathfrak{sl}_{r+1}$. Then the results are known by Lemma 3.14.

Assume $\mu^{(p)} \in [\![\overline{\delta}_1, \overline{\delta}_r]\!]$ and let $s \in \{1, \dots, p-2\}$ be a positive integer.
If $p = 2$ or if for all $s$, either $\alpha^{(s)} + \alpha^{(p)} \in -\Delta_+$ or $\alpha^{(s)} + \alpha^{(p)} \notin \Delta \cup \{0\}$, then by Lemma 4.15, the degree of polynomial $A_{\underline{\boldsymbol{\mu}}}$ only depends on the degree of polynomial $A_{\underline{\boldsymbol{\mu}}^{\#}}$. Let $\underline{i}^{\#} := \mathrm{ht}(\underline{\boldsymbol{\mu}}^{\#})$. Then one can argue as for the $\mathfrak{sl}_{r+1}$ case (cf. Lemma 3.14) and the height $\underline{i}$ can be express in term of $\underline{i}^{\#}$. By doing the same kind of reasoning, we get the statement.

Hence it remains to verify the case when there is a positive integer $s \in \{1, \dots, p-2\}$ such that either $\alpha^{(s)} + \alpha^{(p)} \in \Delta_+$ or $\alpha^{(s)} = -\alpha^{(p)}$.

**I. Assume $\boldsymbol{\alpha^{(s)} + \alpha^{(p)} \in \Delta_+}$.**
Lemma 4.15 shows that

$$\begin{aligned} A_{\underline{\boldsymbol{\mu}}}(X_1, \ldots, X_{m-1}) = \bar{K}^{\#}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1, \ldots, X_{p-2}, X_{p-1} + X_p, X_{p+1}, \ldots, X_{m-1}) \\ + \sum_{a=1}^{N} K^{\star a}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{X}^{\star a}), \end{aligned} \tag{4.12}$$

where $\bar{K}^{\#}_{\underline{\boldsymbol{\mu}}} = \begin{cases} K^{\#}_{\underline{\boldsymbol{\mu}}} & \text{if } \alpha^{(p-1)} + \alpha^{(p)} \neq 0, \\ (P_{\alpha^{(p-1)}}(X_{p-1}) - c_{p-1}) & \text{if } \alpha^{(p-1)} + \alpha^{(p)} = 0. \end{cases}$
If $\deg A_{\underline{\boldsymbol{\mu}}} \leqslant \deg A_{\underline{\boldsymbol{\mu}}^{\#}}$ (respectively, $\deg A_{\underline{\boldsymbol{\mu}}} \leqslant \deg A_{\underline{\boldsymbol{\mu}}^{\#}} + 1$) for $\alpha^{(p-1)} + \alpha^{(p)} \neq 0$ (respectively, $\alpha^{(p-1)} + \alpha^{(p)} = 0$) then using the same strategy as in Lemma 3.14, which is by expressing the heights of $\underline{i}$ in term of $\underline{i}^{\#}$, we get the statement.
Therefore we can assume that $\deg A_{\underline{\boldsymbol{\mu}}} > \deg A_{\underline{\boldsymbol{\mu}}^{\#}}$ (respectively, $\deg A_{\underline{\boldsymbol{\mu}}} > \deg A_{\underline{\boldsymbol{\mu}}^{\#}} + 1$) if $\alpha^{(p-1)} + \alpha^{(p)} \neq 0$ (respectively, $\alpha^{(p-1)} + \alpha^{(p)} = 0$). This assumption means that for some $a \in \{1, \ldots, N\}$, $\deg A_{\underline{\boldsymbol{\mu}}^{\star a}} \geqslant \deg A_{\underline{\boldsymbol{\mu}}}$.

By Lemma 2.7, the possibilities for $-\alpha^{(p)}$ and $\alpha^{(s)}$ which satisfy the assumption $\alpha^{(s)} + \alpha^{(p)} \in \Delta_+$ are the following:

(1) $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_l - \overline{\delta}_i$, with $l < j < i$,

(2) $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_j - \overline{\delta}_l$, with $j < l < i$,

(3) $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_i - \overline{\delta}_l$, with $i < j$ and $i < l$,

(4) $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_i - \delta_l$, with $i < j < l$,

(5) $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_l - \delta_j$, with $l < i < j$.

Note that in all those cases, for all $a \in \{1, \ldots, N\}$, $m - p + s \leqslant \mathrm{ht}(\underline{\boldsymbol{\mu}}^{\star a}) \leqslant m - 1$. Possibly changing the numbering of the equivalence class $\underline{\boldsymbol{\mu}}^{\star a}$, one can assume throughout this proof that $\underline{\boldsymbol{\mu}}^{\star 1} = [(\underline{\mu}^{\star 1}, \underline{\alpha}^{\star 1})]$ is the equivalence class of the star paths with the longest length and set $\underline{i} := \mathrm{ht}(\underline{\boldsymbol{\mu}})$ and $\underline{i}' := \mathrm{ht}(\underline{\boldsymbol{\mu}}^{\star 1})$.

(1) First we give the proof for the case when $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_l - \overline{\delta}_i$, with $l < j < i$.

Recall from Lemma 4.3, for all $(\underline{\mu}, \underline{\alpha})^{\star a} \in \underline{\boldsymbol{\mu}}^{\star a}$, $a \in \{1, \ldots, N\}$,

$$(\underline{\mu}, \underline{\alpha})^{\star a} = (\underline{\mu}', \underline{\alpha}') \star \big((\delta_l, \delta_j), (\alpha^{(s)} + \alpha^{(p)})\big) \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}''),$$

where $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$ is a path of length $\leqslant p-s-1$ between $\delta_j$ and $\delta_i$ whose roots $\underline{\tilde{\alpha}}^{\star a}$ are sums among $(\alpha^{(p-1)}, \dots, \alpha^{(s+1)})$, $\underline{\alpha}' = (\alpha^{(1)}, \dots, \alpha^{(s-1)})$ and $\underline{\alpha}'' = (\alpha^{(p+1)}, \dots, \alpha^{(m)})$. We have $\mathrm{ht}(\underline{\mu}^{\star 1}) = (i_1, \dots, i_{s-1}, i_s + i_p, i_{p-1}, \dots, i_{s+1}, i_{p+1}, \dots, i_m)$.

Set $\underline{i}' = (i'_1, \dots, i'_{m-1})$, so $i'_s = i_s + i_p$. Note that $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_l - \overline{\delta}_i$, with $l < j < i$. Thus,

$$l = k + i'_1 + \cdots + i'_{s-1}, j = l + i'_s \text{ and } \; i = k + i'_1 + \cdots + i'_{p-1}.$$

Hence, for all $(\underline{\mu}, \underline{\alpha}) \in \boldsymbol{\mu}$ the heights in $\underline{i}$ can be expressed in term of $\underline{i}'$ as follows (see Figure 4.9 for an illustration) :

$$\begin{aligned}
&i_1 = i'_1, \;\dots,\; i_{s-1} = i'_{s-1},\; i_{s+1} = i'_{p-1},\; \dots,\; i_{p-1} = i'_{s+1},\; i_{p+1} = i'_p,\; \dots,\; i_{m-1} = i'_{m-2},\\
&i_s = 2r - 2k - (i+l) + 1 = 2r + 1 - 2k - 2(i'_1 + \cdots + i'_{s-1}) - (i'_s + \cdots + i'_{p-1}),\\
&i_p = i'_s - i_s = 2k - 2r - 1 + 2(i'_1 + \cdots + i'_s) + (i'_{s+1} + \cdots + i'_{p-1}).
\end{aligned}$$

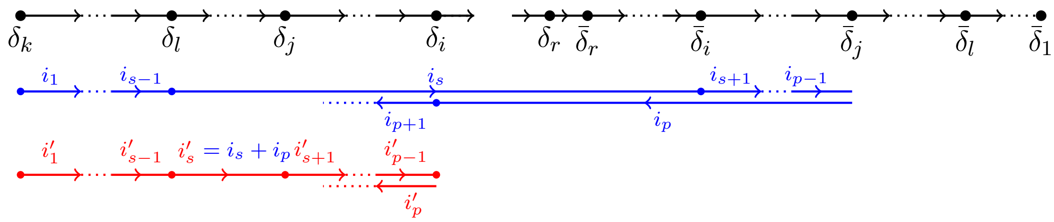


**FIGURE 4.9** – $\underline{i} = \mathrm{ht}(\underline{\mu})$ and $\underline{i}' := \mathrm{ht}(\underline{\mu}^{\star 1})$ where $\alpha^{(p)} = \overline{\delta}_j - \delta_i$, $\alpha^{(s)} = \delta_l - \overline{\delta}_i$, $l < j < i$.

We get,

$$\begin{aligned}
\widetilde{T}_{\underline{d},\boldsymbol{\mu}}(k) &= \sum_{\substack{(\underline{\mu},\underline{\alpha}) \in \mathscr{P}_m(\delta_k),\\ \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!],\, (\underline{\mu},\underline{\alpha}) \in \boldsymbol{\mu}}} i_1^{d_1} \dots i_{m-1}^{d_{m-1}}\\
&= \sum_{\substack{(\underline{\mu},\underline{\alpha})^{\star 1} \in \mathscr{P}_{m-1}(\delta_k),\\ \underline{\mu}^{\star 1} \in [\![\overline{\delta}_1, \delta_k]\!],\\ (\underline{\mu},\underline{\alpha})^{\star 1} \in \boldsymbol{\mu}^{\star 1}}} ({i'}_1)^{d_1} \cdots ({i'}_{s-1})^{d_{s-1}} (2r + 1 - 2k - 2(i'_1 + \cdots + i'_{s-1}) - (i'_s + \cdots + i'_{p-1}))^{d_s}\\
&\quad \times (i'_{p-1})^{d_{s+1}} \cdots ({i'}_{s+1})^{d_{p-1}} (2k - 2r - 1 + 2(i'_1 + \cdots + i'_s) + (i'_{s+1} + \cdots + i'_{p-1}))^{d_p}\\
&\quad \times (i'_p)^{d_{p+1}} \cdots ({i'}_{m-2})^{d_{m-1}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{(\underline{\mu},\underline{\alpha})^{\star 1}\in\mathscr{P}_{m-1}(\delta_k),\\ \underline{\mu}^{\star 1}\in[\![\delta_1,\delta_k]\!],\\ (\underline{\mu},\underline{\alpha})^{\star 1}\in\underline{\boldsymbol{\mu}}^{\star 1}}} ({i'}_1)^{d_1}\cdots({i'}_{s-1})^{d_{s-1}}(i'_{p-1})^{d_{s+1}}\cdots({i'}_{s+1})^{d_{p-1}}(i'_p)^{d_{p+1}}\cdots({i'}_{m-2})^{d_{m-1}}\\
&\quad\times\sum_{\substack{\underline{q}\in\mathbb{N}^p\\ |\underline{q}|=d_s}}\frac{(d_s)!}{q_1!\dots q_p!}(-1)^{d_s-q_p}(2)^{q_1+\cdots+q_{s-1}}(i'_1)^{q_1}\cdots({i'}_{p-1})^{q_{p-1}}\\
&\quad\times\sum_{\substack{\underline{q}'\in\mathbb{N}^p\\ |\underline{q}'|=d_p}}\frac{(d_p)!}{q'_1!\dots q'_p!}(2)^{q'_1+\cdots+q'_s}(2k-2r-1)^{q'_p}(i'_1)^{q'_1}\cdots({i'}_{p-1})^{q'_{p-1}}\\
&= \sum_{\substack{\underline{q}\in\mathbb{N}^p\\ |\underline{q}|=d_s}}\sum_{\substack{\underline{q}'\in\mathbb{N}^p\\ |\underline{q}'|=d_p}}\frac{(d_s)!}{q_1!\dots q_p!}\frac{(d_p)!}{q'_1!\dots q'_p!}(-1)^{d_s}(2)^{q_1+\cdots+q_{s-1}+q'_1+\cdots+q'_s}\ (2k-2r-1)^{q_p+q'_p}\\
&\quad\times\sum_{\substack{(\underline{\mu},\underline{\alpha})^{\star 1}\in\mathscr{P}_{m-1}(\delta_k),\\ \underline{\mu}^{\star 1}\in[\![\delta_1,\delta_k]\!],\\ (\underline{\mu},\underline{\alpha})^{\star 1}\in\underline{\boldsymbol{\mu}}^{\star 1}}}(i'_1)^{d_1+q_1+q'_1}\cdots({i'}_{s-1})^{d_{s-1}+q_{s-1}+q'_{s-1}}\ ({i'}_s)^{q_s+q'_s}\\
&\quad\times({i'}_{s+1})^{d_{p-1}+q_{s+1}+q'_{s+1}}\cdots({i'}_{p-1})^{d_{s+1}+q_{p-1}+q'_{p-1}}({i'}_p)^{d_{p+1}}\cdots({i'}_{m-2})^{d_{m-1}}.
\end{aligned}
\tag{4.13}
$$

Assume that for some $a'\in\{1,\dots,N\}$ the equivalence class $\underline{\boldsymbol{\mu}}^{\star a'}=[(\underline{\mu}^{\star a'},\underline{\alpha}^{\star a'})]$ satisfies $\deg A_{\underline{\boldsymbol{\mu}}^{\star a'}}\geqslant\deg A_{\underline{\boldsymbol{\mu}}}$. According to the proof of Lemma 4.3, there exists a partition $(t_1,\dots,t_{n'}), t_i>0$, of $p-s-1$ such that

$$
\begin{aligned}
i^\star_{s+1} &= i'_{s+1}+\cdots+i'_{s+t_1},\\
i^\star_{s+2} &= i'_{s+t_1+1}+\cdots+i'_{(s+t_1+t_2)},\\
&\ \vdots\\
i^\star_{s+n'} &= i'_{s+t_1+\cdots+t_{n'-1}+1)}+\cdots+i'_{(p-1)}.
\end{aligned}
$$

Hence $\mathrm{ht}(\underline{\mu}^{\star a'})=(i'_1,\dots,i'_{s-1},i'_s,i^\star_{s+1},\dots,i^\star_{s+n'},i'_p,\dots,i'_{m-1})$ with the length of $\underline{\boldsymbol{\mu}}^{\star a'}$, denoted by $m^\star$, is equal to $m-p+s+n'$. Here $i^\star_{s+j}$ denotes the height of the root $(\underline{\tilde{\alpha}}^{\star a'})^{(s+j)}$.

By Lemma 4.18, for each $\underline{d}'=(d'_{s+1},\dots,d'_{p-1})$ there is a polynomial $Q_{n',\underline{t},\underline{d}'}\in\mathbb{C}[X_1,\dots,X_{n'}]$, $\underline{t}=(t_1,\dots,t_{n'})$, of degree $d'_{s+1}+\cdots+d'_{p-1}+(p-s-1)-n'$ such that

$$
\sum_{\substack{1\leqslant j\leqslant n'\\ 1+t_1+\cdots+t_{j-1}\leqslant l\leqslant t_1+\cdots+t_j\\ \sum_l i'_{s+l}=i^\star_{s+j}}}(i'_{s+1})^{d'_{s+1}}\cdots(i'_{p-1})^{d'_{p-1}}=Q_{n',\underline{t},\underline{d}'}(i^\star_{s+1},\dots,i^\star_{s+n'}).
$$

Set $\underline{d}'_{\underline{q},\underline{q}'} = (d_{p-1} + q_{s+1} + q'_{s+1}, \dots, d_{s+1} + q_{p-1} + q'_{p-1})$, $d'_{\underline{q},\underline{q}'} = |\underline{d}'_{\underline{q},\underline{q}'}|$ and write

$$Q_{n',\underline{t},\underline{d}'_{\underline{q},\underline{q}'}}(X_1, \dots, X_{n'}) = \sum_{\substack{\underline{j}=(j_1,\dots,j_{n'}) \\ |\underline{j}| \leqslant d'_{\underline{q},\underline{q}'}+p-s-1-n'}} C_{\underline{j}} X_1^{j_1} \cdots X_{n'}^{j_{n'}}.$$

It is a polynomial of degree $d_{p-1}+q_{s+1}+q'_{s+1}+\dots+d_{s+1}+q_{p-1}+q'_{p-1}+p-s-1-n'$. Hence,

$$\begin{aligned}\widetilde{T}_{\underline{d},\underline{\boldsymbol{\mu}}}(k) = &\sum_{\substack{\underline{q}\in\mathbb{N}^p \\ |\underline{q}|=d_s}} \sum_{\substack{\underline{q}'\in\mathbb{N}^p \\ |\underline{q}'|=d_p}} \frac{(d_s)!}{q_1!\dots q_p!} \frac{(d_p)!}{q'_1!\dots q'_p!} (-1)^{d_s} (2)^{q_1+\dots+q_{s-1}+q'_1+\dots+q'_s} \ (2k-2r-1)^{q_p+q'_p} \\ \times &\sum_{\substack{(\underline{\mu},\underline{\alpha})^{\star a'} \in \mathscr{P}_{m^\star}(\delta_k), \\ \underline{\mu}^{\star a'} \in [\![\delta_1,\delta_k]\!], \\ (\underline{\mu},\underline{\alpha})^{\star a'} \in \underline{\mu}^{\star a'}}} (i'_1)^{d_1+q_1+q'_1} \dots ({i'}_{s-1})^{d_{s-1}+q_{s-1}+q'_{s-1}} \ ({i'}_s)^{q_s+q'_s} \\ \times &\sum_{\substack{\underline{j}=(j_1,\dots,j_{n'}) \\ |\underline{j}| \leqslant d'_{\underline{q},\underline{q}'}+p-s-1-n'}} C_{\underline{j}}\, (i^\star_{s+1})^{j_1} \cdots (i^\star_{s+n'})^{j_{n'}} \ ({i'}_p)^{d_{p+1}} \cdots ({i'}_{m-2})^{d_{m-1}}.\end{aligned}$$

Set $\underline{d}_{\underline{q},\underline{q}'} = (d_1+q_1+q'_1, \dots, d_{s-1}+q_{s-1}+q'_{s-1}, q_s+q'_s, j_1, \dots, j_{n'}, d_{p+1}, \dots, d_{m-1})$, with $|\underline{d}_{\underline{q},\underline{q}'}| = d + (p-s-1) - n' - q_p - q'_p$, and

$$\begin{aligned}\widetilde{T}_{\underline{d}_{\underline{q},\underline{q}'},\underline{\boldsymbol{\mu}}^{\star a'}}(k) = &\sum_{\substack{(\underline{\mu},\underline{\alpha})^{\star a'} \in \mathscr{P}_{m^\star}(\delta_k), \\ \underline{\mu}^{\star a'} \in [\![\delta_1,\delta_k]\!], \\ (\underline{\mu},\underline{\alpha})^{\star a} \in \underline{\mu}^{\star a'}}} (i'_1)^{d_1+q_1+q'_1} \dots ({i'}_{s-1})^{d_{s-1}+q_{s-1}+q'_{s-1}} \ ({i'}_s)^{q_s+q'_s} \\ &\times (i^\star_{s+1})^{j_1} \cdots (i^\star_{s+n'})^{j_{n'}} \ ({i'}_p)^{d_{p+1}} \cdots ({i'}_{m-2})^{d_{m-1}}.\end{aligned}$$

Then

$$\begin{aligned}\widetilde{T}_{\underline{d},\underline{\boldsymbol{\mu}}}(k) = &\sum_{\substack{\underline{q}\in\mathbb{N}^p \\ |\underline{q}|=d_s}} \sum_{\substack{\underline{q}'\in\mathbb{N}^p \\ |\underline{q}'|=d_p}} \sum_{\substack{\underline{j}=(j_1,\dots,j_{n'}) \\ |\underline{j}| \leqslant d'_{\underline{q},\underline{q}'}+p-s-1-n'}} \frac{(d_s)!}{q_1!\dots q_p!} \frac{(d_p)!}{q'_1!\dots q'_p!} C_{\underline{j}} (-1)^{d_s} (2)^{q_1+\dots+q_{s-1}+q'_1+\dots+q'_s} \\ \times &\ (2k-2r-1)^{q_p+q'_p} \widetilde{T}_{\underline{d}_{\underline{q},\underline{q}'},\underline{\boldsymbol{\mu}}^{\star a'}}(k). \qquad (4.14)\end{aligned}$$

By the induction hypothesis applied to $m^\star$ and $\underline{\boldsymbol{\mu}}^{\star a'}$, there exist polynomials $T_{\underline{d}_{\underline{q},\underline{q}'},\underline{\boldsymbol{\mu}}^{\star a'}} \in \mathbb{C}[X_1, \dots, X_{m^\star-1}]$ with

$$\begin{aligned}\deg(T_{\underline{d}_{\underline{q},\underline{q}'},\underline{\boldsymbol{\mu}}^{\star a'}}) &\leqslant d + (p-s-1) - n' - q'_p - q_p + m^\star - \deg A_{\underline{\boldsymbol{\mu}}^{\star a'}} \\ &\leqslant d + m - q_p - q'_p - 1 - \deg A_{\underline{\boldsymbol{\mu}}} < d + m - q_p - q'_p - \deg A_{\underline{\boldsymbol{\mu}}},\end{aligned}$$

since $\deg A_{\underline{\boldsymbol{\mu}}^{\star a'}} \geqslant \deg A_{\underline{\boldsymbol{\mu}}}$, such that $T_{\underline{d}_{\underline{q},\underline{q}'},\underline{\boldsymbol{\mu}}^{\star a'}}(k) = \widetilde{T}_{\underline{d}_{\underline{q},\underline{q}'},\underline{\boldsymbol{\mu}}^{\star a'}}(k)$ for all $k$ such that $\{(\underline{\mu}, \underline{\alpha})^{\star a'} \in \hat{\mathscr{P}}_{m^\star}(\delta_k) \mid \underline{\mu}^{\star a'} \in [\![\overline{\delta}_1, \delta_k]\!],\ (\underline{\mu}, \underline{\alpha})^{\star a'} \in \underline{\boldsymbol{\mu}}^{\star a'}\}$ is nonempty.

Set

$$T_{\underline{d},\underline{\boldsymbol{\mu}}}(k) = \sum_{\substack{\underline{q}\in\mathbb{N}^p \\ |\underline{q}|=d_s}} \sum_{\substack{\underline{q}'\in\mathbb{N}^p \\ |\underline{q}'|=d_p}} \sum_{\substack{\underline{j}=(j_1,\dots,j_{n'}) \\ |\underline{j}|\leqslant d'_{\underline{q},\underline{q}'}+p-s-1-n'}} \frac{(d_s)!}{q_1!\dots q_p!}\frac{(d_p)!}{q'_1!\dots q'_p!} C_{\underline{j}}(-1)^{d_s}(2)^{q_1+\cdots+q_{s-1}+q'_1+\cdots+q'_s}$$
$$\times\ (2k-2r-1)^{q_p+q'_p} T_{\underline{d}_{\underline{q},\underline{q}'},\underline{\boldsymbol{\mu}}^{\star a'}}(k).$$

Then by the induction hypothesis and (4.14), we have $T_{\underline{d},\underline{\boldsymbol{\mu}}}(k) = \widetilde{T}_{\underline{d},\underline{\boldsymbol{\mu}}}(k)$ and $T_{\underline{d},\underline{\boldsymbol{\mu}}}$ is a polynomial of degree $< d + m - \deg A_{\underline{\boldsymbol{\mu}}}$ for all $k$ such that $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!],\ (\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}\}$ is nonempty.

It remains to verify that $T_{\underline{d},\underline{\boldsymbol{\mu}}}(k) = 0$ when

$$\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}\} = \varnothing.$$

Observe that this set is never empty if the set $\{(\underline{\mu}, \underline{\alpha})^{\star 1} \in \hat{\mathscr{P}}_{m-1}(\delta_k) \mid \underline{\mu}^{\star 1} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha})^{\star 1} \in \underline{\boldsymbol{\mu}}^{\star 1}\}$ is nonempty. For any $(\underline{\mu}, \underline{\alpha})^{\star 1} \in \underline{\boldsymbol{\mu}}^{\star 1}$ where $\mathrm{ht}(\underline{\mu}^{\star 1}) = \underline{i}'$, we have $i'_s = \delta_l - \delta_j = \varepsilon_l - \varepsilon_j$, the source of $i'_{s+1}$ is $\delta_j = \varepsilon_j$ and the target of $i'_{p-1}$ is $\delta_i = \varepsilon_i$. Thus we can always reconstruct the initial path $(\underline{\mu}, \underline{\alpha})$ by taking $i_s = \delta_l - \overline{\delta}_i = \varepsilon_l + \varepsilon_i$ and $i_p = \overline{\delta}_j - \delta_i = -\varepsilon_j - \varepsilon_i$. This is possible since $(\alpha^{\star 1})^{(1)}, \dots, (\alpha^{\star 1})^{(p-1)}$ is entirely contained in $[\![\delta_r, \delta_k]\!]$.

Hence, the set $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}\}$ is empty if the set $\{(\underline{\mu}, \underline{\alpha})^{\star 1} \in \hat{\mathscr{P}}_{m-1}(\delta_k) \mid \underline{\mu}^{\star 1} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha})^{\star 1} \in \underline{\boldsymbol{\mu}}^{\star 1}\} = \varnothing$. Furthermore, the set $\{(\underline{\mu}, \underline{\alpha})^{\star a'} \in \hat{\mathscr{P}}_{m^\star}(\delta_k) \mid \underline{\mu}^{\star a'} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha})^{\star a'} \in \underline{\boldsymbol{\mu}}^{\star a'}\}$ is empty too. Our induction hypothesis says that, in this case,

$$T_{\underline{d},\underline{\boldsymbol{\mu}}}(k) = T_{\underline{d}',\underline{\boldsymbol{\mu}}^{\star a'}}(k) = 0,$$

for any $\underline{d}' \in \mathbb{Z}_{\geqslant 0}^{m^\star}$. This proves the lemma for case I(1).

Observe that our above strategy works for case I(5) as well, since by Lemma 4.3 we also have

$$(\underline{\mu}, \underline{\alpha})^{\star a} = (\underline{\mu}', \underline{\alpha}') \star \big((\delta_l, \delta_j), (\alpha^{(s)} + \alpha^{(p)})\big) \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}''),$$

and so the arguments are quite similar. Since the calculations are similar, we omit the detail. Hence we conclude the case I(5) as in the first case.

(2) Assume that $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_j - \overline{\delta}_l$, with $j < l < i$. Recall from Lemma 4.3, for all $(\underline{\mu}, \underline{\alpha})^{\star a} \in \underline{\boldsymbol{\mu}}^{\star a}$, $a \in \{1, \dots, N\}$,

$$(\underline{\mu}, \underline{\alpha})^{\star a} = (\underline{\mu}', \underline{\alpha}') \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}''),$$

where $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$ is a path of length $\leqslant p - s$ between $\delta_j$ and $\delta_i$ whose roots $\underline{\tilde{\alpha}}^{\star a}$ are sums among $(\alpha^{(p-1)}, \dots, \alpha^{(s+1)}, \alpha^{(s)} + \alpha^{(p)})$, $\underline{\alpha}' = (\alpha^{(1)}, \dots, \alpha^{(s-1)})$ and $\underline{\alpha}'' = (\alpha^{(p+1)}, \dots, \alpha^{(m)})$. We have $\mathrm{ht}(\underline{\mu}^{\star 1}) = (i_1, \dots, i_{s-1}, i_{p-1}, \dots, i_{s+1}, i_s + i_p, i_{p+1}, \dots, i_m)$.

Set $\underline{i}' = (i'_1, \dots, i'_{m-1})$, so $i'_{p-1} = i_s + i_p$. Note that $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_j - \overline{\delta}_l$, with $j < l < i$. Thus,

$$j = k + i'_1 + \dots + i'_{s-1}, i = l + i'_{p-1} \text{ and } l = k + i'_1 + \dots + i'_{p-2}.$$

Hence, for all $(\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}$ the heights $\underline{i} := \mathrm{ht}(\underline{\mu})$ can be expressed in term of $\underline{i}'$ as follows (see Figure 4.10 for an illustration) :

$$\begin{aligned}
&i_1 = i'_1, \ \dots, \ i_{s-1} = i'_{s-1}, \ i_{s+1} = i'_{p-1}, \ \dots, \ i_{p-1} = i'_{s+1}, \ i_{p+1} = i'_p, \ \dots, \ i_{m-1} = i'_{m-2}, \\
&i_s = 2r - 2k - (j + l) + 1 = 2r + 1 - 2k - 2(i'_1 + \dots + i'_{s-1}) - (i'_s + \dots + i'_{p-2}), \\
&i_p = i'_{p-1} - i_s = 2k - 2r - 1 + 2(i'_1 + \dots + i'_{s-1}) + (i'_s + \dots + i'_{p-1}).
\end{aligned}$$

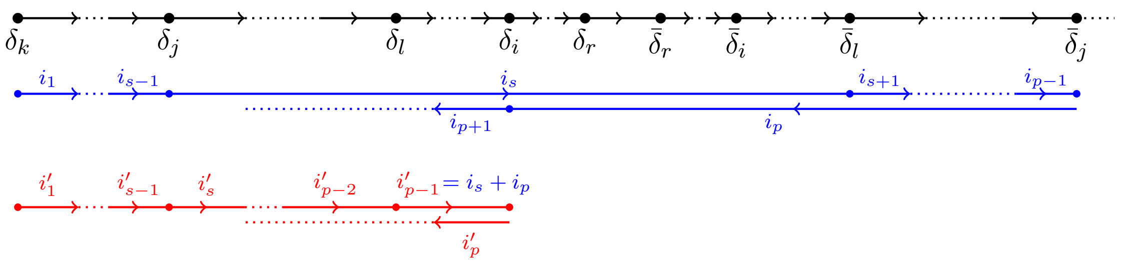



**FIGURE 4.10** – $\underline{i} = \mathrm{ht}(\underline{\mu})$ and $\underline{i}' := \mathrm{ht}(\underline{\mu}^{\star 1})$ where $\alpha^{(p)} = \overline{\delta}_j - \delta_i$, $\alpha^{(s)} = \delta_j - \overline{\delta}_l$, $j < l < i$.

With the same manner as in (4.13) we get,

$$\begin{aligned}
\widetilde{T}_{\underline{d}, \underline{\boldsymbol{\mu}}}(k) = \sum_{\substack{\underline{q} \in \mathbb{N}^{p-1} \\ |\underline{q}| = d_s}} \sum_{\substack{\underline{q}' \in \mathbb{N}^p \\ |\underline{q}'| = d_p}} C_{\underline{q}, \underline{q}'} \ (2k - 2r - 1)^{q_{p-1} + q'_p} \sum_{\substack{(\underline{\mu}, \underline{\alpha})^{\star 1} \in \mathscr{\hat{P}}_{m-1}(\delta_k), \\ \underline{\mu}^{\star 1} \in [\![\overline{\delta}_1, \delta_k]\!], \\ (\underline{\mu}, \underline{\alpha})^{\star 1} \in \underline{\boldsymbol{\mu}}^{\star 1}}} (i'_1)^{d_1 + q_1 + q'_1} \\
\times \dots \times \ ({i'}_s)^{d_{p-1} + q_s + q'_s} \dots ({i'}_{p-1})^{q'_{p-1}} ({i'}_p)^{d_{p+1}} \dots ({i'}_{m-2})^{d_{m-1}}. \qquad (4.15)
\end{aligned}$$

Assume that the equivalence class $\boldsymbol{\underline{\mu}}^{\star a'} = [(\underline{\mu}^{\star a'}, \underline{\alpha}^{\star a'})]$ satisfies $\deg A_{\boldsymbol{\underline{\mu}}^{\star a'}} \geqslant \deg A_{\boldsymbol{\underline{\mu}}}$, for some $a' \in \{1, \ldots, N\}$. By Lemma 4.3, there exists a partition $(t_1, \ldots, t_{n'}), t_i > 0$, of $p - s$ such that

$$\begin{aligned} i^\star_s &= i'_s + \cdots + i'_{s+t_1}, \\ &\vdots \\ i^\star_{s+n'-2} &= i'_{s+t_1+\cdots+t_{(n'-2)}+1} + \cdots + i'_{s+t_1+\ldots+t_{n'-1}}, \\ i^\star_{s+n'-1} &= i'_{s+t_1+\ldots+t_{n'-1}+1} + \cdots + i'_{p-1}. \end{aligned}$$

Hence $\mathrm{ht}(\underline{\mu}^{\star a'}) = (i'_1, \ldots, i'_{s-1}, i^\star_s, i^\star_{s+1}, \ldots, i^\star_{s+n'-1}, i'_p, \ldots, i'_{m-1})$ and the length of $\boldsymbol{\underline{\mu}}^{\star a'}$, denoted by $m^\star$, is equal to $m - (p - s) + n' - 2$. Here $i^\star_{s+j}$ denotes the height of the root $(\underline{\tilde{\alpha}}^{\star a'})^{(s+j)}$.

By Lemma 4.18 and (4.15) we get,

$$\begin{aligned} \widetilde{T}_{\underline{d},\boldsymbol{\underline{\mu}}}(k) = \sum_{\substack{\underline{q}\in\mathbb{N}^{p-1} \\ |\underline{q}|=d_s}} \sum_{\substack{\underline{q}'\in\mathbb{N}^{p} \\ |\underline{q}'|=d_p}} C_{\underline{q},\underline{q}'} \, (2k-2r-1)^{q_{p-1}+q'_p} \sum_{\substack{(\underline{\mu},\underline{\alpha})^{\star a'} \in \hat{\mathscr{P}}_{m^\star}(\delta_k), \\ \underline{\mu}^{\star a'} \in [\![\overline{\delta}_1, \delta_k]\!], \\ (\underline{\mu},\underline{\alpha})^{\star a'} \in \boldsymbol{\underline{\mu}}^{\star a'}}} (i'_1)^{d_1+q_1+q'_1} \cdots \\ \times (i'_{s-1})^{d_{s-1}+q_{s-1}+q'_{s-1}} \, Q_{n',\underline{t},\underline{d}'_{\underline{q},\underline{q}'}}(i^\star_s, \ldots, i^\star_{s+n'-1})(i'_p)^{d_{p+1}} \cdots (i'_{m-2})^{d_{m-1}}, \end{aligned}$$

where $Q_{n',\underline{t},\underline{d}'_{\underline{q},\underline{q}'}}$ is a polynomial of degree $d_{s+1} + \cdots + d_{p-1} + q_s + \cdots + q_{p-2} + q'_s + \cdots + q'_{p-1} + p - s - n'$. Repeated steps as in I(1) enables us to write

$$T_{\underline{d},\boldsymbol{\underline{\mu}}}(k) = \sum_{\substack{\underline{q}\in\mathbb{N}^{p-1} \\ |\underline{q}|=d_s}} \sum_{\substack{\underline{q}'\in\mathbb{N}^{p} \\ |\underline{q}'|=d_p}} \sum_{\substack{\underline{j}=(j_1,\ldots,j_{n'}) \\ |\underline{j}|\leqslant d'_{\underline{q},\underline{q}'}+p-s-1-n'}} C_{\underline{j}} C_{\underline{q},\underline{q}'} \, (2k-2r-1)^{q_{p-1}+q'_p} T_{\underline{d}'_{\underline{q},\underline{q}'},\boldsymbol{\underline{\mu}}^{\star a'}}(k).$$

By the same kind of reasoning as in first case , we have $T_{\underline{d},\boldsymbol{\underline{\mu}}}(k) = \widetilde{T}_{\underline{d},\boldsymbol{\underline{\mu}}}(k)$ and $T_{\underline{d},\boldsymbol{\underline{\mu}}}$ is a polynomial of degree $< d + m - \deg A_{\boldsymbol{\underline{\mu}}}$ for all $k$ such that $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!],\ (\underline{\mu}, \underline{\alpha}) \in \boldsymbol{\underline{\mu}}\}$ is nonempty.

It remains to verify that $T_{\underline{d},\boldsymbol{\underline{\mu}}}(k) = 0$ when

$$\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha}) \in \boldsymbol{\underline{\mu}}\} = \varnothing.$$

Observe that this set is never empty if $\{(\underline{\mu}, \underline{\alpha})^{\star 1} \in \hat{\mathscr{P}}_{m-1}(\delta_k) \mid \underline{\mu}^{\star 1} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha})^{\star 1} \in \boldsymbol{\underline{\mu}}^{\star 1}\}$ is nonempty. For any $(\underline{\mu}, \underline{\alpha})^{\star 1} \in \boldsymbol{\underline{\mu}}^{\star 1}$ where $\mathrm{ht}(\underline{\mu}^{\star 1}) = \underline{i}'$, we have $i'_{p-1} = \delta_l - \delta_j = \varepsilon_l - \varepsilon_j$, the source of $i'_s$ is $\delta_j = \varepsilon_j$. Thus we can always reconstruct the initial path

$(\underline{\mu}, \underline{\alpha})$ by taking $i_s = \delta_j - \overline{\delta}_l = \varepsilon_j + \varepsilon_l$ and $i_p = \overline{\delta}_j - \delta_i = -\varepsilon_j - \varepsilon_i$. This is possible since $(\alpha^{\star 1})^{(1)}, \ldots, (\alpha^{\star 1})^{(p-1)}$ is entirely contained in $[\![\delta_r, \delta_k]\!]$.

Hence, the set $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}\}$ is empty if $\{(\underline{\mu}, \underline{\alpha})^{\star 1} \in \hat{\mathscr{P}}_{m-1}(\delta_k) \mid \underline{\mu}^{\star 1} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha})^{\star 1} \in \underline{\boldsymbol{\mu}}^{\star 1}\} = \varnothing$. Furthermore, the set $\{(\underline{\mu}, \underline{\alpha})^{\star a'} \in \hat{\mathscr{P}}_{m^\star}(\delta_k) \mid \underline{\mu}^{\star a'} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha})^{\star a'} \in \underline{\boldsymbol{\mu}}^{\star a'}\}$ is empty too. Our induction hypothesis says that, in this case,

$$T_{\underline{d}, \underline{\boldsymbol{\mu}}}(k) = T_{\underline{d}', \underline{\boldsymbol{\mu}}^{\star a'}}(k) = 0,$$

for any $\underline{d}' \in \mathbb{Z}_{\geqslant 0}^{m^\star}$. This proves the lemma for case I(2).

(3) Assume that $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_i - \overline{\delta}_l$, with $i < j$ and $i < l$.
Recall from Lemma 4.3, for all $(\underline{\mu}, \underline{\alpha})^{\star a} \in \underline{\boldsymbol{\mu}}^{\star a}$, $a \in \{1, \ldots, N\}$,

$$(\underline{\mu}, \underline{\alpha})^{\star a} = (\underline{\mu}', \underline{\alpha}') \star (\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'', \underline{\alpha}''),$$

where $(\underline{\tilde{\mu}}^{\star a}, \underline{\tilde{\alpha}}^{\star a})$ is a path of length $\leqslant p - s$ between $\delta_j$ and $\delta_i$ whose roots $\underline{\tilde{\alpha}}^{\star a}$ are sums among $(\alpha^{(p-1)}, \ldots, \alpha^{(s+1)}, \alpha^{(s)} + \alpha^{(p)})$, $\underline{\alpha}' = (\alpha^{(1)}, \ldots, \alpha^{(s-1)})$ and $\underline{\alpha}'' = (\alpha^{(p+1)}, \ldots, \alpha^{(m)})$.

Note that the order of roots in $\underline{\alpha}^{\star 1}$ in this case may differ than in the case I(2), depending on the situation of $\alpha^{(p-1)}$ and $\alpha^{(p)}$ as described in the proof of Lemma 4.3(3) and (4). But in all that events $\{\alpha'_s, \ldots, \alpha'_{p-1}\} = \{\alpha^{(p-1)}, \ldots, \alpha^{(s+1)}, \alpha^{(s)} + \alpha^{(p)}\}$. So we can always express the heights in $\underline{i}$ in term of $\underline{i}'$. With the same arguments as in (4.15) we get similar equation and so we omit the detail.

Assume that for some $a' \in \{1, \ldots, N\}$ the equivalence class $\underline{\boldsymbol{\mu}}^{\star a'} = [(\underline{\mu}^{\star a'}, \underline{\alpha}^{\star a'})]$ satisfies $\deg A_{\underline{\boldsymbol{\mu}}^{\star a'}} \geqslant \deg A_{\underline{\boldsymbol{\mu}}}$. Note also that in this case the roots in $\underline{\tilde{\alpha}}^{\star a'}$ is a sum amongs $(\alpha'_s, \ldots, \alpha'_{p-1})$, but not necessary in the sequential order. Then we are doing the similar calculation and so we omit the detail. Hence we conclude as in I(2).

Observe that our method above will work for case I(4) as well and so we omit the detail. Hence we conclude for this case as in I(2).

It remains to verify that $T_{\underline{d}, \underline{\boldsymbol{\mu}}}(k) = 0$ when

$$\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}\} = \varnothing.$$

First, we consider the case where $\mathrm{ht}(\mu^{\star 1}) = (i_1, \ldots, i_{s-1}, i_{p-1}, \ldots, i_{s+1}, i_s + i_p, i_{p+1}, \ldots, i_m)$. The other case will work similarly and so we omit the detail.

Observe that the set in the statement is never empty if the set $\{(\underline{\mu}, \underline{\alpha})^{\star 1} \in \hat{\mathscr{P}}_{m-1}(\delta_k) \mid \underline{\mu}^{\star 1} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha})^{\star 1} \in \underline{\boldsymbol{\mu}}^{\star 1}\}$ is nonempty. For any $(\underline{\mu}, \underline{\alpha})^{\star 1} \in \underline{\boldsymbol{\mu}}^{\star 1}$

where $\mathrm{ht}(\underline{\mu}^{\star 1}) = \underline{i}'$, we have $i'_{p-1} = \overline{\delta}_l - \overline{\delta}_j = \varepsilon_j - \varepsilon_l$, the source of $i'_s$ is $\delta_i = \varepsilon_i$ and the target of $i'_{p-2}$ is $\overline{\delta}_l = -\varepsilon_l$. Thus we can always reconstruct the initial path $(\underline{\mu}, \underline{\alpha})$ by taking $i_s = \delta_i - \delta_l = \varepsilon_i - \varepsilon_l$ and $i_p = \overline{\delta}_i - \overline{\delta}_j = \varepsilon_j - \varepsilon_i$.

Hence, the set $\{(\underline{\mu}, \underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha}) \in \underline{\boldsymbol{\mu}}\}$ is empty if $\{(\underline{\mu}, \underline{\alpha})^{\star 1} \in \hat{\mathscr{P}}_{m-1}(\delta_k) \mid \underline{\mu}^{\star 1} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha})^{\star 1} \in \underline{\boldsymbol{\mu}}^{\star 1}\} = \varnothing$. Furthermore, the set $\{(\underline{\mu}, \underline{\alpha})^{\star a'} \in \hat{\mathscr{P}}_{m^\star}(\delta_k) \mid \underline{\mu}^{\star a'} \in [\![\overline{\delta}_1, \delta_k]\!], (\underline{\mu}, \underline{\alpha})^{\star a'} \in \underline{\boldsymbol{\mu}}^{\star a'}\}$ is empty too. Our induction hypothesis says that, in this case,

$$T_{\underline{d}, \underline{\boldsymbol{\mu}}}(k) = T_{\underline{d}', \underline{\boldsymbol{\mu}}^{\star a'}}(k) = 0,$$

for any $\underline{d}' \in \mathbb{Z}^{m^\star}_{\geqslant 0}$.

**II. Assume $\boldsymbol{\alpha^{(s)} = -\alpha^{(p)}}$.**

Lemma 4.15 shows that

$$\begin{aligned} A_{\underline{\boldsymbol{\mu}}}(X_1, \ldots, X_{m-1}) = \bar{K}^{\#}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\#}}(X_1, \ldots, X_{p-2}, X_{p-1} + X_p, \ldots, X_{m-1}) \\ + (P_{\alpha^{(s)}}(X_s) - c_s) \sum_{a=1}^{N} K^{\star a}_{\underline{\boldsymbol{\mu}}} A_{\underline{\boldsymbol{\mu}}^{\star a}}(\underline{X}^{\star a}), \end{aligned} \tag{4.16}$$

where $\bar{K}^{\#}_{\underline{\boldsymbol{\mu}}} = \begin{cases} K^{\#}_{\underline{\boldsymbol{\mu}}} & \text{if } \alpha^{(p-1)} + \alpha^{(p)} \neq 0, \\ (P_{\alpha^{(p-1)}}(X_{p-1}) - c_{p-1}) & \text{if } \alpha^{(p-1)} + \alpha^{(p)} = 0. \end{cases}$

If $\deg A_{\underline{\boldsymbol{\mu}}} \leqslant \deg A_{\underline{\boldsymbol{\mu}}^{\#}}$ (respectively, $\deg A_{\underline{\boldsymbol{\mu}}} \leqslant \deg A_{\underline{\boldsymbol{\mu}}^{\#}} + 1$) for $\alpha^{(p-1)} + \alpha^{(p)} \neq 0$ (respectively, $\alpha^{(p-1)} + \alpha^{(p)} = 0$) then using the same strategy as in Lemma 3.14, which is by expressing the heights of $\underline{i}$ in term of $\underline{i}^{\#}$, we get the statement.

Therefore we can assume that $\deg A_{\underline{\boldsymbol{\mu}}} > \deg A_{\underline{\boldsymbol{\mu}}^{\#}}$ (respectively, $\deg A_{\underline{\boldsymbol{\mu}}} > \deg A_{\underline{\boldsymbol{\mu}}^{\#}} + 1$) if $\alpha^{(p-1)} + \alpha^{(p)} \neq 0$ (respectively, $\alpha^{(p-1)} + \alpha^{(p)} = 0$). This assumption means that for some $a \in \{1, \ldots, N\}$, $\deg A_{\underline{\boldsymbol{\mu}}^{\star a}} + 1 \geqslant \deg A_{\underline{\boldsymbol{\mu}}}$.

By Lemma 2.7, the possibilities for $\alpha^{(p)}$ and $\alpha^{(s)}$ which satisfying the assumptions $\alpha^{(s)} + \alpha^{(p)} = 0$ are the following:

(1) $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_j - \overline{\delta}_i$, with $j < i$,

(2) $\alpha^{(p)} = \overline{\delta}_i - \overline{\delta}_j$ and $\alpha^{(s)} = \delta_i - \delta_j$, with $i < j$.

Note that in all those cases, for all $a \in \{1, \ldots, N\}$, $m - p + s \leqslant \mathrm{ht}(\underline{\boldsymbol{\mu}}^{\star a}) \leqslant m - 2$. Possibly changing the numbering of the equivalence class $\underline{\boldsymbol{\mu}}^{\star a}$, one can assume throughout this proof that $\underline{\boldsymbol{\mu}}^{\star 1} = [(\underline{\mu}^{\star 1}, \underline{\alpha}^{\star 1})]$ is the equivalence class of the star paths with the longest length and set $\underline{i} := \mathrm{ht}(\underline{\boldsymbol{\mu}})$ and $\underline{i}' := \mathrm{ht}(\underline{\boldsymbol{\mu}}^{\star 1})$.

(1) We consider the case where $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_j - \overline{\delta}_i$, with $j < i$.

Recall from Lemma 4.4, for all $(\underline{\mu},\underline{\alpha})^{\star a} \in \underline{\boldsymbol{\mu}}^{\star a}$, $a \in \{1,\dots,N\}$,

$$(\underline{\mu},\underline{\alpha})^{\star a} = (\underline{\mu}',\underline{\alpha}') \star (\underline{\tilde{\mu}}^{\star a},\underline{\tilde{\alpha}}^{\star a}) \star (\underline{\mu}'',\underline{\alpha}''),$$

where $(\underline{\tilde{\mu}}^{\star a},\underline{\tilde{\alpha}}^{\star a})$ is a path of length $\leqslant p-s-1$ between $\delta_j$ and $\delta_i$ whose roots $\underline{\tilde{\alpha}}^{\star a}$ are sums among $(\alpha^{(p-1)},\dots,\alpha^{(s+1)})$, $\underline{\alpha}' = (\alpha^{(1)},\dots,\alpha^{(s-1)})$ and $\underline{\alpha}'' = (\alpha^{(p+1)},\dots,\alpha^{(m)})$. We have $\mathrm{ht}(\underline{\mu}^{\star 1}) = (i_1,\dots,i_{s-1},i_{p-1},\dots,i_{s+1},i_{p+1},\dots,i_m)$.

Set $\underline{i}' = (i'_1,\dots,i'_{m-2})$. Note that $\alpha^{(p)} = \overline{\delta}_j - \delta_i$ and $\alpha^{(s)} = \delta_j - \overline{\delta}_i$, with $j < i$. Thus,

$$j = k + i'_1 + \cdots + i'_{s-1} \quad \text{and} \quad i = k + i'_1 + \cdots + i'_{p-2}.$$

Hence, for all $(\underline{\mu},\underline{\alpha}) \in \underline{\boldsymbol{\mu}}$ the heights in $\underline{i}$ can be expressed in term of $\underline{i}'$ as follows (see Figure 4.11 for an illustration) :

$$\begin{aligned}
&i_1 = i'_1,\ \dots,\ i_{s-1} = i'_{s-1},\ i_{s+1} = i'_{p-2},\ \dots,\ i_{p-1} = i'_s,\ i_{p+1} = i'_{p-1},\ \dots,\ i_{m-1} = i'_{m-3},\\
&i_s = 2r + 1 - 2k - 2(i'_1 + \cdots + i'_{s-1}) - (i'_s + \cdots + i'_{p-2}),\\
&i_p = -i_s = 2k - 2r - 1 + 2(i'_1 + \cdots + i'_{s-1}) + (i'_{s+1} + \cdots + i'_{p-2}).
\end{aligned}$$

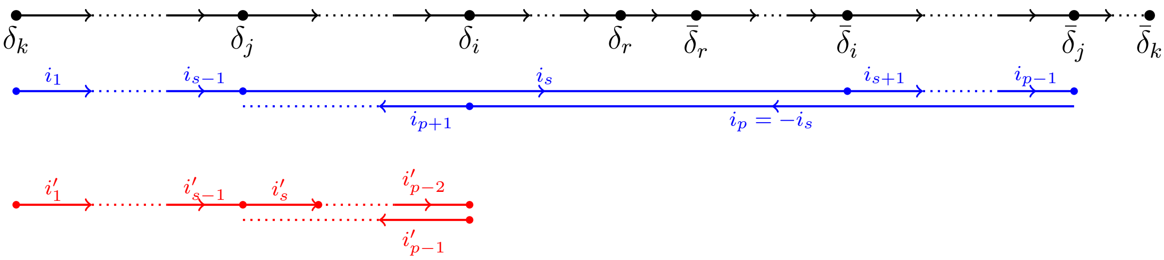



**FIGURE 4.11** – $\underline{i} = \mathrm{ht}(\underline{\mu})$ and $\underline{i}' := \mathrm{ht}(\underline{\mu}^{\star 1})$ where $\alpha^{(p)} = \overline{\delta}_j - \delta_i$, $\alpha^{(s)} = \delta_j - \overline{\delta}_i$, $j < i$.

With the same arguments as in (4.13) we get,

$$\begin{aligned}
\widetilde{T}_{\underline{d},\underline{\boldsymbol{\mu}}}(k) = \sum_{\substack{\underline{q}\in\mathbb{N}^{p-1}\\ |\underline{q}|=d_s}} \sum_{\substack{\underline{q}'\in\mathbb{N}^{p-1}\\ |\underline{q}'|=d_p}} C_{\underline{q},\underline{q}'}\ (2k-2r-1)^{q_{p-1}+q'_{p-1}} \sum_{\substack{(\underline{\mu},\underline{\alpha})^{\star 1}\in\mathscr{P}_{m-2}(\delta_k),\\ \underline{\mu}^{\star 1}\in[\![\overline{\delta}_1,\delta_k]\!],\\ (\underline{\mu},\underline{\alpha})^{\star 1}\in\underline{\boldsymbol{\mu}}^{\star 1}}} (i'_1)^{d_1+q_1+q'_1}\\
\times\cdots\times\ (i'_s)^{d_{p-1}+q_s+q'_s}\cdots(i'_{p-2})^{d_{s+1}+q_{p-2}+q'_{p-2}}(i'_{p-1})^{d_{p+1}}\cdots(i'_{m-3})^{d_{m-1}}. \quad (4.17)
\end{aligned}$$

Assume that the equivalence class $\underline{\boldsymbol{\mu}}^{\star a'} = [(\underline{\mu}^{\star a'}, \underline{\alpha}^{\star a'})]$ satisfies $\deg A_{\underline{\boldsymbol{\mu}}^{\star a'}} + 1 \geqslant \deg A_{\underline{\boldsymbol{\mu}}}$, for some $a' \in \{1, \dots, N\}$. According to the proof of Lemma 4.4, there exists a partition $(t_1, \dots, t_{n'}), t_i > 0$, of $p - s - 1$ such that

$$\begin{aligned} i_s^\star &= i'_s + \cdots + i'_{s+t_1}, \\ &\vdots \\ i^\star_{s+n'-2} &= i'_{s+t_1+\cdots+t_{(n'-2)}+1} + \cdots + i'_{s+t_1+\ldots+t_{n'-1}}, \\ i^\star_{s+n'-1} &= i'_{s+t_1+\ldots+t_{n'-1}+1} + \cdots + i'_{p-2}. \end{aligned}$$

Hence $\mathrm{ht}(\underline{\mu}^{\star a'}) = (i'_1, \dots, i'_{s-1}, i^\star_s, i^\star_{s+1}, \dots, i^\star_{s+n'-1}, i'_{p-1}, \dots, i'_{m-3})$ and the length of $\underline{\boldsymbol{\mu}}^{\star a'}$, denoted by $m^\star$, is equal to $m - p + s + n' - 1$. Here $i^\star_{s+j}$ denotes the height of the root $(\underline{\tilde{\alpha}}^{\star a'})^{(s+j)}$.

On account of Lemma 4.18 and (4.17) and from the arguments of above cases it follows that

$$\begin{aligned} \widetilde{T}_{\underline{d},\underline{\boldsymbol{\mu}}}(k) = &\sum_{\substack{\underline{q}\in\mathbb{N}^{p-1}\\ |\underline{q}|=d_s}} \sum_{\substack{\underline{q}'\in\mathbb{N}^{p-1}\\ |\underline{q}'|=d_p}} C_{\underline{q},\underline{q}'}\ (2k-2r-1)^{q_{p-1}+q'_{p-1}} \sum_{\substack{(\underline{\mu},\underline{\alpha})^{\star a'}\in\mathscr{P}_{m^\star}(\delta_k),\\ \underline{\mu}^{\star a'}\in[\![\overline{\delta}_1,\delta_k]\!],\\ (\underline{\mu},\underline{\alpha})^{\star a'}\in\underline{\boldsymbol{\mu}}^{\star a'}}} (i'_1)^{d_1+q_1+q'_1}\cdots \\ &\times (i'_{s-1})^{d_{s-1}+q_{s-1}+q'_{s-1}} \sum_{\substack{\underline{j}=(j_1,\dots,j_{n'})\\ |\underline{j}|\leqslant d'_{\underline{q},\underline{q}'}+p-s-1-n'}} C_{\underline{j}}(i^\star_s)^{j_1}\cdots(i^\star_{s+n'-1})^{j_{n'}}(i'_{p-1})^{d_{p+1}}\cdots(i'_{m-3})^{d_{m-1}}, \end{aligned}$$

where $d'_{\underline{q},\underline{q}'} = d_{s+1}+q_{p-2}+q'_{p-2}+\cdots+d_{p-1}+q_s+q'_s$. Set $\underline{d}_{\underline{q},\underline{q}'} = (d_1+q_1+q'_1, \dots, d_{s-1}+q_{s-1}+q'_{s-1}, j_1, \dots, j_{n'}, d_{p+1}, \dots, d_{m-1})$, with $|\underline{d}_{\underline{q},\underline{q}'}| = d + (p-s-1) - n' - q_{p-1} - q'_{p-1}$, and

$$\widetilde{T}_{\underline{d}_{\underline{q},\underline{q}'},\underline{\boldsymbol{\mu}}^{\star a'}}(k) \sum_{\substack{(\underline{\mu},\underline{\alpha})^{\star a'}\in\mathscr{P}_{m^\star}(\delta_k),\\ \underline{\mu}^{\star a'}\in[\![\overline{\delta}_1,\delta_k]\!],\\ (\underline{\mu},\underline{\alpha})^{\star a'}\in\underline{\boldsymbol{\mu}}^{\star a'}}} (i'_1)^{d_1+q_1+q'_1}\cdots(i^\star_s)^{j_1}\cdots(i^\star_{s+n'-1})^{j_{n'}}(i'_{p-1})^{d_{p+1}}\cdots(i'_{m-3})^{d_{m-1}}.$$

Then

$$\widetilde{T}_{\underline{d},\underline{\boldsymbol{\mu}}}(k) = \sum_{\substack{\underline{q}\in\mathbb{N}^{p-1}\\ |\underline{q}|=d_s}} \sum_{\substack{\underline{q}'\in\mathbb{N}^{p-1}\\ |\underline{q}'|=d_p}} \sum_{\substack{\underline{j}=(j_1,\dots,j_{n'})\\ |\underline{j}|\leqslant d'_{\underline{q},\underline{q}'}+p-s-1-n'}} C_{\underline{j}} C_{\underline{q},\underline{q}'}(2k-2r-1)^{q_{p-1}+q'_{p-1}} \widetilde{T}_{\underline{d}_{\underline{q},\underline{q}'},\underline{\boldsymbol{\mu}}^{\star a'}}(k). \quad (4.18)$$

By the induction hypothesis applied to $m^\star$ and $\underline{\boldsymbol{\mu}}^{\star a'}$, there exist polynomials

$T_{\underline{d}_{\underline{q},\underline{q}'},\underline{\boldsymbol{\mu}}^{\star a'}} \in \mathbb{C}[X_1,\ldots,X_{m^\star-1}$ with

$$\begin{aligned}\deg(T_{\underline{d}_{\underline{q},\underline{q}'},\underline{\boldsymbol{\mu}}^{\star a'}}) &\leqslant d+(p-s-1)-n'-q_{p-1}-q'_{p-1}+m^\star-\deg A_{\underline{\boldsymbol{\mu}}^{\star a'}}\\ &\leqslant d+m-q_{p-1}-q'_{p-1}-2-\deg A_{\underline{\boldsymbol{\mu}}} < d+m-q_{p-1}-q'_{p-1}-\deg A_{\underline{\boldsymbol{\mu}}},\end{aligned}$$

since $\deg A_{\underline{\boldsymbol{\mu}}^{\star a'}}+1 \geqslant \deg A_{\underline{\boldsymbol{\mu}}}$ by the assumption, such that $T_{\underline{d}_{\underline{q},\underline{q}'},\underline{\boldsymbol{\mu}}^{\star a'}}(k) = \widetilde{T}_{\underline{d}_{\underline{q},\underline{q}'},\underline{\boldsymbol{\mu}}^{\star a'}}(k)$ for all $k$ such that $\{(\underline{\mu},\underline{\alpha})^{\star a'} \in \hat{\mathscr{P}}_{m^\star}(\delta_k) \mid \underline{\mu}^{\star a'} \in [\![\overline{\delta}_1,\delta_k]\!],\ (\underline{\mu},\underline{\alpha})^{\star a'} \in \underline{\boldsymbol{\mu}}^{\star a'}\}$ is nonempty.

Set

$$T_{\underline{d},\underline{\boldsymbol{\mu}}}(k) = \sum_{\substack{\underline{q}\in\mathbb{N}^{p-1}\\ |\underline{q}|=d_s}}\ \sum_{\substack{\underline{q}'\in\mathbb{N}^{p-1}\\ |\underline{q}'|=d_p}}\ \sum_{\substack{\underline{j}=(j_1,\ldots,j_{n'})\\ |\underline{j}|\leqslant d'_{\underline{q},\underline{q}'}+p-s-1-n'}} C_{\underline{j}}C_{\underline{q},\underline{q}'}(2k-2r-1)^{q_{p-1}+q'_{p-1}}T_{\underline{d}_{\underline{q},\underline{q}'},\underline{\boldsymbol{\mu}}^{\star a'}}(k).$$

Then by the induction hypothesis and (4.18), we have $T_{\underline{d},\underline{\boldsymbol{\mu}}}(k) = \widetilde{T}_{\underline{d},\underline{\boldsymbol{\mu}}}(k)$ and $T_{\underline{d},\underline{\boldsymbol{\mu}}}$ is a polynomial of degree $< d+m-\deg A_{\underline{\boldsymbol{\mu}}}$ for all $k$ such that $\{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1,\delta_k]\!],\ (\underline{\mu},\underline{\alpha}) \in \underline{\boldsymbol{\mu}}\}$ is nonempty.

If $\{(\underline{\mu},\underline{\alpha}) \in \hat{\mathscr{P}}_m(\delta_k) \mid \underline{\mu} \in [\![\overline{\delta}_1,\delta_k]\!], (\underline{\mu},\underline{\alpha}) \in \underline{\boldsymbol{\mu}}\} = \varnothing$ then by the same kind of reasoning as in the proof for case I(1), we get $T_{\underline{d},\underline{\boldsymbol{\mu}}}(k) = 0$. This proves the lemma for case II(1).

Observe that our strategy above works for case II(2) as well, and so the arguments are quite similar. Since the calculations are similar, we omit the detail. Hence we conclude the case II(2) as in the first case. □

### 4.2.4 Elements of $\mathscr{E}_m$ with zeroes

We consider in this subsection the elements of $\mathscr{E}_m$ with zeroes. We adopt all the notations and definition as in Section 3.2.4.

This paragraph is about the computation of the sum of all paths $(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}$ that are obtained from $(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})$ by "gluing the loop" as in Section 3.2.4.

**Lemma 4.21.** *Let $m \in \mathbb{Z}_{>0}$ and $\underline{\boldsymbol{\mu}} \in \mathscr{E}_m$ with $n$ zeroes in positions $j_1,\ldots,j_n$ ($n \leqslant m$). There are some classes $\underline{\tilde{\boldsymbol{\mu}}}_1,\ldots,\underline{\tilde{\boldsymbol{\mu}}}_N$ without zero of length $\ell_h := \ell(\underline{\tilde{\boldsymbol{\mu}}}_h) \leqslant m-n$ and polynomial $B_{\underline{\boldsymbol{\mu}}}$ of degree $\leqslant n$ such that*

$$\sum_{\underline{\beta}\in\Pi_{\underline{\mu}}(\underline{j})} \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = B_{\underline{\boldsymbol{\mu}}}(k)\sum_{h=1}^{N} K_{\underline{\tilde{\boldsymbol{\mu}}}_h}A_{\underline{\tilde{\boldsymbol{\mu}}}_h}(\tilde{\imath}_1,\ldots,\tilde{\imath}_{\ell_h-1}),$$

*where* $\mathrm{ht}(\underline{\tilde{\mu}}_h) = (\tilde{i}_1, \ldots, \tilde{i}_{\ell_h})$. *Moreover, we have*

$$B_{\underline{\boldsymbol{\mu}}}(X) = \sum_{j=0}^{n} C_{\underline{\boldsymbol{\mu}}}^{(n-j)}(\tilde{\imath}_1, \ldots, \tilde{\imath}_{m-n-1}) X^j,$$

*where* $C_{\underline{\boldsymbol{\mu}}}^{(0)} = (-1)^n$ *and* $C_{\underline{\boldsymbol{\mu}}}^{(l)} \in \mathbb{C}[X_1, \ldots, X_{m-n-1}]$ *has total degree* $\leqslant l$ *for* $l = 1, \ldots, n$. *In particular, if* $n = 0$, *we have* $B_{\underline{\boldsymbol{\mu}}}(X) = 1$.

*Proof.* We prove the statement by induction on the number of the loops $n$. Set $\underline{j} = (j_1, \ldots, j_n)$ and let $\underline{\beta} := (\beta^{(1)}, \ldots, \beta^{(n)})$ be in $\Pi_{\underline{\mu}^{(\underline{j})}}$.

First of all, observe that if $n = 0$, then the result is known by Lemma 4.15 and Remark 4.17.
If $n = m$ then either $(\underline{\mu}, \underline{\alpha}) = ((\delta_k); (\Pi_{\delta_k})^m)$ or $(\underline{\mu}, \underline{\alpha}) = \left((\overline{\delta}_k); (\Pi_{\overline{\delta}_k})^m\right)$. By Lemma 2.4, there is a polynomial $B_{\underline{\boldsymbol{\mu}}}$ of degree $m$ such that

$$\mathrm{wt}(\underline{\mu}, \underline{\alpha}) = B_{\underline{\boldsymbol{\mu}}}(k),$$

and so the lemma is true for $n = m$. Hence it remains to prove for $n = 1, \ldots, m-1$. For $n = 1$, we have $\underline{j} = (j_1)$ for some $j_1 = 1, \ldots, m$.

- Assume $j_1 = 1$ then $\mu^{(j_1)} = \delta_k$.
  For some $k \in \{2, \ldots, r\}$, we have $\Pi_{\underline{\mu}^{(\underline{j})}} = \{\beta_{k-1}, \beta_k\}$. Hence by Lemma 4.6, Lemma 2.3 and Remark 4.17,

$$\begin{aligned}
\sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) &= \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{(1);\{\beta_{k-1}\}}\right) + \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{(1);\{\beta_k\}}\right) \\
&= \left(-\langle \rho, \varpi_{k-1}^\sharp \rangle + \langle \rho, \varpi_k^\sharp \rangle\right) \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) = \left(\langle \rho, \varpi_k^\sharp - \varpi_{k-1}^\sharp \rangle\right) \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) \\
&= (r - k + 1)\mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) = (r + 1 - k) \sum_{h=1}^{N} K_{\underline{\tilde{\boldsymbol{\mu}}}_h} A_{\underline{\tilde{\boldsymbol{\mu}}}_h}(\tilde{\imath}_1, \ldots, \tilde{\imath}_{\ell_h - 1}).
\end{aligned}$$

  For $k = 1$, we have $\Pi_{\underline{\mu}^{(\underline{j})}} = \{\beta_1\}$. Hence by Lemma 4.6, Lemma 2.3 and Remark 4.17,

$$\begin{aligned}
\sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) &= \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{(1);\{\beta_1\}}\right) = (\langle \rho, \varpi_1^\sharp \rangle) \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) \\
&= \left(\frac{1}{2}(2r - 1 + 1)\right) \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) = (r) \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})
\end{aligned}$$

$$= (r+1-k)\sum_{h=1}^{N} K_{\underline{\tilde{\mu}}_h} A_{\underline{\tilde{\mu}}_h}(\tilde{\imath}_1, \ldots, \tilde{\imath}_{\ell_h-1}).$$

Hence by setting $B_{\underline{\boldsymbol{\mu}}}(X) := r+1-X$, we get the statement.

- Assume $\mu^{(j_1)} = \delta_s$ for some $s \in \{2, \ldots, r\}$ and $s \neq k$. Note that $s = k + \sum_{t=1}^{j_1-1} i_t$ and $\Pi_{\underline{\mu}^{(j)}} = \{\beta_{s-1}, \beta_s\}$. By Lemma 4.6, Lemma 2.3 and Remark 4.17,

$$\sum_{\underline{\beta}\in\Pi_{\underline{\mu}^{(j)}}} \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{(j_1);\{\beta_{s-1}\}}\right) + \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{(j_1);\{\beta_s\}}\right)$$
$$= \left(-\langle\rho, \varpi^\sharp_{s-1}\rangle + 1 + \langle\rho, \varpi^\sharp_s\rangle\right)\ \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) = \left(\langle\rho, \varpi^\sharp_s - \varpi^\sharp_{s-1}\rangle + 1\right)\ \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})$$
$$= (r-s+2)\ \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) = \left(r - k - \sum_{t=1}^{j_1-1} i_t + 2\right) \sum_{h=1}^{N} K_{\underline{\tilde{\mu}}_h} A_{\underline{\tilde{\mu}}_h}(\tilde{\imath}_1, \ldots, \tilde{\imath}_{\ell_h-1}).$$

Note that
$$\sum_{t=1}^{j_1-1} i_t = \sum_{t=1}^{\tilde{j}} \tilde{\imath}_t,$$
where $\tilde{j} = j_1 - 1$. Set
$$B_{\underline{\boldsymbol{\mu}}}(X) := r - X - \sum_{t=1}^{\tilde{j}} \tilde{\imath}_t + 2.$$

Then we prove the statement in this case.

- Assume $\mu^{(j_1)} = \overline{\delta}_s$.
For $s \in \{2, \ldots, r-1\}$, note that
$$s = 2r - k - \sum_{t=1}^{j_1-1} i_t + 1 \quad \text{and} \quad \Pi_{\underline{\mu}^{(j)}} = \{\beta_{s-1}, \beta_s\}.$$

By Lemma 4.6, Lemma 2.3 and Remark 4.17, we have

$$\sum_{\underline{\beta}\in\Pi_{\underline{\mu}^{(j)}}} \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{(j_1);\{\beta_{s-1}\}}\right) + \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{(j_1);\{\beta_s\}}\right)$$
$$= \left(\langle\rho, \varpi^\sharp_{s-1}\rangle - 1 - \langle\rho, \varpi^\sharp_s\rangle + 2\right)\ \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) = \left(\langle\rho, \varpi^\sharp_{s-1} - \varpi^\sharp_s\rangle + 1\right)\ \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})$$

$$= (s-r)\ \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) = \left(r - k - \sum_{t=1}^{j_1-1} i_t + 1\right) \sum_{h=1}^{N} K_{\underline{\tilde{\boldsymbol{\mu}}}_h} A_{\underline{\tilde{\boldsymbol{\mu}}}_h}(\tilde{\imath}_1, \dots, \tilde{\imath}_{\ell_h-1}).$$

For $s = 1$, $\Pi_{\underline{\mu}^{(\underline{j})}} = \{\beta_1\}$, thus

$$\begin{aligned}\sum_{\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) &= \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{(j_1);\{\beta_1\}}\right) = (\langle -\rho, \varpi_1^\sharp\rangle)\ \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) \\ &= (-r)\ \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) = (s-1-r)\ \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) \\ &= \left(r - k - \sum_{t=1}^{j_1-1} i_t + 1\right) \sum_{h=1}^{N} K_{\underline{\tilde{\boldsymbol{\mu}}}_h} A_{\underline{\tilde{\boldsymbol{\mu}}}_h}(\tilde{\imath}_1, \dots, \tilde{\imath}_{\ell_h-1}).\end{aligned}$$

Note that

$$\sum_{t=1}^{j_1-1} i_t = \sum_{t=1}^{\tilde{j}} \tilde{\imath}_t,$$

where $\tilde{j} = j_1 - 1$. By setting

$$B_{\underline{\boldsymbol{\mu}}}(X) := r - X - \sum_{t=1}^{\tilde{j}} \tilde{\imath}_t + 1,$$

we prove the statement in this case.

- Assume $\mu^{(j_1)} = \overline{\delta}_r$.
  Note that $\Pi_{\underline{\mu}^{(\underline{j})}} = \{\beta_{r-1}, \beta_r\}$. By Lemma 4.6, Lemma 2.3 and Remark 4.17, we have

$$\begin{aligned}&\sum_{\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{(j_1);\{\beta_{r-1}\}}\right) + \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{(j_1);\{\beta_r\}}\right) \\ &= \left(\langle \rho, \varpi_{r-1}^\sharp\rangle - 1 - \langle \rho, \varpi_r^\sharp\rangle + 2\right)\ \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) = \left(\langle \rho, \varpi_{r-1}^\sharp - \varpi_r^\sharp\rangle + 1\right)\ \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) \\ &= (-(r-r+1)+1) = 0.\end{aligned}$$

  Since 0 is a polynomial of degree $\leqslant n$ then we get the statement.

By the above observation, the lemma is true for $n = 1$. Let $n \geqslant 2$ and assume the lemma is true for any $n' \in \{1, \dots, n-1\}$.
Set $\underline{j} := (j_1, \dots, j_n)$ the position of loops and let $\underline{\beta} := (\beta^{(1)}, \dots, \beta^{(n)})$ be in $\Pi_{\underline{\mu}^{(\underline{j})}}$. We

are doing the similar calculation as Lemma 4.6, thus

$$\sum_{\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = \left(\sum_{\alpha^{(j_1)}\in\Pi_{\mu^{(j_1)}}} \langle \mu^{j_1}, \alpha^{(j_1)}\rangle \left(\langle \rho, \varpi^{\sharp}_{\alpha^{(j_1)}}\rangle - c_{\alpha^{(j_1)}}\right)\right) \sum_{\underline{\beta}'\in\Pi_{\underline{\mu}^{(\underline{j}')}}} \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j}';\underline{\beta}'}\right),$$

where $\underline{j}' = (j_2,\dots,j_n)$, $c_{\alpha^{(j_1)}} := \sum_{i=1}^{j_1-1}\langle \alpha^{(i)}, \varpi^{\sharp}_{\alpha^{(j_1)}}\rangle$ and $(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j}';\underline{\beta}'}$ is a weighted path with $n-1$ loops. Set $\underline{\boldsymbol{\mu}}' := [(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j}';\underline{\beta}'}]$. So, the induction hypothesis applied to $\underline{\boldsymbol{\mu}}'$ gives,

$$\begin{aligned}
&\sum_{\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right)\\
&= \left(\sum_{\alpha^{(j_1)}\in\Pi_{\mu^{(j_1)}}} \langle \mu^{j_1}, \alpha^{(j_1)}\rangle \left(\langle \rho, \varpi^{\sharp}_{\alpha^{(j_1)}}\rangle - c_{\alpha^{(j_1)}}\right)\right) B_{\underline{\mu}'}(k) \sum_{h=1}^{N} K_{\underline{\tilde{\mu}}'_h} A_{\underline{\tilde{\mu}}'_h}(\tilde{\imath}_1,\dots,\tilde{\imath}_{\ell_h-1})\\
&= \left(\sum_{\alpha^{(j_1)}\in\Pi_{\mu^{(j_1)}}} \langle \mu^{j_1}, \alpha^{(j_1)}\rangle \left(\langle \rho, \varpi^{\sharp}_{\alpha^{(j_1)}}\rangle - c_{\alpha^{(j_1)}}\right)\right) B_{\underline{\mu}'}(k) \sum_{h=1}^{N} K_{\underline{\tilde{\mu}}_h} A_{\underline{\tilde{\mu}}_h}(\tilde{\imath}_1,\dots,\tilde{\imath}_{\ell_h-1})
\end{aligned}$$

since $\underline{\tilde{\boldsymbol{\mu}}}'_h = \underline{\tilde{\boldsymbol{\mu}}}_h$, $h=1,\dots,N$. Set

$$\tilde{B}_{\underline{\boldsymbol{\mu}}}(k) := \left(\sum_{\alpha^{(j_1)}\in\Pi_{\mu^{(j_1)}}} \langle \mu^{j_1}, \alpha^{(j_1)}\rangle \left(\langle \rho, \varpi^{\sharp}_{\alpha^{(j_1)}}\rangle - c_{\alpha^{(j_1)}}\right)\right) B_{\underline{\boldsymbol{\mu}}'}(k).$$

With the same arguments as for the case $n=1$ and the induction hypothesis applied to $\underline{\boldsymbol{\mu}}'$, we see that there exists a polynomial $B_{\underline{\boldsymbol{\mu}}}$ of degree $\leqslant n$ with leading term is $(-1)^n X^n$, and the coefficient of $B_{\underline{\boldsymbol{\mu}}}(X)$ in $X^j$, $j\leqslant n$, is a polynomial in the variable $\tilde{\imath}_1,\dots,\tilde{\imath}_{m-n-1}$, of total degree $\leqslant n-j$ such that $\tilde{B}_{\underline{\boldsymbol{\mu}}}(k) = B_{\underline{\boldsymbol{\mu}}}(k)$. □

*Example* 4.22. Let $n\in\mathbb{Z}_{\geqslant 0}$, with $n\in\{1,\dots,m\}$, and $\underline{\boldsymbol{\mu}}\in\mathscr{E}_m$ with $n$ zeroes in positions $j_1<\cdots<j_n$.

1. Let $(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})$ be a weighted path without loop, with $\underline{\tilde{\mu}} = (\delta_k,\bar{\delta}_s,\bar{\delta}_l,\delta_s,\delta_k)$ with $k<l<s<r$. By Example 4.16, there are some classes $\underline{\tilde{\boldsymbol{\mu}}}_1$ and $\underline{\tilde{\boldsymbol{\mu}}}_2$ such that

$$\mathrm{wt}\left(\underline{\tilde{\mu}},\underline{\tilde{\alpha}}\right) = \sum_{h=1}^{2} K_{\underline{\boldsymbol{\mu}}_h} A_{\underline{\tilde{\boldsymbol{\mu}}}_h}(\tilde{\imath}_1,\dots,\tilde{\imath}_{\ell_h-1}).$$

Set $\underline{j} := (2,5)$ and let $\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j})}}$ with $\Pi_{\underline{\mu}^{(\underline{j})}} = \{\beta_{s-1}, \beta_s\} \times \{\beta_{s-1}, \beta_s\}$. Set $\underline{\boldsymbol{\mu}} := \{(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\}_{\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j})}}}$.
We have

$$\sum_{\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\{\beta_{s-1},\beta_{s-1}\}}\right) + \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\{\beta_{s-1},\beta_{s}\}}\right)$$
$$+ \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\{\beta_{s},\beta_{s-1}\}}\right) + \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\{\beta_{s},\beta_{s}\}}\right).$$

With the same arguments as Lemma 4.8, we have

$$\begin{aligned}
\mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\{\beta_{s-1},\beta_{s-1}\}}\right) &= (\langle\rho, \varpi^{\sharp}_{s-1}\rangle - 1)(-\langle\rho, \varpi^{\sharp}_{s-1}\rangle + 1)\mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}});\\
\mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\{\beta_{s-1},\beta_{s}\}}\right) &= (\langle\rho, \varpi^{\sharp}_{s-1}\rangle - 1)(\langle\rho, \varpi^{\sharp}_{s-1}\rangle)\mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}});\\
\mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\{\beta_{s},\beta_{s-1}\}}\right) &= (2 - \langle\rho, \varpi^{\sharp}_{s}\rangle)(-\langle\rho, \varpi^{\sharp}_{s-1}\rangle + 1)\mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}});\\
\mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\{\beta_{s},\beta_{s}\}}\right) &= (2 - \langle\rho, \varpi^{\sharp}_{s}\rangle)(\langle\rho, \varpi^{\sharp}_{s}\rangle)\mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}).
\end{aligned}$$

Hence by Lemma 2.3, we get

$$\begin{aligned}
\sum_{\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) &= \left(\langle\rho, \varpi^{\sharp}_{s-1} - \varpi^{\sharp}_{s}\rangle + 1\right)\left(\langle\rho, \varpi^{\sharp}_{s} - \varpi^{\sharp}_{s-1}\rangle + 1\right)\mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})\\
&= \left((s-r)(r-s+2)\right)\mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})\\
&= (r - k - \tilde{\imath}_1 + 1)\left(r - k - \sum_{t=1}^{3}\tilde{\imath}_t + 2\right)\mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})\\
&= \Big(\left((r+1)(r+2) - (2r+3)\tilde{\imath}_1 - (r+1)\tilde{\imath}_2 - (r+1)\tilde{\imath}_3 + \tilde{\imath}_1^2 + \tilde{\imath}_1\tilde{\imath}_2 + \tilde{\imath}_1\tilde{\imath}_3\right)k^0\\
&\quad + \left(-(2r+3) - 2\tilde{\imath}_1 + \tilde{\imath}_2 + \tilde{\imath}_3\right)k^1 + k^2\Big)\left(\sum_{h=1}^{2} K_{\underline{\tilde{\boldsymbol{\mu}}}_h} A_{\underline{\tilde{\boldsymbol{\mu}}}_h}(\tilde{\imath}_1, \ldots, \tilde{\imath}_{\ell_h-1})\right)\\
&= \sum_{j=0}^{2} C^{2-j}_{\underline{\boldsymbol{\mu}}}(\tilde{\imath}_1, \ldots, \tilde{\imath}_3)k^j\left(K_{\underline{\tilde{\boldsymbol{\mu}}}_1} A_{\underline{\tilde{\boldsymbol{\mu}}}_1}(\tilde{\imath}_1, \tilde{\imath}_2) + K_{\underline{\tilde{\boldsymbol{\mu}}}_2} A_{\underline{\tilde{\boldsymbol{\mu}}}_2}(\tilde{\imath}_1, \tilde{\imath}_2)\right).
\end{aligned}$$

By setting

$$B_{\underline{\boldsymbol{\mu}}}(X) := \sum_{j=0}^{2} C^{2-j}_{\underline{\boldsymbol{\mu}}}(\tilde{\imath}_1, \ldots, \tilde{\imath}_3)X^j,$$

we get that $B_{\underline{\boldsymbol{\mu}}}$ is a polynomial of degree 2 with leading term $X^2$ and the coefficient of $B_{\underline{\boldsymbol{\mu}}}(X)$ in $X^j$, $j \leqslant 2$, is a polynomial in the variable $\tilde{\imath}_1, \ldots, \tilde{\imath}_3$ of total degree $\leqslant 2-j$. Thus,

$$\sum_{\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j})}}} \mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = B_{\underline{\boldsymbol{\mu}}}(k)\left(K_{\underline{\tilde{\boldsymbol{\mu}}}_1}A_{\underline{\tilde{\boldsymbol{\mu}}}_1}(\tilde{\imath}_1,\tilde{\imath}_2)+K_{\underline{\tilde{\boldsymbol{\mu}}}_2}A_{\underline{\tilde{\boldsymbol{\mu}}}_2}(\tilde{\imath}_1,\tilde{\imath}_2)\right).$$

2. Let $(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})$ be a weighted path without loop with $\underline{\tilde{\mu}}=(\delta_k,\delta_s,\bar{\delta}_l,\bar{\delta}_j,\bar{\delta}_s,\delta_j,\delta_k)$, $1<k<s<j<l$, and $\mathrm{ht}(\underline{\mu})=(\tilde{\imath}_1,\tilde{\imath}_2,\tilde{\imath}_3,\tilde{\imath}_4,\tilde{\imath}_5,\tilde{\imath}_6)$. By Example 4.16, we have

$$\mathrm{wt}\left(\underline{\tilde{\mu}},\underline{\tilde{\alpha}}\right)=(\tilde{\imath}_4+1)A_{\underline{\tilde{\boldsymbol{\mu}}}^{\#}}(\tilde{\imath}_1,\tilde{\imath}_2,\tilde{\imath}_3)+\sum_{h=1}^{5}K_{\underline{\boldsymbol{\mu}}_h}A_{\underline{\tilde{\boldsymbol{\mu}}}_h}(\tilde{\imath}_1,\ldots,\tilde{\imath}_{\ell_h-1}),$$

where

$$\underline{\tilde{\boldsymbol{\mu}}}^{\#}=[\delta_k,\delta_s,\bar{\delta}_l,\bar{\delta}_j];\qquad \underline{\tilde{\boldsymbol{\mu}}}_1=[\delta_k,\delta_s,\delta_l,\bar{\delta}_j];\qquad \underline{\tilde{\boldsymbol{\mu}}}_2=[\delta_k,\delta_s,\bar{\delta}_j];$$
$$\underline{\tilde{\boldsymbol{\mu}}}_3=[\delta_k,\delta_s,\delta_l,\bar{\delta}_l,\bar{\delta}_j];\qquad \underline{\tilde{\boldsymbol{\mu}}}_4=[\delta_k,\delta_s,\delta_j,\delta_l,\bar{\delta}_j];\qquad \underline{\tilde{\boldsymbol{\mu}}}_5=[\delta_k,\delta_s,\delta_j,\bar{\delta}_j].$$

Observe that $(\tilde{\imath}_4+1)A_{\underline{\tilde{\boldsymbol{\mu}}}^{\#}}(\tilde{\imath}_1,\tilde{\imath}_2,\tilde{\imath}_3)=A_{[\delta_k,\delta_s,\bar{\delta}_l,\bar{\delta}_j,\bar{\delta}_i,\bar{\delta}_j]}(\tilde{\imath}_1,\ldots,\tilde{\imath}_5)$.
Set $\underline{\tilde{\boldsymbol{\mu}}}_6:=[\delta_k,\delta_s,\bar{\delta}_l,\bar{\delta}_j,\bar{\bar{\delta}}_i,\bar{\delta}_j]$, we get

$$\mathrm{wt}\left(\underline{\tilde{\mu}},\underline{\tilde{\alpha}}\right)=\sum_{h=1}^{6}K_{\underline{\tilde{\boldsymbol{\mu}}}_h}A_{\underline{\tilde{\boldsymbol{\mu}}}_h}(\tilde{\imath}_1,\ldots,\tilde{\imath}_{\ell_h-1}).$$

Set $\underline{j}:=(1,4,7)$ and let $\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j})}}$ with $\Pi_{\underline{\mu}^{(\underline{j})}}=\{\beta_{k-1},\beta_k\}\times\{\beta_{l-1},\beta_l\}\times\{\beta_{s-1},\beta_s\}$. Set $\underline{\boldsymbol{\mu}}:=\{(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\}_{\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j})}}}$.
With the same arguments as Lemma 4.8 and Lemma 4.6 and by Lemma 2.3, we get

$$\begin{aligned}
&\sum_{\underline{\beta}\in\Pi_{\underline{\mu}^{(\underline{j})}}}\mathrm{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right)\\
&=\left(\langle\rho,\varpi_k^{\sharp}-\varpi_{k-1}^{\sharp}\rangle\right)\left(\langle\rho,\varpi_{l-1}^{\sharp}-\varpi_l^{\sharp}\rangle+1\right)\left(\langle\rho,\varpi_{s-1}^{\sharp}-\varpi_s^{\sharp}\rangle+1\right)\mathrm{wt}(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})\\
&=((r-k+1)(l-r)(s-r))\,\mathrm{wt}(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})\\
&=(r-k+1)\left(r-k-\sum_{t=1}^{2}\tilde{\imath}_t+1\right)\left(r-k-\sum_{t=1}^{4}\tilde{\imath}_t+1\right)\mathrm{wt}(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})\\
&=\big(((r+1)^3+(r+1)^2(-2\tilde{\imath}_1-2\tilde{\imath}_2-\tilde{\imath}_3-\tilde{\imath}_4)\\
&\qquad+(r+1)(\tilde{\imath}_1^2+2\tilde{\imath}_1\tilde{\imath}_2+\tilde{\imath}_1\tilde{\imath}_3+\tilde{\imath}_1\tilde{\imath}_4+\tilde{\imath}_2^2+\tilde{\imath}_2\tilde{\imath}_3+\tilde{\imath}_2\tilde{\imath}_4)\big)\,k^0
\end{aligned}$$

$$
\begin{aligned}
&+ \left(-3(r+1)^2 + (r+1)(4\tilde{\imath}_1 + 4\tilde{\imath}_2 + 2\tilde{\imath}_3 + 2\tilde{\imath}_4)\right. \\
&\left(-\tilde{\imath}_1^2 - 2\tilde{\imath}_1\tilde{\imath}_2 - \tilde{\imath}_1\tilde{\imath}_3 - \tilde{\imath}_1\tilde{\imath}_4 - \tilde{\imath}_2^2 - \tilde{\imath}_2\tilde{\imath}_4 - \tilde{\imath}_2\tilde{\imath}_4\right) k^1 \\
&\left. + \left(3(r+1) - 2\tilde{\imath}_1 - 2\tilde{\imath}_2 - \tilde{\imath}_3 - \tilde{\imath}_4\right) k^2 + (-1)k^3\right) \\
= &\sum_{j=0}^{3} C_{\underline{\boldsymbol{\mu}}}^{3-j}(\tilde{\imath}_1, \ldots, \tilde{\imath}_4) k^j \sum_{h=1}^{6} K_{\underline{\tilde{\boldsymbol{\mu}}}_h} A_{\underline{\tilde{\boldsymbol{\mu}}}_h}(\tilde{\imath}_1, \ldots, \tilde{\imath}_{\ell_h - 1})),
\end{aligned}
$$

where the $K_{\underline{\tilde{\boldsymbol{\mu}}}_h}$'s are some constants. By setting

$$
B_{\underline{\boldsymbol{\mu}}}(X) := \sum_{j=0}^{3} C_{\underline{\boldsymbol{\mu}}}^{3-j}(\tilde{\imath}_1, \ldots, \tilde{\imath}_4) X^j,
$$

we get that $B_{\underline{\boldsymbol{\mu}}}$ is a polynomial of degree 3 with leading term $(-1)X^3$ and the coefficient of $B_{\underline{\boldsymbol{\mu}}}(X)$ in $X^j$, $j \leqslant 3$, is a polynomial in the variable $\tilde{\imath}_1, \ldots, \tilde{\imath}_4$ of total degree $\leqslant 3 - j$. Thus,

$$
\sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j})}}} \operatorname{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = B_{\underline{\boldsymbol{\mu}}}(k) \left(\sum_{h=1}^{6} K_{\underline{\tilde{\boldsymbol{\mu}}}_h} A_{\underline{\tilde{\boldsymbol{\mu}}}_h}(\tilde{\imath}_1, \ldots, \tilde{\imath}_{\ell_h - 1})\right).
$$

3. Let $r = 4$ and $(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})$ be a weighted path without loop with $\underline{\tilde{\mu}} = (\delta_2, \bar{\delta}_4, \delta_2)$. Set $\underline{j} := (2)$ and let $\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j})}}$ with $\Pi_{\underline{\mu}^{(\underline{j})}} = \{\beta_3, \beta_4\}$.
We have

$$
\sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{(\underline{j})}}} \operatorname{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = \operatorname{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{(2);\{\beta_3\}}\right) + \operatorname{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{(2);\{\beta_4\}}\right).
$$

Recall for $k \in \{1, \ldots, r\}$,

$$
\langle \rho, \varpi_k^\sharp \rangle = \frac{k}{2}(2r - k + 1).
$$

Thus,

$$
\begin{aligned}
\operatorname{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{(2);\{\beta_3\}}\right) &= \langle \bar{\delta}_4, \check{\beta}_3 \rangle (\langle \rho, \varpi_3^\sharp \rangle - \langle \alpha^{(1)}, \varpi_3^\sharp \rangle) \operatorname{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}); \\
&= (\langle \rho, \varpi_3^\sharp \rangle - 1) \operatorname{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) \\
&= \left((\frac{3}{2}(2 \times 4 - 3 + 1)) - 1\right) \operatorname{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) = 8 \operatorname{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}); \\
\operatorname{wt}\left((\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{(2);\{\beta_4\}}\right) &= \langle \bar{\delta}_4, \check{\beta}_4 \rangle (\langle \rho, \varpi_4^\sharp \rangle - \langle \alpha^{(1)}, \varpi_4^\sharp \rangle) \operatorname{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}});
\end{aligned}
$$

$$= -(\langle \rho, \varpi_4^\sharp \rangle - 2)\mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})$$
$$= \left( (\frac{4}{2}(2 \times 4 - 4 + 1) - 2) \right) \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}) = -8 \ \mathrm{wt}(\underline{\tilde{\mu}}, \underline{\tilde{\alpha}}).$$

Hence,

$$\sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{(j)}}} \mathrm{wt}\left( (\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}} \right) = 0.$$

**Corollary 4.23.** *Let $m \in \mathbb{Z}_{>0}$ and $n \in \{0, \dots, m\}$. Let $\underline{\boldsymbol{\mu}} \in \mathscr{E}_m$ with $n \leqslant m$ zeroes in positions $j_1, \dots, j_n$, and $\underline{\tilde{\boldsymbol{\mu}}}$ as in Lemma 4.21. Then for some polynomial $T_{\underline{\boldsymbol{\mu}}} \in \mathbb{C}[X]$ of degree at most $m$, for all $k \in \{1, \dots, r\}$,*

$$\sum_{(\underline{\tilde{\mu}},\underline{\tilde{\alpha}}) \in \underline{\tilde{\boldsymbol{\mu}}}} \sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{\underline{j}}}} \sum_{\substack{(\underline{\tilde{\mu}},\underline{\tilde{\alpha}}) \in \hat{\mathscr{P}}_{m-n}(\delta_k), \\ \underline{\tilde{\mu}} \in [\![\bar{\delta}_1, \delta_k]\!], \, (\underline{\tilde{\mu}},\underline{\tilde{\alpha}}) \in \underline{\tilde{\boldsymbol{\mu}}}}} \mathrm{wt}\left( (\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}} \right) = T_{\underline{\boldsymbol{\mu}}}(k).$$

If $n = 0$ has no zero, then $T_{\underline{\boldsymbol{\mu}}}$ is the polynomial provided by Lemma 4.20. If $n = m$, then $\mathrm{ht}(\underline{\boldsymbol{\mu}}) = (\underline{0})$ and $T_{\underline{\boldsymbol{\mu}}} = T_m$ is the polynomial provided by Lemma 2.4. So, in these two cases, the statement is known. Also, our notations is compatible with the notation of what follows Lemma 4.20.

*Proof.* Let $k \in \{1, \dots, r\}$. By Lemma 4.15, Remark 4.17, Lemma 4.20 and Lemma 4.21, we have,

$$\sum_{(\underline{\tilde{\mu}},\underline{\tilde{\alpha}}) \in \underline{\tilde{\boldsymbol{\mu}}}} \sum_{\underline{\beta} \in \Pi_{\underline{\mu}^{\underline{j}}}} \sum_{\substack{(\underline{\tilde{\mu}},\underline{\tilde{\alpha}}) \in \hat{\mathscr{P}}_{m-n}(\delta_k), \\ \underline{\tilde{\mu}} \in [\![\bar{\delta}_1, \delta_k]\!], \, (\underline{\tilde{\mu}},\underline{\tilde{\alpha}}) \in \underline{\tilde{\boldsymbol{\mu}}}}} \mathrm{wt}\left( (\underline{\tilde{\mu}}, \underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}} \right)$$
$$= \sum_{\substack{(\underline{\tilde{\mu}},\underline{\tilde{\alpha}}) \in \hat{\mathscr{P}}_{m-n}(\delta_k), \\ \underline{\tilde{\mu}} \in [\![\bar{\delta}_1, \delta_k]\!], \, (\underline{\tilde{\mu}},\underline{\tilde{\alpha}}) \in \underline{\tilde{\boldsymbol{\mu}}}}} B_{\underline{\boldsymbol{\mu}}}(k) \sum_{h=1}^{N} K_{\underline{\tilde{\boldsymbol{\mu}}}_h} A_{\underline{\tilde{\boldsymbol{\mu}}}_h}(\tilde{\imath}_1, \dots, \tilde{\imath}_{\ell_h - 1})$$
$$= \sum_{\substack{(\underline{\tilde{\mu}},\underline{\tilde{\alpha}}) \in \hat{\mathscr{P}}_{m-n}(\delta_k), \\ \underline{\tilde{\mu}} \in [\![\bar{\delta}_1, \delta_k]\!], \, (\underline{\tilde{\mu}},\underline{\tilde{\alpha}}) \in \underline{\tilde{\boldsymbol{\mu}}}}} \sum_{j=0}^{n} C_{\underline{\boldsymbol{\mu}}}^{(n-j)}(\tilde{\imath}_1, \dots, \tilde{\imath}_{m-n-1}) k^j \sum_{h=1}^{N} K_{\underline{\tilde{\mu}}_h} A_{\underline{\tilde{\mu}}_h}(\tilde{\imath}_1, \dots, \tilde{\imath}_{\ell_h - 1})$$
$$= \sum_{\substack{(\underline{\tilde{\mu}},\underline{\tilde{\alpha}}) \in \hat{\mathscr{P}}_{m-n}(\delta_k), \\ \underline{\tilde{\mu}} \in [\![\bar{\delta}_1, \delta_k]\!], \, (\underline{\tilde{\mu}},\underline{\tilde{\alpha}}) \in \underline{\tilde{\boldsymbol{\mu}}}}} \sum_{j=0}^{n} \sum_{\substack{\underline{d}_j = (d_1, \dots, d_{m-n-1}), \\ d_1 + \dots + d_{m-n-1} \leq n-j}} C_{\underline{d},j} \tilde{\imath}_1^{d_1} \cdots \tilde{\imath}_{m-n-1}^{d_{m-n-1}} k^j A_{\underline{\tilde{\mu}}}(\tilde{\imath}_1, \dots, \tilde{\imath}_{m-n-1}).$$

Set

$$\tilde{T}_{\underline{d}_j,\underline{\tilde{\boldsymbol{\mu}}}} = \sum_{\substack{(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})\in\hat{\mathscr{P}}_{m-n}(\delta_k),\\ \underline{\tilde{\mu}}\in[\![\bar{\delta}_1,\delta_k]\!],\,(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})\in\underline{\tilde{\boldsymbol{\mu}}}}} \tilde{\imath}_1^{d_1}\cdots\tilde{\imath}_{m-n-1}^{d_{m-n-1}} A_{\underline{\tilde{\boldsymbol{\mu}}}}(\tilde{\imath}_1,\ldots,\tilde{\imath}_{m-n-1}).$$

Then by Lemma 4.20, there are some polynomials $T_{\underline{d}_j,\underline{\tilde{\boldsymbol{\mu}}}}$ of degree at most $(n-j)+(m-n)=m-j$, such that

$$\sum_{(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})\in\underline{\tilde{\boldsymbol{\mu}}}}\ \sum_{\beta\in\Pi^j_{\underline{\mu}}}\ \sum_{\substack{(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})\in\hat{\mathscr{P}}_{m-n}(\delta_k),\\ \underline{\tilde{\mu}}\in[\![\bar{\delta}_1,\delta_k]\!],\,(\underline{\tilde{\mu}},\underline{\tilde{\alpha}})\in\underline{\tilde{\mu}}}} \operatorname{wt}\left((\underline{\tilde{\mu}},\underline{\tilde{\alpha}})_{\underline{j};\underline{\beta}}\right) = \sum_{j=0}^{n}\ \sum_{\substack{\underline{d}_j=(d_1,\ldots,d_{m-n-1}),\\ d_1+\cdots+d_{m-n-1}\le n-j}} C_{\underline{d},j}k^jT_{\underline{d}_j,\underline{\tilde{\boldsymbol{\mu}}}}(k).$$

Moreover, if $j<n$, then $T_{\underline{d}_j,\underline{\tilde{\boldsymbol{\mu}}}}$ has degree $<m-j$. By setting

$$T_{\underline{\boldsymbol{\mu}}}(X) := \sum_{j=0}^{n}\ \sum_{\substack{\underline{d}_j=(d_1,\ldots,d_{m-n-1}),\\ d_1+\cdots+d_{m-n-1}\le n-j}} C_{\underline{d},j}X^jT_{\underline{d}_j,\underline{\tilde{\boldsymbol{\mu}}}}(X),$$

we have that $T_{\underline{\boldsymbol{\mu}}}$ is a polynomial of degree at most $m-j+j=m$. □

## 4.3 Proof of Theorem 7

We first prove Lemma 4.10 as a consequence of Corollary 4.23.

*Proof of Lemma 4.10.* According to Theorem 4.1, the paths starting from $\bar{\delta}_k$ have weights entirely contained in $[\![\bar{\delta}_1,\bar{\delta}_k]\!]$. So the sum

$$\sum_{\substack{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\bar{\delta}_k)\\ \underline{\mu}\in[\![\bar{\delta}_1,\bar{\delta}_k]\!]}} \operatorname{wt}(\underline{\mu},\underline{\alpha})$$

can be computed exactly as in the $\mathfrak{sl}_{r+1}$ case, and the result are known by Lemma 3.19. Hence there is a polynomial $\hat{T}_{2,m}$ of degree at most $m$ such that for all $k\in\{1,\ldots,r\}$,

$$\sum_{\substack{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\bar{\delta}_k)\\ \underline{\mu}\in[\![\bar{\delta}_1,\bar{\delta}_k]\!]}} \operatorname{wt}(\underline{\mu},\underline{\alpha}) = \hat{T}_{2,m}(k).$$

So it remains to consider the paths starting from $\delta_k$ and contained in $[\![\bar{\delta}_1,\delta_k]\!]$.

By Corollary 4.23, we have for all $k \in \{1, \ldots, r\}$,

$$\sum_{\substack{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\delta_k)\\ \underline{\mu}\in[\![\bar{\delta}_1,\delta_k]\!]}} \mathrm{wt}(\underline{\mu},\underline{\alpha}) = \sum_{\underline{\boldsymbol{\mu}}\in\mathscr{E}_m} \sum_{\substack{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\delta_k),\\ \underline{\mu}\in[\![\bar{\delta}_1,\delta_k]\!],\ (\underline{\mu},\underline{\alpha})\in\underline{\boldsymbol{\mu}}}} \mathrm{wt}(\underline{\mu},\underline{\alpha}) = \sum_{\underline{\boldsymbol{\mu}}\in\mathscr{E}_m} T_{\underline{\boldsymbol{\mu}}}(k).$$

Set

$$\hat{T}_{1,m} := \sum_{\underline{\boldsymbol{\mu}}\in\mathscr{E}_m} T_{\underline{\boldsymbol{\mu}}} \in \mathbb{C}[X].$$

By Corollary 4.23, $T_{\underline{\boldsymbol{\mu}}}$ has degree at most $m$. Therefore $\hat{T}_{1,m}$ has degree at most $m$ and satisfies the condition of the lemma. □

We are now in a position to prove Theorem 7

*Proof of Theorem 7.* By Lemma 1.19, we have

$$\mathrm{ev}_\rho(\widehat{\mathrm{d}p}_{m,k}) = \sum_{\mu\in P(\delta)_k} \sum_{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\mu)} \mathrm{wt}(\underline{\mu},\underline{\alpha})\langle\mu,\check{\beta}_k\rangle.$$

Remember from Section 2.3 that

$$P(\delta)_k = \{\delta_k, \delta_{k+1}, \bar{\delta}_k, \bar{\delta}_{k+1}\}, \quad k = 1, \ldots, r-1, \quad \text{and} \quad P(\delta)_r = \{\delta_r, \bar{\delta}_r\}.$$

Hence, for $k = 1, \ldots, r-1$

$$\begin{aligned}\mathrm{ev}_\rho(\widehat{\mathrm{d}p}_{m,k}) = &\sum_{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\delta_k)} \mathrm{wt}(\underline{\mu},\underline{\alpha}) - \sum_{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\delta_{k+1})} \mathrm{wt}(\underline{\mu},\underline{\alpha}) \\ &+ \sum_{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\bar{\delta}_{k+1})} \mathrm{wt}(\underline{\mu},\underline{\alpha}) - \sum_{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\bar{\delta}_k)} \mathrm{wt}(\underline{\mu},\underline{\alpha}),\end{aligned}$$

and

$$\mathrm{ev}_\rho(\widehat{\mathrm{d}p}_{m,r}) = \sum_{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\delta_r)} \mathrm{wt}(\underline{\mu},\underline{\alpha}) - \sum_{(\underline{\mu},\underline{\alpha})\in\hat{\mathscr{P}}_m(\bar{\delta}_r)} \mathrm{wt}(\underline{\mu},\underline{\alpha}).$$

Let $\hat{T}_{1,m}$ and $\hat{T}_{2,m}$ be as in Lemma 4.10 and set

$$\hat{Q}_m := \hat{T}_{1,m}(X) - \hat{T}_{1,m}(X+1) + \hat{T}_{2,m}(X+1) - \hat{T}_{2,m}(X) \tag{4.19}$$

(Note that $\hat{T}_{1,m}(r+1) = \hat{T}_{2,m}(r+1) = 0$). Then $\hat{Q}_m$ is a polynomial of degree at most $m-1$, and we have

$$\operatorname{ev}_\rho(\widehat{\mathrm{d}p_m}) = \operatorname{ev}_\rho\left(\frac{1}{m!}\sum_{k=1}^{r}\widehat{\mathrm{d}p}_{m,k}\otimes\varpi_k^\sharp\right) = \frac{1}{m!}\sum_{k=1}^{r}\operatorname{ev}_\rho\left(\widehat{\mathrm{d}p}_{m,k}\right)\varpi_k^\sharp = \frac{1}{m!}\sum_{k=1}^{r}\hat{Q}_m\varpi_k^\sharp.$$

Moreover, $\hat{Q}_1 = 2$. □

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# Index of notations

# Index

**Résumé :** il existe plusieurs filtrations intéressantes définies sur la sous-algèbre de Cartan d'une algèbre de Lie simple complexe issues de contextes très variés : l'une est la filtration principale qui provient du dual de Langlands, une autre provient de l'algèbre de Clifford associée à une forme bilinéaire invariante non-dégénérée, une autre encore provient de l'algèbre symétrique et la projection de Chevalley, deux autres enfin proviennent de l'algèbre enveloppante et des projections de Harish-Chandra. Il est connu que toutes ces filtrations coïncident. Ceci résulte des travaux de Rohr, Joseph et Alekseev-Moreau. La relation remarquable entre les filtrations principale et de Clifford fut essentiellement conjecturée par Kostant. L'objectif de ce mémoire de thèse est de proposer une nouvelle démonstration de l'égalité entre les filtrations symétrique et enveloppante pour une algèbre de Lie simple de type $A$ ou $C$. Conjointement au résultat et Rohr et le théorème d'Alekseev-Moreau, ceci fournit une nouvelle démonstration de la conjecture de Kostant, c'est-à-dire une nouvelle démonstration du théorème de Joseph. Notre démonstration est très différente de la sienne. Le point clé est d'utiliser une description explicite des invariants via la représentation standard, ce qui est possible en types $A$ et $C$. Nous décrivons alors les images de leurs différentielles en termes d'objects combinatoires, appelés *des chemins pondérés*, dans le graphe cristallin de la représentation standard. Les démonstrations pour les types $A$ et $C$ sont assez similaires, mais de nouveaux phénomènes apparaissent en type $C$, ce qui rend la démonstration nettement plus délicate dans ce cas.
Mots-clé : filtration principale, projections de Chevalley et de Harish-Chandra, chemins pondérés.

**Abstract :** There are several interesting filtrations on the Cartan subalgebra of a complex simple Lie algebra coming from very different contexts: one is *the principal filtration* coming from the Langlands dual, one is coming from the Clifford algebra associated with a non-degenerate invariant bilinear form, one is coming from the symmetric algebra and the Chevalley projection, and two other ones are coming from the enveloping algebra and Harish-Chandra projections. It is known that all these filtrations coincide. This results from a combination of works of several authors (Rohr, Joseph, Alekseev-Moreau). The remarkable connection between the principal filtration and the Clifford filtration was essentially conjectured by Kostant. The purpose of this thesis is to establish a new correspondence between the enveloping filtration and the symmetric filtration for a simple Lie algebra of type $A$ or $C$. Together with Rohr's result and Alekseev-Moreau theorem, this provides another proof of Kostant's conjecture for these types, that is, a new proof of Joseph's theorem. Our proof is very different from his approach. The starting point is to use an explicit description of invariants via the standard representation which is possible in types $A$ and $C$. Then we describe the images of their differentials by the *generalised Chevalley and Harish-Chandra projections* in term of combinatorial objects, called *weighted paths*, in the crystal graph of the standard representation. The proofs for types $A$ and $C$ are quite similar, but there are new phenomenons in type $C$ which makes the proof much more tricky in this case.
Keywords : principal filtration, Chevalley and Harish-Chandra projections, weighted paths.