(φ-Γ) Modules

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1 Introduction

Let us start with the most basic example of perfectoid tilting. Let K be the p-adic completion of $\mathbb{Q}_p(\zeta_{p^\infty})$ with ring of integers K° being the p-adic completion of $\mathbb{Z}_p[\zeta_{p^\infty}]$. Noting that $\frac{x^p-1}{x-1} \equiv (x-1)^p \mod p$, we find that

$$K^\circ/(p) = \mathbb{Z}_p[\zeta_{p^\infty}]/(p) \cong \mathbb{F}_p[X^{\frac{1}{p^\infty}}]/(X-1)^{p-1} \cong \mathbb{F}_p[T^{\frac{1}{p^\infty}}]/(T^{p-1})$$

where this chain of isomorphisms take ζ_p to X, and X to $T+1$. Now we have a well defined morphism

$$\begin{aligned} \left(\mathbb{F}_p[T^{\frac{1}{p^\infty}}]/(T^{p-1})\right)^{\text{perf}} &\to \varprojlim_n \mathbb{F}_p[T^{\frac{1}{p^\infty}}]/(T^{(p-1)p^n}) \\ (x_n \mod T^{p-1})_{n\geq 0} &\mapsto (x_n^{p^n} \mod T^{(p-1)p^n})_{0\geq 1} \end{aligned}$$

whose inverse is given by sending $(y_0 \mod T^{p-1}, y_1 \mod T^{p(p-1)}, \dots)$ to $(y_0 \mod , T^{p-1}, y_1^{\frac{1}{p}} \mod T^{p-1}, \dots)$. Hence combining the above calculations, we get an isomorphism

$$(K^\circ)^\flat \xrightarrow{\sim} \mathbb{F}_p\langle T^{\frac{1}{p^\infty}}\rangle \quad (T\text{-adic completion})$$

Recall that we have a sharp map, which is a multiplicative map

$$\sharp : K^{\circ,\flat} = \varprojlim_{\varphi} K^\circ/(p) \xrightarrow{\sim} \varprojlim_{x\mapsto x^p} K^\circ \xrightarrow{\text{pr}_0} K^\circ$$

sending $(x_n \mod p)_{n\geq 0}$ to Recall that we have a commutative diagram

$$\begin{tikzcd} W(K^{\circ,\flat}) \arrow[r, two heads, "\theta"] & K^\circ \\ K^{\circ,\flat} \arrow[u, "{[\cdot]}"] \arrow[ur, "\sharp"'] & \end{tikzcd}$$

where θ is a surjective ring homomorphism called the Fontaine map, and concretely, $\theta : \sum_{n=0}^\infty p^n[x_n] \mapsto \sum_{n=0}^\infty p^n \sharp x_n$.

2 Idea Behind (φ, Γ)-modules

3 Characteristic p-theory

As we have seen the strategy is to work with fields of the form $E = k((u))$, i.e., the field of Laurent series with coefficients in a perfect field k of characteristic $p > 0$. The reason for this is that the category of Galois representations associated to a field of characteristic p E can be interpreted as a category of vector spaces over the separable closure E_s equipped with a Frobenius morphism φ.

For $V \in \mathrm{Rep}_{\mathbb{F}_p}(G_E)$, define $D_E(V) = (E_s \otimes V)^{G_E}$ equipped with a φ_E-semilinear endomorphism $\varphi_{D_E}(V)$ induced by $\varphi_{E_s} \otimes 1$. For $M \in \Phi\mathrm{M}_{\mathrm{E}}^{\text{ét}}$ define $V_E(M)$ to be the $\mathbb{F}_p$-vector space

$$(E_s \otimes_E M)^{\varphi=1}$$

with it's G_E-action induced by the G_E-action on E and $\varphi = \varphi_{E_s} \otimes \varphi_M$. We need to check that these functors we just defined take value in the right category and are inverse of each other. The main ingredient for this is the famous Hilbert's 90 Theorem.

Theorem 3.1 (Hilbert's 90)**.** *Suppose that V is a E_s-vector space with a continuous semilinear action of G_E (for the discrete topology), then the E_s-linear map*

$$\begin{aligned} E_s \otimes_E V^{G_E} &\to V \\ \lambda \otimes v &\mapsto \lambda v \end{aligned}$$

is an isomorphism.

Proof. Let us first show that this map is injective, i.e., an E-free family in V^{G_E} is E_s-free in V. So let $v_1, \ldots, v_n \in V^{G_E}$ be a free family, and $\lambda_1, \ldots, \lambda_n \in E_s$ be such that $\sum_i \lambda_i v_i = 0$. Suppose that $\lambda_j \neq 0$ for some $1 \leq j \leq n$, and let M/E be a finite extension containing λ_j. Then by non-degeneracy of the Trace pairing, there exists $x \in M$ such that $\mathrm{Tr}_{M/E}(x\lambda_j) \neq 0$. But for $\sigma \in \mathrm{Gal}(M/E)$, we have that:

$$0 = \sigma(\sum_i x\lambda_i v_i) = \sum_i \sigma(x\lambda_i)\sigma(v_i) = \sum_i \sigma(x\lambda_i)v_i$$

summing over $\sigma \in \mathrm{Gal}(M/K)$, we get

$$\sum_i \mathrm{Tr}(x\lambda_i)v_i = 0$$

which contradicts the freeness hypothesis.

To prove surjectivity, we need to show that V^{G_E} generates V as an E_s-vector space. For $v \in V \setminus \{0\}$, let V_v be the E_s-sub-vector space generated by the orbit of v. If we show that $V_v^{G_E}$ generates V_v, then we win.
Since the action is continuous, there exists a finite extension M/E such that $v \in V^{G_M}$. Let $\lambda_1, \ldots, \lambda_n$ is a basis for M, $\{\sigma_1, \ldots, \sigma_n\} = \mathrm{Gal}(M/K)$. We know that the matrix $(\sigma_j(\lambda_i))_{i,j}$ is invertible. Since V_v is E_s-linearly generated by the $\sigma_j(v)$, it is generated by the $\sum_j \sigma_j(\lambda_i)\sigma_j(v) = \sum_\sigma \sigma(\lambda_i v) \in V^{G_E}$.

□

Lemma 3.2. *1. For $V \in \mathrm{Rep}_{\mathbb{F}_p}(G_E)$, the E-vector space $D_E(V)$ is finite dimensional of dimension $\dim_{\mathbb{F}_p} V$, and the E-linearlisation of $\varphi_{D_E(V)}$ is an isomorphism.*

2. *For $M \in \Phi\mathrm{M}_{\mathrm{E}}^{\acute{e}t}$, the $\mathbb{F}_p$-vector space $V_E(M)$ is finite dimensional with dimension at most* $\dim_E M$.

Proof. **1)** By Hilbert's 90 theorem applied to the E_s-vector space $E_s \otimes_{\mathbb{F}_p} V$ equipped with the diagonal G_E-action (which is continuous for the discrete topology since it is continuous on each term), we get that the natural morphism

$$E_s \otimes_E D_E(V) = E_s \otimes_E (E_s \otimes_{\mathbb{F}_p} V)^{G_E} \to E_s \otimes_{\mathbb{F}_p} V \tag{3.1}$$

is an isomorphism. This shows that $D_E(V)$ is a finite dimensional E-vector space of dimension $\dim_{\mathbb{F}_p} V$. To show that the linear map $\mathrm{Lin}(\varphi) : \varphi_E^*(D_E(V)) \to D_E(V)$ is an isomorphism, note that we have $E_s \otimes_E \varphi_E^*(D_E(V)) \cong \varphi_{E_s}^*(E_s \otimes_E D_E(V))$ sending $e_s \otimes (e \otimes d)$ to $e_s e \otimes (1 \otimes d)$ inducing a commutiative diagram

$$\begin{tikzcd}
E_s \otimes_{\varphi,E_s} (E_s \otimes_E D_E(V)) \arrow[drr, "\mathrm{Lin}(\varphi)"] & & \\
E_s \otimes_E (E \otimes_{\varphi,E} D_E(V)) \arrow[u, "\cong"] \arrow[rr, "\mathrm{id} \otimes \mathrm{Lin}(\varphi)"'] & & E_s \otimes_E D_E(V)
\end{tikzcd}$$

this shows that we can check that $\mathrm{Lin}(\varphi)$ is an isomorphism after extension of scalars to E_s. Since we have that $E_s \otimes_E D_E(V) \cong E_s \otimes_{\mathbb{F}_p} V$ compatibly with the action of φ (which is on the first term of the left hand side), it suffices to check that the morphism

$$E_s \otimes_{\varphi,E_s} (E_s \otimes_{\mathbb{F}_p} V) \to E_s \otimes_{\mathbb{F}_p} V$$

is an isomorphism. But this is just $\otimes_{\mathbb{F}_p} V$ applied to the map $E_s \otimes_{\varphi,E_s} E_s \to E_s$ which is obviously an isomorphism. Hence the result.
2) Now we prove that $V_E(M)$ is a finite dimensional $\mathbb{F}_p$-vector space. For this, it suffices to check that the natural morphism

$$E_s \otimes_{\mathbb{F}_p} V_E(M) = E_s \otimes_{\mathbb{F}_p} (E_s \otimes_E M)^{\varphi=1} \to E_s \otimes_E M \tag{3.2}$$

is injective. For this, it suffices to show that every $\mathbb{F}_p$-free family in $V_E(M)$ is E_s-free in $E_s \otimes_E M$. Suppose that this is not true and choose a family $v_1, \dots, v_r \in V_E(M)$ providing a counterexample with a minimal r. Then we have

$$\sum_i a_i v_i = 0$$

for some non-zero $a_i \in E_s$ (by minimality). We can suppose that $a_1 = 1$, which gives $v_1 = -\sum_{i>1} a_i v_i$. Applying φ, we get:

$$v_1 = \varphi(v_1) = -\sum_{i>1} \varphi(a_i)\varphi(v_i) = -\sum_{i>1} \varphi(a_i) v_i$$

Hence we get that $\sum_{i>1}(a_i - \varphi(a_i))v_i = 0$ which by minimality of r can only be true if $a_i = \varphi(a_i)$ for all $i > 1$, i.e., $a_i \in \mathbb{F}_p$ which contradicts the freeness of the v_i over $\mathbb{F}_p$. $\square$

Now we are able to prove the equivalence of categories we are seeking.

Theorem 3.3. *The functors D_E and V_E are exact, rank preserving and induce an equivalence of categories*

$$\left\{\begin{matrix} G_E\text{-representations on finite} \\ \text{dimensional } \mathbb{F}_p\text{-vector spaces} \end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix} \varphi\text{-modules} \\ \text{over } E \end{matrix}\right\}$$

Proof. By the morphism in (3.1), we have

$$V_E \circ D_E(V) = (E_s \otimes_E D_E(V))^{\varphi=1} \cong (E_s \otimes_{\mathbb{F}_p} V)^{\varphi=1} = V$$

this shows that $V_E \circ D_E \cong \mathrm{id}$.
Now let us analyse $D_E \circ V_E$. Passing to G_E-invariants in (3.2), we get an injection

$$D_E \circ V_E(M) = (E_s \otimes_{\mathbb{F}_p} V_E(M))^{G_E} \hookrightarrow (E_s \otimes_E M)^{G_E} = M$$

So to prove that $D_E \circ V_E \cong id$, we only need to prove that $\#V_E(M) = p^d$ for $d = \dim_E M$. To do this, note that we have an isomorphism

$$V_E(M) \xrightarrow{\sim} \mathrm{Hom}_{E,\varphi}(M^\vee, E_s)$$

Choose a basis $\{m_1^\vee, \ldots, m_d^\vee\}$ of $M^\vee$ and $\varphi_{M^\vee}(m_j^\vee) = \sum_i c_{ij} m_i^\vee$ with $(c_{ij})_{i,j} \in \mathrm{Gl}_d(E)$. In general, an E-linear map $M^\vee \to E_s$ is given by $m_i^\vee \to x_i \in E_s$ for each i, and the compatibility with the φ-action imposes the equality $x_j^p = \sum_i c_{ij} x_i$ for all j. Therefore, we have an identification $V_E(M) = \mathrm{Hom}_{E\text{-alg}}(A, E_s)$ where

$$A = E[X_1, \ldots, X_d]/(X_j^p - \sum_i c_{ij} X_i)_{1 \le j \le d}$$

A is a finite E-algebra of rank p^d and we want to prove that the set of its E_s-valued points has size equal to $p^d = \dim_E A$. This amounts to saying that A is an étale E-algebra which can be checked by proving the vanishing of $\Omega_{A/E}$. But by direct computation, we have

$$\Omega_{A/E} = (\oplus_j A \cdot \mathrm{d}X_j)/(\sum_j c_{ij} X_i)_{1 \le j \le d}$$

but since the matrix $(c_{ij})_{i,j}$ is invertible in E hence in A, the vanishing follows. □

Using this result, we are able to describe the category $\mathrm{Rep}_{\mathbb{Z}_p}(G_E)$ of continuous G_E-representations on finitely generated $\mathbb{Z}_p$-modules by successive approximation. To do this, we need a process that allows us to go from characteristic p objects to characteristic 0 objects. The main tool in our disposal is use of Witt vectors, but recall that we are dealing with fields E that are not necessarily perfect for which the Witt vectors $W(E)$ do not behave well. Nevertheless, to carry on with the theory, we need to suppose that there exists a complete discrete valuation ring $\mathcal{O}_\mathcal{E}$ of characteristic 0, with uniformizer p, fraction field $\mathcal{E}$, and residue field E. Let us also assume that this ring is equipped with an endomorphism $\varphi : \mathcal{O}_\mathcal{E} \to \mathcal{O}_\mathcal{E}$ lifting the Frobenius morphism on E. We will construct this ring in the specific cases that we will need.

Definition 3.4. The category $\Phi\mathrm{M}^{\text{ét}}_{\mathcal{O}_\mathcal{E}}$ of étale φ-modules over $\mathcal{O}_\mathcal{E}$ consists of pairs $(\mathcal{M}, \varphi_\mathcal{M})$ where $\mathcal{M}$ is a finitely generated $\mathcal{O}_\mathcal{E}$-module and $\varphi_\mathcal{M}$ is a φ-semilinear endomorphism of $\mathcal{M}$ whose $\mathcal{O}_\mathcal{E}$-linearisation $\varphi^*_{\mathcal{O}_\mathcal{E}}(\mathcal{M}) \to \mathcal{M}$ is an isomorphism.

Consider the (unique) maximal unramified extension $\mathcal{E}^{\mathrm{unr}}$ of $\mathcal{E}$ with ring of integers $\mathcal{O}_\mathcal{E}^{\mathrm{unr}}$. The ring $\mathcal{O}_\mathcal{E}^{\mathrm{unr}}$ is the strict Hensialisation of $\mathcal{O}_\mathcal{E}$ and it is a discrete valuation ring with uniformizer p and residue field E_s. By lemma [Sta18, $Tag04GP$], if $f : \mathcal{O}_\mathcal{E} \to \mathcal{O}_\mathcal{E}$ is a local map whose reduction $\overline{f} : E \to E$ lifts to a map $\overline{f}' : E_s \to E_s$, there exists a unique local map $f' : \mathcal{O}_\mathcal{E}^{\mathrm{unr}} \to \mathcal{O}_\mathcal{E}^{\mathrm{unr}}$ lifting f. Applying this to φ, we get a unique local morphism still denoted φ on $\mathcal{O}_\mathcal{E}^{\mathrm{unr}}$ lifting φ and reducing to the Frobenius morphism on E_s.
Likewise, taking $f = \mathrm{id}$ and $\overline{f}'$ to be an element of $\mathrm{Gal}(E_s/E)$, we also get an action of G_E on $\mathcal{O}_\mathcal{E}^{\mathrm{unr}}$ which is simply the identification of $\mathrm{Gal}(\mathcal{E}^{\mathrm{unr}}/\mathcal{E})$. Moreover, this action of G_E is continuous and commutes with φ by uniqueness of the lift f'. Taking the p-adic completion $\widehat{\mathcal{O}_\mathcal{E}^{\mathrm{unr}}}$ we get the same properties. In fact we have the following useful properties

Lemma 3.5. *[BC09, 3.2.4] We have the following identities*

$$\widehat{\mathcal{O}_{\mathcal{E}}^{unr}}^{G_E} = \mathcal{O}_{\mathcal{E}} \qquad \widehat{\mathcal{E}^{unr}}^{G_E} = \mathcal{E}$$
$$\widehat{\mathcal{O}_{\mathcal{E}}^{unr}}^{\varphi=1} = \mathbb{Z}_p \qquad \widehat{\mathcal{E}^{unr}}^{\varphi=1} = \mathbb{Q}_p$$

Proof. Note that we only need to prove the integral claims. The inclusions $\mathcal{Z}_p \hookrightarrow \widehat{\mathcal{O}_{\mathcal{E}}^{\mathrm{unr}}}^{\varphi=1}$ and $\mathcal{O}_{\mathcal{E}} \hookrightarrow \widehat{\mathcal{O}_{\mathcal{E}}^{\mathrm{unr}}}^{G_E}$ are local maps between p-adically separated and complete local rings. So it suffices to show the equality modulo p^n for $n \geq 1$. We do this by induction, so let us first check this for $n = 1$.
Taking the G_E-invariants of the exact sequence

$$0 \to \widehat{\mathcal{O}_{\mathcal{E}}^{\mathrm{unr}}} \xrightarrow{\times p} \widehat{\mathcal{O}_{\mathcal{E}}^{\mathrm{unr}}} \to E_s \to 0$$

we get an injection $\widehat{\mathcal{O}_{\mathcal{E}}^{\mathrm{unr}}}^{G_E}/(p) \hookrightarrow E_s^{G_E} = E$ which is bijective since the image contains $\mathcal{O}_{\mathcal{E}}/(p) = E$, showing that $\mathcal{O}_{\mathcal{E}} \to \widehat{\mathcal{O}_{\mathcal{E}}^{\mathrm{unr}}}^{G_E}/(p)$ is surjective. Similarly, given that $E_s^{\varphi=1} = \mathbb{F}_p$, we get the same conclusion for $\mathbb{Z}_p$.
Now suppose that $n > 1$ and that $\mathcal{O}_{\mathcal{E}} \to \widehat{\mathcal{O}_{\mathcal{E}}^{\mathrm{unr}}}^{G_E}/(p^{n-1})$ is surjective. Let $\xi \in \widehat{\mathcal{O}_{\mathcal{E}}^{\mathrm{unr}}}^{G_E}$, we need to show that there exists $x \in \mathcal{O}_{\mathcal{E}}$ such that $x \equiv \xi \mod p^n$. By the induction hypothesis, we can choose $c \in \mathcal{O}_{\mathcal{E}}$ such that $\xi = c + p^{n-1}\xi'$ with $c \in \mathcal{O}_{\mathcal{E}}$ and $\xi' \in \widehat{\mathcal{O}_{\mathcal{E}}^{\mathrm{unr}}}^{G_E}$. Applying the case $n = 1$ to ξ', we get an element $c' \in \mathcal{O}_{\mathcal{E}}$ such that $\xi' \equiv c' \mod p$. Hence $\xi \equiv c + p^{n-1}c' \mod p^n$, with $c + p^{n-1}c' \in \mathcal{O}_{\mathcal{E}}$ which is what we wanted to prove. The second case follows using the same argument. $\square$

Now we are able to state the main theorem of this section which is due to Fontaine:

Theorem 3.6. *There are covariant naturally quasi-inverse equivalences of abelian categories*

$$D_{\mathcal{E}} : \mathrm{Rep}_{\mathbb{Z}_p}(G_E) \to \Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}}}^{\acute{e}t}, \qquad V_{\mathcal{E}} : \Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}}}^{\acute{e}t} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_E)$$

defined by

$$D_{\mathcal{E}}(V) = (\widehat{\mathcal{O}_{\mathcal{E}}^{unr}}^{G_E} \otimes_{\mathbb{Z}_p} V)^{G_E} \qquad V_{\mathcal{E}}(M) = (\widehat{\mathcal{O}_{\mathcal{E}}^{unr}}^{G_E} \otimes_{\mathcal{O}\ \mathcal{E}} M)^{\varphi=1}$$

References

[BC09] O. Brinon and B. Conrad. Cmi summer school notes on p-adic hodge theory. 2009.

[Sta18] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2018.