Directed animals as branching-annihilating particles systems, through intertwining

Olivier Hénard*, Édouard Maurel-Segala† and Arvind Singh‡

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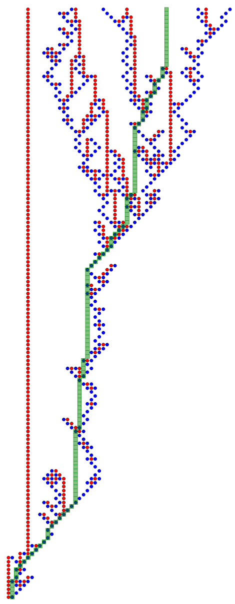

Abstract

We consider the uniform infinite pyramid (UIP) defined in a previous article (Hénard, Maurel-Segala, Singh 2024) as the local limit of uniformly sampled directed animals on the square lattice, when their size grows to infinity. We interpret this random object as an interacting particle system and show that it satisfies a remarkable intertwining property: the particle system representing the UIP, which has long-range interactions, is the trace of a simple branching/annihilating system with only local interactions. We prove similar statements for the uniform infinite non-negative pyramid (IHP+) and the critical Boltzmann half-pyramid (BHP).

Keywords: directed animals; particle system; intertwining.

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*LMO, Université Paris-Saclay. Email: `olivier.henard@universite-paris-saclay.fr`

†LMO, Université Paris-Saclay. Email: `edouard.maurel-segala@universite-paris-saclay.fr`

‡CNRS and LMO, Université Paris-Saclay. Email: `arvind.singh@universite-paris-saclay.fr`. Work partially supported by ANR 19-CE40-0025 ProGraM

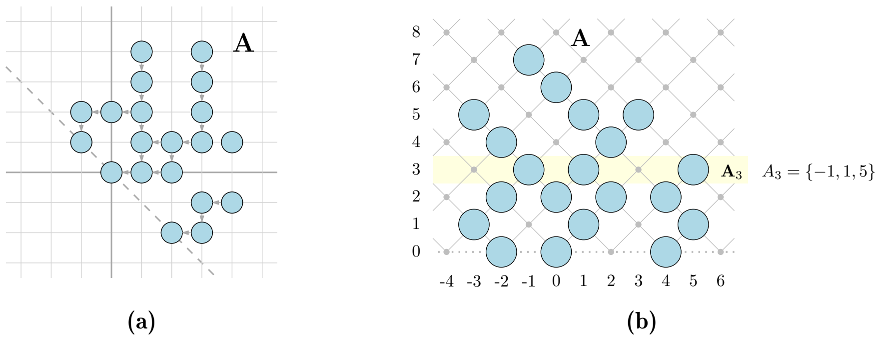


Figure 1: (a) A finite directed animal. Arrows between vertices are used to represent the adjacency relation. (b). The same animal after rotation and scaling of the lattice. The animal is seen as a particle system: here, the particles alive at time 3 are highlighted in yellow.

1 Introduction

This paper is concerned with the study of a particle system appearing as the local limit of (uniformly sampled) directed animals on the square lattice. Let us briefly recall the definition of these sets. We call *directed animal* (on the square lattice) a finite subset $\mathbf{A} \subset \{(x, y) \in \mathbb{Z}^2 : x + y \geqslant 0\}$ such that every vertex $\mathbf{v} \in \mathbf{A}$ which is not on the boundary line $\{(x, y) \in \mathbb{Z}^2 : x + y = 0\}$ has at least one neighbour either immediately on its left or immediately below it (cf. Figure 1 (a)). The set of vertices on the boundary line are called the *sources* of the animal.

The study of directed animals began with Dhar [6, 8, 7], who discovered that, contrary to their (much more complicated) undirected counterparts, it is possible to count them and obtain explicit formulas for the number of directed animals with a given size and a given source set. This model subsequently attracted much attention from the combinatorics and probabilistic communities. We refer the reader to Flajolet and Sedgewick's book [10], Bousquet-Mélou's ICM invited lecture [5], as well as [17, 11, 2, 18, 3, 4, 13, 1] and the references therein for additional details.

In this paper, we are more specifically interested in looking at *infinite* random directed animals. These random sets are obtained as the limit of uniformly sampled finite directed animals when their size grows to infinity. These limits exist and are described in a previous paper by the authors [12]. Let us quickly recall the construction of the main objects of our study.

First, as is customary when working with directed animals, we rotate the lattice by 45° counter-clockwise and then scale it by a factor of $\sqrt{2}$ (see Figure 1 (b)). Therefore, the sources of an animal $\mathbf{A}$ are now the vertices of $\mathbf{A}$ with y-coordinate equal to 0. More generally, we can talk about the *layer* n of $\mathbf{A}$, denoted $\mathbf{A}_n$, defined as the subset of vertices with y-coordinate equal to n, and we write A_n for the set of x-coordinates of these vertices. We observe that, necessarily, $A_n \subset 2\mathbb{Z}$ if n is even and $A_n \subset 2\mathbb{Z} + 1$ if n is odd.

Given a subset $C \subset \mathbb{Z}$, we say that C is *admissible* if it is finite and either $C \subset 2\mathbb{Z}$ or $C \subset 2\mathbb{Z}+1$. If it also holds that $C \subset \mathbb{N}$, then we say that C is *positive admissible*. Furthermore, we define

$$[C] := (C+1) \cup (C-1).$$

We observe that, if C is admissible, then any subset of $[C]$ is also admissible. It also follows from the definition of a directed animal that $A_{n+1} \subset [A_n]$ for all n.

With these notations, we can interpret an animal $\mathbf{A}$ as a particle system $(A_n, n \geqslant 0)$ on $\mathbb{Z}$ where A_0 is the initial position of the particles and, more generally, A_n is the set of positions of the particles alive at time n in the system.[1] Furthermore, by the defining property of animals, given A_n, the set of possible positions for the particles at time $n+1$ is $[A_n]$. One of the main results of [12] (*cf.* Theorem 1 and Corollary 5) states that the particle system representing the uniform infinite animal (UIP), defined as the limit of uniformly sampled finite directed animals, is a Markov process with a remarkably explicit transition kernel.

Given a finite set $C = \{c_1 < c_2 < \ldots < c_\ell\}$, we write $|C| := \ell$ for the size of this set and, provided that it is non-empty, we define its *energy* $\eta(C)$ by

$$\eta(C) := \prod_{i=1}^{\ell-1}(c_{i+1} - c_i - 1),$$

with the usual convention $\prod_1^0 = 1$.

Definition 1 (**UIP**). *The uniform infinite pyramid (UIP[2]) is the Markov particle system $\bar{A} = (\bar{A}_n, n \geqslant 0)$ starting from a non-empty admissible set $\bar{A}_0$ and with transition kernel*

$$\bar{Q}(C, D) := \mathbb{P}(\bar{A}_{n+1} = D \mid \bar{A}_0, \ldots, \bar{A}_n = C) = \frac{\eta(D)}{3^{|C|}\eta(C)} \tag{1}$$

for all $D \subset [C]$ with $D \neq \emptyset$.

Together with the UIP, there are two other Markov particle systems of special interest. The first one corresponds to a finite, non-negative[3] directed animal sampled according to the critical Boltzmann law:

Definition 2 (**BHP**). *The critical Boltzmann pyramid (BHP) is the Markov particle system $A = (A_n, n \geqslant 0)$ stating from a positive admissible set A_0 and with transition kernel*

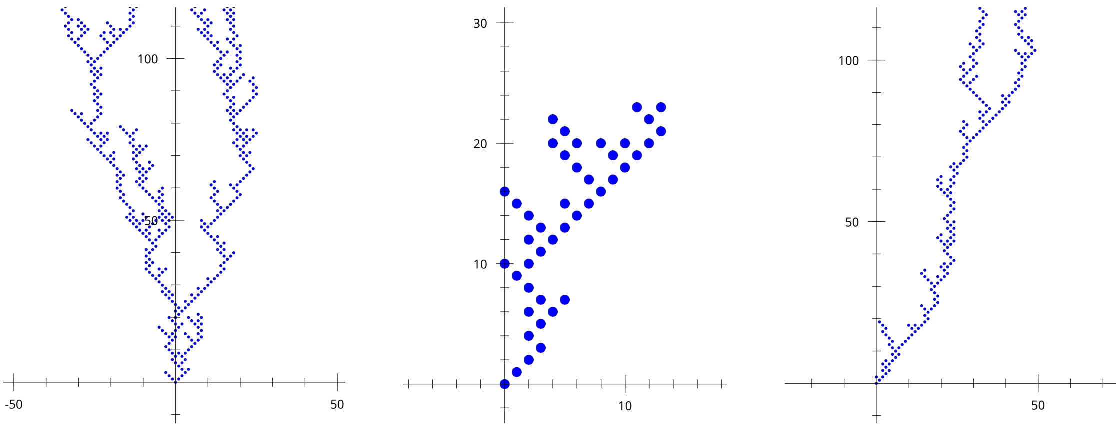
$$Q(C, D) := \mathbb{P}(A_{n+1} = D \mid A_0, \ldots, A_n = C) = \begin{cases} \frac{1}{(\min C+1)\eta(C)3^{|C|}} & \text{for } D = \emptyset, \\ \frac{(\min D+1)\eta(D)}{(\min C+1)\eta(C)3^{|C|}} & \text{for } D \neq \emptyset \text{ and } D \subset [C] \cap \mathbb{N}, \end{cases} \tag{2}$$

*which is absorbed when becoming empty (*i.e.*, if $A_n = \emptyset$, then $A_{n+1} = \emptyset$).*

[1] There is at most one particle per site. When the animal $\mathbf{A}$ is finite, the particle system ultimately becomes extinct, *i.e.*, $A_n = \emptyset$ for n large enough. For infinite animals, there is always at least one particle present at all times.

[2] The 'P' in UIP refers to *pyramid*, which is the name given to an animal starting with a single source. We keep it here for consistency with [12] and do likewise for the BHP and the UIP+. The Markov viewpoint we adopt here allows for general starting configurations, though.

[3] We say that an animal is non-negative if the x-coordinates of its vertices are all non-negative.



(a) UIP **(b)** BHP **(c)** UIP+

Figure 2: Simulations of the particle system of the UIP, BHP and UIP+. Here, we start with a single source at 0 ($i.e$ we construct pyramids).

The third Markov system is obtained in the same way as the UIP, but now taking the local limit of uniformly sampled non-negative directed animals.

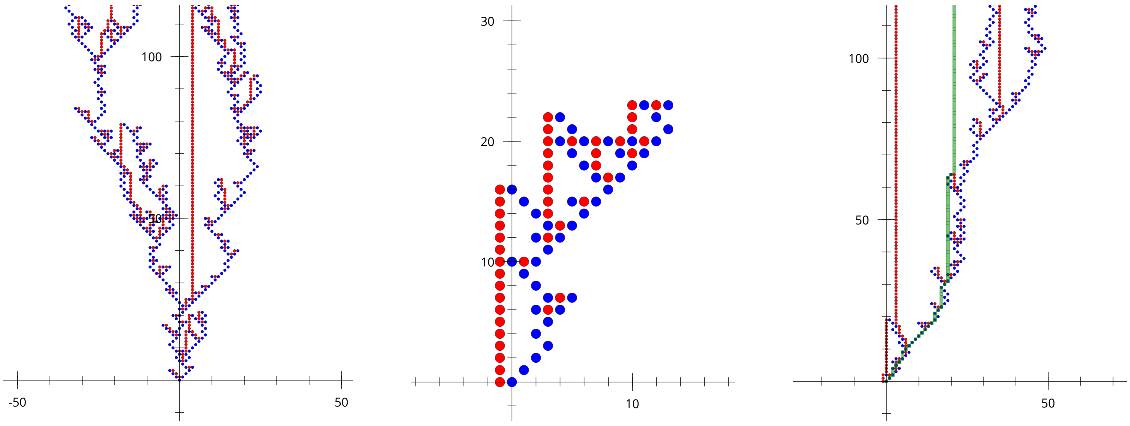
Definition 3 **(UIP+).** *The uniform infinite non-negative pyramid (UIP+) is the Markov particle system $\bar{A}^+ = (\bar{A}_n^+, n \geqslant 0)$ starting from a non-empty, positive, admissible set $\bar{A}_0^+$ and with transition kernel*

$$\bar{Q}^+(C, D) := \mathbb{P}(\bar{A}_{n+1}^+ = D \,|\, \bar{A}_0^+, \dots, \bar{A}_n^+ = C) = \frac{(\min D + 1)(\max D + 2)\eta(D)}{(\min C + 1)(\max C + 2)\eta(C) 3^{|C|}} \tag{3}$$

for all $D \subset [C] \cap \mathbb{N}$ with $D \neq \emptyset$.

See Figure 2 for a simulation of these processes. Some comments are in order. First, by definition, the UIP is simply the particle system where, given $\bar{A}_n$, the configuration $\bar{A}_{n+1}$ at the next step is chosen with a probability proportional to its energy $\eta(\bar{A}_{n+1})$ among all admissible configurations (which are the non-empty subsets of $[\bar{A}_n]$). Notice that the partition function appearing in the denominator of (1) takes a remarkably simple form $\sum_{D \subset [\bar{A}_n], D \neq \emptyset} \eta(D) = 3^{|\bar{A}_n|} \eta(\bar{A}_n)$.

The formula for the BHP is similar to that for the UIP, with the addition of a $\min D + 1$ multiplicative factor. Noticing that $(\min D + 1)\eta(D) = \eta(D \cup \{-2\})$, we can interpret the BHP dynamics as selecting a configuration at random, proportional to its energy computed after adding an immortal invisible particle at position -2 (which does not reproduce). We also point out that, even though it is not immediate from the definition of its Markov kernel, the BHP always goes extinct a.s. This follows immediately from the fact that the BHP is a finite animal whose size N is chosen according to the critical Boltzmann weights, and then sampled uniformly at random among all non-negative animals with N vertices; see again [12] for additional details.



(a) UIP **(b)** BHP **(c)** UIP+

Figure 3: Simulations of the branching-annihilating particle systems for the UIP, BHP, and UIP+. The particles are colored in blue. The anti-particles are shown in red (and the muon for the UIP+ is represented as a green square). By keeping only the blue particles, we recover exactly the UIP (a), BHP (b), and UIP+ (c) simulated in Figure 3.

The UIP+ arises as the local limit of directed animals conditioned to stay non-negative. This conditioning translates, for the kernels of the particle systems, as a conditioning obtained via Doob's h-transform with $h(D) := (\min D + 1)(\max D + 2)$, which is harmonic for the UIP; see equation (49) of [12].

Thus, because the energy function η is defined as a product of the distance (minus 1) between particles, it follows that each particle x interacts only with its left ℓ and right r neighbors through the term $(x - \ell - 1)(r - x - 1)$. Consequently, the interaction between particles has infinite range. The aim of this paper is to show that, in fact, this interaction may be understood as being created/mediated by invisible "anti-particles". In the case of the UIP, we will prove that we can construct a new system composed of particles and anti-particles, where two particles of opposite types annihilate when they collide, and such that the trace of this system, after erasing all the anti-particles (keeping only the particles), is exactly the original UIP. The defining feature of this new model is that all particles and anti-particles evolve independently, following nearest-neighbor displacement rules. Similar results hold for the BHP and for the UIP+ [4]. The exact statement of our main result is provided in Theorem 1. See Figure 3 for an illustration of these branching/annihilating particle systems.

The fact that particle systems associated with directed animals appear as traces of larger Markov systems is a manifestation of intertwining properties enjoyed by the kernels of our processes. This is reminiscent of the intertwining property for Dyson's interacting Brownian motion [19], as well as of the intertwining property of Bessel processes together with their future infimum [14]. We provide additional comments on this analogy throughout the paper. However, it is noteworthy to keep in mind that we are not working with a "conservative" system,

[4]For the UIP+, we will also introduce a new (single and eternal) particle called the "muon".

in the sense that the number of particles fluctuates over time. We are not aware of any other branching/annihilating processes having this intertwining property.

Concerning the genealogy of the notion, the word “intertwining” for Markov kernels first appeared in a technical report by Yor [20], then was quickly adopted by Diaconis and Fill [9], who first studied the notion *per se* in an abstract setting. Long before, though, the notion had been used without being named, as in the seminal paper of Rogers and Pitman on Markov functions [15], where it is used to prove the infamous 2M-X Pitman theorem. In our discrete-time setup, a convenient and very readable reference is the set of lecture notes by Swart [16].

2 Branching-Annihilating Particle System

2.1 States of the Particle System

Definition 4 (Simple and Critical Configurations)**.** *A configuration $\varkappa$ of our system consists of a finite arrangement of a set $A = \{a_1, \ldots, a_k\}$ of particles $\bullet$ and a set $B = \{b_1, \ldots, b_p\}$ of anti-particles $\circ$ on $\mathbb{Z}$, satisfying the following properties:*

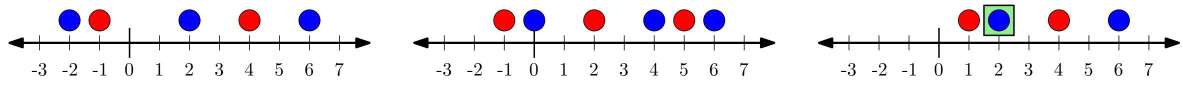
- *A and B are disjoint.*
- *All the particles $\bullet$ share the same parity: they are all located on either even sites or odd sites.*
- *There is strict alternation between particles and anti-particles:*
  - *At most one particle or anti-particle occupies any given site.*
  - *No two consecutive particles are allowed without exactly one antiparticle between them, and vice versa.*

If $|A| = |B| + 1$ (i.e., the alternation of particles $\bullet$ and antiparticles $\circ$ starts and ends with a particle $\bullet$), the configuration $\varkappa = (A, B)$ is called simple admissible*, and the set of all such configurations is denoted by $\bar{\mathcal{S}}$.*

If $|A| = |B|$ and $-1 \leqslant \min(B) < \min(A)$, the configuration $\varkappa$ is called critical admissible*, and the set of all such configurations is denoted by $\mathcal{S}$.*

Remark 1. By construction, there can never be more than one particle or anti-particle on a single site. Additionally, due to the alternation between particles and antiparticles, we have $|B| \in \{|A|-1, |A|, |A|+1\}$. In fact, the difference $|A|-|B|$ remains constant for a given model. For the UIP, we are in the so-called *simple* case where $|A| = |B| + 1$. In contrast, the *critical* case $|A| = |B|$ arises when studying the BHP.

Definition 5 (Positive configuration)**.** *If (A, B) is critical admissible, we sometimes consider an additional type of particle: the* muon *$\overrightarrow{\Upsilon} \in \mathbb{Z}$, which is unique in a configuration. The muon satisfies the condition that there exists a particle $a \in A$ and an anti-particle $b \in B$ such that the interval (b, a) contains no particles or antiparticles, and $b < \overrightarrow{\Upsilon} \leqslant a$.*



(a) Simple configuration

(b) Critical configuration

(c) Positive configuration

Figure 4: Examples of configurations. Particles are represented in blue, anti-particles in red, and the muon is shown as a green box. Notice that the muon may overlap with a particle (but not with an anti-particle).

Such a configuration $\varkappa = (A, B, \vec{\Upsilon})$ *is called* positive admissible, *and we denote the set of all such configurations by* $\bar{\mathcal{S}}^+$.

Definition 6. *We call* $\varkappa$ *a configuration if it is either a simple or critical admissible configuration* (A, B), *or a positive admissible configuration* $(A, B, \vec{\Upsilon})$. *In all cases, we denote by* $\pi(\varkappa) = A$ *the set of particles.*

Remark 2. The distinction between simple, critical, and positive configurations arises from the fact that each of these sets will be invariant under the branching/annihilating dynamics that we will define in the next section. Consequently, a model starting, for example, with $A = \{0\}$ and $B = \emptyset$ will remain simple admissible at all times.

Proposition 1. *Let* $A = \{a_1 < \cdots < a_k\}$ *be a non-empty admissible set.*

(a) *If* $A \subset \mathbb{N}$, *we have*

$$(\min(A) + 1)\eta(A) = |\{B : (A, B) \text{ is critical admissible }\}| = |\{\varkappa \in \mathcal{S} : \pi(\varkappa) = A\}|.$$

(b) $\eta(A) = |\{B : (A, B) \text{ is simple admissible }\}| = |\{\varkappa \in \bar{\mathcal{S}} : \pi(\varkappa) = A\}|.$

(c) *if* $A \subset \mathbb{N}$, *we have*

$$\begin{aligned} \frac{(\min(A) + 1)(\max(A) + 2)}{2}\eta(A) &= |\{(B, \vec{\Upsilon}) : (A, B, \vec{\Upsilon}) \text{ is positive admissible }\}| \\ &= |\{\varkappa \in \bar{\mathcal{S}}^+ : \pi(\varkappa) = A\}|. \end{aligned}$$

Proof. Statements (a) and (b) are straighforward. We just need to prove (c). We must place an anti-particle between each consecutive particles. With the convention $a_0 = -2$, it comes that

$$
\begin{aligned}
&|\{(B,\overrightarrow{\Upsilon}) : (A,B,\overrightarrow{\Upsilon}) \text{ is positive-admissible }\}|,\\
&=\sum_{p=1}^{k}\prod_{1\leqslant i\leqslant k: q\neq p}(a_i-a_{i-1}-1)|\{(b,\overrightarrow{\Upsilon}) : a_{p-1}<b<\overrightarrow{\Upsilon}\leqslant a_p\}|,\\
&=\sum_{p=1}^{k}\prod_{1\leqslant i\leqslant k: q\neq p}(a_i-a_{i-1}-1)\frac{(a_p-a_{p-1}-1)(a_p-a_{p-1})}{2},\\
&=\eta(A\cup\{-2\})\sum_{p=1}^{k}\frac{a_p-a_{p-1}}{2},\\
&=(\min(A)+1)\eta(A)(\max(A)+2)\frac{1}{2}.
\end{aligned}
$$

□

2.2 Transitions of the particle system

We now define a dynamics for the particles and anti-particles that preserves the stability of simple (resp. critical, resp. positive) configurations. We begin with the simple and critical cases, which are easier to handle as they do not involve the muon.

Definition 7 (Dynamics for Simple and Critical Configurations)**.** *Let $\varkappa = (A, B)$ be a simple or critical configuration, and suppose that for each $a \in A$, we are given a choice $\xi_a \in \{-1, 0, 1\}$. We will define a new configuration $\varkappa^\xi$. To handle the case where several particles/anti-particles may temporarily occupy the same site, we introduce two counting functions, μ_A^ξ and μ_B^ξ, which map $\mathbb{Z}$ to $\mathbb{N}$ and count the new particles and antiparticles, respectively:*

- $\mu_A^\xi = \sum_{a\in A:\xi_a\in\{-1,0\}} \delta_{a-1} + \sum_{a\in A:\xi_a\in\{0,1\}} \delta_{a+1}$,
- $\mu_B^\xi = \sum_{b\in B} \delta_b + \sum_{a\in A:\xi_a=0} \delta_a$.

This setup corresponds to keeping the previous anti-particles (the first term in μ_B^ξ) and for each $a \in A$:

- *If $\xi_a = 1$, place a particle at $a + 1$,*
- *If $\xi_a = -1$, place a particle at $a - 1$,*
- *If $\xi_a = 0$, place a particle at both $a + 1$ and $a - 1$, and an antiparticle at a.*

Finally, for each site with an antiparticle, if there are particles on the same site, remove the antiparticle and one of the particles. More compactly, we define the new configuration as follows:

$$
A^\xi = \{x : (\mu_A^\xi - \mu_B^\xi)(x) = 1\},
$$
$$
B^\xi = \{x : (\mu_A^\xi - \mu_B^\xi)(x) = -1\}.
$$

The positive case is similar, but with a twist to deal with the muon.

Definition 8 (Dynamic for Positive Configuration)**.** *The process for the positive configuration follows the same general structure as the simple and critical cases, with a modification to account for the muon. We proceed as follows: we first define the functions μ_A^ξ and μ_B^ξ as before, and then we introduce the following adjustments:*

- $\vec{\Upsilon}^\xi = \vec{\Upsilon} + \mathbb{1}_{\vec{\Upsilon} \in A, \xi_{\vec{\Upsilon}} \in \{-1,0\}}$,
- $\mu_{\vec{\Upsilon}}^\xi = \mathbb{1}_{\vec{\Upsilon} \in A, \xi_{\vec{\Upsilon}} = -1}(\delta_{\vec{\Upsilon}+1} - \delta_{\vec{\Upsilon}-1})$,
- $\nu_A^\xi = \mu_A^\xi + \mu_{\vec{\Upsilon}}^\xi$.

The key modifications are as follows:

- *The muon, $\vec{\Upsilon}$, remains in the same position unless it occupies the same site as a particle that jumps to the left or reproduces. In that case, the muon jumps to the right.*
- *$\mu_{\vec{\Upsilon}}^\xi$ acts as a correction to μ_A^ξ. Specifically, if a particle on the site $\vec{\Upsilon}$ attempts to jump to the left, the muon prevents this behavior and forces the particle to jump to the right instead.*

Finally, we apply the same rules for the configuration after the adjustments:

$$A^\xi = \{x : (\nu_A^\xi - \mu_B^\xi)(x) = 1\},$$

$$B^\xi = \{x : (\nu_A^\xi - \mu_B^\xi)(x) = -1\}.$$

The next proposition is key to proving our main result.

Proposition 2.

(a) *If the configuration $\varkappa$ is in $\mathcal{S}$ (resp. $\bar{\mathcal{S}}$, resp. $\bar{\mathcal{S}}^+$), then for any $\xi \in \{-1,0,1\}^{|A|}$ the same is true for $\varkappa^\xi$.*

(b) *If $A, \varkappa'$ are such that $A' := \pi(\varkappa') \subset [A]$ and $\varkappa'$ is in $\mathcal{S}$ (resp. $\bar{\mathcal{S}}$, resp. $\bar{\mathcal{S}}^+$), then there exists at most one $\varkappa$ in $\mathcal{S}$ (resp. $\bar{\mathcal{S}}$, resp. $\bar{\mathcal{S}}^+$) and at most one $\xi \in \{-1,0,1\}^{|A|}$ such that $\pi(\varkappa) = A$ and $\varkappa^\xi = \varkappa'$.*

Remark 3. We will see later that in fact, for (b), there is also always existence of the couple $\varkappa, \xi$, but we don't need it right now, and a direct proof would be trickier at this point.

Proof. Statement (a) is straightforward. We now prove (b). We first prove the result in the case of simple/critical configurations. By construction, we have

$$\mu := \mu_A^\xi - \mu_B^\xi = \sum_{a \in A'} \delta_a - \sum_{b \in B'} \delta_b.$$

Furthermore, there is an antiparticle on layer A' directly above a particle on layer A if and only if that particle has reproduced:

$$\{a : \xi_a = 0\} = A \cap B'.$$

For all other particles, there is nothing directly above them (remember that other particles are at least distance 2 apart), and they either jumped to the right or to the left without anyone crossing them. If they jumped to the right, there will be one more particle than antiparticles to the right of them on the next layer; otherwise, the number will be the same:

$$\{a : \xi_a = 1\} = \{a \in A \setminus B' : \sum_{x>a} \mu(x) = 1\},$$

$$\{a : \xi_a = -1\} = \{a \in A \setminus B' : \sum_{x>a} \mu(x) = 0\}.$$

Hence, we have completely recovered ξ. From there, we deduce μ_A^ξ, which is a function of A and ξ, and $\mu_B^\xi = \mu_A^\xi - \mu$. Finally, we get that

$$B = \{x \notin A : \mu_B^\xi(x) = 1\}.$$

We now turn our attention to the case of positive configurations. Here we need to tweak the previous construction slightly to account for the muon $\vec{\Upsilon}$. If we know $\vec{\Upsilon}'$, we know that $\vec{\Upsilon} \in \{\vec{\Upsilon}', \vec{\Upsilon}' - 1\}$. If it was in $\vec{\Upsilon}' - 1$, it must have jumped, but this was only possible if there was a particle in $\vec{\Upsilon}' - 1$. Conversely, if there was a particle in $\vec{\Upsilon}' - 1$, it was not possible for $\vec{\Upsilon}$ to be in $\vec{\Upsilon}'$ since to the left of $\vec{\Upsilon}$, there is always an antiparticle before any particles. So we can recover the position of the muon:

$$\vec{\Upsilon} = \vec{\Upsilon}' - \mathbb{1}_{\vec{\Upsilon}' - 1 \in A}.$$

With this in mind, we reconstruct the rest:

$$\mu := \sum_{a \in A'} \delta_a - \sum_{b \in B'} \delta_b$$

$$\{a : \xi_a = 0\} = A \cap B'$$

$$\{a : \xi_a = 1, a \neq \vec{\Upsilon}\} = \{a \in A \setminus B' : \sum_{x>a} \mu(x) = 1\}$$

$$\{a : \xi_a = -1, a \neq \vec{\Upsilon}\} = \{a \in A \setminus B' : \sum_{x>a} \mu(x) = 0\}$$

and if $\vec{\Upsilon} \in A$:

$$\xi_{\vec{\Upsilon}} = (\mathbb{1}_{\vec{\Upsilon}' = \vec{\Upsilon}} - \mathbb{1}_{\vec{\Upsilon}' = \vec{\Upsilon} + 1}) \mathbb{1}_{\vec{\Upsilon} \notin B'}.$$

Thus, we have completely recovered ξ. Finally, we know $\mu_{\vec{\Upsilon}}^\xi$, and from there, we deduce μ_B^ξ, and then B. □

2.3 Random dynamics of the particle systems

From the previous results, we can now define a random dynamic for the system. Given a configuration $\varkappa_n$, we choose independently, from the past, a sequence ξ_n uniformly distributed in $\{-1,0,1\}^{|A_n|}$, where $A_n = \pi(\varkappa_n)$ is the set of particles in configuration $\varkappa_n$. We then update the configuration as follows:

$$\varkappa_{n+1} = \varkappa_n^{\xi_n}.$$

By Proposition 2 (a), the sets $\mathcal{S}$, $\bar{\mathcal{S}}$, and $\bar{\mathcal{S}}^+$ remain stable under this dynamic. Thus, this procedure defines three Markov kernels: P, $\bar{P}$, and $\bar{P}^+$, evolving on simple, critical, and positive configurations, respectively.

Theorem 1.

1. *Suppose that $\bar{A}_0 \subset \mathbb{Z}$ is a non-empty admissible set and $\bar{B}_0$ has uniform distribution in $\{\bar{B} : (\bar{A}_0, \bar{B}) \in \bar{\mathcal{S}}\}$, then the Markov chain $(\bar{A}_n, \bar{B}_n)_{n\geqslant 0}$ with kernel $\bar{P}$ on $\bar{\mathcal{S}}$ started from $(\bar{A}_0, \bar{B}_0)$ is such that:*
   - *(a) $(\bar{A}_n, n \geqslant 0\}$ has the law of the UIP started from the source set $\bar{A}_0$ i.e. it is a Markov chain with kernel $\bar{Q}$ defined by (1).*
   - *(b) For all n, given $(\bar{A}_0, \ldots, \bar{A}_n)$ the law of $\bar{B}_n$ is uniform on $\{\bar{B} : (\bar{A}_n, \bar{B}) \in \bar{\mathcal{S}}\}$.*
2. *Suppose that $A_0 \subset \mathbb{N}$ is a non-empty admissible set and that B_0 has uniform distribution in $\{B : (A_0, B) \in \mathcal{S}\}$, then the Markov chain $(A_n, B_n)_{n\geqslant 0}$ with kernel P on $\mathcal{S}$ started from (A_0, B_0) is such that:*
   - *(a) $(A_n, n \geqslant 0)$ has the law of the BHP started from the source set A_0 i.e. it is a Markov chain with kernel Q defined by (2).*
   - *(b) for all n, given $(A_0, \ldots, A_n)$ the law of B_n is uniform on $\{B : (A_n, B) \in \mathcal{S}\}$.*
3. *Suppose that $\bar{A}_0^+ \subset \mathbb{N}$ is a non-empty admissible set and that $(\bar{B}_0^+, \vec{\Upsilon}_0)$ has uniform distribution in $\{(\bar{B}^+, \vec{\Upsilon}) : (\bar{A}_0^+, \bar{B}^+, \vec{\Upsilon}) \in \bar{\mathcal{S}}^+\}$, then the Markov chain $(\bar{A}_n^+, \bar{B}_n^+, \vec{\Upsilon})_{n\geqslant 0}$ with kernel $\bar{P}^+$ on $\bar{\mathcal{S}}^+$ started from $(\bar{A}_0^+, \bar{B}_0^+, \vec{\Upsilon})$ is such that:*
   - *(a) $(\bar{A}_n^+, n \geqslant 0)$ has the law of the UIP+ started from the source set $\bar{A}_0^+$ i.e. it is a Markov chain with kernel $\bar{Q}^+$ defined by (3).*
   - *(b) for all n, given $(\bar{A}_0^+, \ldots, \bar{A}_n^+)$ the law of $(\bar{B}_n^+, \vec{\Upsilon})$ is uniform on the set $\{(\bar{B}^+, \vec{\Upsilon}) : (\bar{A}_n^+, \bar{B}^+, \vec{\Upsilon}) \in \bar{\mathcal{S}}^+\}$.*
   - *(c) Furthermore, the process $(\bar{A}_n^+, \vec{\Upsilon})_{n\in\mathbb{N}}$ is also, y itself, a Markov chain living on the state space*

$$\{(A = \{a_1 < \cdots < a_k\}, \vec{\Upsilon} \in \mathbb{N}) : \vec{\Upsilon} \in \sqcup_{i=2}^{k} [\![a_{i-1} + 2; a_i]\!]\}$$

with transition kernel Q^* defined by:

$$Q^\star((A,\vec{\Upsilon}),(A',\vec{\Upsilon}'))$$
$$= \frac{\eta(A',\vec{\Upsilon}')(\min(A')+1)}{3^{|A|}\eta(A,\vec{\Upsilon})(\min(A)+1)}\mathbb{1}_{(\vec{\Upsilon}=\vec{\Upsilon}' \text{ or } \vec{\Upsilon}'-1=\vec{\Upsilon}\in A)}\mathbb{1}_{(A'\subset[A] \text{ and } A\neq\emptyset)} \quad (4)$$

where, denoting by i the index such that $\vec{\Upsilon}\in[\![a_{i-1}+2;a_i]\!]$, we have set

$$\eta(A,\vec{\Upsilon}) := \frac{\vec{\Upsilon}-a_{i-1}-1}{a_i-a_{i-1}-1}\eta(A).$$

Proof. The three cases have almost identical proofs. Let us prove the first assertion for the UIP. Following Swart [16], it is enough to prove the property for $n=1$ and then conclude by induction. So, given $\bar{A}_0$, we choose $\bar{B}_0$ uniformly such that $\varkappa_0 = (\bar{A}_0,\bar{B}_0)$ is in $\bar{\mathcal{S}}$. Then we choose ξ_1 uniformly in $\{-1;0;1\}$ and we define $(\bar{A}_1,\bar{B}_1) = \varkappa_1 := \varkappa_0^{\xi_1}$. We now compute the law of $(\bar{A}_1,\bar{B}_1)$. By construction $\bar{A}_1\subset[\bar{A}_0]$ and

$$\mathbb{P}((\bar{A}_1,\bar{B}_1)=(a_1,b_1))$$
$$= \sum_{\bar{B}_0:(\bar{A}_0,\bar{B}_0)\in\bar{\mathcal{S}}} \frac{1}{|\{\bar{B}:(\bar{A}_0,\bar{B})\in\bar{\mathcal{S}}\}|} \sum_{\xi\in\{-1;0;1\}^{|\bar{A}_0|}} \frac{1}{|\{-1;0;1\}^{|\bar{A}_0|}|}\mathbb{1}_{(\bar{A}_0,\bar{B}_0)^\xi=(a_1,b_1)}. \quad (5)$$

But now, according to the (b) of Proposition 2 there is at most one couple $(\bar{B}_0,\xi)$ such that $(\bar{A}_0,\bar{B}_0)^\xi = (a_1,b_1)$ so that

$$\mathbb{P}((\bar{A}_1,\bar{B}_1)=(a_1,b_1)) \leqslant \frac{1}{\eta(\bar{A}_0)3^{|\bar{A}_0|}}\mathbb{1}_{a_1\subset[\bar{A}_0]}$$

where we used Proposition 1 which tells us that $|\{\bar{B}_0:(\bar{A}_0,\bar{B}_0)\in\bar{\mathcal{S}}\}| = \eta(\bar{A}_0)$.

Now we sum over all possible values of b_1 such that $(a_1,b_1)\in\bar{S}$, since we know from Proposition 1 that there are $\eta(a_1) = |\{b_1:(a_1,b_1)\in\bar{\mathcal{S}}\}|$ possibilities, we find that

$$\mathbb{P}(\bar{A}_1=a_1) \leqslant \frac{\eta(\bar{A}_1)}{\eta(\bar{A}_0)3^{|\bar{A}_0|}}\mathbb{1}_{a_1\subset[\bar{A}_0]} = \bar{Q}(\bar{A}_0,a_1).$$

But since Q is a kernel, summing on a_1 gives 1 and since the left hand side is a probability measure it must also gives 1. This means that all the inequality above is, in fact, an equality and therefore:

$$\mathbb{P}((\bar{A}_1,\bar{B}_1)=(a_1,b_1)) = \bar{Q}(\bar{A}_0,a_1)\frac{1}{|\{b_1:(a_1,b_1)\in\bar{\mathcal{S}}\}|}$$

i.e. starting from B uniformly distributed and applying the dynamic, we conlude that $\bar{A}$ evolves according to kernel $\bar{Q}$ and $\bar{B}$ remains uniformly distributed at all times.

The proof of the second and third statements for the BHP and the UIP+ respectively are similar. We leave out the details for the reader. □

Let us observe that, as a byproduct of the proof, we have established the existence of a solution for statement (b) of Proposition 2:

Corollary 1. *If $A, \varkappa'$ is such that $A' := \pi(\varkappa') \subset [A]$ and $\varkappa'$ is in $\mathcal{S}$ (resp. $\bar{\mathcal{S}}$ resp $\bar{\mathcal{S}}^+$) then there exists unique $\varkappa$ in $\mathcal{S}$ (resp. $\bar{\mathcal{S}}$ resp $\bar{\mathcal{S}}^+$) and a unique $\xi \in \{-1; 0; 1\}^{|A|}$ such that $\pi(\varkappa) = A$ and $\varkappa^\xi = \varkappa'$.*

Remark 4. We have constructed a simple dynamic on sets of particles and anti-particles, where the projection obtained by deleting the anti-particles maps to the UIP, BHP, or UIP+ models. The inclusion of the anti-particles B arises naturally from the computations in Proposition 1. However, the choice of dynamic for the anti-particles is only one of several possible options. We have opted for what is arguably the simplest dynamic: the anti-particles remain stationary.

Nonetheless, we could have chosen a different dynamic for the anti-particles. For instance, we could allow the anti-particles to move according to independent lazy simple random walks. More formally, consider a simple configuration

$$\varkappa = (A = \{a_1 < \cdots < a_k\}, B = \{b_1 < \cdots < b_{k-1}\}),$$

where $a_i < b_i < a_{i+1}$ for each i, and define a new dynamic as follows. As before, we choose ξ uniformly at random in $\{-1, 0, 1\}^{|A|}$ and, independently, we pick $V_1, \ldots, V_{k-1}$ i.i.d., such that

$$\mathbb{P}(V_i = 1) = \mathbb{P}(V_i = -1) = u, \qquad \mathbb{P}(V_i = 0) = 1 - 2u,$$

for some parameter $0 \leqslant u \leqslant \frac{1}{2}$. We then define the new positions of the anti-particles as

$$\hat{b}_i = \begin{cases} b_i + V_i & \text{if } a_i < b_i + V_i < a_{i+1}, \\ a_i + 1 & \text{if } b_i + V_i < a_i + 1, \\ a_{i+1} - 1 & \text{if } a_{i+1} - 1 < b_i + V_i. \end{cases}$$

Thus, the new configuration for the anti-particles is $\hat{B} = \hat{b}_1 < \cdots < \hat{b}_{k-1}$, and the updated configuration is

$$\varkappa' = (A, \hat{B})^\xi.$$

This dynamic, where the anti-particles evolve according to lazy random walks, also preserves the intertwining property with the UIP. In fact, looking back at the proof of Theorem 1, the only essential property for the dynamic of the anti-particles B is that each anti-particle b_i remains uniformly distributed in the interval $[\![a_i + 1; a_{i+1} - 1]\!]$. Even more exotic behaviors can be considered for the anti-particles, yet, as long as this condition is satisfied, the results will hold.

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