The de Rham period ring, B_{dR}

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Introduction : We present the definition and first properties of the field B_{dR}. This field, and its associated valuation ring B_{dR}^+ were introduced by Fontaine in the 1980's. Their are not only necessary to formulate a comparison between étale and de Rham cohomology, but also provide a good context to study p-adic periods (which are the p-adic analogue of the numbers called periods in $\mathbb{C}$).

Let us fix a prime number p.

1 Introduction

As usual in p-adic Hodge theory, we consider a discrete valuation field K (for instance a finite extension of $\mathbb{Q}_p$) of mixed characteristic, and with perfect residue field k of characteristic p. We denote by $W(k)$ the ring of Witt vectors over k, K_0 the fraction field of $W(k)$, $\overline{K}$ an algebraic closure of K with residue field $\overline{k}$, C the completion of $\overline{K}$ (that is $C = \mathbb{C}_p$ if K is a finite extension of $\mathbb{Q}_p$) and $C^\flat := \lim_\phi C/p$ its tilt, $G_K := \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K and $\mathcal{O}_K$, $\mathcal{O}_{\overline{K}}$, $\mathcal{O}_C$ and $\mathcal{O}_{C^\flat}$ the rings of integers of the fields K, $\overline{K}$, C and $C^\flat$.

We also let $A_{\mathrm{inf}} := W(\mathcal{O}_{C^\flat})$ and $A_{\mathrm{inf},K} := A_{\mathrm{inf}} \otimes_{\mathcal{O}_K} K = A_{\mathrm{inf}}[\frac{1}{p}]$. Remark that the map $\theta : A_{\mathrm{inf}} \to \mathcal{O}_C$ extends uniquely into a morphism of K-algebras $\theta_{\mathbb{Q}} : A_{\mathrm{inf},K} \to C$. The morphisms θ and $\theta_{\mathbb{Q}}$ are surjective, with principal kernel generated by an element $\xi \in A_{\mathrm{inf}}$. We are now able to define the following *de Rham period ring*:

$$B_{\mathrm{dR}}^+ := \lim_n (A_{\mathrm{inf}}/\xi^n[1/p]) = (A_{\mathrm{inf},K}/(\ker\theta_{\mathbb{Q}})^n).$$

The field B_{dR} is then defined as the fraction field of B_{dR}^+. The field B_{dR} is useful to state the étale-de Rham comparison theorem:

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Theorem 1.1. *Let X be a proper smooth algebraic variety over K. Then there is a G_K-equivariant isomorphism of filtered B_{dR}-vector spaces:*

$$H^*_{dR}(X/K) \otimes_K B_{dR} \cong H^*_{\acute{e}t}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{dR}.$$

Remark in this statement that the Galois action is induced on the left-hand side by the Galois action on B_{dR} and is trivial on de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(X/K)$, while it is given by transport de structure on étale cohomology $\mathrm{H}^*_{\acute{\mathrm{e}}\mathrm{t}}(X_{\overline{K}}, \mathbb{Z}_p)$. Remark similarly that the filtration of B_{dR} is induced by the "t-adic filtration" on B_{dR}, by the Hodge filtration on de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(X/K)$, and is trivial on étale cohomology $\mathrm{H}^*_{\acute{\mathrm{e}}\mathrm{t}}(X_{\overline{K}}, \mathbb{Z}_p)$.

In fact, one can prove that the G_K-action on B_{dR} satisfies $B_{\mathrm{dR}}^{G_K} = K$, so one can recover the de Rham cohomology of X from its étale cohomology:

$$\mathrm{H}^*_{\mathrm{dR}}(X/K) \cong (\mathrm{H}^*_{\acute{\mathrm{e}}\mathrm{t}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}})^{G_K}.$$

From this statement and the following equality of dimensions: $\dim_K \mathrm{H}^*_{\mathrm{dR}}(X/K) = \dim_{\mathbb{Q}_p} \mathrm{H}^*_{\acute{\mathrm{e}}\mathrm{t}}(X_{\overline{K}}, \mathbb{Q}_p)$, one can say that étale cohomology $\mathrm{H}^*_{\acute{\mathrm{e}}\mathrm{t}}(X_{\overline{K}}, \mathbb{Q}_p)$ is, as a $\mathbb{Q}_p$-vector space, a *de Rham representation* of G_K.

2 B_{dR}, first properties

By definition, B^+_{dR} is the ξ-adic completion of $A_{\mathrm{inf},K}$. In particular, $A_{\mathrm{inf},K} = A_{\mathrm{inf}}[1/p]$ is a dense subring of B^+_{dR}.

Proposition 2.1. *For each integer $n \geqslant 0$, we have $(ker\theta_{\mathbb{Q}})^n \cap A_{inf} = (ker\theta)^n$, and $\cap_{n\geqslant 0}(ker\theta_{\mathbb{Q}})^n = 0$. In particular B^+_{dR} is a complete discrete valuation ring with residue field C.*

Proof. The first claim follows from the characterization of A_{inf} as the universal pro-nilpotent p-complete thickening of $\mathcal{O}_C$. Since p is a nonzero divisor in A_{inf}, the second claim reduces to proving that $\cap_{n\geqslant 0}(\xi)^n = 0$ in A_{inf}, which is true since ξ is a generator of the kernel of θ and a nonzero divisor.

The completeness (which includes by definition being separated) of B^+_{dR} follows from $\cap_{n\geqslant 0}(\mathrm{ker}\theta_{\mathbb{Q}})^n = 0$. Finally, the residue field of B^+_{dR} identifies with $B^+_{\mathrm{dR}}/(\xi) \cong A_{\mathrm{inf},K}/(\xi)$, which is isomorphic to C since $\theta_{\mathbb{Q}}$ is surjective. ◆

Proposition 2.2. *There is a Galois equivariant inclusion $\overline{K} \hookrightarrow B^+_{dR}$.*

In fact, Colmez proved (in 1994) that $\overline{K}$ is dense inside B^+_{dR} via this embedding.

Proof. There is a natural inclusion $\overline{k} \hookrightarrow \mathcal{O}_{C^\flat}$ sending $x \mapsto (x^{p^{-n}})$, which induces an inclusion $W(\overline{k}) \hookrightarrow A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat})$, and thus an inclusion $W(\overline{k})[1/p] \hookrightarrow B^+_{\mathrm{dR}}$. Now, any element of $\overline{K}$ is a root of a monic polynomial with coefficients in $W(\overline{k})[1/p]$. Such a polynomial splits completely in C; but C is the residue field of B^+_{dR}, thus b Hensel's lemma, it also splits completely in B^+_{dR}. This proves there is a unique inclusion $\overline{K} \hookrightarrow B^+_{\mathrm{dR}}$. The Galois-equivariance follows from the construction. ◆

We will now construct an element $t \in B^+_{\mathrm{dR}}$, and show this is a uniformizer for the discrete valuation of B^+_{dR}. In particular, this will prove the relation $B_{\mathrm{dR}} = B^+_{\mathrm{dR}}[1/t]$, and the fact that t and ξ differ by a unit in B^+_{dR}.

Let $\varepsilon \in \mathcal{O}_{C^\flat}$ be a (non-trivial) system of p-power roots of unity in $\mathcal{O}_{C^\flat}$. By definition of A_{inf}, this produces an element $[\varepsilon] \in A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat})$ (that is, the Teichmüller lift of $\varepsilon \in \mathcal{O}_{C^\flat}$). In fact, doing this for all such elements ε of $\mathbb{Z}_p(1)$[a] defines a natural embedding of $\mathbb{Z}_p(1) \hookrightarrow A_{\mathrm{inf}}^{\times}$. Now we define the map:

$$\varepsilon \mapsto t := \log([\varepsilon]) = \sum_{n\geqslant 1}(-1)^{n+1}\frac{([\varepsilon]-1)^n}{n}.$$

This sum converges in the maximal ideal of B_{dR}^+: indeed, ξ divides $[\varepsilon]-1$, and B_{dR}^+ is the ξ-adic completion of $A_{\mathrm{inf}}[1/p]$. In particular, this identifies $\mathbb{Z}_p(1)$ with a subgroup of the additive group B_{dR}^+.

Proposition 2.3. *With the previous notations, t is a uniformizer of B_{dR}^+.*

Proof. We already know that $t \in \xi B_{\mathrm{dR}}^+$, so we just need to prove that $t \notin \xi^2 B_{\mathrm{dR}}^+$. It suffices to prove that $[\varepsilon]-1 \notin \xi^2 B_{\mathrm{dR}}^+$, or equivalently that $[\varepsilon]-1 \notin \xi^2 A_{\mathrm{inf}}$. This can be done by technical computation on valuations, or by identifying $A_{\mathrm{inf}}/(\xi)^2$ with a construction involving the de Rham complex of $\mathcal{O}_{\overline{K}}$ over $\mathcal{O}_K$ (done in Fontaine's "Le corps des périodes p-adiques", 1994). ◆

We can thus identify the ξ-adic filtration on B_{dR} with its t-adic filtration. The purpose of defining t instead of ξ is that the action of t is (by definition, thanks to the previous construction) given by the cyclotomic character. In particular, the graded pieces for this filtrations are indexed by the integers $i \in \mathbb{Z}$, and we have the identification $\mathrm{gr}^i B_{\mathrm{dR}} \cong C \cdot t^i = C(i)$ for any $i \in \mathbb{Z}$. This last equality, where $C(i) := \mathbb{Z}_p(i) \otimes_{\mathbb{Z}_p} C$ is the i^{th} Tate twist of C, follows from the description of the G_K-action on t as being given by the cyclotomic character χ. The Hodge-Tate period ring B_{HT} is defined as the graded ring associated to this filtration:

$$B_{\mathrm{HT}} := \bigoplus_{i\in\mathbb{Z}} C(i) = \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i B_{\mathrm{dR}}.$$

Proposition 2.4. *There is a natural isomorphism $B_{dR}^{G_K} \cong (B_{dR}^+)^{G_K} \cong K$.*

Proof. The Ax-Sen-Tate theorem shows that $C^{G_K} = K$, and $C(n)^{G_K} = 0$ if $n \neq 0$. The result thus follows from studying the filtration on B_{dR}, whose graded pieces are equal to $\mathrm{gr}^n B_{\mathrm{dR}} \cong C(n)$, for $n \in \mathbb{Z}$. ◆

Remark 2.5. The period ring A_{inf}, and then the de Rham period rings B_{dR} and B_{dR}^+, can be defined more generally for any perfectoid ring R instead of just the ring of integers $\mathcal{O}_{C^\flat}$ of $C^\flat$. In this generality, the fundamental exact sequence of p-adic Hodge theory can be proved ([AMMN19]), and involves the pro-étale cohomology of the generic fibre of R. Moreover, the de Rham period ring $B_{\mathrm{dR}}^+(R)$ can be identified with the Hodge completion $(\widehat{L\Omega}_R)_{\mathbb{Q}_p}$ of *derived de Rham cohomology* $L\Omega_R$ of R, as developed in [Bha12]. Under this identification, the ξ-adic filtration on B_{dR}^+ corresponds to the Hodge filtration on the completed derived de Rham cohomology.

[a] Recall that $\mathbb{Z}_p(1)$ is defined as the Tate module $T_p(\overline{K}^{\times}) = \lim_n \overline{K}^{\times}[p^n]$ of the p-divisible group $\overline{K}^{\times}$ of roots of unity.