§ 1. Introduction to p-divisible groups and Dieudonné modules

What is p-divisible group?

Abelian Schemes

A an S-abelian scheme

$\downarrow$ proper, smooth, group S-scheme, with geo. connected fibres

S a scheme

Ex. Elliptic curves over S

Ex. $\mathrm{Pic}^0_{X/S}$, X/S proper, smooth curve with geo. connected fibres.

A/S family of abelian varieties parametrized by S

$A/\mathbb{C} \rightsquigarrow \mathbb{C}^g/\mathbb{Z}^{2g}$. $(g = \dim_S A)$

p-div group associated to A/S

$G_n = A[p^n]$ the kernel of $A \xrightarrow{\cdot p^n} A$

Then G_n is a finite flat p^n-torsion group S-scheme of order p^{2gn}.

Inductive system: $G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \cdots \hookrightarrow G_n \hookrightarrow G_{n+1}$

where $G_n = G_{n+1}[p^n]$.

Defn. A p-divisible group (Barsotti-Tate group) of height h over S is an inductive system $G = (G_n)_{n\geq 1}$ such that

(1). G_n is p^n-torsion of order p^{nh} $(h = 2g)$

(2). $G_n \xrightarrow{\sim} G_{n+1}[p^n] \hookrightarrow G_{n+1}$ gives the transition maps.

Tate modules associated to G

Inverse system. $\cdots \xrightarrow{\cdot p} G_{n+1} \xrightarrow{\cdot p} G_n \xrightarrow{\cdot p} \cdots \xrightarrow{\cdot p} G_2 \xrightarrow{\cdot p} G_1$

If $S = \mathrm{Spec}\, k$.

Defn. $T_p(G) = \varprojlim_{n\geq 1} G_n(\bar{k})$

Ex: $k = \bar{k}$, $T_p(A[p^\infty]) = \begin{cases} \mathbb{Z}_p^{2g} & \text{if } p \neq \operatorname{char} k \\ \mathbb{Z}_p^{\leq 2g} & \text{if } p = \operatorname{char} k \end{cases}$.

$$\boxed{\begin{array}{ccccc} \text{Abelian Schemes} & \overset{①}{\rightsquigarrow} & p\text{-Divisible Groups} & \overset{②}{\rightsquigarrow} & \text{Linear Algebra} \\ A & \longmapsto & A[p^\infty] & \longmapsto & A[p^\infty](k) \ (\text{Tate module}) \end{array}}$$

This process will capture important geometric information of A.

K discrete valuation field of char 0, with perfect residue field of char p

$$A/K \text{ abelian variety} \overset{①}{\rightsquigarrow} G = A[p^\infty] \overset{②}{\rightsquigarrow} T_pG = \varprojlim_{n\geq 1} A[p^n](\overline{K}) \cong \mathbb{Z}_p^{2g}$$

$$G = A[\ell^\infty] \rightsquigarrow T_\ell G = \varprojlim_{n\geq 1} A[\ell^n](\overline{K}) \quad \circlearrowleft \text{ Galois action } G_K \quad (\ell \neq p)$$

Ex 1. (Étale cohomology)

$$H^i_{ét}(A, \mathbb{Z}_p/\mathbb{Z}_\ell) \quad \Lambda^i_{\mathbb{Z}_p} H^1_{ét}(A, \mathbb{Z}_p/\mathbb{Z}_\ell) \cong \mathbb{Z}_\ell^{2g}$$

$$H^1_{ét}(A, \mathbb{Z}_p(\mathbb{Z}_\ell)) = \mathrm{Hom}_{\mathbb{Z}_p}(T_pG, \mathbb{Z}_p(\mathbb{Z}_\ell)) \cong \mathbb{Z}_p^{2g} / \mathbb{Z}_\ell^{2g}$$

Ex 2. (Hodge-Tate decomposition)

$$0 \to H^1(A, \mathcal{O}_A) \otimes_K \widehat{\overline{K}} \to H^1_{ét}(A, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \widehat{\overline{K}} \to H^0(A, \Omega^1_{A/K}) \otimes_K \widehat{\overline{K}}(-1) \to 0$$

If A has good reduction, i.e.

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{A} \arrow[d] \\ K \arrow[r] & \mathcal{O}_K \end{tikzcd} \quad \tilde{A} \text{ abelian scheme}$$

then $\tilde{A}$ gives a p-divisible group $\tilde{G}/\mathcal{O}_K$

Tate (1967) used $\tilde{G}$ to prove HT-decomposition for A.

Ex 3. (Criteria for good reduction)

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Néron-Ogg-Shafarevich

(Ab Var) $\quad A$ has good reduction $\quad \begin{array}{c}\tilde{A} \\ | \\ \mathcal{O}_K \\ (0,p)\end{array}$ using information at $\ell \neq p$ $\quad \iff \quad \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell G$ is an unramified G_K-repn. $\quad$ (Linear Alg) $(\ell \neq p)$

Grothendieck $\Updownarrow$ $\qquad\qquad\qquad \Updownarrow$

后 $\quad G_\ell$ has good reduction $\quad G_p \quad \begin{array}{c}\tilde{G}_\ell, \tilde{G}_p \\ | \\ \mathcal{O}_K\end{array}$ $\quad \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_pG$ is a crystalline G_K-repn $\quad (\ell = p)$ $\quad$ (Fontaine-Breuil-Kisin)

$(\ell, p\text{-div grp})$

using information at p ?

最后.

To reach FBK, we need to understand p-div grp $\tilde{\mathcal{A}}/\mathcal{O}_K$ by linear alg.

Now the problem moves from generic fibre to integral level, to residue field.

① p-divisible grp over perfect field k of char $p>0$ $\xleftrightarrow{\text{Dieudonné}}$ Dieudonné modules

deformation of p-div grp where p is locally nilpotent $\xleftrightarrow{\text{Grothendieck-Messing}}$ Dieudonné crystals

Next goal

(p-div grp over $W(k)$ $\xleftrightarrow{\text{Barsotti, Honda, Fontaine}}$ Honda systems)

② p-div grp over $\mathcal{O}_K$ $\xleftrightarrow{\text{Breuil-Kisin}}$ Breuil-Kisin modules of Basotti-Tate type.