Cohomology of alg varieties

X proper smooth algebraic variety over $K \hookrightarrow \mathbb{C}$

- The Betti Cohomology
$$H^*_B(X(\mathbb{C}), \mathbb{Z})$$

- The de Rham cohomology
$$H^*_{dR}(X(\mathbb{C}))$$ with Hodge filtration

Have an isomorphism

$$H^*_{dR}(X) \longrightarrow H^*_B(X(\mathbb{C}), \mathbb{Z})$$

Image of the form

$$\left(f(z) + \frac{1}{z^2}\, g\left(\frac{1}{z}\right)\right) dz$$

Thus cokernel of the form

$$\left\langle \frac{1}{z}\, dz \right\rangle .$$

Can be explained better by Mayer Vietoris sequence.

$$H^2(\mathbb{P}^1(\mathbb{C}), \mathbb{Z}) \cong H^1(\mathbb{A}^1 \setminus 0, \mathbb{Z}) \cong \mathbb{Z}\langle \gamma \rangle$$

$$\vdots$$

$$H^2_{dR}(\mathbb{P}^1(\mathbb{C})) \longrightarrow H^2(\mathbb{P}^1(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

$$\frac{dz}{z} \longmapsto \int_\gamma \frac{dz}{z} = 2\pi i$$

Therefore isomorphism not possible over $\mathbb{Z}$. Needed to extend scalars to $\mathbb{C}$.

X sm. proj. $/K$

Another cohomology

$H^i_{et}(X_{\bar{K}}, \mathbb{Z}_\ell) \circlearrowleft Gal(\bar{K}/K)$

How are $H^i_{et}(X_{\bar{K}}, \mathbb{Z}_\ell)$ and $H^i_{dR}(X/K)$ related?

Take a p-adic field K mixed char complete discrete valuation field with perfect residue field

eg $\rightarrow$
- $K/\mathbb{Q}_p$ < finite
- completion of max unramified ext^n of $\mathbb{Q}_p$

Thm — $X/_K$ proper smooth alg var

There is an isomorphism of filtered B_{dR}-resp with $\mathrm{Gal}(\bar{K}/K)$ action

$$H^*_{dR}(X/K) \otimes_K B_{dR} \xrightarrow{\sim} H^*_{et}(X_{\bar{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{dR}$$

Rem — (cyclotomic character) $\chi : G_K \longrightarrow \mathbb{Z}_p^\times$ be the character defined by $g\zeta = \zeta^{\chi(g)}$ for all p-power roots of unity $\zeta \in \bar{K}^\times$.

$$H^2_{et}(\mathbb{P}^1_{\bar{K}}, \mathbb{Z}_p) = \chi^{-1} \text{ as } \mathrm{Gal}(\bar{K}/K)\text{-module}$$

So the ring B_{dR} contains an element "$2\pi i$" and G_K acts on "$\mathbb{Q}_p\langle 2\pi i\rangle$" by the character χ^{-1}.

Let $C = \hat{\bar{K}}$ wrt norm topology. We will show that G_K does not act by χ^{-1} on any non-zero subspace of C. So C does not contain any element "$2\pi i$". Cannot replace B_{dR} with C

- Want $B_{dR} \circlearrowleft \mathrm{Gal}(\overline{K}/K)$

filtration

- $B_{dR}^{G_K} = K$

Thus can recover de Rham cohomology

$$H^*_{dR}(X/K) \cong \left(H^*_{et}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{dR}\right)^{G_K}$$

One can check

$$\dim_K H^*_{dR}(X/K) = \dim_{\mathbb{Q}_p} H^*_{ét}(X_{\overline{K}}, \mathbb{Q}_p)$$

An arbitrary fd $\mathbb{Q}_p$-vsp V of G_K satisfies

$$\dim_K (V \otimes_{\mathbb{Q}_p} B_{dR})^{G_K} \leq \dim_{\mathbb{Q}_p} V$$

A $\mathbb{Q}_p$-vsp V is called B_{dR}-admissible or de Rham if

$$\dim_K (V \otimes_{\mathbb{Q}_p} B_{dR})^{G_K} = \dim_{\mathbb{Q}_p} V$$

Case $X = \mathbb{P}^1 \quad K = \mathbb{Q} \hookrightarrow \mathbb{C}$

$$H^2_{dR}(\mathbb{P}^1(\mathbb{C}))$$

$$H^2_{dR}(\mathbb{P}^1(\mathbb{C})) = H^0_{dR}(\mathbb{P}^1(\mathbb{C}), \Omega^2) \oplus H^1_{dR}(\mathbb{P}^1(\mathbb{C}), \Omega^1) \oplus H^2_{dR}(\mathbb{P}^1(\mathbb{C}), \Omega^0)$$

$$H^1_{dR}(\mathbb{P}^1(\mathbb{C}), \Omega^1) = \left\langle \frac{dz}{z} \right\rangle$$

$$k[z]\,dz \oplus k[z^{-1}]\frac{dz}{z^2} \qquad \frac{-}{z}$$

$$\longrightarrow k[z, z^{-1}]\,dz$$

$$\longrightarrow H^1(X, \Omega_X) \longrightarrow 0$$

Tate's 1966 paper on p-div groups.

Tate establishes a Hodge like decomposition of the Tate module of p-div group on $\mathcal{O}_K$.

More precisely, $\mathcal{G}$ be a p-div group, on $\mathcal{O}_K$.

Let $T_p(\mathcal{G}) = \varprojlim_n \mathcal{G}[p^n](\bar{K})$ be the Tate module and $V_p(\mathcal{G}) = T_p(\mathcal{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Then Tate proves a Hodge-like G_K-equivariant decomposition

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} V_p(\mathcal{G}) \simeq \left(\mathbb{C}_p \otimes \mathcal{O}_K\, \omega_{\mathcal{G}^\vee}\right) \oplus \left(\mathbb{C}_p(\chi_{cycl}^{-1}) \otimes_{\mathcal{O}_K} \omega_{\mathcal{G}}^\vee\right)$$

$\mathcal{G}^\vee$ cartier dual of $\mathcal{G}$

ω _ cotangent space

$$\mathbb{C}_p(\chi^{-1}_{cycl}) = \mathbb{C}_p\langle e\rangle$$

$$g(\lambda e) = g\lambda\, \chi^{-1}_{cycl}(g)\, e$$

In particular, if $A/_K$ abelian variety good reduction:

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^1_{ét}(A_{\overline{K}}, \mathbb{Q}_p)$$

$$\cong \left(\mathbb{C}_p \otimes_K H^1(A; \mathcal{O}_A)\right) \oplus \left(\mathbb{C}_p(\chi_{cycl}^{-1}) \otimes H^0(A, \Omega^1_{A/K})\right)$$

$\Omega^0_{A/K}$ sheaf of Kal...

Goal:

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^r_{ét}(X_{\overline{K}}, \mathbb{Q}_p)$$

$$\cong \bigoplus_{a+b=r} \mathbb{C}_p(\chi_{cycl}^{-a}) \otimes_K H^b(X, \Omega^a_{X/K})$$

Rings of p-adic periods — $K/\mathbb{Q}_p < \infty$

K perf'd

$$C = \widehat{K}, \quad C^\flat, \quad \mathcal{O}_{C^\flat}$$

$$A_{inf} := W(\mathcal{O}_{C^\flat})$$

$$\begin{tikzcd}
\theta : A_{inf} \arrow[r] & \mathcal{O}_C \\
\mathcal{O}_{C^\flat} \arrow[u, "{[\,]}"] \arrow[ur] &
\end{tikzcd}
\qquad a \mapsto [a]$$

$\ker(\theta) = (\xi)$ $\quad$ ξ primitive deg 1

$$\xi = [(p, p^{1/p}, p^{1/p^2}, \dots)] - p \qquad (p, \dots) \in \mathcal{O}_{C^\flat}$$

$$= \frac{[\varepsilon] - 1}{[\varepsilon]^{1/p} - 1} = 1 + [\varepsilon] + \dots + [\varepsilon]^{\frac{p-1}{p}}$$

$$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_{C^\flat}$$

$$A^{o}_{crys} := A_{inf}\left[\frac{\xi^2}{2!}, \frac{\xi^3}{3!}, \dots\right]$$ pd envelope of $\ker(\theta)$ in A_{inf}

$$A^{+}_{cry} := \lim_{n} A^{o}_{cry} / p^n$$

$$B^+_{crys} := A^+_{crys}[1/p]$$

$$t := \log[\varepsilon] = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} ([\varepsilon] - 1)^n$$

$$\in A^+_{crys}$$

$$g \cdot t = \sum \frac{(-1)^{n+1}}{n} (g \cdot [\varepsilon] - 1)^n$$

$$= \sum \frac{(-1)^{n+1}}{n} ([\varepsilon]^{\chi(g)} - 1)^n$$

$$= \log [\varepsilon]^{\chi(g)}$$

$$= \chi(g) \log [\varepsilon] = \chi(g) t$$

therefore recover $\langle\langle 2\pi i \rangle\rangle$.

$$B_{crys} := B^+_{crys}[1/t]$$

$\curvearrowleft$ $\circlearrowright$ φ from Ainf

G_K action

$$A_{\inf,K} = A_{\inf} \otimes_{\mathcal{O}_{K_0}} K$$

$$\begin{array}{cl} K & \\ | & \Big) \text{ totally ramified} \\ K & \\ | & \Big) \text{ unramified} \\ \mathbb{Q}_p & \end{array}$$

$$\theta_K : A_{\inf,K} \longrightarrow C$$

$\ker(\theta_K)$ principal

$$\bigcap_{n \geq 1} \left(\ker(\theta_K)\right)^m = \{0\}$$

$$B^+_{dR/K} := \lim_n A_{\inf,K} / (\ker \theta_K)^n$$

<u>Prop</u> — (a) B^+_{dR}/K complete d