A SYLLABUS OF THE WORKSHOP ON W-ALGEBRAS

MELBOURNE, NOVEMBER 28 - DECEMBER 2, 2016

Part 1. General presentation of the workshop

The goal of this workshop is to introduce the theory of vertex algebras and affine W-algebras, which are certain vertex algebras, with emphasis on their geometrical aspects.

Overview of theory, and goals of the workshop

Roughly speaking, a vertex algebra is a vector space V, endowed with a distinguished vector, the vacuum vector, and the vertex operator map from V to the space of formal Laurent series with linear operators on V as coefficients. These data satisfy a number of axioms and have some fundamental properties as, for example, an analogue to the Jacobi identity, locality and associativity. Although the definition is purely algebraic, the above axioms have deep geometric meaning. They reflect the fact that vertex algebras give an algebraic framework of the two-dimensional conformal field theory. The connections of this topic with other branches of mathematics and physics include algebraic geometry (moduli spaces), representation theory (modular representation theory, geometric Langlands correspondence), two dimensional conformal field theory, string theory (mirror symmetry) and four dimensional gauge theory (AGT conjecture).

The (affine) W-algebras were first introduced by Zamolodchikov in the 1980s in physics and then developed by Fateev-Lukyanov, Feigin-Frenkel, Kac-Roan-Wakimoto and others. The finite W-algebras, the finite dimensional analogs of W-algebras, were introduced by Premet. They go back to Kostant's works in the 1970s who studied some particular cases. The W-algebras were extensively studied by physicists in 1990s in connection with two dimensional conformal field theory. More recently, the appearance of the AGT conjecture in physics led many researchers in mathematics towards W-algebras. In the meantime, the finite W-algebras have caught attention for different reasons that are mostly related with more classical problems of representation theory.

The nicest vertex algebras are those which are both rational and lisse (or C_2-cofinite). The rationality means the completely irreducibility of modules. The lisse condition is a certain finiteness condition as explained next paragraph. If a vertex algebra V is rational and lisse, then it gives rise to a rational conformal field theory. In particular, the characters of simple V-modules form vector valued modular functions, and moreover, the category of V-modules forms a modular tensor category, so that one can associate with it an invariant of knots.

To each vertex algebra V one can naturally attach a certain Poisson variety X_V called the associated variety of V. A vertex algebra V is called lisse if $\dim X_V = 0$. Lisse vertex algebras are natural generalizations of finite-dimensional algebras and possess remarkable properties. For instance, the modular invariance of characters still holds without the rationality assumption.

In fact the geometry of the associated variety often reflects some algebraic properties of the vertex algebras V. More generally, vertex algebras whose associated variety has only finitely symplectic leaves, are also of great interest for several reasons that will be addressed in the workshop.

Important examples of vertex algebras are those coming from affine Kac-Moody algebra, which are called affine vertex algebras. They play a crucial role in the representation theory of affine Kac-Moody algebras, and of W-algebras. In the case that V is a simple affine vertex algebra, its associated variety is an invariant and conic subvariety of the corresponding simple Lie algebra. It plays an analog role to the associated variety of primitive ideals of the enveloping algebra of simple Lie algebras. However, associated varieties of affine vertex algebras are not necessarily contained in the nilpotent cone and it is difficult to describe them in general.

In fact, although associated varieties seem to be significant also in connection with the recent study of four dimensional superconformal field theory, their general description is fairly open, except in a few cases.

The affine W-algebras are certain vertex algebras associated with nilpotent elements of simple Lie algebras. They can be regarded as affinizations of finite W-algebras, and can also be considered as generalizations of affine Kac-Moody algebras and Virasoro algebras. They quantize the arc space of the Slodowy slices associated with nilpotent elements. The study of affine W-algebras began with the work of Zamolodchikov in 1985. Mathematically, affine W-algebras are defined by the method of quantized Drinfeld-Sokolov reduction that was discovered by Feigin and Frenkel in the 1990s. The general definition of affine W-algebras were given by Kac, Roan and Wakimoto in 2003. Affine W-algebras are related with integrable systems, the two-dimensional conformal field theory and the geometric Langlands program. The most recent developments in representation theory of affine W-algebras were done by Kac-Wakimoto and Arakawa.

Since they are not finitely generated by Lie algebras, the formalism of vertex algebras is necessary to study them. The study of affine W-algebras will be the ultimate goal of the workshop. In this context, associated varieties of W-algebras, and their quotients, are important tools to understand some properties, such as the lisse condition and even the rationality condition.

It is only quite recently that the study of associated varieties of vertex algebras and their arc spaces, has been more intensively developed. In this workshop we wish to highlight this aspect of the theory of vertex algebras which seems to be very promising. In particular, the workshop will include open problems on associated varieties of W-algebras raised by recent works of Tomoyuki Arakawa and Anne Moreau.

Overview of lectures

One of the first interesting examples of non-commutative vertex algebras are the affine vertex algebras associated with affine Kac-Moody algebras which play a crucial role in the representation theory of affine Kac-Moody algebras, and of W-algebras. For this reason the workshop will start with an introduction to affine Kac-Moody algebras and their representations (Lectures 1).

We will introduce the notion of vertex algebras in Lectures 2, and discuss some important related objects as Zhu's C_2 algebras, Zhu's algebras and Zhu's functors. The Zhu's functor gives a correspondence between the theory of modules over a vertex algebra and the representation theory of its Zhu's algebra. This correspondence is particularly well-understood in the case of the universal affine vertex algebras, where Zhu's algebras are enveloping algebras of the corresponding finite-dimensional simple Lie algebras.

The W-algebras are certain vertex algebras associated with nilpotent elements of a simple Lie algebra. Zhu's algebras of W-algebras are finite W-algebras. The later are certain generalizations of the enveloping algebra of a simple Lie algebra. They can be defined through the BRST cohomology associated with nilpotent elements. So the definition and properties of (finite and affine) W-algebras are deeply related to the geometry of of nilpotent orbits. We will explain in Lectures 3 the definition of finite W-algebras by BRST reduction (= a form of quantized Hamiltonian reduction) after outlining basics on nilpotent orbits.

Any vertex algebra is naturally filtered and the corresponding graded algebra is a Poisson vertex algebra. Moreover, the spectrum of the Zhu's C_2 algebra, which is a generating ring of this graded algebra, is what we call the associated variety. Its geometry gives important information on the vertex algebra as we wish to illustrate in this workshop. A nice way to construct Poisson vertex algebras is to consider the coordinate ring of the arc space of a Poisson variety. Actually, strong relations exists, at least conjecturally, between the arc space of the associated variety and the singular support of a vertex algebra, that is, the spectrum of the corresponding graded algebra. All these aspects will be discussed in Lectures 4.

The last series of lectures (Lectures 5) will be about affine W-algebras. They are defined by a certain BRST reduction, called the quantum Drinfeld-Sokolov reduction, associated with nilpotent elements. Rational W-algebras and lisse W-algebras are particularly interesting classes of W-algebras. The rationality and the lisse conditions, and some other properties will be considered. Associated varieties of affine W-algebras, and their quotients, will be also discussed.

The lectures will include exercises and open problems sessions.

Part 2. **Description of the lectures**

The workshop will be roughly divided into five series of lectures that we describe below. Each of these series of lectures will consist in several talks given either by participants, or by one of the mentors, Tomoyuki Arakawa and Anne Moreau.

A lecture note which could serve as a working base for the participants of the workshop is in preparation.

1. Lectures "Introduction to affine Lie algebras, and their representations"

Content. Rapid review on semi-simple Lie algebras, and their corresponding combinatorics. Affine Kac-Moody algebras. Representations of affine Kac-Moody algebras, category $\mathcal{O}$. Integrable and admissible representations.

Exercises sessions.

Organization and speakers. These series of lectures can be given by participants.

▶ These lectures are expected on the first day, after an overview of theory and a presentation of the goals of the workshop.

References

[Hernandez-lectures] D. Hernandez. An introduction to affine Kac-Moody algebras. Preprint http://www.ctqm.au.dk/research/MCS/Hernandeznotes.pdf, 2006.

[Kac] V. G. Kac. Infinite-dimensional Lie algebras. Third edition. Cambridge University Press, Cambridge, 1990.

[Humphreys] James Humphreys. Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$. *Graduate Studies in Mathematics*, 94. American Mathematical Society, Providence, RI, 2008.

[Moody-Pianzola] R. Moody and A. Pianzola. Lie algebras with triangular decompositions. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1995.

[Carter] R. W. Carter. Lie algebras of finite and affine type. Cambridge Studies in Advanced Mathematics, 96. Cambridge University Press, Cambridge, 2005.

[Tauvel-Yu] P. Tauvel and R. W. T. Lie algebras and algebraic groups. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.

[Kac-Wakimoto08] Victor G. Kac and Minoru Wakimoto. On rationality of W-algebras. *Transform. Groups*, 13(3-4):671–713, 2008.

Comments on the references. Kac's book [Kac] is a reference in this topic.

We think that David Hernandez's lectures [Hernandez-lectures] can be a good reference to start learning on the topic.

Humphreys' book [Humphreys] is good to learn ring-theoretic properties of category $\mathcal{O}$, but it deals only with the finite-dimensional Lie algebras. For affine Kac-Moody algebras setting, we refer to Moody-Pianzola's book [Moody-Pianzola].

Carter's book [Carter] is probably a bit thick, but the advantage is that its first part covers all backgrounds on finite Lie algebras. Also, the book contains many data, explicit examples and detailed proofs.

Tauvel-Yu's book [Tauvel-Yu] does not deal with affine Lie algebras but we feel it can be a good reference throughout the workshop since it contains almost all necessary basics on algebraic geometry, algebraic groups, simple Lie algebras and nilpotent orbits in simple Lie algebras.

Admissible representations are important for affine W-algebras. The definition is not available in the above references. For the definition of admissible representations we refer to [Kac-Wakimoto08].

2. Lectures "Vertex algebras and Zhu functors, the canonical filtration and Zhu's C_2-algebras"

Content. Definition of vertex algebras. Borcherds identity and other properties. First examples of vertex algebras: commutative vertex algebras, the Virasoro vertex algebra, affine vertex algebras (associated with affine Kac-Moody algebras), etc. Modules over vertex algebras. The canonical filtration, Zhu's C_2 algebras, Zhu's algebras and Zhu's functors. Connections with the representations of affine Kac-Moody algebras.

Exercises session and open problems session.

Organization and speakers. Most of these lectures can be given by participants. Perhaps it may be better that the last part (about Zhu's functors and applications) will be given by one of the mentors, Tomoyuki Arakawa or Anne Moreau.

▶ These lectures are expected on the first and second days.

References

[Kac] V. G. Kac. Vertex algebras for beginners. Second edition. University Lecture Series, 10. American Mathematical Society, Providence, RI, 1998.

[Frenkel-BenZvi] E. Frenkel and D. Ben-Zvi. Vertex algebras and algebraic curves. Mathematical Surveys and Monographs, 88. American Mathematical Society, Providence, RI, 2001.

[Kac-lectures] V. G. Kac. Introduction to vertex algebras, Poisson vertex algebras, and integrable Hamiltonian PDE. Preprint `http://arxiv.org/pdf/1512.00821v1.pdf`.

[Arakawa-lectures] T. Arakawa, Introduction to W-algebras and their representation theory. Preprint `http://arxiv.org/pdf/1605.00138v1.pdf`.

[Arakawa12] Tomoyuki Arakawa. A remark on the C_2 cofiniteness condition on vertex algebras. *Math. Z.*, 270(1-2):559–575, 2012.

Comments on the references. The main references on the topic are [Kac] and [Frenkel-BenZvi]. More precisely, for the definition of vertex algebras, first examples and modules over vertex algebras, we refer to Chapters 1–4 of [Frenkel-BenZvi] (or [Kac]). We also advise the lectures notes by Victor Kac [Kac-lectures] for a first approach.

Concerning the canonical filtration, Zhu's C_2 algebras, Zhu's algebras and Zhu's functors, and connections to the representations of affine Kac-Moody algebras, we refer to the lectures notes by Tomoyuki Arakawa [Arakawa-lectures], and reference therein (in particular, see [Arakawa12] for further details).

3. Lectures "BRST cohomology, quantum Hamiltonian reduction, geometry of nilpotent orbits and finite W-algebras"

Content.
1. Review on nilpotent elements in a simple Lie algebra (Jacobson-Morosov Theorem and $\mathfrak{sl}_2$-triples, weighted Dynkin diagrams) and nilpotent orbits (Chevalley order, Hasse diagram, regular, sub-regular, minimal nilpotent orbits, etc.).
2. Poisson algebras, Poisson varieties and Hamiltonian reduction. Poisson structure on Slodowy slices associated with $\mathfrak{sl}_2$-triples.
3. BRST cohomology, quantum Hamiltonian reduction and definition of finite W-algebras. Primitive ideals and representations of finite W-algebras.
4. **Advanced topics**. If time allows, we will also discuss on some aspects of the singularities of nilpotent orbit closures (such as normality questions, branchings and nilpotent Slodowy slices) that can be useful for the open problems session.

Exercises session and open problems session.

Organization and speakers. The part concerning nilpotent elements and nilpotent orbits, and maybe the part concerning Poisson algebras, Poisson varieties and Hamiltonian reduction, can be given by participants.

Lacking references on the topic, the part concerning BRST cohomology, quantum Hamiltonian reduction and corresponding definition of W-algebras will be given by Anne Moreau. This part will include discussions on primitive ideals and representations of finite W-algebras.

If we have the time for advanced topics on the singularities of nilpotent orbit closures, the lecture will be given by Anne Moreau.

► These lectures are expected on the second and third day.

References

[Collingwood-McGovern] D. Collingwood and W.M. McGovern. Nilpotent orbits in semisimple Lie algebras. Van Nostrand Reinhold Co. New York, 65, 1993.

[Tauvel-Yu] P. Tauvel and R. W. T. Lie algebras and algebraic groups. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.

[Wang-lectures] W. Wang. Nilpotent orbits and finite W-algebras. Geometric representation theory and extended affine Lie algebras, 71–105, Fields Inst. Commun., 59, *Amer. Math. Soc.*, Providence, RI, 2011.

[Moreau-lectures] A. Moreau, Nilpotent orbits and finite W-algebras. Preprint http://www-math.sp2mi.univ-poitiers.fr/~amoreau/Kent2014-Walg.pdf.

[Vaisman] I. Vaisman. Lectures on the geometry of Poisson manifolds. Progress in Mathematics, 118, Birkhäuser Verlag, Basel, 1994.

[Chriss-Ginzburg] N. Chriss and V. Ginzburg. Representation theory and complex geometry. Reprint of the 1997 edition. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2010.

[LaurentGengoux-Pichereau-Vanhaecke] C. Laurent-Gengoux, A. Pichereau and P. Vanhaecke. Poissonstructures. Springer,Heidelberg, 347, 2013.

[Arakawa-Moreau-lectures] T. Arakawa and A. Moreau. Lectures on W-algebras (workshop in Melbourne). In preparation.

[Arakawa-lectures] T. Arakawa, Introduction to W-algebras and their representation theory. Preprint http://arxiv.org/pdf/1605.00138v1.pdf.

[DeSole-Kac06] A. De Sole and V. G. Kac. Finite vs affine W-algebras. *Jpn. J. Math.* 1(1):137–261, 2006.

[Brundan-Goodwin-Kleshchev08] J. Brundan, S. Goodwin and A. Kleshchev, Highest weight theory for finite W-algebras. *Int. Math. Res. Not.* 15, 2008.

[Arakawa15b] Tomoyuki Arakawa. Rationality of W-algebras: principal nilpotent cases. *Ann. Math.*, 182(2):565–694, 2015.

[Brundan-Kleshchev-memoirs] J. Brundan and A. Kleshchev. Representations of shifted Yangians and finite W-algebras. Mem. Amer. Math. Soc. 196 (2008), 107 pp.

[Losev10] I. Losev. Finite W-algebras. Proceedings of the International Congress of Mathematicians. Volume III, 1281–1307, Hindustan Book Agency, New Delhi, 2010, 16–02.

[Henderson15] A. Henderson. Singularities of nilpotent orbit closures. *Rev. Roumaine Math. Pures Appl.* 60(4):441–469, 2015.

[Juteau-et-al15] B. Fu, D. Juteau, P. Levy and E. Sommers. Generic singularities of nilpotent orbit closures. arXiv:1502.05770[math.RT], to appear in *Adv. Math.*.

[Arakawa-Moreau16] T. Arakawa and A. Moreau. On the irreducibility of associated varieties of W-algebras. Preprint http://arxiv.org/pdf/1608.03142.pdf.

Comments on the references. Our references for nilpotent elements and nilpotent orbits in simple Lie algebras are [Collingwood-McGovern, Tauvel-Yu]. We also recommend the lecture notes by Wang [Wang-lectures] and Anne Moreau [Moreau-lectures] (which are both written in order to introduce finite W-algebras).

Concerning symplectic geometry, Poisson algebras and Poisson varieties, a standard reference is [Vaisman]; see [LaurentGengoux-Pichereau-Vanhaecke] for a more recent book on the topic. Chapter 1 of [Chriss-Ginzburg] is also good for an introduction (it does not cover Hamiltonian reduction). In order to prepare the lecture, and to explain the Poisson structure on Slodowy slices, we advise to read [Moreau-lectures, Section 9], and references therein.

A reference on BRST cohomology and the corresponding definition of finite W-algebras is [Arakawa-lectures], but it deals with only regular nilpotent element cases in type A. We hope that [Arakawa-Moreau-lectures] will be a better reference. In the meantime, we refer to [DeSole-Kac06, Brundan-Goodwin-Kleshchev08, Arakawa15b]. For the representation theory of finite W-algebras, we refer to [Brundan-Kleshchev-memoirs] and [Losev10].

Possible references for the singularities of nilpotent orbit closures (advanced topics) are [Henderson15, Juteau-et-al15, Arakawa-Moreau16], and references therein.

4. Lectures "Geometry of jet schemes, Poisson vertex algebras and associated varieties of vertex algebras"

Content.
1. Definitions of jet schemes and arc spaces, geometrical aspects. The coordinate ring of the arc space of an algebraic variety is naturally a commutative vertex algebra: description of the vertex algebra structure in this case.
1. Definitions of vertex Lie algebras and Poisson vertex algebras. Examples of the graded algebra associated with any vertex algebra (filtered by the canonical filtration), and of the coordinate ring of the arc space of a Poisson variety.
3. Definitions of the associated variety and the singular support of a vertex algebra. First properties and examples, definition of the lisse (or C_2-cofinite) condition. Connections between the arc space of the associated variety and the singular support.

Exercises session and open problems session.

Organization and speakers. These series of lectures will be mostly done by the mentors. More precisely, the part about jet schemes and arc space will be presented by Anne Moreau, except if some of the participants agrees to do it. The parts about Poisson vertex algebras and associated variety will be presented by Tomoyuki Arakawa and/or Anne Moreau.

► These lectures are expected on the third and fourth days.

References

[Ein-Mustata09] L. Ein and M. Mustaţă, *Jet schemes and singularities*, Proc. Sympos. Pure Math., **80**, Part 2, Amer. Math. Soc., Providence, (2009).

[Ishii11] S. Ishii, *Geometric properties of jet schemes*, Comm. Algebra **39** (2011), n°5, 1872–1882.

[Moreau-Yu16] Anne Moreau and Rupert Wei Tze Yu. Jet schemes of the closure of nilpotent orbits. *Pacific J. Math.*, 281(1):137–183, 2016.

[Frenkel-BenZvi] E. Frenkel and D. Ben-Zvi. Vertex algebras and algebraic curves. Mathematical Surveys and Monographs, 88. American Mathematical Society, Providence, RI, 2001.

[Kac-lectures] V. G. Kac. Introduction to vertex algebras, Poisson vertex algebras, and integrable Hamiltonian PDE. Preprint `http://arxiv.org/pdf/1512.00821v1.pdf`.

[Arakawa-lectures] T. Arakawa, Introduction to W-algebras and their representation theory. Preprint `http://arxiv.org/pdf/1605.00138v1.pdf`.

[Arakawa-Moreau-lectures] T. Arakawa and A. Moreau. Lectures on W-algebras (workshop in Melbourne). In preparation.

[Arakawa12] Tomoyuki Arakawa. A remark on the C_2 cofiniteness condition on vertex algebras. *Math. Z.*, 270(1-2):559–575, 2012.

[Arakawa15a] Tomoyuki Arakawa. Associated varieties of modules over Kac-Moody algebras and C_2-cofiniteness of W-algebras. *Int. Math. Res. Not.* 2015(22): 11605-11666, 2015.

[Arakawa-Moreau16] T. Arakawa and A. Moreau. On the irreducibility of associated varieties of W-algebras. Preprint `http://arxiv.org/pdf/1608.03142.pdf`.

Comments on the references. About the geometry of jet schemes and arc spaces we refer to [Ein-Mustata09, Sections 1 and 2], [Ishii11] and references therein; see also [Moreau-Yu16, Section 2].

We refer to Chapter 15 of [Frenkel-BenZvi], [Kac-lectures, Lectures 5] for vertex Lie algebras and Poisson vertex algebras. The Poisson vertex algebra structure on arc spaces is dealt with only in [Arakawa-lectures, §3.4] (and reference therein); see also [Arakawa-Moreau-lectures].

For the definition of associated variety, see [Arakawa12]. For interesting recent examples of associated varieties, see [Arakawa15a, Arakawa-Moreau16] and references therein. These last two references are advanced research articles so they are perhaps difficult to read for a first approach.

5. Lectures "Affine W-algebras, rationality of W-algebras, and chiral differential operators on groups"

Content. Definition of affine W-algebras, via the BRST complex. Modules over affine W-algebras, and the BRST functor. Associated varieties of W-algebras, and their quotients. Rationality of W-algebras. Chiral differential operators on groups. TQFT associated with vertex algebras.

The content of these lectures will highly depend on the remaining time.

Exercises session and open problems session.

Organization and speakers. These series of lectures will be given by Tomoyuki Arakawa.

▶ These lectures are expected on the fourth and fifth days.

References

[Frenkel-BenZvi] E. Frenkel and D. Ben-Zvi. Vertex algebras and algebraic curves. Mathematical Surveys and Monographs, 88. American Mathematical Society, Providence, RI, 2001.

[Arakawa-lectures] T. Arakawa, Introduction to W-algebras and their representation theory. Preprint http://arxiv.org/pdf/1605.00138v1.pdf.

[Arakawa-Moreau-lectures] T. Arakawa and A. Moreau. Lectures on W-algebras (workshop in Melbourne). In preparation.

[Arakawa05] Tomoyuki Arakawa. Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture. *Duke Math. J.*, 130(3):435–478, 2005.

[Arakawa07] Tomoyuki Arakawa. Representation theory of W-algebras. *Invent. Math.*, 169(2):219–320, 2007.

[Arakawa11] Tomoyuki Arakawa. Representation theory of W-algebras, II. In *Exploring new structures and natural constructions in mathematical physics*, volume 61 of *Adv. Stud. Pure Math.*, pages 51–90. Math. Soc. Japan, Tokyo, 2011.

[Arakawa15a] Tomoyuki Arakawa. Associated varieties of modules over Kac-Moody algebras and C_2-cofiniteness of W-algebras. *Int. Math. Res. Not.* 2015(22): 11605-11666, 2015.

[Arakawa15b] Tomoyuki Arakawa. Rationality of W-algebras: principal nilpotent cases. *Ann. Math.*, 182(2):565–694, 2015.

[Arakawa16] Tomoyuki Arakawa. Rationality of admissible affine vertex algebras in the category $\mathcal{O}$. *Duke Math. J.*, 165(1), 67–93, 2016.

[Arakawa-Moreau16] T. Arakawa and A. Moreau. On the irreducibility of associated varieties of W-algebras. Preprint http://arxiv.org/pdf/1608.03142.pdf.

Comments on the references. For the definition of affine W-algebras, we refer to Chapter 14 of [Frenkel-BenZvi] (it only deals with the regular nilpotent elements case) or to [Arakawa-lectures] (it only deals with the regular nilpotent elements case in $\mathfrak{sl}_n$).

For more about affine W-algebras (their representations, rationality questions, corresponding associated varieties, etc), we refer to the papers of Tomoyuki Arakawa and references therein.