Bouttier-Di Francesco-Guitter bijection, tricks and admissibility.

The goal of this exercise session is to present a bijection, due to Bouttier, Di Francesco and Guitter, between bipartite planar maps and a class of labeled trees. Exercises 1 to 3 follow article [1]. In exercise 5, we use this bijection, and the intermediate combinatorial results of exercise 4, to deduce the admissibility criterion of weight sequences.

Exercise 1: From maps to planar mobiles.

Consider $m_\bullet = (m, \delta)$ a bipartite planar map having a distinguished vertex δ. Perform the following operations:

(i) Color in white every vertex of $m_\bullet$ and label them by their distances to δ.
(ii) Draw a black vertex inside each face.
(iii) In each face consider the clockwise order around it. Note that each edge is travelled in its two directions when we follow the clockwise order of its two adjacent faces.
(iv) For a face f and a white vertex adjacent to it, draw a new edge inside f between this vertex and the black vertex living inside f if the next white vertex in the clockwise order around f has a smaller label. Perform this drawing without crossings.
(v) Erase the edges of m.

We obtain a graph embedded on the plane (it is not connected).

1. Apply operation (i) to (v) to this pointed bipartite planar map:

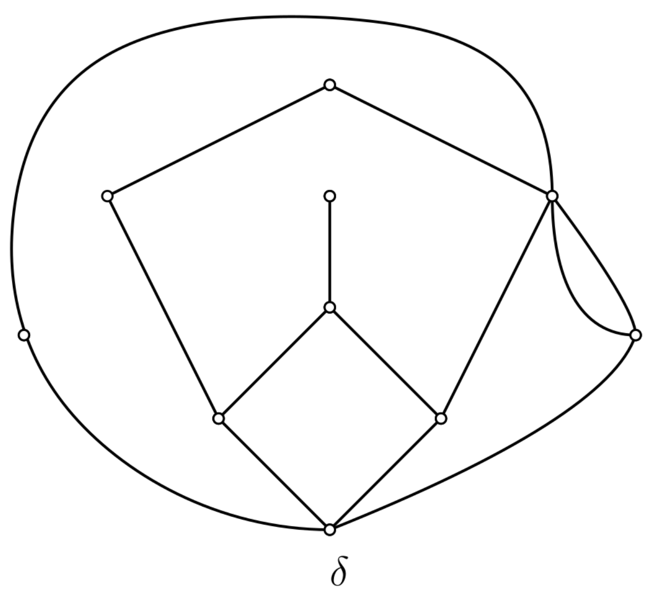

Figure 1: A pointed bipartite planar map

2. Show that δ is isolated on the resulting graph.

(vi) Erase δ.

We denote by Mob the concatenation of the operation (i) to (vi) (keeping the labels of the white vertices).

3. For every vertex u of m, let $A(u)$ denote the label of u (its distance to δ in m). Using the fact that m is bipartite show that if u, v are two neighboring vertices of m then:

$$|A(u) - A(v)| = 1.$$

4. Show that for each face f of m, the degree of the black vertex associated to f on $\mathrm{Mob}(m_\bullet)$ is half of the perimeter of f.
5. Let v be a black vertex, and u_1, u_2 two neighbors of v consecutive in the clockwise order on $\mathrm{Mob}(m_\bullet)$. Show that:

$$A(u_2) \geq A(u_1) - 1.$$

We call this property (P).
6. Show that $\mathrm{Mob}(m_\bullet)$ does not have loops and using Euler's formula deduce that $\mathrm{Mob}(m_\bullet)$ is a planar tree.

$*$ A mobile is a tree with black and white vertices such that all the neighbors of a black vertex are white and viceversa. A well-labeled mobile is a mobile such that all the white vertices have a label and verify property (P). Finally we say that a well-labeled mobile is standard if its minimum label is 1 and that it is planar if it is equipped with a clockwise order (an embedding on the plane or the sphere). Note that, for every pointed bipartite planar map $m_\bullet$, $\mathrm{Mob}(m_\bullet)$ is a standard well-labeled planar mobile.

Exercise 2: From mobiles to maps.

Consider $\mathcal{T}$ a standard well-labeled planar mobile. All we are going to do here works for general well-labeled planar mobiles just by translating all the labels by the minimum label $+1$. A corner of $\mathcal{T}$ is an angular sector delimited by a white vertex and two consecutives edges around this vertex. We define the following operations:

(i') Label each corner by the label of the associated white vertex.
(ii') For each corner C with label $A \geq 2$, denote by $s(C)$ the first encountered vertex in the clockwise order with label $A-1$. Draw an edge between the vertex associated to C and the vertex associated to $s(C)$ without hitting the edges of $\mathcal{T}$ and following the contour (clockwise order) of $\mathcal{T}$.

1. Show that the edges constructed in step (ii') are well defined and that we can draw them all without crossings.

(iii') Draw a new white vertex with label 0 in the infinite face.
(iv') For each corner C with label 1, draw an edge between the vertex associated to C and the new vertex with label 0 without hitting the other edges.

2. Show that this operation can be done without crossings.

(v') Erase all the edges of $\mathcal{T}$ and all the black vertices. Denote by m the resulting embedded graph.
(vi') Point m at the unique vertex with label 0 and denote by $m_\bullet$ the resulting pointed embedded graph.

3. Show that m is connected. So $m_\bullet$ is a pointed planar map.
4. Show that m is bipartite.
5. We denote by BFG the concatenation of operations (i') to (vi'). Apply BFG to this (page 3)

standard well-labeled planar mobile:

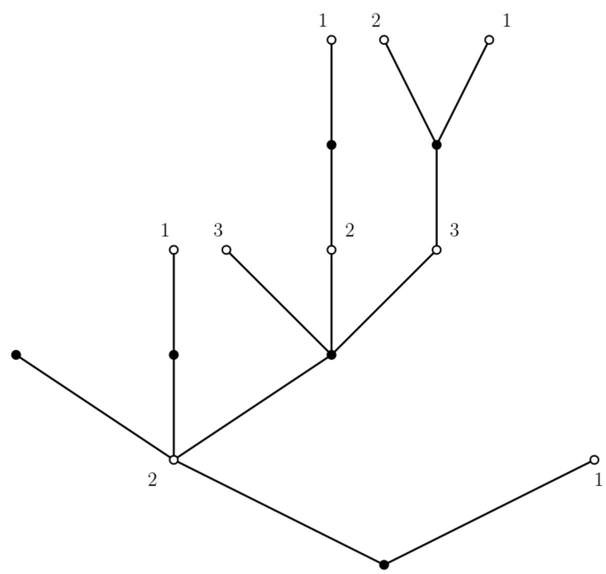

Figure 2: A standard well-labeled planar mobile.

Exercise 3: Bijection.

1. Consider $m_\bullet$ a pointed bipartite map.

1.1. Show that $\#\text{Edges}(m_\bullet) = \#\text{Edges}(\text{Mob}(m_\bullet))$ and deduce that $\#\text{Edges}(m_\bullet) = \#\text{Corners}(\text{Mob}(m_\bullet))$.
1.2. Consider e an edge of $m_\bullet$. e connects two corners of $\text{Mob}(m_\bullet)$. Let $\text{Mob}^e(m_\bullet)$ be the map $\text{Mob}(m_\bullet)$ where we add the edge e. Note that $\text{Mob}^e(m_\bullet)$ has two faces and set $\mathcal{C}$ the face not containing δ.
Show that there is a unique vertex on $\mathcal{C}$ realizing the minimum on $\mathcal{C}$ and that this point is one of the extremities of e.
1.3. Deduce that $\text{BFG}(\text{Mob}(m_\bullet)) = m_\bullet$.

2. Consider $\mathcal{T}$ a standard well-labeled planar mobile.
Fix v a black vertex of $\mathcal{T}$. For every C, C' two corners such that their associated vertices u_1, u_2 are successive neighbors of v and are in clockwise order, draw:

- The edge of $\text{BGF}(\mathcal{T})$ going from C to $s(C)$.
- For every integer $i \leq A(u_2) - A(u_1) - 1$, draw the edge going from $s^i(C')$ to $s^{i+1}(C')$.

2.1 What happens if we erase all the edges of $\mathcal{T}$ containing v?
2.2 Deduce that $\text{Mob}(\text{BGF}(\mathcal{T})) = \mathcal{T}$.

A plane tree is a rooted tree (at a vertex) with an ordering for the children of each vertex. Note that this definition is equivalent to the definition of a planar tree with a rooted edge. We say that a mobile is a plane mobile if its tree structure is a plane tree.

Exercise 4: Counting bridges and Janson & Stefansson's trick.

For every integer $l \geq 1$, let $\mathcal{B}_l$ be the set of l-tuple $(x_1, ..., x_l)$ of integers such that:

$$\sum_{i=1}^{l} x_i = l.$$

1. Show that

$$\#\mathcal{B}_l = \binom{2l-1}{l}.$$

2. Let $\mathcal{T}$ be a plane mobile rooted at a white vertex. We denote by ρ the root. Perform the following operations:

- Denote by $r_1, ..., r_j$ the children of ρ (in clockwise order) and by convention set $r_0 = r_{j+1} = \rho$. Following the contour of the mobile, draw without crossings, an edge going from r_i to r_{i+1} for every $i \in [\![1, j]\!]$.

- Let $u \neq \rho$ be a white vertex. Denote by $u_1, ..., u_j$ its children and by u_0 its parent. For every $i \in [\![0, j]\!]$, draw an edge going from u_i to u_{i+1} following the contour of the mobile and without crossings. Where by convention we take $u_{j+1} = u$.

- Erase the edges of $\mathcal{T}$, we obtain a tree as a consequence of Euler's formula. Root it at the black vertex r_1.

We denote by JS this transformation, it was introduced by Janson and Stefansson in [2].

2.1. What happens with the white vertices after performing JS?
2.2. Let v be a black vertex of $\mathcal{T}$. Show that the number of children of v on JS($\mathcal{T}$) is $\deg_{\mathcal{T}}(v)$.
2.3. Show that the map JS defines a bijection between plane mobiles rooted at a white vertex and plane trees where all the leaves are white and the rest of the vertices are black.

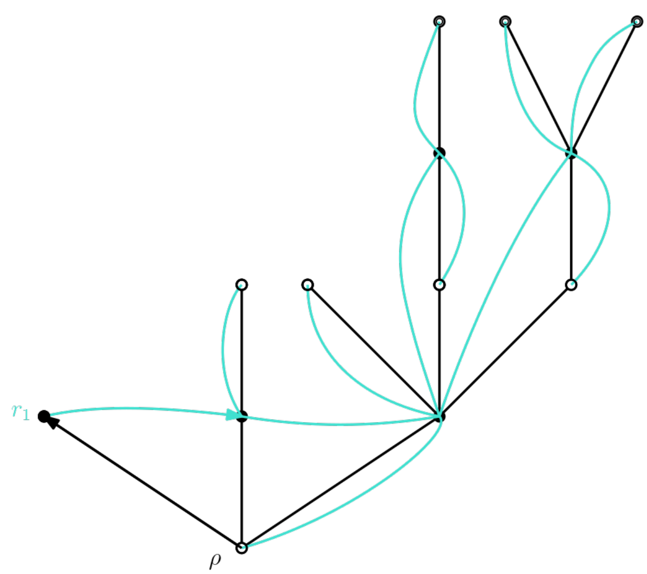

Figure 3: Janson & Stefansson's trick

$*$ The goal of exercise 5 is to show that if a sequence $q := (q_k)_{k\geq 0}$ of non negative numbers is admissible then the function:

$$f_q(x) := 1 + \sum_{k=1}^{\infty} \binom{2k-1}{k} q_k x^k$$

has a fix point. The reciprocal was proven during the second lecture of Nicolas Curien.

Exercise 5: Enumeration results (following Nicolas Curien's notes).

Fix l a positive integer. Set $N(k) := \binom{2k-1}{k}$, for every $k \geq 1$, and $N_0 := 1$.

1. Show that the map BFG defines a bijection between pointed bipartite planar maps with a distinguished face with perimeter $2l$ and standard well-labeled planar mobiles pointed at a black vertex with degree l.

Let $m_\bullet^*$ be a pointed planar map with a root edge. We impose that the root face (the face incident on the right of the root edge) has perimeter $2l$.

Perform the following operations:

(vii) Unroot the map $m_\bullet^*$ (forget where is the root edge) but keep track of the root face. Denote by $\mathcal{T}$ the standard well-labeled planar mobile $\mathrm{Mob}(m_\bullet^*)$ rooted at the black vertex living inside the root face.
$(viii)$ Erase the root of $\mathcal{T}$, v_0, and the edges containing it. We obtain a collection of l standard well-labeled mobiles. This collection is cyclically ordered. Root each new mobile at its unique white vertex which was a neighbor of v_0 in $\mathcal{T}$. Remark that the new mobiles are plane mobiles.
(ix) To order them, choose uniformly at random one of the new mobiles to be the first one. Translate each label by $-$ the label of the root of the first mobile.

We denote by $\mathrm{Mob}^*(m_\bullet^*) := (\mathcal{T}_1, ..., \mathcal{T}_p)$ the resulting forest of well-labeled pointed plane mobiles.

Fix $q = (q_k)_{k\in\mathbb{N}^*}$ a sequence of non negative numbers (such that $q \neq 0$). Let w_q denote the Boltzman measure associated to q. By convention we set $q_0 := 1$.

2. For $i \in [\![1, p]\!]$, let u_i be the root of $\mathcal{T}_i$ and $u_{p+1} = u_1$. Note that:

$$\forall i \in [\![1, p]\!],\ A(u_{i+1}) \geq A(u_i) - 1 \quad (P').$$

We say that an ordered collection of l well-labeled plane mobiles rooted at a white vertex is a standard well-labeled forest of rooted plane mobiles if the root of the first mobile is 0 and satisfies (P'). We extend Janson & Stefansson's trick to well-labeled forest of mobiles by erasing the labels and performing Janson & Stefansson's trick at each mobile of the forest (and keeping the order between the trees).

3. Show that the image measure of w_q on $\mathcal{M}_0^{(l)}$ by $\mathrm{JS} \circ \mathrm{Mob}^*$ is the measure $\tilde{w}_q$ defined by:

$$\tilde{w}_q(\mathcal{F}) = 2N(l) \prod_{u\in \mathrm{Vertex}(\mathcal{F})} N(\kappa_u)\, q_{\kappa_u}$$

for every forest of l plane trees $\mathcal{F}$ (there is a direct bijection between trees and trees with leaves colored in white). Where $\mathrm{Vertex}(\mathcal{F})$ stands for the set of vertices of $\mathcal{F}$ and for every vertex u, κ_u stands fo the number of children of u.
4. Show that q is admissible if and only if:

$$\sum_{T\in \mathrm{Tree}} \prod_{u\in \mathrm{Vertex}(T)} N(\kappa_u)\, q_{\kappa_u} < \infty$$

and deduce that $f_q(x) = x$ has a positive solution.
5. Suppose that q is admissible and write Z_q for smallest fix point of f_q. Take $(T_1, ..., T_l)$ a random variable

distributed according to:

$$\frac{\tilde{w}_q}{w_q(\mathcal{M}_0^{(l)})}.$$

Show that $(T_1, ..., T_l)$ are iid Galton-Watson trees with offspring distribution:

$$\forall k \in \mathbb{N},\ \mu_q(k) = Z_q^{k-1}\, N(k)\, q_k.$$

6. Deduce that:

$$W_0^{(l)} := w_q(\mathcal{M}_0^{(l)}) = 2N(l)\, Z_q^l.$$

References

[1] J. Bouttier, P. Di Francesco, and E. Guitter. Planar maps as labeled mobiles. *Electron. J. Combin.*, (2004).

[2] S. Janson and S.Ö.Stefánsson. Scaling limits of random planar maps with a unique large face. *The Annals of Probability*, (2015).