Sets with large intersection and ubiquity

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Abstract

A central problem motivated by Diophantine approximation is to determine the size properties of subsets of $\mathbb{R}^d$ ($d \in \mathbb{N}$) of the form

$$F_\varphi = \{x \in \mathbb{R}^d \mid \|x - x_i\| < \varphi(r_i) \text{ for infinitely many } i \in I\}$$

where $\|\cdot\|$ denotes an arbitrary norm, I a denumerable set, $(x_i, r_i)_{i\in I}$ a family of elements of $\mathbb{R}^d \times (0,\infty)$ and φ a nonnegative nondecreasing function defined on $[0,\infty)$. We show that if F_{Id}, where Id denotes the identity function, has full Lebesgue measure in a given nonempty open subset V of $\mathbb{R}^d$, the set F_φ belongs to a class $\mathrm{G}^h(V)$ of sets with large intersection in V with respect to a given gauge function h. We establish that this class is closed under countable intersections and that each of its members has infinite Hausdorff g-measure for every gauge function g which increases faster than h near zero. In particular, this yields a sufficient condition on a gauge function g such that a given countable intersection of sets of the form F_φ has infinite Hausdorff g-measure. In addition, we supply several applications of our results to Diophantine approximation. For any nonincreasing sequence ψ of positive real numbers converging to zero, we investigate the size and large intersection properties of the sets of all points that are ψ-approximable by rationals, by rationals with restricted numerator and denominator and by real algebraic numbers. This enables us to refine the analogs of Jarník's theorem for these sets. We also study the approximation of zero by values of integer polynomials and deduce several new results concerning Mahler's and Koksma's classifications of real transcendental numbers.

1. *Introduction*

In Diophantine approximation, one is often interested in determining the size properties of subsets of $\mathbb{R}^d$ ($d \in \mathbb{N}$) of the form

$$F_\varphi = \left\{x \in \mathbb{R}^d \;\middle|\; \|x - x_i\| < \varphi(r_i) \text{ for infinitely many } i \in I\right\} \tag{1·1}$$

where $\|\cdot\|$ denotes an arbitrary norm, I a denumerable set, $(x_i, r_i)_{i\in I}$ a family of elements of $\mathbb{R}^d \times (0,\infty)$ and φ a nonnegative nondecreasing function defined on $[0,\infty)$. When this set has Lebesgue measure zero, one usually seeks its Hausdorff dimension and even aims at determining its Hausdorff h-measure $\mathcal{H}^h$ (see Section 2 for the definition) for every gauge function h when a more accurate information is needed.

Definition. A gauge function is a nondecreasing function h defined on $[0,\varepsilon]$ for some $\varepsilon > 0$ and such that $\lim_{0^+} h = h(0) = 0$. The set of gauge functions is denoted by $\mathfrak{D}$.

In this paper we show that a very simple hypothesis on $(x_i, r_i)_{i\in I}$ suffices to ensure that the set F_φ enjoys a *large intersection* property. As a by-product, we obtain a sufficient condition on $h \in \mathfrak{D}$ such that a given countable intersection of sets of the form (1·1) has infinite Hausdorff h-measure.

The prototype of all sets of the form (1·1) is the set

$$J_{1,\tau} = \left\{ x \in \mathbb{R} \;\middle|\; \left| x - \frac{p}{q} \right| < q^{-\tau} \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\}$$

of all real numbers that are τ-approximable by rationals ($\tau > 0$). A well-known theorem of Dirichlet ensures that $J_{1,\tau} = \mathbb{R}$ if $\tau \leqslant 2$, see [**22**]. In the opposite case, $J_{1,\tau}$ has Lebesgue measure zero and Jarník and Besicovitch established that its Hausdorff dimension is $2/\tau$, see [**10**, **25**]. In addition, K. Falconer [**20**] proved that $J_{1,\tau}$ enjoys a large intersection property in the sense that it belongs to the class $\mathcal{G}^{2/\tau}(\mathbb{R})$. Recall that the class $\mathcal{G}^s(\mathbb{R}^d)$ of sets with large intersection of Hausdorff dimension at least a given $s \in (0,d]$ is the collection of all G_δ-subsets F of $\mathbb{R}^d$ that satisfy

$$\dim \bigcap_{n=1}^{\infty} f_n(F) \geqslant s \tag{1·2}$$

for every sequence $(f_n)_{n\in\mathbb{N}}$ of similarities, where dim stands for Hausdorff dimension. It is the maximal class of G_δ-sets of Hausdorff dimension at least s that is closed under countable intersections and similarities. K. Falconer introduced it in order to supply a general setting for various families of sets of dimension at least s which enjoy the remarkable property that countable intersections of the sets also have dimension at least s, see [**20**]. This property is somewhat counterintuitive because the dimension of the intersection of two subsets of $\mathbb{R}^d$ of dimensions d_1 and d_2 respectively is usually expected to be $d_1 + d_2 - d$, see [**21**, Chapter 8].

The set $J_{1,\tau}$ can be generalized in the following manner. Let $\psi = (\psi(q))_{q\in\mathbb{N}}$ be a nonincreasing sequence of positive real numbers converging to zero and let

$$K_{d,\psi} = \left\{ x \in \mathbb{R}^d \;\middle|\; \left\| x - \frac{p}{q} \right\| < \psi(q) \text{ for infinitely many } (p,q) \in \mathbb{Z}^d \times \mathbb{N} \right\}$$

be the set of all points that are ψ-approximable by rationals. Note that this set is of the form (1·1). Khintchine [**27**] established that it has full (resp. zero) Lebesgue measure in $\mathbb{R}^d$ if $\sum_q \psi(q)^d q^d = \infty$ (resp. $< \infty$). Jarník [**26**] described more precisely the size properties of $K_{d,\psi}$. Specifically he determined its Hausdorff h-measure for h in the set $\mathfrak{D}_d$ that is defined as follows.

Definition. Let $\mathfrak{D}_d$ be the set of all $h \in \mathfrak{D}$ such that $r \mapsto h(r)/r^d$ is positive and nonincreasing on $(0,\varepsilon]$ for some $\varepsilon > 0$.

It is easy to check that any function in $\mathfrak{D}_d$ is continuous in a neighborhood of zero. For $g, h \in \mathfrak{D}_d$, let us write $g \prec h$ if g/h monotonically tends to infinity at zero. Moreover, let Id denote the identity function. The result of Jarník, later refined by V. Beresnevich, D. Dickinson and S. Velani [**7**], is the following: for every $h \in \mathfrak{D}_d$ with $h \prec \mathrm{Id}^d$, the set $K_{d,\psi}$ has infinite (resp. zero) Hausdorff h-measure if $\sum_q h(\psi(q)) q^d = \infty$ (resp. $< \infty$).

This criterion yields the Hausdorff dimension $s_{d,\psi}$ of $K_{d,\psi}$. On top of that, the results of Section 4 enable to show that $K_{d,\psi}$ belongs to the class $\mathcal{G}^{s_{d,\psi}}(\mathbb{R}^d)$ of sets with large intersection when $s_{d,\psi} > 0$. However, this property is not really satisfactory for the following reasons. While ψ is a refinement of $(q^{-\tau})_{q\in\mathbb{N}}$, the class $\mathcal{G}^{s_{d,\psi}}(\mathbb{R}^d)$ of K. Falconer which contains $K_{d,\psi}$ cannot give a precise account of the potential complexity of ψ. More generally, the fact that a given set belongs to $\mathcal{G}^s(\mathbb{R}^d)$ $(0 < s \leqslant d)$ shows that it enjoys a large intersection property and is of dimension at least s, but yields no more accurate information as regards its size.

This is mainly to cope with that problem that we introduce in Section 2 new classes of sets with large intersection which are finer than those of K. Falconer. To be specific, given a gauge function $h \in \mathfrak{D}_d$ and a nonempty open subset V of $\mathbb{R}^d$, we define a class $\mathrm{G}^h(V)$ of sets with large intersection in V with respect to h. The class $\mathrm{G}^h(V)$ is closed under countable intersections and each of its members has infinite Hausdorff g-measure, for every gauge function $g \in \mathfrak{D}_d$ with $g \prec h$. Moreover, the classes $\mathrm{G}^h(V)$ allow to perform a precise study of the large intersection properties of $K_{d,\psi}$. Indeed, Theorem 5 in Section 4 shows that for every gauge function $h \in \mathfrak{D}_d$ and every nonempty open subset V of $\mathbb{R}^d$, the set $K_{d,\psi}$ belongs to $\mathrm{G}^h(V)$ if and only if $\sum_q h(\psi(q))q^d$ diverges. This result also completely describes the size properties of $K_{d,\psi}$. Specifically, for $h \in \mathfrak{D}$, let

$$h_d : r \mapsto r^d \inf_{\rho\in(0,r]} \frac{h(\rho)}{\rho^d}. \tag{1·3}$$

If $\sum_q h_d(\psi(q))q^d = \infty$ (resp. $< \infty$), then $\mathcal{H}^h(K_{d,\psi} \cap V) = \mathcal{H}^h(V)$ (resp. $= 0$) for every open V. This is an extension of Jarník's theorem to all the gauge functions in $\mathfrak{D}$.

Furthermore, since the set L_1 of all Liouville numbers may be expressed as a countable intersection of sets of the form $K_{1,\psi}$, we can infer its size and large intersection properties, thereby improving a result of L. Olsen and D. Renfro [**31**, **32**]. Note that L_1 cannot belong to any class of K. Falconer since its Hausdorff dimension is zero. Thus it is necessary to use the classes $\mathrm{G}^h(V)$ in order to study the large intersection properties of this set.

A. Baker and W. Schmidt [**2**] as well as V. Beresnevich [**3**] and Y. Bugeaud [**12**, **15**] investigated the size properties of another generalization of $J_{1,\tau}$, where the real numbers are approached by real algebraic numbers rather than rationals only. For $n \in \mathbb{N}$, let $\mathbb{A}_n$ be the set of all real algebraic numbers of degree at most n. The height $H(a)$ of $a \in \mathbb{A}_n$ is the maximum of the absolute values of the coefficients of its minimal defining polynomial over $\mathbb{Z}$. Let $\psi = (\psi(q))_{q\in\mathbb{N}}$ be a nonincreasing sequence of positive real numbers converging to zero and let

$$A_{n,\psi} = \left\{x \in \mathbb{R} \;\middle|\; |x - a| < \psi(H(a)) \text{ for infinitely many } a \in \mathbb{A}_n\right\}$$

be the set of all real numbers that are ψ-approximable by real algebraic numbers of degree at most n. V. Beresnevich [**3**] established a Khintchine-type result for $A_{n,\psi}$. This set has full (resp. zero) Lebesgue measure in $\mathbb{R}$ if $\sum_h \psi(h)h^n = \infty$ (resp. $< \infty$). Moreover, Y. Bugeaud [**12**] proved an analog of Jarník's theorem: for every $g \in \mathfrak{D}_1$ with $g \prec \mathrm{Id}$, the set $A_{n,\psi}$ has infinite (resp. zero) Hausdorff g-measure if $\sum_h g(\psi(h))h^n = \infty$ (resp. $< \infty$). In [**15**] he also showed that $A_{n,\psi}$ enjoys a large intersection property when ψ is the product of $(h^{-\omega-1})_{h\in\mathbb{N}}$, for $\omega \geqslant n$, and a logarithmic correction. As a complement, Theorem 9 in Section 4 shows that for every gauge function $g \in \mathfrak{D}_1$ and every nonempty open subset V of $\mathbb{R}$, the set $A_{n,\psi}$ belongs to $\mathrm{G}^g(V)$ if and only if $\sum_h g(\psi(h))h^n$ diverges. This yields new results concerning Koksma's classification of real transcendental numbers [**28**]. In addition, Theorem 9 completely describes the size properties of $A_{n,\psi}$: for every $g \in \mathfrak{D}$ and every open V, we have $\mathcal{H}^g(A_{n,\psi} \cap V) = \mathcal{H}^g(V)$ (resp. $= 0$) if $\sum_h g_1(\psi(h))h^n = \infty$ (resp. $< \infty$), where g_1 is defined as in (1·3).

The previous examples are studied in detail in Section 4. We also investigate the size and large intersection properties of the sets which occur in simultaneous inhomogeneous Diophantine approximation, in Diophantine approximation with restrictions and in the context of the approximation of zero by values of integer polynomials. This last problem is related to Mahler's classification of real transcendental numbers [**30**].

Let us consider the set F_φ given by (1·1) again. By covering F_φ appropriately, it is usually obvious to supply a sufficient condition on a gauge function h to ensure that it has zero Hausdorff h-measure. Conversely, it is often much more difficult to provide a sufficient condition on h such that F_φ has infinite h-measure. This problem was basically solved by Y. Bugeaud [**14**] in the case where the family $(x_i, r_i)_{i\in I}$ comes from an *optimal regular system* of points. In dimension $d = 1$, Y. Bugeaud even proved in [**15**] that F_φ enjoys a large intersection property. Under the same hypothesis, A. Baker and W. Schmidt [**2**] had given an accurate lower bound on the Haudorff dimension of F_φ. Likewise, the problem was solved by V. Beresnevich, D. Dickinson and S. Velani [**7**] in the case where $(x_i, r_i)_{i\in I}$ forms a *ubiquitous system*. A similar notion had been introduced by M. Dodson, B. Rynne and J. Vickers [**16**] in order to give a lower bound on the dimension of F_φ. In addition, J.-M. Aubry and S. Jaffard [**1**, **24**] investigated the problem with a view to performing the multifractal analysis of some random processes. The trouble is that the notions of optimal regular system and of ubiquitous system in the sense of [**7**] are rather technical. In addition, whereas the large intersection properties of F_φ have been exhibited in some particular cases (e.g. the set $J_{1,\tau}$ of all real numbers that are τ-approximable by rationals [**20**] or the set of all real numbers that are approached by the points of an optimal regular system [**15**]), they have never been investigated systematically.

We show in this paper that a very simple hypothesis on the family $(x_i, r_i)_{i\in I}$ suffices to ensure that F_φ always belongs to a certain class $\mathrm{G}^h(V)$ of sets with large intersection. On the one hand, this supplies a sufficient condition on $g \in \mathfrak{D}$ such that F_φ has infinite Hausdorff g-measure. On the other hand, this enables to investigate the size properties of a countable intersection of sets of the form (1·1). The aforementioned hypothesis is that the set

$$F_{\mathrm{Id}} = \left\{x \in \mathbb{R}^d \;\middle|\; \|x - x_i\| < r_i \text{ for infinitely many } i \in I\right\}$$

has full Lebesgue measure in a given nonempty open subset V of $\mathbb{R}^d$. In this case, $(x_i, r_i)_{i\in I}$ is called a *homogeneous ubiquitous system* in V. This kind of property can easily be established if a Khintchine-type result holds. By way of illustration, the family $(p/q, \psi(q))_{(p,q)\in\mathbb{Z}^d\times\mathbb{N}}$ is a homogeneous ubiquitous system in $\mathbb{R}^d$ if $\sum_q \psi(q)^d q^d = \infty$ by virtue of Khintchine's theorem. In the same vein, $(a, \psi(H(a)))_{a\in\mathbb{A}_n}$ is a homogeneous ubiquitous system in $\mathbb{R}$ if $\sum_h \psi(h)h^n = \infty$. More generally, a result of V. Beresnevich [**4**] enables to prove that an optimal regular system of points leads to a homogeneous ubiquitous system. Thus the results of this paper are relevant in all the situations where optimal regular systems of points arise.

Theorem 2 in Section 3 shows that if $(x_i, r_i)_{i\in I}$ is a homogeneous ubiquitous system in a fixed nonempty open subset V of $\mathbb{R}^d$, the set F_φ lies in the class $\mathrm{G}^h(V)$ for every gauge function $h \in \mathfrak{D}_d$ and every function φ which coincides with the pseudo-inverse function of $h^{1/d}$ in a neighborhood of zero. Basically, this theorem systematically converts a Khintchine-type result into a large intersection property. For example, it implies that $K_{d,\psi} \in \mathrm{G}^h(V)$ if $\sum_q h(\psi(q))q^d = \infty$. In practice, Theorem 2 also enables to effortlessly deduce a Jarník-type result from a Khintchine-type one. This way, we establish that $\mathcal{H}^h(K_{d,\psi} \cap V) = \mathcal{H}^h(V)$ for every $h \in \mathfrak{D}$ with $\sum_q h_d(\psi(q))q^d = \infty$. We refer to Section 4 for many other applications of Theorem 2.

The paper is organized as follows. We define in Section 2 the class $\mathrm{G}^h(V)$ of sets with large intersection in a given nonempty open subset V of $\mathbb{R}^d$ with respect to a given gauge function $h \in \mathfrak{D}_d$ and we state its main properties. This is the purpose of Proposition 1 and Theorem 1. In Section 3 we introduce the notion of homogeneous ubiquitous system in a given nonempty open subset V of $\mathbb{R}^d$ and we state Theorem 2, which describes the large intersection properties of the set F_φ defined by (1·1) when $(x_i, r_i)_{i\in I}$ is a homogeneous ubiquitous system. Section 4 provides several applications to the theory of Diophantine approximation. We investigate the size and large intersection properties of the set of all points that are ψ-approximable by rationals (in the homogeneous case as well as in the inhomogeneous one), by rationals with restricted numerator and denominator and by real algebraic numbers. Moreover, we study the approximation of zero by values of integer polynomials and we deduce some results concerning Mahler's and Koksma's classifications of real transcendental numbers. Section 5 and Section 6 are devoted to proving Theorem 1, Theorem 2 and ancillary results.

2. *Sets with large intersection*

Recall that the set $\mathfrak{D}$ of gauge functions is defined at the beginning of Section 1. For every $h \in \mathfrak{D}$, the Hausdorff h-measure of $F \subseteq \mathbb{R}^d$ is given by

$$\mathcal{H}^h(F) = \lim_{\delta \downarrow 0} \uparrow \inf_{\substack{F \subseteq \bigcup_p U_p \\ |U_p| < \delta}} \sum_{p=1}^{\infty} h(|U_p|).$$

The infimum is taken over all sequences $(U_p)_{p\in\mathbb{N}}$ of sets with $F \subseteq \bigcup_p U_p$ and $|U_p| < \delta$ for all $p \in \mathbb{N}$, where $|\cdot|$ denotes diameter. Note that $\mathcal{H}^h$ is a Borel measure on $\mathbb{R}^d$, see [**33**]. The Hausdorff dimension of a nonempty set $F \subseteq \mathbb{R}^d$ is defined by

$$\dim F = \sup\{s \in (0,d) \mid \mathcal{H}^{\mathrm{Id}^s}(F) = \infty\} = \inf\{s \in (0,d) \mid \mathcal{H}^{\mathrm{Id}^s}(F) = 0\}$$

with the convention that $\sup \emptyset = 0$ and $\inf \emptyset = d$, see [**21**].

The classes of sets with large intersection considered by K. Falconer in [**20**] only refer to the functions Id^s, where $s \in (0,d]$. Those introduced in this section are associated with the functions that belong to the set $\mathfrak{D}_d$ defined in Section 1. The classes of K. Falconer also necessarily refer to a large intersection property in the whole space $\mathbb{R}^d$ because all similarities come into play in their definition, see (1·2). However, large intersection properties are often investigated in a subset of $\mathbb{R}^d$, see Section 3 and Section 4. Thus the classes that we introduce are not defined using similarities. Instead, we make use of outer net measures analogous to those which arise in the study of the $\mathcal{M}^s_\infty$-dense construction of K. Falconer [**19**] or characterize the classes $\mathcal{G}^s(\mathbb{R}^d)$, see [**20**, Theorem B].

Given an integer $c \geqslant 2$, let Λ_c be the collection of all c-adic cubes of $\mathbb{R}^d$, that is, sets of the form $\lambda = c^{-j}(k + [0,1)^d)$ for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$. The integer j is the generation of λ, denoted by $\langle\lambda\rangle_c$. For any $h \in \mathfrak{D}_d$, the set of all $\varepsilon \in (0,1]$ such that h is nondecreasing on $[0,\varepsilon]$ and $r \mapsto h(r)/r^d$ is nonincreasing on $(0,\varepsilon]$ is nonempty. Let ε_h denote its supremum and let $\Lambda_{c,h}$ be the set of all cubes $\lambda \in \Lambda_c$ with $|\lambda| < \varepsilon_h$. The outer net measure associated with $h \in \mathfrak{D}_d$ is defined by

$$\forall F \subseteq \mathbb{R}^d \qquad \mathcal{M}^h_\infty(F) = \inf_{(\lambda_p)_{p\in\mathbb{N}} \in R_{c,h}(F)} \sum_{p=1}^{\infty} h(|\lambda_p|) \tag{2·1}$$

where $R_{c,h}(F)$ is the set of all sequences $(\lambda_p)_{p\in\mathbb{N}}$ in $\Lambda_{c,h} \cup \{\emptyset\}$ enjoying $F \subseteq \bigcup_p \lambda_p$. The outer measure $\mathcal{M}^h_\infty$ is related to $\mathcal{H}^h$, see [**33**, Theorem 49]. In particular, if a subset F of $\mathbb{R}^d$ enjoys $\mathcal{M}^h_\infty(F) > 0$ then $\mathcal{H}^h(F) > 0$. We can now define the classes of sets with large intersection we are interested in. Recall that a G_δ-set is one that may be expressed as a countable intersection of open sets.

Definition. Let $h \in \mathfrak{D}_d$ and let V be a nonempty open subset of $\mathbb{R}^d$. The class $\mathrm{G}^h(V)$ of subsets of $\mathbb{R}^d$ with large intersection in V with respect to h is the collection of all G_δ-subsets F of $\mathbb{R}^d$ such that $\mathcal{M}^g_\infty(F \cap U) = \mathcal{M}^g_\infty(U)$ for every $g \in \mathfrak{D}_d$ enjoying $g \prec h$ and every open set $U \subseteq V$.

Remarks. Proposition 13 in Section 5 shows that $\mathrm{G}^h(V)$ depends on the choice of neither the integer c nor the norm $\mathbb{R}^d$ is endowed with, even if they affect the construction of $\mathcal{M}^g_\infty$ for any $g \in \mathfrak{D}_d$ with $g \prec h$.

Y. Bugeaud [**15**] generalized the classes of K. Falconer as follows. For any gauge function h, $\mathcal{G}^h(\mathbb{R}^d)$ denotes the class of all G_δ-subsets F of $\mathbb{R}^d$ such that

$$\mathcal{H}^g\left(\bigcap_{n=1}^{\infty} f_n(F)\right) = \infty \tag{2·2}$$

for every sequence $(f_n)_{n\in\mathbb{N}}$ of similarities and every gauge function $g \prec h$. However, the stability of $\mathcal{G}^h(\mathbb{R}^d)$ under countable intersections is proven in [**15**] for only very specific gauge functions h. First, the gauge functions considered by Y. Bugeaud are strictly increasing and concave in a neighborhood of the origin. Second, in order to prove the implication $(b) \Rightarrow (c)$ of [**15**, Theorem 6] just by adapting the proof of [**20**, Theorem B] as Y. Bugeaud did, it is necessary to assume that the gauge functions enjoy a certain scaling property.

The next proposition follows immediately from the definition of $\mathrm{G}^h(V)$.

PROPOSITION 1. *Let $h \in \mathfrak{D}_d$ and let V be a nonempty open subset of $\mathbb{R}^d$. Then*
(a) $\mathrm{G}^{h_1}(V) \supseteq \mathrm{G}^{h_2}(V)$ *for every $h_1, h_2 \in \mathfrak{D}_d$ with $h_1 \prec h_2$;*
(b) $\mathrm{G}^h(V_1) \supseteq \mathrm{G}^h(V_2)$ *for every nonempty open sets $V_1, V_2 \subseteq \mathbb{R}^d$ with $V_1 \subseteq V_2$;*
(c) $\mathrm{G}^h(V) = \bigcap_g \mathrm{G}^g(V)$ *where $g \in \mathfrak{D}_d$ enjoys $g \prec h$;*
(d) $\mathrm{G}^h(V) = \bigcap_U \mathrm{G}^h(U)$ *where U is a nonempty open subset of V;*
(e) *every G_δ-set which contains a set of $\mathrm{G}^h(V)$ also belongs to $\mathrm{G}^h(V)$;*
(f) $F \cap U \in \mathrm{G}^h(U)$ *for every $F \in \mathrm{G}^h(V)$ and every nonempty open set $U \subseteq V$.*

As shown by the following theorem, the class $\mathrm{G}^h(V)$ enjoys the same kind of stability properties as the class $\mathcal{G}^s(\mathbb{R}^d)$ of K. Falconer, see [**20**, Theorem A].

THEOREM 1. *Let $h \in \mathfrak{D}_d$ and let V be a nonempty open subset of $\mathbb{R}^d$. Then*
(a) *the class $\mathrm{G}^h(V)$ is closed under countable intersections;*
(b) *the set $f^{-1}(F)$ belongs to $\mathrm{G}^h(V)$ for every bi-Lipschitz mapping $f : V \to \mathbb{R}^d$ and every set $F \in \mathrm{G}^h(f(V))$;*

(c) *every set* $F \in \mathrm{G}^h(V)$ *enjoys* $\mathcal{H}^g(F) = \infty$ *for every* $g \in \mathfrak{D}_d$ *with* $g \prec h$ *and in particular* $\dim F \geqslant s_h = \sup\{s \in (0,d) \mid \mathrm{Id}^s \prec h\}$.

Theorem 1 is proven in Section 5. For any $h \in \mathfrak{D}_d$, it enables us to relate $\mathrm{G}^h(\mathbb{R}^d)$ to the classes of K. Falconer and those of Y. Bugeaud. Indeed, for $F \in \mathrm{G}^h(\mathbb{R}^d)$, this theorem shows that (2·2) holds for every sequence $(f_n)_{n\in\mathbb{N}}$ of similarities and every $g \in \mathfrak{D}_d$ with $g \prec h$. In particular (1·2) holds with $s = s_h$. Thus $\mathrm{G}^h(\mathbb{R}^d)$ is included in the class $\mathcal{G}^h(\mathbb{R}^d)$ of Y. Bugeaud and is strictly included in the class $\mathcal{G}^{s_h}(\mathbb{R}^d)$ of K. Falconer if $s_h > 0$.

Let us mention another important consequence of Theorem 1. Let $h \in \mathfrak{D}_d$ and let V be a nonempty open subset of $\mathbb{R}^d$. For every sequence $(F_n)_{n\in\mathbb{N}}$ of sets of the class $\mathrm{G}^h(V)$,

$$\forall g \in \mathfrak{D}_d \qquad g \prec h \quad \Longrightarrow \quad \mathcal{H}^g\left(\bigcap_{n=1}^{\infty} F_n\right) = \infty.$$

Hence $\dim \bigcap_n F_n \geqslant s_h$. In addition, if the dimension of F_n is at most s_h for some $n \in \mathbb{N}$, the previous intersection is of dimension s_h.

Since the classes $\mathrm{G}^h(V)$ are defined for $h \in \mathfrak{D}_d$ only, we shall restrict ourselves to the gauge functions in $\mathfrak{D}_d$ when we investigate large intersection properties. Moreover, when we study size properties, we should consider all the gauge functions in $\mathfrak{D}$ because the Hausdorff measures are defined for all these functions. The following result shows that it generally suffices to consider the gauge functions in $\mathfrak{D}_d$. Recall that for any $h \in \mathfrak{D}$ the function h_d is defined by (1·3) with $h_d(0) = 0$ and observe that h_d coincides with h in a neighborhood of the origin when $h \in \mathfrak{D}_d$.

PROPOSITION 2. *For every* $h \in \mathfrak{D}$, *we have* $h_d \in \mathfrak{D}_d \cup \{0\}$. *Moreover, there is a real number* $\kappa \geqslant 1$ *such that for every* $h \in \mathfrak{D}$ *and every* $F \subseteq \mathbb{R}^d$,

$$\mathcal{H}^{h_d}(F) \leqslant \mathcal{H}^h(F) \leqslant \kappa \mathcal{H}^{h_d}(F).$$

The proof of Proposition 2 is omitted because it is a straightforward extension of that of [**32**, Lemma 2.2] to any dimension $d \in \mathbb{N}$. Note that for $h \in \mathfrak{D}$, if $h(r)/r^d$ tends to infinity as $r \to 0$, every nonempty open subset of $\mathbb{R}^d$ has infinite Hausdorff h-measure. Otherwise, $h_d(r) = \mathrm{O}(r^d)$ as $r \to 0$ so that $\mathcal{H}^{h_d}$ is finite on every compact subset of $\mathbb{R}^d$. By Proposition 2, the measure $\mathcal{H}^h$ is also finite on compacts. Since it is a translation invariant Borel measure, it coincides up to a multiplicative constant with the Lebesgue measure on the Borel subsets of $\mathbb{R}^d$.

3. *Homogeneous ubiquity*

Let I denote a denumerable set and let $\mathcal{S}_d(I)$ be the set of all families $(x_i, r_i)_{i\in I}$ of elements of $\mathbb{R}^d \times (0,\infty)$ such that

$$\sup_{i\in I} r_i < \infty \qquad \text{and} \qquad \forall m \in \mathbb{N} \quad \#\left\{ i \in I \;\middle|\; \|x_i\| < m \text{ and } r_i > \frac{1}{m} \right\} < \infty.$$

This last condition is equivalent to the fact that for every bounded set $E \subseteq \mathbb{R}^d$ and every $\varepsilon > 0$ there are at most finitely many $i \in I$ satisfying $x_i \in E$ and $r_i > \varepsilon$.

Let $(x_i, r_i)_{i\in I} \in \mathcal{S}_d(I)$ and let $\varphi : [0,\infty) \to \mathbb{R}$ be a nonnegative nondecreasing function. We shall study the large intersection properties of

$$F_\varphi = \left\{ x \in \mathbb{R}^d \;\middle|\; \|x - x_i\| < \varphi(r_i) \text{ for infinitely many } i \in I \right\}. \tag{3·1}$$

Note that F_φ depends on φ only through its local behavior at zero. Indeed, $F_{\tilde{\varphi}} = F_\varphi$ for every nonnegative nondecreasing function $\tilde{\varphi} : [0,\infty) \to \mathbb{R}$ that coincides with φ in a neighborhood of the origin. Moreover F_φ is a G_δ-set since $F_\varphi = \bigcap_{n=1}^\infty \bigcup_{n'=n}^\infty B(x_{i_{n'}}, \varphi(r_{i_{n'}}))$ for any enumeration $(i_n)_{n\in\mathbb{N}}$ of I, with the convention that every open ball $B(x,r)$ with center $x \in \mathbb{R}^d$ is empty if its radius r vanishes. Theorem 2 below states that, under certain hypotheses on φ, F_φ is a set with large intersection if the family $(x_i, r_i)_{i\in I}$ satisfies the following definition.

Definition. Let I be a denumerable set and let V be a nonempty open subset of $\mathbb{R}^d$. A family $(x_i, r_i)_{i\in I} \in \mathcal{S}_d(I)$ is called a homogeneous ubiquitous system in V if the set F_{Id} given by (3·1) with $\varphi = \mathrm{Id}$ has full Lebesgue measure in V.

Remarks. As shown by Proposition 15 in Section 6, if $(x_i, r_i)_{i\in I} \in \mathcal{S}_d(I)$ is a homogeneous ubiquitous system in V, so is $(x_i, \kappa r_i)_{i\in I}$ for every $\kappa > 0$. Thus the fact that $(x_i, r_i)_{i\in I} \in \mathcal{S}_d(I)$ is a homogeneous ubiquitous system in V does not depend on the choice of the norm $\mathbb{R}^d$ is endowed with.

Let $(x_i, r_i)_{i\in I} \in \mathcal{S}_d(I)$ be a homogeneous ubiquitous system in V. It is also a homogeneous ubiquitous system in every nonempty open subset of V. Furthermore, for any $\eta > 0$ the set $I_{\eta,V}$ of all $i \in I$ such that $x_i \in V$ and $r_i \leqslant \eta$ is denumerable and $(x_i, r_i)_{i\in I_{\eta,V}} \in \mathcal{S}_d(I_{\eta,V})$ is a homogeneous ubiquitous system in V.

As an example, for any integer $c \geqslant 2$ the family $(kc^{-j}, c^{-j})_{(j,k)\in\mathbb{N}\times\mathbb{Z}^d}$ is a homogeneous ubiquitous system in $\mathbb{R}^d$. Likewise, by Dirichlet's theorem, for every $x \in \mathbb{R}^d$ there are infinitely many $(p,q) \in \mathbb{Z}^d \times \mathbb{N}$ such that $\|x - p/q\|_\infty < q^{-1-1/d}$, where $\|\cdot\|_\infty$ denotes the supremum norm, see [**22**, Theorem 200]. It follows that $(p/q, q^{-1-1/d})_{(p,q)\in\mathbb{Z}^d\times\mathbb{N}}$ is a homogeneous ubiquitous system in $\mathbb{R}^d$.

It turns out that the optimal regular systems of points, which are common in the theory of Diophantine approximation, also yield homogeneous ubiquitous systems. The notion of regular system was introduced by A. Baker and W. Schmidt [**2**] and refined by V. Beresnevich [**4**] in the following manner. Let V be the cartesian product of d real nonempty open intervals, let A be a denumerable subset of V and let $N : A \to (0,\infty)$ be a *height* function. Then (A, N) is called a regular system in V if for some $\kappa > 0$ and every open cube $\beta \subseteq V$,

$$\exists t_\beta > 0 \quad \forall t > t_\beta \quad \exists A_{\beta,t} \subseteq A \cap \beta \quad \begin{cases} \# A_{\beta,t} \geqslant \kappa |\beta|^d t^d \\ \forall a \in A_{\beta,t} \quad N(a) \leqslant t \\ \forall a, a' \in A_{\beta,t} \quad a \neq a' \Rightarrow \|a - a'\| \geqslant 1/t \end{cases}.$$

In addition, a regular system (A, N) is called optimal if for all open cubes $\beta \subseteq V$,

$$\exists \kappa'_\beta > 0 \quad \forall t > t_\beta \qquad \#\{a \in A \cap \beta \mid N(a) \leqslant t\} \leqslant \kappa'_\beta t^d.$$

Examples include the points with rational coordinates, the real algebraic numbers of bounded degree and the algebraic integers of bounded degree, associated with suitable height functions, see [**3**, **6**, **12**, **11**, **14**]. Let us now consider an optimal regular system (A, N) in V. The results of [**4**] lead to the fact that $(a, \psi(N(a)))_{a\in A}$ is a homogeneous ubiquitous system in V for every nonincreasing function $\psi : (0,\infty) \to (0,\infty)$ that satisfies $\sum_n \psi(n)^d n^{d-1} = \infty$ and tends to zero at infinity.

We can now state the main result of this section. Recall that every gauge function $h \in \mathfrak{D}_d$ is continuous and nondecreasing on $[0, \varepsilon_h)$, positive on $(0, \varepsilon_h)$ and vanishes at zero. The pseudo-inverse function of $h^{1/d}$ is defined by

$$\forall r \in [0, h^{1/d}({\varepsilon_h}^-)) \qquad (h^{1/d})^{-1}(r) = \inf\{\rho \in [0,\varepsilon_h) \mid h^{1/d}(\rho) \geqslant r\}$$

where $h^{1/d}({\varepsilon_h}^-) = \sup_{[0,\varepsilon_h)} h^{1/d} > 0$. Observe that the function $(h^{1/d})^{-1}$ is nonnegative and nondecreasing on $[0, h^{1/d}({\varepsilon_h}^-))$. Let $[(h^{1/d})^{-1}]$ be the set of all nonnegative nondecreasing functions $\varphi : [0,\infty) \to \mathbb{R}$ that coincide with it in a neighborhood of zero. The next theorem shows that the set F_φ given by (3·1) for $\varphi \in [(h^{1/d})^{-1}]$ enjoys a large intersection property.

THEOREM 2. *Let I be a denumerable set, let V be a nonempty open subset of $\mathbb{R}^d$ and let $(x_i, r_i)_{i\in I} \in \mathcal{S}_d(I)$ be a homogeneous ubiquitous system in V. Then*

$$\forall h \in \mathfrak{D}_d \quad \forall \varphi \in [(h^{1/d})^{-1}] \qquad F_\varphi \in \mathrm{G}^h(V).$$

This result is proven in Section 6. It is to be compared with the *mass transference principle* established by V. Beresnevich and S. Velani in [**8**], which discusses the Hausdorff h-measure of F_φ when F_{Id} has full Lebesgue measure in $\mathbb{R}^d$. Nevertheless, none of these results implies the other one.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in $[0,1)^d$ and let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers converging to zero. Assume that $\limsup_n B(x_n, r_n)$ has Lebesgue measure 1. Then, J.-M. Aubry and S. Jaffard [**1**] established that the set

$$\limsup_{n\to\infty} \bigcup_{k\in\mathbb{Z}^d} B(k + x_n, {r_n}^t)$$

lies in the class $\mathcal{G}^{d/t}(\mathbb{R}^d)$ of K. Falconer for every $t \in [1,\infty)$ and $d = 1$. They applied this result to perform the multifractal analysis of a model of random wavelet series. Theorem 2 enables to strengthen it and extend it to any d. To this end, observe that the family $(k + x_n, r_n)_{(n,k)\in\mathbb{N}\times\mathbb{Z}^d} \in \mathcal{S}_d(\mathbb{N}\times\mathbb{Z}^d)$ is a homogeneous ubiquitous system in $\mathbb{R}^d$, so that the aforementioned set belongs to the class $\mathrm{G}^{\mathrm{Id}^{d/t}}(\mathbb{R}^d)$, which is obviously included in $\mathcal{G}^{d/t}(\mathbb{R}^d)$. We use this in [**17**] in order to study the size and large intersection properties of the Hölder singularity sets of various random processes.

4. *Applications to Diophantine approximation*

As an application of the previous results, we review some of the sets that arise in classical Diophantine approximation and study their size and large intersection properties. Let ψ be a nonincreasing sequence of positive reals numbers converging to zero. We provide a complete description of the size properties of the set of all points that are ψ-approximable by rationals (or real algebraic numbers, etc.). More precisely, we compute its Hausdorff h-measure for every gauge function $h \in \mathfrak{D}$. We also describe its large intersection properties. To be specific, we determine for which gauge functions $h \in \mathfrak{D}_d$ and which nonempty open set V it belongs to the class $\mathrm{G}^h(V)$. As a by-product, we obtain new results concerning the set of all Liouville numbers. At the end of this section, we also study how well zero may be approximated by integer polynomials evaluated at a given point. Mahler's and Koksma's classifications of real transcendental numbers are investigated as well.

4·1. *Simultaneous homogeneous approximation*

The distance from a point $y \in \mathbb{R}^d$ to $\mathbb{Z}^d$ is given by $|y|_{\mathbb{Z}^d} = \min_{k\in\mathbb{Z}^d} \|y - k\|$. For any $q_0 \in \mathbb{N}$, let $\mathbb{N}_{q_0} = \{q_0, q_0+1, \ldots\}$ denote the set of all integers greater than or equal to

q_0 and let Ψ_{q_0} be the set of all nonincreasing sequences $\psi = (\psi(q))_{q\in\mathbb{N}_{q_0}}$ of positive real numbers converging to zero. Given $\psi \in \Psi_{q_0}$, a point $x \in \mathbb{R}^d$ is called ψ-approximable by rationals if $|qx|_{\mathbb{Z}^d} < q\psi(q)$ holds for infinitely many $q \in \mathbb{N}_{q_0}$. The set $K_{d,\psi}$ of all these points was first studied by Khintchine [**27**] and is of the form (3·1) since

$$K_{d,\psi} = \left\{ x \in \mathbb{R}^d \;\middle|\; \left\| x - \frac{p}{q} \right\| < \psi(q) \text{ for infinitely many } (p,q) \in \mathbb{Z}^d \times \mathbb{N}_{q_0} \right\}.$$

In the particular case where $\psi(q) = q^{-\tau}$, for $\tau > 0$ and $q \in \mathbb{N}$, any point of $J_{d,\tau} = K_{d,\psi}$ is called τ-approximable by rationals. As an obvious consequence of Dirichlet's theorem, $J_{d,\tau} = \mathbb{R}^d$ if $\tau \leqslant 1 + 1/d$. Moreover, Jarník [**26**] proved that the Hausdorff dimension of $J_{d,\tau}$ is $(d+1)/\tau$ for any $\tau > 1 + 1/d$. For $d = 1$ this result was previously established in [**25**] and independently proven by Besicovitch [**10**]. Theorem 2 yields the following complementary result.

Proposition 3. *The set $J_{d,\tau}$ belongs to $\mathrm{G}^{\mathrm{Id}^{\frac{d+1}{\tau}}}(\mathbb{R}^d)$ for every $\tau > 1 + 1/d$.*

Proof. This follows from Theorem 2 since $\mathrm{Id}^{\frac{d+1}{\tau}} \in \mathfrak{D}_d$ and $(p/q, q^{-1-1/d})_{(p,q)\in\mathbb{Z}^d\times\mathbb{N}}$ is a homogeneous ubiquitous system in $\mathbb{R}^d$ owing to Dirichlet's theorem. □

In order to improve Proposition 3 and extend it to general sequences ψ, we need to recall Khintchine's theorem [**27**].

Theorem 3 (Khintchine). *Let $q_0 \in \mathbb{N}$ and $\psi \in \Psi_{q_0}$. If $\sum_q \psi(q)^d q^d = \infty$ (resp. $< \infty$), then $K_{d,\psi}$ has full (resp. zero) Lebesgue measure in $\mathbb{R}^d$.*

Khintchine's theorem ensures that $(p/q, \psi(q))_{(p,q)\in\mathbb{Z}^d\times\mathbb{N}_{q_0}}$ is a homogeneous ubiquitous system in $\mathbb{R}^d$ if $\sum_q \psi(q)^d q^d = \infty$. For example, $(p/q, q^{-1}(q\log q)^{-1/d})_{(p,q)\in\mathbb{Z}^d\times\mathbb{N}_2}$ is a homogeneous ubiquitous system in $\mathbb{R}^d$. When $\sum_q \psi(q)^d q^d < \infty$ the set $K_{d,\psi}$ has Lebesgue measure zero. However, its size properties can be investigated thanks to the following result of Jarník [**26**].

Theorem 4 (Jarník). *Let $q_0 \in \mathbb{N}$, let $\psi \in \Psi_{q_0}$ and let $h \in \mathfrak{D}_d$ with $h \prec \mathrm{Id}^d$. If $\sum_q h(\psi(q))q^d = \infty$ (resp. $< \infty$), then $K_{d,\psi}$ has infinite (resp. zero) Hausdorff h-measure.*

Remark. The previous statements of Khintchine's and Jarník's theorems are due to V. Beresnevich, D. Dickinson and S. Velani [**7**]. The original ones actually included some extra assumptions on ψ.

Theorem 2 yields the following result, which is both a refinement of Khintchine's and Jarník's theorems and an improvement on Proposition 3. Recall that for any $h \in \mathfrak{D}$ the function h_d is defined by (1·3).

Theorem 5. *Let $q_0 \in \mathbb{N}$, $\psi \in \Psi_{q_0}$, $h \in \mathfrak{D}$ and let V be an open subset of $\mathbb{R}^d$. Then*

$$\begin{cases} \sum_q h_d(\psi(q))q^d = \infty & \Longrightarrow \quad \mathcal{H}^h(K_{d,\psi} \cap V) = \mathcal{H}^h(V) \\ \sum_q h_d(\psi(q))q^d < \infty & \Longrightarrow \quad \mathcal{H}^h(K_{d,\psi} \cap V) = 0 \end{cases}.$$

Moreover, for $h \in \mathfrak{D}_d$ and $V \neq \emptyset$,

$$K_{d,\psi} \in \mathrm{G}^h(V) \qquad \Longleftrightarrow \qquad \sum_q h(\psi(q))q^d = \infty.$$

Proof. Let us assume that $\sum_q h_d(\psi(q))q^d < \infty$. It is straightforward to check that $\mathcal{H}^{h_d}(K_{d,\psi} \cap V) = 0$ because

$$K_{d,\psi} = \mathbb{Z}^d + \limsup_{q\to\infty} \bigcup_{p\in\{0,\ldots,q-1\}^d} B\left(\frac{p}{q}, \psi(q)\right).$$

Proposition 2 then ensures that $\mathcal{H}^h(K_{d,\psi} \cap V) = 0$.

Suppose that $V \neq \emptyset$, $h \in \mathfrak{D}_d$ and $\sum_q h(\psi(q))q^d < \infty$. There is a gauge function $\overline{h} \in \mathfrak{D}_d$ with $\overline{h} \prec h$ and $\sum_q \overline{h}(\psi(q))q^d < \infty$. Using $\overline{h}$ rather than h above, we obtain $\mathcal{H}^{\overline{h}}(K_{d,\psi} \cap V) = 0$. Theorem 1 implies that $K_{d,\psi} \notin \mathrm{G}^h(V)$.

Let us assume that $V \neq \emptyset$, $h \in \mathfrak{D}_d$ and $\sum_q h(\psi(q))q^d = \infty$. Let $\tilde{h}$ be a nondecreasing function defined on $[0,\infty)$ which coincides with h in a neighborhood of zero. As ψ tends to zero, $\sum_q \tilde{h}(\psi(q))q^d = \infty$. In addition, $(\tilde{h}^{1/d}(\psi(q)))_{q\in\mathbb{N}_{q_0}}$ belongs to Ψ_{q_0}, so Khintchine's theorem implies that $(p/q, \tilde{h}^{1/d}(\psi(q)))_{(p,q)\in\mathbb{Z}^d\times\mathbb{N}_{q_0}}$ is a homogeneous ubiquitous system in $\mathbb{R}^d$. Theorem 2 ensures that for any $\varphi \in [(\tilde{h}^{1/d})^{-1}]$, the set of all $x \in \mathbb{R}^d$ such that $\|x - p/q\| < \varphi(\tilde{h}^{1/d}(\psi(q)))$ for infinitely many $(p,q) \in \mathbb{Z}^d \times \mathbb{N}_{q_0}$ belongs to $\mathrm{G}^{\tilde{h}}(\mathbb{R}^d)$. Since $\varphi(\tilde{h}^{1/d}(r)) \leqslant r$ for every $r > 0$ small enough, this set is included in $K_{d,\psi}$. Proposition 1 implies that $K_{d,\psi} \in \mathrm{G}^h(V)$.

Let $h \in \mathfrak{D}$ with $\sum_q h_d(\psi(q))q^d = \infty$. Note that $h_d \in \mathfrak{D}_d$ owing to Proposition 2. In order to show that $\mathcal{H}^h(K_{d,\psi} \cap V) = \mathcal{H}^h(V)$, we can obviously assume that V is nonempty. If $h_d \prec \mathrm{Id}^d$, there is a gauge function $\underline{h} \in \mathfrak{D}_d$ with $h_d \prec \underline{h}$ and $\sum_q \underline{h}(\psi(q))q^d = \infty$. Using $\underline{h}$ instead of h above, we obtain $K_{d,\psi} \in \mathrm{G}^{\underline{h}}(V)$. Theorem 1 then implies that $\mathcal{H}^{h_d}(K_{d,\psi} \cap V) = \infty = \mathcal{H}^{h_d}(V)$. Proposition 2 leads to $\mathcal{H}^h(K_{d,\psi} \cap V) = \mathcal{H}^h(V)$. This still holds if $h_d \not\prec \mathrm{Id}^d$. Indeed, in this case, $\sum_q \psi(q)^d q^d = \infty$ and $\mathcal{H}^h$ coincides up to a multiplicative constant with the Lebesgue measure on the Borel subsets of $\mathbb{R}^d$, so the result follows from Khintchine's theorem. □

Theorem 5 enables to study the size and large intersection properties of

$$L_d = \left\{ x \in \mathbb{R}^d \backslash \mathbb{Q}^d \;\middle|\; \forall n \in \mathbb{N}\ \exists (p,q) \in \mathbb{Z}^d \times \mathbb{N}_2 \quad \left\| x - \frac{p}{q} \right\| < \frac{1}{q^n} \right\}.$$

Note that L_1 is the set of all Liouville numbers. Let $h \in \mathfrak{D}$. In dimension $d = 1$, L. Olsen and D. Renfro [**31**, **32**] established that $\mathcal{H}^h(L_d) = 0$ if $h_d(r) = \mathrm{o}(r^s)$ as $r \to 0$ for some $s > 0$ and that $\mathcal{H}^h(L_d \cap V) = \infty$ for every nonempty open subset V of $\mathbb{R}^d$ otherwise. The following corollary ensures that this criterion is still valid if $d \geqslant 2$ and additionally shows that L_d enjoys a large intersection property.

Corollary 4. *Let $h \in \mathfrak{D}$ and let V be a nonempty open subset of $\mathbb{R}^d$. Then*

$$\begin{cases} [\forall s > 0 \quad h_d(r) \neq \mathrm{o}(r^s)] & \Longrightarrow \quad \mathcal{H}^h(L_d \cap V) = \infty \\ [\exists s > 0 \quad h_d(r) = \mathrm{o}(r^s)] & \Longrightarrow \quad \mathcal{H}^h(L_d \cap V) = 0 \end{cases}.$$

Moreover, for $h \in \mathfrak{D}_d$,

$$L_d \in \mathrm{G}^h(V) \qquad \Longleftrightarrow \qquad [\forall s > 0 \quad h(r) \neq \mathrm{o}(r^s)].$$

Proof. Suppose that $h_d(r) = \mathrm{o}(r^s)$ for some $s > 0$. We can assume that $s < d$. Since

$$L_d = (\mathbb{R}^d \backslash \mathbb{Q}^d) \cap \bigcap_{\tau>0} \downarrow J_{d,\tau} \tag{4·1}$$

we have $L_d \subseteq J_{d,2(d+1)/s}$. Moreover, Jarník's theorem ensures that $\mathcal{H}^{\mathrm{Id}^s}(J_{d,2(d+1)/s}) = 0$. As a result, $\mathcal{H}^{h_d}(L_d) = 0$. Proposition 2 yields $\mathcal{H}^h(L_d \cap V) = 0$.

Let us assume that $h \in \mathfrak{D}_d$ and $h(r) = \mathrm{o}(r^s)$ for some $s > 0$. Using $\overline{h} : r \mapsto \sqrt{h(r)}$ rather than h above, we obtain $\mathcal{H}^{\overline{h}}(L_d \cap V) = 0$. Hence $L_d \notin \mathrm{G}^h(V)$ by Theorem 1.

Conversely, assume that $h \in \mathfrak{D}_d$ and $h(r) \neq \mathrm{o}(r^s)$ for all $s > 0$. Let $\tau \in (0, \infty)$ and suppose that $\sum_q h(q^{-\tau})q^d < \infty$. It follows that $u \mapsto h(u^{-\tau})u^d$ is integrable at infinity. Moreover, for $r > 0$ small enough,

$$\int_{\frac{1}{2r^{1/\tau}}}^{\infty} h(u^{-\tau})u^d \mathrm{d}u \geqslant \int_{\frac{1}{2r^{1/\tau}}}^{\frac{1}{r^{1/\tau}}} h(u^{-\tau})u^d \mathrm{d}u \geqslant \frac{1}{2^{d+1}} \cdot \frac{h(r)}{r^{\frac{d+1}{\tau}}}.$$

Thus $h(r) = \mathrm{o}(r^{(d+1)/\tau})$, which is a contradiction. Hence $\sum_q h(q^{-\tau})q^d = \infty$, so that $J_{d,\tau} \in \mathrm{G}^h(V)$ by Theorem 5. In addition, since $\mathbb{R}^d \backslash \mathbb{Q}^d$ is a G_δ-set of full Lebesgue measure in $\mathbb{R}^d$, it also belongs to $\mathrm{G}^h(V)$, owing to Proposition 11 in Section 5. As the intersection in (4·1) can be written as a countable one, $L_d \in \mathrm{G}^h(V)$ by Theorem 1.

Let $h \in \mathfrak{D}$ with $h_d(r) \neq \mathrm{o}(r^s)$ for all $s > 0$. Note that $h_d \in \mathfrak{D}_d$ owing to Proposition 2. Furthermore, there is a function $\underline{h} \in \mathfrak{D}_d$ with $h_d \prec \underline{h}$ and $\underline{h}(r) \neq \mathrm{o}(r^s)$ for all $s > 0$. Using $\underline{h}$ rather than h above, we obtain $L_d \in \mathrm{G}^{\underline{h}}(V)$. Hence $\mathcal{H}^{h_d}(L_d \cap V) = \infty$ by virtue of Theorem 1. Proposition 2 yields $\mathcal{H}^h(L_d \cap V) = \infty$. $\square$

Let V denote a nonempty open subset of $\mathbb{R}^d$ and, for any $n \in \mathbb{N}$, let $f_n : V \to \mathbb{R}^d$ be a bi-Lipschitz mapping. Corollary 4 and Theorem 1 ensure that

$$\tilde{L}_{d,f} = \bigcap_{n=1}^{\infty} {f_n}^{-1}(L_d) \in \mathrm{G}^h(V)$$

for every $h \in \mathfrak{D}_d$ such that $h(r) \neq \mathrm{o}(r^s)$ for all $s > 0$. In addition, $\tilde{L}_{d,f}$ has infinite Hausdorff h-measure in every nonempty open subset U of V. A typical application is the following result.

COROLLARY 5. *There are uncountably many ways to write every point in $\mathbb{R}^d$ as the sum of two points in L_d.*

This corollary generalizes a classical result of Erdős [**18**] which states that every real number can be written as the sum of two Liouville numbers.

4·2. *Simultaneous inhomogeneous approximation*

Let $b \in \mathbb{R}^d$, $q_0 \in \mathbb{N}$ and $\psi \in \Psi_{q_0}$. A classical generalization of the previous problem is to study the size and large intersection properties of the set $K^b_{d,\psi}$ of all points $x \in \mathbb{R}^d$ such that $|qx - b|_{\mathbb{Z}^d} < q\psi(q)$ holds for infinitely many $q \in \mathbb{N}_{q_0}$. For $b = 0$ this is obviously the set of all points that are ψ-approximable by rationals. In addition, it is straightforward to check that $K^b_{d,\psi}$ is of the form (3·1) since

$$K^b_{d,\psi} = \left\{ x \in \mathbb{R}^d \;\middle|\; \left\| x - \frac{b+p}{q} \right\| < \psi(q) \text{ for infinitely many } (p,q) \in \mathbb{Z}^d \times \mathbb{N}_{q_0} \right\}.$$

W. Schmidt [**34**] established the analog of Khintchine's theorem: $K^b_{d,\psi}$ has full (resp. zero) Lebesgue measure in $\mathbb{R}^d$ if $\sum_q \psi(q)^d q^d = \infty$ (resp. $< \infty$). Moreover J. Levesley [**29**] proved the analog of the Jarník-Besicovitch theorem and Y. Bugeaud [**14**] established the analog of Jarník's theorem. By imitating the proof of Theorem 5 and using Schmidt's theorem instead of Khintchine's, it is straightforward to establish the following result.

Theorem 6. *Let $b \in \mathbb{R}^d$, $q_0 \in \mathbb{N}$, $\psi \in \Psi_{q_0}$, $h \in \mathfrak{D}$ and let V be an open subset of $\mathbb{R}^d$. Then*

$$\begin{cases} \sum_q h_d(\psi(q))q^d = \infty & \Longrightarrow \quad \mathcal{H}^h(K^b_{d,\psi} \cap V) = \mathcal{H}^h(V) \\ \sum_q h_d(\psi(q))q^d < \infty & \Longrightarrow \quad \mathcal{H}^h(K^b_{d,\psi} \cap V) = 0 \end{cases}.$$

Moreover, for $h \in \mathfrak{D}_d$ and $V \neq \emptyset$,

$$K^b_{d,\psi} \in \mathrm{G}^h(V) \quad \Longleftrightarrow \quad \sum\nolimits_q h(\psi(q))q^d = \infty.$$

Let $(b_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$, let V be a nonempty open set and let $h \in \mathfrak{D}$ with $\sum_q h_d(\psi(q))q^d = \infty$. If $h_d \prec \mathrm{Id}^d$, there is a gauge function $\underline{h} \in \mathfrak{D}_d$ which satisfies $h_d \prec \underline{h}$ and $\sum_q \underline{h}(\psi(q))q^d = \infty$. For each $n \in \mathbb{N}$, we have $K^{b_n}_{d,\psi} \in \mathrm{G}^{\underline{h}}(V)$ by Theorem 6. Theorem 1 and Proposition 2 then yield

$$\mathcal{H}^h\left(V \cap \bigcap_{n=1}^{\infty} K^{b_n}_{d,\psi}\right) = \mathcal{H}^h(V).$$

This still holds if $h_d \not\prec \mathrm{Id}^d$. Indeed, in this case, $\sum_q \psi(q)^d q^d = \infty$ and $\mathcal{H}^h$ coincides up to a multiplicative constant with the Lebesgue measure on the Borel subsets of $\mathbb{R}^d$, so the result follows from Schmidt's theorem. Moreover, taking $h \in \mathfrak{D}_d$ with $h \prec \mathrm{Id}^d$, $V = \mathbb{R}^d$ and $b_n = b$ for all $n \in \mathbb{N}$, we find out that the technical assumptions made by Y. Bugeaud in [**14**] to establish the analog of Jarník's theorem are unnecessary.

4·3. *Approximation with restrictions*

G. Harman studied the approximation of real numbers by rationals whose numerator and denominator belong to a given subset of $\mathbb{N}$, see [**23**]. By way of illustration, let us assume that this subset is the set $\mathbb{P}$ of all prime numbers. Let $q_0 \in \mathbb{N}_2$, $\psi \in \Psi_{q_0}$ and

$$\Pi_\psi = \left\{ x \in [0,\infty) \;\middle|\; \left| x - \frac{p}{q} \right| < \psi(q) \text{ for infinitely many } (p,q) \in \mathbb{P} \times (\mathbb{P} \cap \mathbb{N}_{q_0}) \right\}.$$

This set is of the form (3·1). Moreover, G. Harman [**23**] showed that it has full (resp. zero) Lebesgue measure in $(0,\infty)$ if $\sum_q \psi(q)q/(\log q)^2 = \infty$ (resp. $< \infty$). The following result can easily be established by imitating the proof of Theorem 5 and using [**23**, Theorem 6.7] instead of Khintchine's theorem. Recall that for any gauge function $h \in \mathfrak{D}$, the function h_1 is defined by (1·3).

Theorem 7. *Let $q_0 \in \mathbb{N}_2$, $\psi \in \Psi_{q_0}$, $h \in \mathfrak{D}$ and let V be an open subset of $\mathbb{R}$. Then*

$$\begin{cases} \sum_q h_1(\psi(q))q/(\log q)^2 = \infty & \Longrightarrow \quad \mathcal{H}^h(\Pi_\psi \cap V) = \mathcal{H}^h((0,\infty) \cap V) \\ \sum_q h_1(\psi(q))q/(\log q)^2 < \infty & \Longrightarrow \quad \mathcal{H}^h(\Pi_\psi \cap V) = 0 \end{cases}.$$

Moreover, for $h \in \mathfrak{D}_1$ and $V \neq \emptyset$,

$$\Pi_\psi \in \mathrm{G}^h(V) \quad \Longleftrightarrow \quad \sum\nolimits_q h(\psi(q))q/(\log q)^2 = \infty \quad \text{and} \quad V \subseteq (0,\infty).$$

Theorem 7 leads to [**7**, Theorem 14] which states that the set $\Pi_\psi \cap [0,1]$ has infinite (resp. zero) Hausdorff h-measure for any gauge function $h \in \mathfrak{D}_1$ such that $h \prec \mathrm{Id}$ and $\sum_q h(\psi(q))q/(\log q)^2 = \infty$ (resp. $< \infty$).

By adapting the proof of Theorem 5 and using the results of [**23**, Chapter 6], one can show that the statement of Theorem 7 remains valid if Π_ψ is replaced by the set of all positive real numbers that are ψ-approximable by rationals whose numerator and denominator are properly represented as the sum of two squares and if the considered series is $\sum_q h_1(\psi(q))q/\log q$. Likewise, the statement of Theorem 7 is still valid if Π_ψ is replaced by the set of all positive reals that are ψ-approximable by rationals whose numerator is prime and whose denominator is the sum of two squares (or vice versa) and if the considered series is $\sum_q h(\psi(q))q/(\log q)^{3/2} = \infty$. Furthermore, Theorem 2 along with [**23**, Theorem 6.2] implies that sets with large intersection still occur in the case where the numerator and the denominator of the rational approximates are restricted to more general subsets of $\mathbb{N}$.

4·4. *Approximation by real algebraic numbers and Koksma's classification of real transcendental numbers*

Let $\mathbb{A}$ be the set of all real algebraic numbers and let $H(a)$ denote the height of $a \in \mathbb{A}$, that is, the maximum of the absolute values of the coefficients of its minimal defining polynomial over $\mathbb{Z}$. Let $h_0 \in \mathbb{N}$ and $\psi \in \Psi_{h_0}$. For each $n \in \mathbb{N}$, let $\mathbb{A}_{n,h_0}$ denote the set of all real algebraic numbers of degree at most n and of height at least h_0. Any real number x is called ψ-approximable by real algebraic numbers of degree at most n if it lies in

$$A_{n,\psi} = \left\{ x \in \mathbb{R} \;\middle|\; |x-a| < \psi(H(a)) \text{ for infinitely many } a \in \mathbb{A}_{n,h_0} \right\}.$$

This set is clearly of the form (3·1). The following result of V. Beresnevich [**3**, **7**] is the analog of Khintchine's theorem for $A_{n,\psi}$.

THEOREM 8 (V. Beresnevich). *Let $n, h_0 \in \mathbb{N}$ and $\psi \in \Psi_{h_0}$. If $\sum_h \psi(h)h^n = \infty$ (resp. $< \infty$), then $A_{n,\psi}$ has full (resp. zero) Lebesgue measure in $\mathbb{R}$.*

V. Beresnevich, D. Dickinson and S. Velani [**7**], as well as Y. Bugeaud [**12**], established the analog of Jarník's theorem for $A_{n,\psi}$. In addition, Y. Bugeaud [**15**] showed that this set enjoys a large intersection property when $\psi(h)$ is the product of a negative power of h and a logarithmic correction. The following theorem improves these results and can easily be established by imitating the proof of Theorem 5 and using Beresnevich's theorem instead of Khintchine's. Recall that for any $g \in \mathfrak{D}$, the function g_1 is defined as in (1·3).

THEOREM 9. *Let $n, h_0 \in \mathbb{N}$, $\psi \in \Psi_{h_0}$, $g \in \mathfrak{D}$ and let V be an open subset of $\mathbb{R}$. Then*

$$\begin{cases} \sum_h g_1(\psi(h))h^n = \infty & \Longrightarrow \quad \mathcal{H}^g(A_{n,\psi} \cap V) = \mathcal{H}^g(V) \\ \sum_h g_1(\psi(h))h^n < \infty & \Longrightarrow \quad \mathcal{H}^g(A_{n,\psi} \cap V) = 0 \end{cases}.$$

Moreover, for $g \in \mathfrak{D}_1$ and $V \neq \emptyset$,

$$A_{n,\psi} \in \mathrm{G}^g(V) \qquad \Longleftrightarrow \qquad \sum_h g(\psi(h))h^n = \infty.$$

From now on, we assume that $\psi(h) = h^{-\omega-1}$ for each $h \in \mathbb{N}$ and some $\omega > -1$ and we let $U^*_{n,\omega}$ denote the set $A_{n,\psi}$. Moreover, for every $x \in \mathbb{R}$, let $\omega^*_n(x)$ be the supremum of all ω such that $x \in U^*_{n,\omega}$. Koksma [**28**] introduced a classification of the real transcendental numbers x which is based on the quantity

$$\omega^*(x) = \limsup_{n\to\infty} \frac{\omega^*_n(x)}{n}.$$

In particular, a real transcendental number x is said to be an S^*-number if $\omega^*(x) < \infty$. In this case, $\omega^*(x)$ is called the type of x, see [**13**, **15**]. A theorem due to Sprindžuk [**36**] together with a result of E. Wirsing [**37**] shows that Lebesgue-almost every real number x is an S^*-number with $\omega_n^*(x) = n$ for every $n \in \mathbb{N}$. Thus the set

$$\Omega'_\tau = \bigcap_{n=1}^{\infty} \{x \in \mathbb{R} \mid \omega_n^*(x) \geqslant \tau(n+1) - 1\}$$

has full Lebesgue measure in $\mathbb{R}$ if $\tau = 1$. Moreover, A. Baker and W. Schmidt [**2**] established that $\dim \Omega'_\tau = 1/\tau$ for every $\tau \in (1, \infty)$ and it is the lower bound on the dimension which is the hardest to obtain. As we shall explain, the following result is an improvement on this lower bound.

Proposition 6. *Let $\tau \in [1, \infty)$ and $g \in \mathfrak{D}_1$. Assume that $\sum_h g(1/h) h^{-1+1/\tau} = \infty$. Then Ω'_τ belongs to $\mathrm{G}^g(\mathbb{R})$.*

Proof. Note that

$$\Omega'_\tau = \bigcap_{n=1}^{\infty} \bigcap_{-1<\omega<\tau(n+1)-1} \downarrow U^*_{n,\omega}.$$

Let $n \in \mathbb{N}$ and $\omega \in (-1, \tau(n+1) - 1)$. A routine calculation shows that $\sum_h g(h^{-\omega-1}) h^n$ diverges so that $U^*_{n,\omega} \in \mathrm{G}^g(\mathbb{R})$ by Theorem 9. We conclude using Theorem 1 since the previous decreasing intersection can be written as a countable one. □

Let $\tau \in [1, \infty)$. On account of the previous proposition, the set Ω'_τ belongs to $\mathrm{G}^{\mathrm{Id}^{1/\tau}}(\mathbb{R})$. Thus its dimension is at least $1/\tau$, as previously obtained by A. Baker and W. Schmidt. Proposition 6 yields the size and large intersection properties of the sets

$$\tilde{\Omega}_\tau = \{x \in \mathbb{R} \backslash \mathbb{A} \mid \omega^*(x) \geqslant \tau\} \qquad \text{and} \qquad \Omega_\tau = \{x \in \mathbb{R} \backslash \mathbb{A} \mid \omega^*(x) = \tau\}$$

as shown by the following result.

Proposition 7. *Let V be an open subset of $\mathbb{R}$, let $\tau \in [1, \infty)$, let $g \in \mathfrak{D}$ and let*

$$\tau_g = \inf \{t \in (0, \infty) \mid g_1(r) = \mathrm{o}(r^{1/t}) \text{ as } r \to 0\}.$$

If $\tau > \tau_g$, then $\mathcal{H}^g(\tilde{\Omega}_\tau \cap V) = \mathcal{H}^g(\Omega_\tau \cap V) = 0$, else $\mathcal{H}^g(\tilde{\Omega}_\tau \cap V) = \mathcal{H}^g(V)$. In addition, $\mathcal{H}^g(\Omega_\tau \cap V) = \mathcal{H}^g(V)$ if $\tau = \tau_g$. Furthermore, for $g \in \mathfrak{D}_1$ and $V \neq \emptyset$, the set $\tilde{\Omega}_\tau$ contains a set of $\mathrm{G}^g(V)$ if and only if $\tau \leqslant \tau_g$.

Proof. Assume that $\tau > \tau_g$. For $\tau_g < \tau' < \tau'' < \tau$,

$$\tilde{\Omega}_\tau \subseteq \bigcup_{n=1}^{\infty} U^*_{n,\tau''(n+1)-1}$$

and $g_1(r) = \mathrm{o}(r^{1/\tau'})$. Theorem 9 then ensures that $\mathcal{H}^g(U^*_{n,\tau''(n+1)-1}) = 0$. It follows that $\mathcal{H}^g(\tilde{\Omega}_\tau) = 0$. Hence $\mathcal{H}^g(\tilde{\Omega}_\tau \cap V) = \mathcal{H}^g(\Omega_\tau \cap V) = 0$.

Let us suppose that $g \in \mathfrak{D}_1$, $V \neq \emptyset$ and $\tau > \tau_g$. There is a gauge function $\overline{g} \in \mathfrak{D}_1$ such that $\overline{g} \prec g$ and $\tau > \tau_{\overline{g}}$. Using $\overline{g}$ instead of g above gives $\mathcal{H}^{\overline{g}}(\tilde{\Omega}_\tau \cap V) = 0$. Theorem 1 implies that $\tilde{\Omega}_\tau$ cannot contain any set of $\mathrm{G}^g(V)$.

Assume that $g \in \mathfrak{D}_1$, $V \neq \emptyset$ and $\tau \leqslant \tau_g$. Thanks to Theorem 19 and Lemma 15 in [**35**], one easily checks that $\Omega'_\tau \subseteq \tilde{\Omega}_\tau$. In particular, $\tilde{\Omega}_1$ contains the set Ω'_1, which belongs to $\mathrm{G}^g(V)$ by Proposition 6. If $\tau > 1$, note that

$$\bigcap_{1<\tau'<\tau} \downarrow \Omega'_{\tau'} \subseteq \tilde{\Omega}_\tau.$$

For $\tau' \in (1,\tau)$, the series $\sum_h g(1/h)h^{-1+1/\tau'}$ diverges. Otherwise, $u \mapsto g(1/u)u^{-1+1/\tau'}$ would be integrable at infinity and, for $r > 0$ small enough, we would have

$$\frac{g(r)}{r^{1/\tau'}} \leqslant \frac{2^{1/\tau'}}{r} \int_r^{2r} \frac{g(s)}{s^{1/\tau'}}\,\mathrm{d}s \leqslant 2^{1+1/\tau'} \int_{\frac{1}{2r}}^{\infty} g\left(\frac{1}{u}\right) u^{-1+1/\tau'}\mathrm{d}u$$

so that $g(r) = \mathrm{o}(r^{1/\tau'})$, which would contradict the fact that $\tau \leqslant \tau_g$. Proposition 6 then leads to $\Omega'_{\tau'} \in \mathrm{G}^g(V)$. As the previous decreasing intersection can be written as a countable one, Theorem 1 ensures that $\tilde\Omega_\tau$ contains a set of $\mathrm{G}^g(V)$.

Now assume that $g \in \mathfrak{D}$ and $\tau \leqslant \tau_g$. To show that $\mathcal{H}^g(\tilde\Omega_\tau \cap V) = \mathcal{H}^g(V)$, we may suppose that $V \neq \emptyset$ and $g_1 \in \mathfrak{D}_1$, owing to Proposition 2. If $g_1 \prec \mathrm{Id}$, there is a gauge function $\underline{g} \in \mathfrak{D}_1$ such that $g_1 \prec \underline{g}$ and $\tau \leqslant \tau_{\underline{g}}$. Using $\underline{g}$ rather than g above, we obtain that $\tilde\Omega_\tau$ contains a set of $\mathrm{G}^{\underline{g}}(V)$. Theorem 1 and Proposition 2 then yield $\mathcal{H}^g(\tilde\Omega_\tau \cap V) = \mathcal{H}^g(V)$. This still holds if $g_1 \not\prec \mathrm{Id}$. Indeed, in this case, $\tau = \tau_g = 1$ and $\mathcal{H}^g$ coincides up to a multiplicative constant with the Lebesgue measure on the Borel subsets of $\mathbb{R}$. Hence the result follows from the fact that $\tilde\Omega_1$ contains Ω'_1, which has full Lebesgue measure in $\mathbb{R}$.

It remains to establish that $\mathcal{H}^g(\Omega_\tau \cap V) = \mathcal{H}^g(V)$ if $\tau = \tau_g$. To this end, observe that the set of all real transcendental numbers x enjoying $\omega^*(x) > \tau_g$ has Hausdorff g-measure zero, because it is included in

$$\bigcup_{k=1}^{\infty} \uparrow \bigcup_{n=1}^{\infty} U^*_{n,(\tau_g+1/k)(n+1)-1}$$

and because $U^*_{n,(\tau_g+1/k)(n+1)-1}$ has g-measure zero by virtue of Theorem 9. □

Proposition 7 leads to [**2**, Theorem 2] and [**15**, Theorem 3] which respectively state that $\dim \tilde\Omega_\tau = 1/\tau$ and $\dim \Omega_\tau = 1/\tau$ for any $\tau \in [1,\infty)$. In addition, it implies that Ω_τ has infinite $1/\tau$-dimensional Hausdorff measure.

4·5. *Approximation of zero by values of integer polynomials and Mahler's classification of real transcendental numbers*

Koksma's classification of real transcendental numbers is very close to that previously introduced by Mahler [**30**], where the real transcendental numbers x are grouped depending on the accuracy with which integer polynomials evaluated at x approach zero.

Before discussing Mahler's classification, let us consider a more general problem. For any $n \in \mathbb{N}$, let $\mathbb{Z}_n[X]$ be the set of all integer polynomials of degree at most n. Let $H(p)$ denote the height of $p \in \mathbb{Z}_n[X]$, that is, the maximum of the absolute values of its coefficients. Let $h_0 \in \mathbb{N}$ and $\psi \in \Psi_{h_0}$. Let $P_{n,\psi}$ be the set of all real numbers x such that

$$|p(x)| < H(p)\psi(H(p)) \tag{4·2}$$

holds for infinitely many $p \in \mathbb{Z}_n[X]$ with $H(p) \geqslant h_0$. V. Bernik [**9**] and V. Beresnevich [**5**] showed that $P_{n,\psi}$ has Lebesgue measure zero if $\sum_h \psi(h)h^n < \infty$. In the opposite case, V. Beresnevich [**3**] established that $P_{n,\psi}$ has full measure in $\mathbb{R}$. To our knowledge, the size properties of $P_{n,\psi}$ have not been studied any further.

We shall investigate the large intersection properties of $P_{n,\psi}$. Note that this set is not of the form (3·1) when $n \geqslant 2$. However, it is a G_δ-set since

$$P_{n,\psi} = \bigcap_{k=1}^{\infty} \downarrow \bigcup_{k'=k}^{\infty} \left\{ x \in \mathbb{R} \mid |p_{k'}(x)| < H(p_{k'})\psi(H(p_{k'})) \right\}$$

for any enumeration $(p_k)_{k\in\mathbb{N}}$ of the set of all integer polynomials of degree at most n and of height at least h_0.

THEOREM 10. *Let $n, h_0 \in \mathbb{N}$, $\psi \in \Psi_{h_0}$, $g \in \mathfrak{D}_1$ and let V be a nonempty open subset of $\mathbb{R}$. Then*

$$\sum\nolimits_h g(\psi(h))h^n = \infty \qquad \Longrightarrow \qquad P_{n,\psi} \in \mathrm{G}^g(V).$$

Proof. Let $r \in (0,\infty)$, $\kappa = n^2(1+r)^{n-1} > 0$ and $x \in A_{n,\psi/\kappa} \cap (-r,r)$. There are infinitely many real algebraic numbers $a \in \mathbb{A}_{n,h_0}$ with $|x-a| < \psi(H(a))/\kappa$. As ψ tends to zero, we may assume that all these numbers enjoy $|x-a| \leqslant 1$. Let a be such an algebraic number and let $p_a = \sum_q b_q X^q \in \mathbb{Z}_n[X]$ denote its minimal defining polynomial. We have

$$p_a(x) = p_a(x) - p_a(a) = (x-a)\sum_{q=1}^{n} b_q \sum_{k=0}^{q-1} x^{q-1-k}a^k$$

together with $|x^{q-1-k}a^k| \leqslant r^{q-1-k}(1+r)^k \leqslant (1+r)^{n-1}$ for every $k \in \{0,\ldots,q-1\}$ and $q \in \{1,\ldots,n\}$. Therefore

$$|p_a(x)| \leqslant |x-a|H(p_a)\kappa < H(p_a)\psi(H(p_a)).$$

Hence (4·2) holds for $p = p_a$. As a result, (4·2) holds for infinitely many $p \in \mathbb{Z}_n[X]$ with $H(p) \geqslant h_0$. Thus $x \in P_{n,\psi}$. This yields $A_{n,\psi/\kappa} \cap (-r,r) \subseteq P_{n,\psi}$.

Since $\sum_h g(\psi(h)/\kappa)h^n = \infty$ and ψ/κ is nonincreasing and converges to zero, Theorem 9 ensures that $A_{n,\psi/\kappa} \in \mathrm{G}^g(V)$. As a consequence,

$$\mathcal{M}^f_\infty(P_{n,\psi} \cap U) \geqslant \mathcal{M}^f_\infty(A_{n,\psi/\kappa} \cap (-r,r) \cap U) \geqslant \mathcal{M}^f_\infty((-r,r) \cap U)$$

for every gauge function $f \in \mathfrak{D}_1$ with $f \prec g$ and every open $U \subseteq V$. Finally, the increasing sets lemma [**33**, Theorem 52] for the outer net measure $\mathcal{M}^f_\infty$ implies that the right-hand side tends to $\mathcal{M}^f_\infty(U)$ as $r \to \infty$. $\square$

To link what precedes with Mahler's classification, we assume that $\psi(h) = h^{-\omega-1}$ for all $h \in \mathbb{N}$ and some $\omega > -1$ and we let $U_{n,\omega}$ denote the set $P_{n,\psi}$. In addition, for every $x \in \mathbb{R}$, let $\omega_n(x)$ be the supremum of all ω such that $x \in U_{n,\omega}$. Mahler's classification of the real transcendental numbers x is based on

$$\omega(x) = \limsup_{n\to\infty} \frac{\omega_n(x)}{n}.$$

In particular, a real number x is algebraic if and only if $\omega(x) = 0$ and a real transcendental number x such that $0 < \omega(x) < \infty$ is called an S-number. Dirichlet's principle ensures that every real transcendental number x satisfies $\omega_n(x) \geqslant n$ for all $n \in \mathbb{N}$ and Sprindžuk [**36**] established that Lebesgue-almost every real number x is an S-number with $\omega_n(x) = n$ for all $n \in \mathbb{N}$.

Mahler's classification is closely related to Koksma's. Specifically, any S-number is an S^*-number and vice versa, see [**35**, Theorem 22]. Furthermore, for any real transcendental number x and every integer n, we have $\omega^*_n(x) \leqslant \omega_n(x)$, so that $\omega^*(x) \leqslant \omega(x)$. Proposition 7 then implies that

$$\mathcal{H}^g(\{x \in V \backslash \mathbb{A} \mid \omega(x) \geqslant \tau\}) = \mathcal{H}^g(V).$$

for every $g \in \mathfrak{D}$, every open set V and every real number $\tau \in [1,\tau_g]$. In particular, for any $\tau \in [1,\infty)$, the set of all real transcendental numbers x such that $\omega(x) \geqslant \tau$ has infinite

$1/\tau$-dimensional Hausdorff measure, thereby being of Hausdorff dimension at least $1/\tau$ as previously stated by [**2**, Theorem 4].

5. *Proof of Theorem* 1

For $\lambda \in \Lambda_c$ and $F \subseteq \mathbb{R}^d$, let $R_c^\lambda(F)$ be the set of all sequences $(\lambda_p)_{p\in\mathbb{N}}$ in $\Lambda_c \cup \{\emptyset\}$ such that $F \cap \lambda \subseteq \bigsqcup_p \lambda_p \subseteq \lambda$ (i.e. the sets λ_p, $p \in \mathbb{N}$, are disjoint, contained in λ and cover $F \cap \lambda$). Note that $R_c^\lambda(F) \subseteq R_{c,h}(F \cap \lambda)$ for every $\lambda \in \Lambda_{c,h}$ and every $h \in \mathfrak{D}_d$. We begin by establishing a series of lemmas and ancillary results.

LEMMA 8. *Let $h \in \mathfrak{D}_d$. For every $F \subseteq \mathbb{R}^d$ and every $\lambda \in \Lambda_{c,h}$,*

$$\mathcal{M}_\infty^h(F \cap \lambda) = \inf_{(\lambda_p)_{p\in\mathbb{N}} \in R_c^\lambda(F)} \sum_{p=1}^{\infty} h(|\lambda_p|).$$

Proof. Let $F \subseteq \mathbb{R}^d$ and $\lambda \in \Lambda_{c,h}$. Recall that $\mathcal{M}_\infty^h(F \cap \lambda)$ is given by (2·1). Let $(\lambda_p)_{p\in\mathbb{N}} \in R_{c,h}(F \cap \lambda)$. Let P be the set of all $p \in \mathbb{N}$ such that $\lambda_p \cap \lambda \neq \emptyset$ and $\lambda_p \neq \lambda_{p'}$ for all $p' < p$. Then, let P' be the set of all $p \in P$ such that $\lambda_p \not\subseteq \lambda_{p'}$ for all $p' \in P\backslash\{p\}$. If $P' = \{p_0\}$ for some p_0 with $\lambda_{p_0} \supseteq \lambda$, let $\lambda'_{p_0} = \lambda$. If not, let $\lambda'_p = \lambda_p$ for all $p \in P'$. In addition, let $\lambda'_p = \emptyset$ for all $p \notin P'$. It is now easy to check that $(\lambda'_p)_{p\in\mathbb{N}} \in R_c^\lambda(F)$ and $\sum_p h(|\lambda'_p|) \leqslant \sum_p h(|\lambda_p|)$ because h is nondecreasing on $(0, \varepsilon_h)$. We conclude by taking the infimums in this inequality. □

We shall express the $\mathcal{M}_\infty^h$-mass of small c-adic cubes and of their interior in terms of their diameter. Y. Bugeaud [**15**] asserted that $\mathcal{M}_\infty^h(\lambda) = h(|\lambda|)$ for every dyadic cube λ of small diameter when the gauge function h is concave in a neighborhood of zero. However, most classical gauge functions do not satisfy this property. The following lemma shows that no assumption on the concavity of h is actually required.

LEMMA 9. *Let $h \in \mathfrak{D}_d$ and $\lambda \in \Lambda_{c,h}$. Then $\mathcal{M}_\infty^h(\mathrm{int}\lambda) = \mathcal{M}_\infty^h(\lambda) = h(|\lambda|)$.*

Proof. Let F denote the set of all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ such that $x_p \neq kc^{-j}$ for all $p \in \{1, \ldots, d\}$, $j \in \mathbb{N}$ and $k \in \mathbb{Z}$. We only have to show that $\mathcal{M}_\infty^h(\mathrm{int}\lambda) \geqslant h(|\lambda|)$. We first assume that $\mathcal{M}_\infty^h(\mathrm{int}\lambda') < h(|\lambda'|)$ for every c-adic cube $\lambda' \subseteq \lambda$. As a result, $\mathcal{M}_\infty^h(F \cap \lambda') < h(|\lambda'|)$ and, owing to Lemma 8, there exists $(\nu_p^{\lambda'})_{p\in\mathbb{N}} \in R_c^{\lambda'}(F)$ with $\sum_p h(|\nu_p^{\lambda'}|) < h(|\lambda'|)$ and $\nu_p^{\lambda'} \subset \lambda'$ for all p. Consequently, $\sum_p h(|\nu_p^\lambda|) = \alpha h(|\lambda|)$ for some $\alpha \in (0,1)$ and it is straightforward to show by induction on $j \geqslant \langle\lambda\rangle_c$ that there exists $(\lambda_p^j)_{p\in\mathbb{N}} \in R_c^\lambda(F)$ such that $\sum_p h(|\lambda_p^j|) \leqslant \alpha h(|\lambda|)$ and $\langle\lambda_p^j\rangle_c \geqslant j$ or $\lambda_p^j = \emptyset$ for all p. Furthermore, $g(r) = h(r)/r^d$ is nonincreasing on $(0, \varepsilon_h)$ and tends to $\eta \in (0, \infty]$ as $r \to 0$, so $g(|\lambda|) \leqslant \eta$ and for any $\eta' \in (0, \eta)$ there is an integer $j \geqslant \langle\lambda\rangle_c$ such that $h(|\lambda'|) \geqslant \eta'|\lambda'|^d$ for all $\lambda' \in \Lambda_c$ with $\langle\lambda'\rangle_c \geqslant j$. Thus

$$\begin{aligned}\alpha h(|\lambda|) &\geqslant \sum_{p=1}^{\infty} h(|\lambda_p^j|) \geqslant \eta' \sum_{p=1}^{\infty} |\lambda_p^j|^d = \eta'\kappa \sum_{p=1}^{\infty} \mathcal{L}^d(\lambda_p^j) \\ &\geqslant \eta'\kappa\mathcal{L}^d(F \cap \lambda) = \eta'\kappa\mathcal{L}^d(\lambda) = \eta'|\lambda|^d\end{aligned}$$

where $\kappa = |[0,1)^d|^d$ and $\mathcal{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$. Letting $\eta' \to \eta$, we get $g(|\lambda|) \geqslant \eta/\alpha$, which is a contradiction.

Therefore, there is a c-adic cube $\lambda' \subseteq \lambda$ such that $\mathcal{M}_\infty^h(\mathrm{int}\lambda') = h(|\lambda'|)$ and we can assume that $\lambda' \subset \lambda$. Let $(\lambda_p)_{p\in\mathbb{N}} \in R_c^\lambda(\mathrm{int}\lambda)$. For each c-adic cube $\mu \subseteq \lambda$ of generation

$j' = \langle \lambda' \rangle_c$ that contains some nonempty λ_p, the set $\mu \cap \mathrm{int}\lambda$ is covered by the cubes λ_p enjoying $\emptyset \neq \lambda_p \subseteq \mu$, so that

$$h(|\mu|) = h(|\lambda'|) = \mathcal{M}^h_\infty(\mathrm{int}\lambda') = \mathcal{M}^h_\infty(\mathrm{int}\mu) \leqslant \mathcal{M}^h_\infty(\mu \cap \mathrm{int}\lambda) \leqslant \sum_{\lambda_p \subseteq \mu} h(|\lambda_p|).$$

Therefore, these cubes μ together with the cubes λ_p of generation at most $j' - 1$ yield a covering $(\nu_p)_{p\in\mathbb{N}} \in R^\lambda_c(\mathrm{int}\lambda)$ such that $\sum_p h(|\nu_p|) \leqslant \sum_p h(|\lambda_p|)$ and $\langle \nu_p \rangle_c \leqslant j'$ or $\nu_p = \emptyset$ for all p. Each c-adic cube $\mu' \subseteq \lambda$ of generation $j' - 1$ that strictly contains some nonempty ν_p necessarily contains c^d cubes $\nu_{j_1^{\mu'}}, \ldots, \nu_{j_{c^d}^{\mu'}}$ of generation j' and

$$\sum_{q=1}^{c^d} h(|\nu_{j_q^{\mu'}}|) = c^d\, h\left(\frac{|\mu'|}{c}\right) = c^d \left(\frac{|\mu'|}{c}\right)^d g\left(\frac{|\mu'|}{c}\right) \geqslant |\mu'|^d g(|\mu'|) = h(|\mu'|)$$

because g is nonincreasing on $(0, \varepsilon_h)$ and $\mu'/c \leqslant \mu' < \varepsilon_h$. It follows that these cubes μ' together with the cubes ν_p of generation at most $j'-1$ yield a covering $(\nu'_p)_{p\in\mathbb{N}} \in R^\lambda_c(\mathrm{int}\lambda)$ such that $\sum_p h(|\nu'_p|) \leqslant \sum_p h(|\lambda_p|)$ and $\langle \nu'_p \rangle_c \leqslant j' - 1$ or $\nu'_p = \emptyset$ for all p. This process is iterated so as to end up with $h(|\lambda|) \leqslant \sum_p h(|\lambda_p|)$. We conclude thanks to Lemma 8. □

The following lemma is a straightforward adaptation of [**20**, Lemma 1] so we just outline its proof.

LEMMA 10. *Let $h \in \mathfrak{D}_d$, $F \subseteq \mathbb{R}^d$, $C \in (0, 1]$ and $\rho \in (0, \varepsilon_h]$. Let U be an open subset of $\mathbb{R}^d$ such that $\mathcal{M}^h_\infty(F \cap \lambda) \geqslant C\mathcal{M}^h_\infty(\lambda)$ for every c-adic cube $\lambda \subseteq U$ enjoying $|\lambda| < \rho$. Then $\mathcal{M}^h_\infty(F \cap U) \geqslant C\mathcal{M}^h_\infty(U)$.*

Proof. We may assume that $U \neq \emptyset$. Since U is open, it is the union of a family Λ_U of disjoint c-adic cubes of diameter less than ρ. Let $(\lambda_p)_{p\in\mathbb{N}} \in R_{c,h}(F \cap U)$. Let P be the set of all $p \in \mathbb{N}$ such that $\lambda_p \neq \lambda_{p'}$ for all $p' < p$ and let P' be the set of all $p \in P$ such that $\lambda_p \not\subseteq \lambda_{p'}$ for all $p' \in P \backslash \{p\}$. Then, the sets λ_p, $p \in P'$, are disjoint cubes which cover $F \cap U$. Let $\lambda \in \Lambda_U$. Recall that $|\lambda| < \rho \leqslant \varepsilon_h$. If $P_\lambda = \{p \in P' \mid \lambda_p \subset \lambda\} \neq \emptyset$, the cubes μ_p, $p \in P_\lambda$, cover $F \cap \lambda$ so

$$Ch(|\lambda|) = C\mathcal{M}^h_\infty(\lambda) \leqslant \mathcal{M}^h_\infty(F \cap \lambda) \leqslant \sum_{p \in P_\lambda} h(|\lambda_p|)$$

owing to Lemma 9. If P_λ is empty, $\lambda \subseteq \lambda_p$ for some $p \in P'' = P' \backslash \bigcup_{\lambda \in \Lambda_U} P_\lambda$. The cubes $\lambda \in \Lambda_U$ enjoying $P_\lambda \neq \emptyset$ together with the cubes λ_p, $p \in P''$, yield a covering which belongs to $R_{c,h}(U)$ and implies that $C\mathcal{M}^h_\infty(U) \leqslant \sum_p h(|\lambda_p|)$. We conclude using (2·1). □

The next proposition concerns the G_δ-sets of full Lebesgue measure.

PROPOSITION 11. *Let V be a nonempty open subset of $\mathbb{R}^d$ and let F be a G_δ-subset of $\mathbb{R}^d$ of full Lebesgue measure in V. Then $F \in \mathrm{G}^h(V)$ for every $h \in \mathfrak{D}_d$.*

Proof. Let $h \in \mathfrak{D}_d$, let $\lambda \in \Lambda_{c,h}$ with $\lambda \subseteq V$ and let $(\lambda_p)_{p\in\mathbb{N}} \in R^\lambda_c(F)$. Since the function $r \mapsto h(r)/r^d$ is nonincreasing on $(0, \varepsilon_h)$, we have

$$\begin{aligned}\sum_{p=1}^\infty h(|\lambda_p|) &\geqslant \frac{h(|\lambda|)}{|\lambda|^d} \sum_{p=1}^\infty |\lambda_p|^d = \frac{h(|\lambda|)}{|\lambda|^d} \kappa \sum_{p=1}^\infty \mathcal{L}^d(\lambda_p) \\ &\geqslant \frac{h(|\lambda|)}{|\lambda|^d} \kappa \mathcal{L}^d(F \cap \lambda) = \frac{h(|\lambda|)}{|\lambda|^d} \kappa \mathcal{L}^d(\lambda) = h(|\lambda|)\end{aligned}$$

where $\kappa = |[0,1)^d|^d$. Thus Lemma 8 and Lemma 9 give $\mathcal{M}^h_\infty(F\cap\lambda) \geqslant h(|\lambda|) = \mathcal{M}^h_\infty(\lambda)$. We conclude using Lemma 10. $\square$

The following lemma is reminiscent of Lemma 2 and Lemma 3 in [**20**] and involves Lipschitz images of sets.

LEMMA 12. *Let $h \in \mathfrak{D}_d$, let V denote a nonempty open subset of $\mathbb{R}^d$ and let $f : V \to \mathbb{R}^d$ be a bi-Lipschitz mapping. Let $F \subseteq \mathbb{R}^d$, $C \in (0,1]$ and $\rho \in (0, \varepsilon_h]$. Assume that $\mathcal{M}^h_\infty(F\cap U) \geqslant C\mathcal{M}^h_\infty(U)$ for every open set $U \subseteq f(V)$ such that $|U| < \rho$. Then $\mathcal{M}^g_\infty(f^{-1}(F)\cap U) = \mathcal{M}^g_\infty(U)$ for every open set $U \subseteq V$ and every $g \in \mathfrak{D}_d$ with $g \prec h$.*

Proof. Let $\psi : V \to \mathbb{R}^d$ denote a Lipschitz mapping. There exists $n \in \mathbb{N}$ such that $\|\psi(y) - \psi(x)\| \leqslant c^n\|y - x\|$ for all $x, y \in V$. Let $\lambda \in \Lambda_c$. Then $\psi(V \cap \lambda)$, which is of diameter at most $c^n|\lambda|$, is covered by K c-adic cubes of diameter $c^n|\lambda|$, where $K \in \mathbb{N}$ only depends on the dimension d and the choice of $\|\cdot\|$. Hence it is covered by Kc^{nd} cubes of diameter $|\lambda|$. Using this, it is easy to show that $\mathcal{M}^h_\infty(\psi(A)) \leqslant Kc^{nd}\mathcal{M}^h_\infty(A)$ for every $A \subseteq V$. Moreover, there are $n_1, n_2 \in \mathbb{N}$ such that $c^{-n_1}\|x-y\| \leqslant \|f(x)-f(y)\| \leqslant c^{n_2}\|x-y\|$ for all $x, y \in V$. Let $U \subseteq V$ be an open set and let $\mu \subseteq U$ be a c-adic cube of diameter less than $c^{-n_2}\rho$. Then $f(\mathrm{int}\mu) \subseteq f(V)$ is an open set of diameter less than ρ and

$$\mathcal{M}^h_\infty(\mathrm{int}\mu) \leqslant Kc^{n_1 d}\mathcal{M}^h_\infty(f(\mathrm{int}\mu)) \leqslant \frac{Kc^{n_1 d}}{C}\mathcal{M}^h_\infty(f(\mathrm{int}\mu)\cap F).$$

It follows that $\mathcal{M}^h_\infty(\mathrm{int}\mu \cap f^{-1}(F)) \geqslant C'\mathcal{M}^h_\infty(\mathrm{int}\mu)$ where $C' = CK^{-2}c^{-(n_1+n_2)d}$ and Lemma 9 gives $\mathcal{M}^h_\infty(f^{-1}(F)\cap\mu) \geqslant C'h(|\mu|)$. This is true for all $\mu \subseteq U$ of diameter less than $c^{-n_2}\rho$, i.e. of generation at least j, say.

Let $g \in \mathfrak{D}_d$ with $g \prec h$. Let $\varphi = g/h$ and let $\rho' > 0$ denote the supremum of all $x \in (0, \min(\varepsilon_g, \varepsilon_h))$ such that φ is nonincreasing on $(0, x)$. Let $\lambda \subseteq U$ be a c-adic cube of diameter less than ρ'. Since φ tends to infinity at zero, there is an integer $j' \geqslant \max(\langle\lambda\rangle_c, j)$ such that $\varphi(|\lambda'|) \geqslant \varphi(|\lambda|)/C'$ for all $\lambda' \in \Lambda_c$ with $\langle\lambda'\rangle_c \geqslant j'$. Now let $(\lambda_p)_{p\in\mathbb{N}} \in R^\lambda_c(f^{-1}(F))$. For each p with $\langle\lambda_p\rangle_c \leqslant j' - 1$, we have $g(|\lambda_p|) \geqslant \varphi(|\lambda|)h(|\lambda_p|)$. Meanwhile, for each cube $\mu \subseteq \lambda$ of generation j' that is not contained in one of the previous λ_p, $f^{-1}(F)\cap\mu$ is covered by the nonempty sets λ_p enjoying $\lambda_p \subseteq \mu$, so that

$$\sum_{\lambda_p\subseteq\mu} g(|\lambda_p|) \geqslant \frac{\varphi(|\lambda|)}{C'}\sum_{\lambda_p\subseteq\mu} h(|\lambda_p|) \geqslant \frac{\varphi(|\lambda|)}{C'}\mathcal{M}^h_\infty(f^{-1}(F)\cap\mu) \geqslant \varphi(|\lambda|)h(|\mu|).$$

Therefore, these cubes μ along with the cubes λ_p of generation at most $j' - 1$ yield a covering which belongs to $R_{c,h}(\lambda)$ and implies that $\varphi(|\lambda|)\mathcal{M}^h_\infty(\lambda) \leqslant \sum_p g(|\lambda_p|)$. As a result, $\mathcal{M}^g_\infty(\lambda) \leqslant \mathcal{M}^g_\infty(f^{-1}(F)\cap\lambda)$ by Lemma 8 and Lemma 9. We conclude using Lemma 10. $\square$

Lemma 12 leads to the following proposition.

PROPOSITION 13. *Let $h \in \mathfrak{D}_d$ and let V be a nonempty open subset of $\mathbb{R}^d$. Then $\mathrm{G}^h(V)$ depends on the choice of neither $\|\cdot\|$ nor c.*

Proof. Let $c \geqslant 2$ and let $\|\cdot\|_1$, $\|\cdot\|_2$ be two norms on $\mathbb{R}^d$. The corresponding diameters and outer net measures are denoted by $|\cdot|_1$, $|\cdot|_2$ and $\mathcal{M}^h_{\infty,1}$, $\mathcal{M}^h_{\infty,2}$ respectively. Let F be a G_δ-set such that $\mathcal{M}^g_{\infty,1}(F\cap U) = \mathcal{M}^g_{\infty,1}(U)$ for every open set $U \subseteq V$ and every $g \in \mathfrak{D}_d$ enjoying $g \prec h$. To show that the same is true for $\mathcal{M}^g_{\infty,2}$, let U be an open subset of V and let $g \in \mathfrak{D}_d$ with $g \prec h$. Then $f = \sqrt{gh} \in \mathfrak{D}_d$ and $g \prec f \prec h$.

Moreover, let $\kappa = |[0,1)^d|_2/|[0,1)^d|_1$. Thus $|\lambda|_2 = \kappa|\lambda|_1$ for every c-adic cube λ and $\min(1,\kappa)^d f(|\lambda|_1) \leqslant f(|\lambda|_2) \leqslant \max(1,\kappa)^d f(|\lambda|_1)$ if $|\lambda|_2 < \varepsilon_f \min(1,\kappa)$ because $f \in \mathfrak{D}_d$. Let $\lambda \subseteq U$ be such a cube and let $(\lambda_p)_{p\in\mathbb{N}} \in R_c^\lambda(F)$. Since $|\lambda_p|_1 \leqslant |\lambda|_1 = |\lambda|_2/\kappa < \varepsilon_f$ for all p, we have

$$\sum_{p=1}^{\infty} f(|\lambda_p|_2) \geqslant \min(1,\kappa)^d \sum_{p=1}^{\infty} f(|\lambda_p|_1) \geqslant \min(1,\kappa)^d \mathcal{M}^f_{\infty,1}(F\cap\lambda).$$

In addition, $f \prec h$ and $\mathrm{int}\lambda \subseteq V$ is open, so $\mathcal{M}^f_{\infty,1}(F\cap \mathrm{int}\lambda) = \mathcal{M}^f_{\infty,1}(\mathrm{int}\lambda)$. Then Lemma 9 yields $\mathcal{M}^f_{\infty,1}(F\cap\lambda) \geqslant f(|\lambda|_1)$ and $f(|\lambda|_2) = \mathcal{M}^f_{\infty,2}(\lambda)$. Thus

$$\sum_{p=1}^{\infty} f(|\lambda_p|_2) \geqslant \left(\frac{\min(1,\kappa)}{\max(1,\kappa)}\right)^d \mathcal{M}^f_{\infty,2}(\lambda).$$

It follows from Lemma 8 and Lemma 10 that

$$\mathcal{M}^f_{\infty,2}(F\cap U) \geqslant \left(\frac{\min(1,\kappa)}{\max(1,\kappa)}\right)^d \mathcal{M}^f_{\infty,2}(U)$$

and Lemma 12 leads to $\mathcal{M}^g_{\infty,2}(F\cap U) = \mathcal{M}^g_{\infty,2}(U)$ for every open $U \subseteq V$ since $g \prec f$.

Let us endow $\mathbb{R}^d$ with the supremum norm $\|\cdot\|_\infty$ and let $c \geqslant 2$. One easily checks that

$$\forall h \in \mathfrak{D}_d \quad \forall A \subseteq \mathbb{R}^d \qquad \mathcal{H}^h_{\varepsilon_h}(A) \leqslant \mathcal{M}^h_\infty(A) \leqslant (3c^2)^d \mathcal{H}^h_{\varepsilon_h}(A) \tag{5·1}$$

where $\mathcal{H}^h_{\varepsilon_h}$ is the Hausdorff pre-measure defined in terms of coverings by sets of diameter less than ε_h. Let $c_1, c_2 \geqslant 2$. The corresponding outer net measures are denoted by $\mathcal{M}^h_{\infty,1}$, $\mathcal{M}^h_{\infty,2}$ respectively. Let us assume that $\mathcal{M}^g_{\infty,1}(F\cap U) = \mathcal{M}^g_{\infty,1}(U)$ for every open set $U \subseteq V$ and every $g \in \mathfrak{D}_d$ satisfying $g \prec h$ and show that the same is true for $\mathcal{M}^g_{\infty,2}$. Let $g \in \mathfrak{D}_d$ with $g \prec h$ and let $f = \sqrt{gh}$. As $f \prec h$, for every open $U \subseteq V$,

$$\begin{aligned}\mathcal{M}^f_{\infty,2}(F\cap U) &\geqslant \mathcal{H}^f_{\varepsilon_f}(F\cap U) \geqslant (3{c_1}^2)^{-d}\mathcal{M}^f_{\infty,1}(F\cap U) = (3{c_1}^2)^{-d}\mathcal{M}^f_{\infty,1}(U)\\ &\geqslant (3{c_1}^2)^{-d}\mathcal{H}^f_{\varepsilon_f}(U) \geqslant (9{c_1}^2{c_2}^2)^{-d}\mathcal{M}^f_{\infty,2}(U)\end{aligned}$$

thanks to (5·1). We conclude by Lemma 12 since $g \prec f$. $\square$

We can now prove Theorem 1. We begin by establishing (iii). Let $h \in \mathfrak{D}_d$ and let V be a nonempty open subset of $\mathbb{R}^d$. Let $F \in \mathrm{G}^h(V)$. In particular $\mathcal{M}^f_\infty(F) \geqslant \mathcal{M}^f_\infty(V) > 0$ for all $f \in \mathfrak{D}_d$ satisfying $f \prec h$. Thanks to (5·1) we have $\mathcal{H}^f_{\varepsilon_f}(F) > 0$ if $\mathbb{R}^d$ is endowed with the supremum norm, thus $\mathcal{H}^f(F) > 0$. This remains true if $\mathbb{R}^d$ is endowed with any other norm. Hence for every $g \in \mathfrak{D}_d$ enjoying $g \prec h$, we have $\mathcal{H}^{\sqrt{gh}}(F) > 0$ so $\mathcal{H}^g(F) = \infty$.

Let us now establish (ii). Let $f : V \to \mathbb{R}^d$ denote a bi-Lipschitz mapping, let F belong to $\mathrm{G}^h(f(V))$ and let $g \in \mathfrak{D}_d$ with $g \prec h$. For every open $U \subseteq f(V)$, we have $\mathcal{M}^{\sqrt{gh}}(F\cap U) = \mathcal{M}^{\sqrt{gh}}(U)$, so Lemma 12 yields $\mathcal{M}^g_\infty(f^{-1}(F)\cap U) = \mathcal{M}^g_\infty(U)$ for every open $U \subseteq V$. This holds for all $g \in \mathfrak{D}_d$ with $g \prec h$, so the G_δ-set $f^{-1}(F)$ is in $\mathrm{G}^h(V)$.

We end this section by proving (i). Let $(\Phi_k)_{k\in\mathbb{N}}$ be a sequence of sets which belong to $\mathrm{G}^h(V)$. Let $g \in \mathfrak{D}_d$ such that $g \prec h$ and let $f = \sqrt{gh}$. Imitating the proof of [**20**, Lemma 4] and using the increasing sets lemma [**33**, Theorem 52] for the outer net measure $\mathcal{M}^f_\infty$, it is easy to show that $\mathcal{M}^f_\infty(\bigcap_k \Phi_k \cap U) \geqslant 3^{-d}\mathcal{M}^f_\infty(U)$ for every open set $U \subseteq V$. Because $g \prec f$, Lemma 12 yields $\mathcal{M}^g_\infty(\bigcap_k \Phi_k \cap U) = \mathcal{M}^g_\infty(U)$ for every open $U \subseteq V$. This holds for all $g \in \mathfrak{D}_d$ with $g \prec h$ so $\bigcap_k \Phi_k \in \mathrm{G}^h(V)$.

6. *Proof of Theorem* 2

From now on, I is a denumerable set and V is a nonempty open subset of $\mathbb{R}^d$. We begin by establishing some preliminary results.

LEMMA 14. *Let $(x_i, r_i)_{i\in I} \in \mathcal{S}_d(I)$ be a homogeneous ubiquitous system in V and let U be a nonempty bounded open subset of V. Then for all $\rho > 0$ there is a finite subset I' of I such that the closed balls $\bar{B}(x_i, r_i)$, $i \in I'$, are disjoint subsets of U which enjoy*

$$\sum_{i\in I'} \mathcal{L}^d\left(\bar{B}(x_i, r_i)\right) \geqslant \frac{\mathcal{L}^d(U)}{2\cdot 3^d} \qquad \text{and} \qquad \forall i \in I' \quad r_i \leqslant \rho.$$

Proof. Let $\rho > 0$ and let $I_{\rho,U}$ be the set of all $i \in I$ such that $x_i \in U$ and $r_i \leqslant \rho$. One easily checks that $I_{\rho,U}$ is denumerable and $(x_i, r_i)_{i\in I_{\rho,U}} \in \mathcal{S}_d(I_{\rho,U})$ is a homogeneous ubiquitous system in U. Let $(i_n)_{n\in\mathbb{N}}$ be an enumeration of $I_{\rho,U}$. For $\varepsilon > 0$ the set of all $n \in \mathbb{N}$ satisfying $r_{i_n} > \varepsilon$ is finite since it is included in the set of all $i \in I_{\rho,U}$ enjoying $x_i \in U$ and $r_i > \varepsilon$, while U is bounded. Hence r_{i_n} tends to zero as $n \to \infty$ and, up to a reordering, we can assume that the sequence $(r_{i_n})_{n\in\mathbb{N}}$ is nonincreasing. In addition $(x_{i_n}, r_{i_n})_{n\in\mathbb{N}} \in \mathcal{S}_d(\mathbb{N})$ is a homogeneous ubiquitous system in U.

Note that every nonempty open set $U' \subseteq U$ contains a ball $\bar{B}(x_{i_n}, r_{i_n})$. Thus we can let n_1 be the smallest integer such that $\bar{B}(x_{i_{n_1}}, r_{i_{n_1}}) \subseteq U$ and n_k be the smallest integer such that $\bar{B}(x_{i_{n_k}}, r_{i_{n_k}}) \subseteq U\backslash(\bar{B}(x_{i_{n_1}}, r_{i_{n_1}})\cup\ldots\cup\bar{B}(x_{i_{n_{k-1}}}, r_{i_{n_{k-1}}}))$ for every $k \geqslant 2$. Since $r_{i_n} \downarrow 0$ we have

$$U \cap \limsup_{n\to\infty} B(x_{i_n}, r_{i_n}) \subseteq \bigcup_{k=1}^{\infty} \bar{B}(x_{i_{n_k}}, 3r_{i_{n_k}}).$$

Note that the Lebesgue measure of the left-hand side is $\mathcal{L}^d(U) < \infty$. Thus the measure of $\bigcup_{k=1}^{k_1} \bar{B}(x_{i_{n_k}}, 3r_{i_{n_k}})$ is at least $\mathcal{L}^d(U)/2$ for some $k_1 \in \mathbb{N}$. To conclude, we let I' be the set of all i_{n_k} for $k \in \{1, \ldots, k_1\}$. $\square$

Lemma 14 yields the following proposition.

PROPOSITION 15. *Let $(x_i, r_i)_{i\in I} \in \mathcal{S}_d(I)$ be a homogeneous ubiquitous system in V. Then $(x_i, \kappa r_i)_{i\in I}$ is a homogeneous ubiquitous system in V for any $\kappa > 0$.*

Proof. Let $\kappa > 0$. Since $(x_i, \kappa r_i)_{i\in I} \in \mathcal{S}_d(I)$, it remains to show that the set R_κ of all $x \in \mathbb{R}^d$ such that $\|x - x_i\| < \kappa r_i$ for infinitely many $i \in I$ has full Lebesgue measure in V. This is obvious if $\kappa \geqslant 1$ so we assume that $\kappa < 1$.

Let $U \subseteq V$ be a nonempty bounded open set and let $j \in \mathbb{N}$. By Lemma 14 there is a finite subset I_j of I such that the balls $\bar{B}(x_i, r_i) \subseteq U$, $i \in I_j$, are disjoint and enjoy $\sum_{i\in I_j} \mathcal{L}^d\left(\bar{B}(x_i, r_i)\right) \geqslant \mathcal{L}^d(U)/(2\cdot 3^d)$ and $r_i \leqslant 2^{-j}$. Thus the Lebesgue measure of $U \cap R_\kappa$, which contains $\limsup_j \bigsqcup_{i\in I_j} B(x_i, \kappa r_i)$, is at least $\kappa^d \mathcal{L}^d(U)/(2\cdot 3^d)$.

Let us assume that $\mathcal{L}^d(V\backslash R_\kappa) > 0$. Then $\mathcal{L}^d(V_m\backslash R_\kappa) > 0$ for m large enough, where $V_m = V \cap (-m, m)^d$. Furthermore, there exists a compact set $K \subseteq R_\kappa \cap V_m$ such that $\mathcal{L}^d(R_\kappa\cap V_m\backslash K) < \kappa^d \mathcal{L}^d(V_m\backslash R_\kappa)/(2\cdot 3^d)$. Applying what precedes to the bounded open set $U = V_m\backslash K$, we obtain $\mathcal{L}^d(R_\kappa \cap V_m\backslash K) \geqslant \kappa^d \mathcal{L}^d(V_m\backslash K)/(2\cdot 3^d) \geqslant \kappa^d \mathcal{L}^d(V_m\backslash R_\kappa)/(2\cdot 3^d)$ and we end up with a contradiction. Hence R_κ has full Lebesgue measure in V. $\square$

We can now prove Theorem 2. Let $(x_i, r_i)_{i\in I} \in \mathcal{S}_d(I)$ be a homogeneous ubiquitous system in V, let $h \in \mathfrak{D}_d$ and let $\varphi : [0, \infty) \to \mathbb{R}$ be a nonnegative nondecreasing function that coincides with $(h^{1/d})^{-1}$ in a neighborhood of the origin. In addition, let $\varphi_h : [0,\infty) \to \mathbb{R}$ denote a nonnegative nondecreasing function that coincides with $(h^{1/d})^{-1}$ on $[0, h^{1/d}(\varepsilon_h{}^-))$. Note that $\varphi_h(h^{1/d}(r)) \leqslant r$ for every $r \in [0,\varepsilon_h)$ and that $h^{1/d}(\varphi_h(r)) \geqslant r$ for every $r \in [0, h^{1/d}(\varepsilon_h{}^-))$. Moreover, φ and φ_h coincide near zero, so that $F_\varphi = F_{\varphi_h}$. Hence it suffices to show that $F_{\varphi_h} \in \mathrm{G}^h(V)$.

Recall that $r \mapsto h(r)/r^d$ is nonincreasing on $(0,\varepsilon_h)$. Let η denote its limit at zero. We first assume that $\eta \leqslant 1$. As a consequence, $h^{1/d}(r) \leqslant r$ for every $r \in [0,\varepsilon_h)$. Thus $r \leqslant h^{1/d}(\varphi_h(r)) \leqslant \varphi_h(r)$ for every $r \in [0, h^{1/d}(\varepsilon_h{}^-))$, so that $F_{\mathrm{Id}} \subseteq F_{\varphi_h}$. Meanwhile, F_{Id} is a G_δ-set of full Lebesgue measure in V so it belongs to $\mathrm{G}^h(V)$ by Proposition 11. Proposition 1(e) then gives $F_{\varphi_h} \in \mathrm{G}^h(V)$.

From now on, we assume that $\eta > 1$. Let $f \in \mathfrak{D}_d$ with $f \prec h$. There is a real number $\rho \in (0, \min(\varepsilon_h, h^{1/d}(\varepsilon_h{}^-), \varepsilon_f)]$ such that $0 < \varphi_h(r) \leqslant \varphi_h(h^{1/d}(r)) \leqslant r$ for all $r \in (0,\rho)$. Moreover, $\|x\|_\infty/\kappa \leqslant \|x\| \leqslant \kappa\|x\|_\infty$ for all $x \in \mathbb{R}^d$ and some $\kappa \geqslant 1$. Note that $(2/\kappa)^d \leqslant \mathcal{L}^d(B(0,1)) \leqslant (2\kappa)^d$. Let us show that

$$\mathcal{M}^f_\infty(F_{\varphi_h} \cap \lambda) \geqslant \frac{\mathcal{M}^f_\infty(\lambda)}{2 \cdot 48^d \kappa^{4d}} \tag{6·1}$$

for every c-adic cube $\lambda \subseteq V$ with $|\lambda| < \rho$. To this end, we build a generalized Cantor set $K \subseteq F_{\varphi_h} \cap \lambda$ and a measure π supported on K. Let $G_0 = \{\overline{\lambda}\}$ and $\pi(\overline{\lambda}) = f(|\lambda|)$.

Step 1. As f/h tends to infinity at zero, for some $r > 0$ we have

$$\forall r' \in (0,r] \qquad \frac{f(r')}{h(r')} \geqslant 2 \cdot 6^d \kappa^d \frac{f(|\lambda|)}{\mathcal{L}^d(\mathrm{int}\lambda)}. \tag{6·2}$$

Furthermore, owing to Lemma 14 there is a finite subset I' of I such that the balls $\bar{B}(x_i,r_i)$, $i \in I'$, are disjoint subsets of $\mathrm{int}\lambda$ which satisfy

$$\sum_{i \in I'} \mathcal{L}^d\left(\bar{B}(x_i,r_i)\right) \geqslant \frac{\mathcal{L}^d(\mathrm{int}\lambda)}{2 \cdot 3^d} \tag{6·3}$$

and $r_i \leqslant r$. Let $G_1 = \{\bar{B}(x_i, \varphi_h(r_i)/2), i \in I'\}$. Each closed ball $\beta \in G_1$ is associated with the open ball $\tilde{\beta} = B(x_i,r_i)$. Note that $\beta \subseteq B(x_i,\varphi_h(r_i)) \subseteq \tilde{\beta} \subseteq \overline{\tilde{\beta}} \subseteq \mathrm{int}\lambda$ and $|\tilde{\beta}| \leqslant 2h^{1/d}(|\beta|)$ because $r_i < \rho$. We set

$$\forall \beta \in G_1 \qquad \pi(\beta) = \frac{\mathcal{L}^d(\tilde{\beta})}{\sum\limits_{\beta' \in G_1} \mathcal{L}^d(\tilde{\beta}')} \pi(\overline{\lambda}).$$

Let $\beta \in G_1$. We have $\mathcal{L}^d(\tilde{\beta}) \leqslant \kappa^d |\tilde{\beta}|^d \leqslant 2^d \kappa^d h(|\beta|)$. Thus, using (6·2) and (6·3) together with the observation that $|\beta| \leqslant |\tilde{\beta}|/2 \leqslant r$, we obtain

$$\pi(\beta) \leqslant 2 \cdot 6^d \kappa^d \frac{f(|\lambda|)}{\mathcal{L}^d(\mathrm{int}\lambda)} h(|\beta|) \leqslant f(|\beta|).$$

Step 2. As f/h tends to infinity at the origin, for some $r > 0$ we have

$$\forall r' \in (0,r] \qquad \frac{f(r')}{h(r')} \geqslant 2 \cdot 6^d \kappa^d \max_{\beta \in G_1} \frac{f(|\beta|)}{\mathcal{L}^d(\mathrm{int}\beta)}. \tag{6·4}$$

Let $\beta \in G_1$. Lemma 14 yields a finite set $I' \subseteq I$ such that the balls $\bar{B}(x_i,r_i)$, $i \in I'$, are disjoint subsets of $\mathrm{int}\beta$ which enjoy $\sum_{i \in I'} \mathcal{L}^d\left(\bar{B}(x_i,r_i)\right) \geqslant \mathcal{L}^d(\mathrm{int}\beta)/(2 \cdot 3^d)$ and $r_i \leqslant r$. Let $G_2^\beta = \{\bar{B}(x_i,\varphi_h(r_i)/2), i \in I'\}$. Each closed ball $\gamma \in G_2^\beta$ is associated with an open ball $\tilde{\gamma} = B(x_i,r_i)$. We have $\gamma \subseteq B(x_i,\varphi_h(r_i)) \subseteq \tilde{\gamma} \subseteq \overline{\tilde{\gamma}} \subseteq \mathrm{int}\beta$ and $|\tilde{\gamma}| \leqslant 2h^{1/d}(|\gamma|)$. Let

$G_2 = \bigcup_{\beta \in G_1} G_2^\beta$. We set

$$\forall \beta \in G_1 \quad \forall \gamma \in G_2^\beta \qquad \pi(\gamma) = \frac{\mathcal{L}^d(\tilde{\gamma})}{\sum\limits_{\gamma' \in G_2^\beta} \mathcal{L}^d(\tilde{\gamma}')} \pi(\beta).$$

Thanks to (6·4), the bound on $\pi(\beta)$ we obtained at the previous step and the fact that $\mathcal{L}^d(\tilde{\gamma}) \leqslant 2^d \kappa^d h(|\gamma|)$, we get

$$\pi(\gamma) \leqslant 2 \cdot 6^d \kappa^d \frac{f(|\beta|)}{\mathcal{L}^d(\mathrm{int}\beta)} h(|\gamma|) \leqslant f(|\gamma|).$$

Summing-up of the construction. Iterating this procedure, we construct recursively a sequence $(G_q)_{q \geqslant 0}$ of collections of sets which have a π-mass and satisfy the following properties.

(A) We have $G_0 = \{\overline{\lambda}\}$ and $\pi(\overline{\lambda}) = f(|\lambda|)$. In addition, the set $\overline{\lambda}$ contains a finite number of sets of G_1.

(B) For every $q \in \mathbb{N}$, every $\gamma \in G_q$ is a closed ball which contains a finite number of sets of G_{q+1}. Moreover, there are an open ball $\tilde{\gamma}$, a unique closed set $\beta \in G_{q-1}$ and an index $i \in I$ enjoying $\gamma \subseteq B(x_i, \varphi_h(r_i)) \subseteq \tilde{\gamma} \subseteq \overline{\tilde{\gamma}} \subseteq \mathrm{int}\beta$ and $|\gamma| = \varphi_h(|\tilde{\gamma}|/2)$. Furthermore, the closed balls $\overline{\tilde{\gamma}}$, $\gamma \in G_q$, are disjoint.

(C) For every $q \in \mathbb{N}$ and every $\gamma \in G_q$ that is included in $\beta \in G_{q-1}$,

$$\pi(\gamma) = \frac{\mathcal{L}^d(\tilde{\gamma})}{\sum\limits_{\substack{\gamma' \in G_q \\ \gamma' \subseteq \beta}} \mathcal{L}^d(\tilde{\gamma}')} \pi(\beta) \leqslant \frac{2 \cdot 3^d}{\mathcal{L}^d(\mathrm{int}\beta)} \mathcal{L}^d(\tilde{\gamma}) \pi(\beta) \quad \text{and} \quad \pi(\gamma) \leqslant f(|\gamma|).$$

Thus $K = \bigcap_{q=0}^\infty \downarrow \bigcup_{\beta \in G_q} \beta$ is a generalized Cantor set included in $F_{\varphi_h} \cap \lambda$ and π can be extended to all Borel subsets of $\mathbb{R}^d$, thanks to [**21**, Proposition 1.7]. We obtain a finite Borel measure supported on K with total mass $\pi(K) = f(|\lambda|)$.

Scaling properties of π. Let μ be a c-adic subcube of λ. Let us give an upper bound on the π-mass of μ in terms of its diameter. We can assume that μ intersects K (if not, $\pi(\mu) = 0$) and that μ intersects at least two sets of G_{q+1} for some $q \geqslant 0$ (otherwise, (C) would ensure that $\pi(\mu) \leqslant \pi(\gamma_{q+1}) \leqslant f(|\gamma_{q+1}|) \to 0$ as $q \to \infty$, where γ_{q+1} denotes the unique set of G_{q+1} that intersects μ). Let $\beta \in G_q$ ($q \geqslant 0$) be the closed set of largest diameter $|\beta|$ such that μ intersects at least two closed balls of G_{q+1} that are included in β. Note that $\pi(\mu) \leqslant \pi(\beta)$. If $|\mu| \geqslant |\beta|$, using (A), (C) and the fact that f is nondecreasing on $(0, \rho)$, we have $\pi(\mu) \leqslant \pi(\beta) \leqslant f(|\beta|) \leqslant f(|\mu|)$.

We now assume that $|\mu| < |\beta|$ and we let $\gamma_1, \ldots, \gamma_n$ ($n \geqslant 2$) denote the closed balls of G_{q+1} that intersect μ. These balls are contained in β and (C) leads to

$$\pi(\mu) = \sum_{m=1}^n \pi(\mu \cap \gamma_m) \leqslant \frac{2 \cdot 3^d}{\mathcal{L}^d(\mathrm{int}\beta)} \pi(\beta) \sum_{m=1}^n \mathcal{L}^d(\tilde{\gamma}_m).$$

Moreover, because of (B), the closed balls $\overline{\tilde{\gamma}_m}$, $m \in \{1, \ldots, n\}$, are disjoint so that $|\mu| \geqslant (|\tilde{\gamma}_m| - |\gamma_m|)/2$ for every m. Let $y_m \in \mu \cap \gamma_m$. The closed ball (in the sense of the supremum norm) $\bar{B}_\infty(y_m, (|\tilde{\gamma}_m| - |\gamma_m|)/(8\kappa))$ is contained in $\tilde{\gamma}_m$ and is the union of 2^d closed cubes with edge length $(|\tilde{\gamma}_m| - |\gamma_m|)/(8\kappa) \leqslant |\mu|/(4\kappa)$ and y_m as a vertex. Note that one of these cubes is included in $\mu \cap \tilde{\gamma}_m$. As a result,

$$\mathcal{L}^d(\mu \cap \tilde{\gamma}_m) \geqslant \left(\frac{|\tilde{\gamma}_m| - |\gamma_m|}{8\kappa} \right)^d \geqslant \left(\frac{|\tilde{\gamma}_m|}{16\kappa} \right)^d \geqslant \frac{\mathcal{L}^d(\tilde{\gamma}_m)}{16^d \kappa^{2d}}$$

since $|\gamma_m| = \varphi_h(|\tilde{\gamma}_m|/2) \leqslant |\tilde{\gamma}_m|/2$ for $|\tilde{\gamma}_m| < \rho$. It follows that

$$\pi(\mu) \leqslant \frac{2 \cdot 48^d \kappa^{2d}}{\mathcal{L}^d(\mathrm{int}\beta)} \pi(\beta) \sum_{m=1}^{n} \mathcal{L}^d(\mu \cap \tilde{\gamma}_m) = \frac{2 \cdot 48^d \kappa^{2d}}{\mathcal{L}^d(\mathrm{int}\beta)} \pi(\beta) \mathcal{L}^d \left(\mu \cap \bigsqcup_{m=1}^{n} \tilde{\gamma}_m \right)$$
$$\leqslant 2 \cdot 48^d \kappa^{2d} \pi(\beta) \frac{\mathcal{L}^d(\mu)}{\mathcal{L}^d(\mathrm{int}\beta)}.$$

Owing to (A) and (C) we have $\pi(\beta) \leqslant f(|\beta|)$. In addition, note that $\mathcal{L}^d(\mu) \leqslant \kappa^d |\mu|^d$ and $\mathcal{L}^d(\mathrm{int}\beta) \geqslant |\beta|^d/\kappa^d$. Recall also that $|\mu| < |\beta| < \rho$ and that $r \mapsto f(r)/r^d$ is nonincreasing on $(0, \rho)$. Hence $\pi(\mu) \leqslant 2 \cdot 48^d \kappa^{4d} f(|\mu|)$.

We can now prove (6·1). Let $(\lambda_p)_{p\in\mathbb{N}} \in R_c^\lambda(F_{\varphi_h})$. We have

$$\sum_{p=1}^{\infty} f(|\lambda_p|) \geqslant \frac{1}{2 \cdot 48^d \kappa^{4d}} \sum_{p=1}^{\infty} \pi(\lambda_p) \geqslant \frac{1}{2 \cdot 48^d \kappa^{4d}} \pi\left(\bigcup_{p=1}^{\infty} \lambda_p \right) \geqslant \frac{\pi(F_{\varphi_h} \cap \lambda)}{2 \cdot 48^d \kappa^{4d}}.$$

Since the set $F_{\varphi_h} \cap \lambda$ contains K, its π-mass is at least $\pi(K) = f(|\lambda|)$. As a result, $\sum_p f(|\lambda_p|) \geqslant f(|\lambda|)/(2 \cdot 48^d \kappa^{4d})$. Lemma 9 additionally gives $f(|\lambda|) = \mathcal{M}^f_\infty(\lambda)$ and Lemma 8 leads to (6·1). Thus, by Lemma 10, $\mathcal{M}^f_\infty(F_{\varphi_h} \cap U) \geqslant \mathcal{M}^f_\infty(U)/(2 \cdot 48^d \kappa^{4d})$ for every open set $U \subseteq V$. Let $g \in \mathfrak{D}_d$ with $g \prec h$. Then $f = \sqrt{gh} \in \mathfrak{D}_d$ satisfies $g \prec f \prec h$. It follows from the preceding inequality and Lemma 12 that $\mathcal{M}^g_\infty(F_{\varphi_h} \cap U) = \mathcal{M}^g_\infty(U)$ for all open $U \subseteq V$. Hence the G_δ-set F_{φ_h} is in $\mathrm{G}^h(V)$.

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