A slow transient diffusion in a drifted stable potential

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Abstract

We consider a diffusion process X in a random potential $\mathbb{V}$ of the form $\mathbb{V}_x = \mathbb{S}_x - \delta x$, where δ is a positive drift and $\mathbb{S}$ is a strictly stable process of index $\alpha \in (1, 2)$ with positive jumps. Then the diffusion is transient and $X_t / \log^{\alpha} t$ converges in law towards an exponential distribution. This behaviour contrasts with the case where $\mathbb{V}$ is a drifted Brownian motion and provides an example of a transient diffusion in a random potential which is as "slow" as in the recurrent setting.

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1 Introduction

Let $(\mathbb{V}(x),\ x \in \mathbb{R})$ be a two-sided stochastic process defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We call a diffusion in the random potential $\mathbb{V}$ a formal solution X of the S.D.E:

$$\begin{cases} dX_t = d\beta_t - \frac{1}{2}\mathbb{V}'(X_t)dt \\ X_0 = 0, \end{cases}$$

where β is a standard Brownian motion independent of $\mathbb{V}$. Of course, the process $\mathbb{V}$ may not be differentiable (for example when $\mathbb{V}$ is a Brownian motion) and we should formally consider X as a diffusion whose conditional generator given $\mathbb{V}$ is

$$\frac{1}{2} e^{\mathbb{V}(x)} \frac{d}{dx}\left(e^{-\mathbb{V}(x)} \frac{d}{dx}\right).$$

Such a diffusion may be explicitly constructed from a Brownian motion through a random change of time and a random change of scale. This class of processes has

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been widely studied for the last twenty years and bears a close connection with the model of the random walk in random environment (RWRE), see [17] and [12] for a survey on RWRE and [11], [12] for the connection between the two models.

This model exhibits many interesting features. For instance, when the potential process $\mathbb{V}$ is a Brownian motion, the diffusion X is recurrent and Brox [2] proved that $X_t/\log^2 t$ converges to a non-degenerate distribution. Thus, the diffusion is much "slower" than in the trivial case $\mathbb{V} = 0$ (then X is simply a Brownian motion).

We point out that Brox's theorem is the analogue of Sinai's famous theorem for RWRE [13] (see also [4] and [8]). Just as for the RWRE, this result is a consequence of a so-called "localization phenomenon": the diffusion is trapped in some valleys of its potential $\mathbb{V}$. Brox's theorem may also be extended to a wider class of potentials. For instance, when $\mathbb{V}$ is a strictly stable process of index $\alpha \in (0, 2]$, Schumacher [11] proved that

$$\frac{X_t}{\log^\alpha t} \xrightarrow[t\to\infty]{\text{law}} b_\infty,$$

where b_∞ is a non-degenerate random variable, whose distribution depends on the parameters of the stable process $\mathbb{V}$.

There is also much interest concerning the behaviour of X in the transient case. When the potential is a drifted Brownian motion *i.e.* $\mathbb{V}_x = \mathbb{B}_x - \frac{\kappa}{2}x$ where $\mathbb{B}$ is a two-sided Brownian motion and $\kappa > 0$, then the associated diffusion X is transient toward $+\infty$ and its rate of growth is polynomial and depends on κ. Precisely, Kawazu and Tanaka [7] proved that

- If $0 < \kappa < 1$, then $\frac{1}{t^\kappa} X_t$ converges in law towards a Mittag-Leffler distribution of index κ.
- If $\kappa = 1$, then $\frac{\log t}{t} X_t$ converges in probability towards $\frac{1}{4}$.
- If $\kappa > 1$, then $\frac{1}{t} X_t$ converges almost surely towards $\frac{\kappa-1}{4}$.

In particular, when $\kappa < 1$, the rate of growth of X is sub-linear. Refined results on the rates of convergence for this process were later obtained by Tanaka [16] and Hu *et al.* [6].

In fact, this behaviour is not specific to diffusions in a drifted Brownian potential. More generally, it is proved in [15] that if $\mathbb{V}$ is a two-sided Lévy process with no positive jumps and if there exists $\kappa > 0$ such $\mathbf{E}[e^{\kappa \mathbb{V}_1}] = 1$, then the rate of growth of X_t is linear when $\kappa > 1$ and of order t^κ when $0 < \kappa < 1$ (see also [3] for a law of large numbers in a general Lévy potential). These results are the analogues of those previously obtained by Kesten *et al.* [9] for the discrete model of the RWRE.

In this paper, we study the asymptotic behaviour of a diffusion in a drifted stable potential. Precisely, let $(\mathbb{S}_x,\ x \in \mathbb{R})$ denote a two-sided càdlàg stable process with index $\alpha \in (1, 2)$. By two-sided, we mean that

(a) The process $(\mathbb{S}_x,\ x \geq 0)$ is strictly stable with index $\alpha \in (1, 2)$, in particular $\mathbb{S}_0 = 0$.

(b) For all $x_0 \in \mathbb{R}$, the process $(\mathbb{S}_{x+x_0} - \mathbb{S}_{x_0},\ x \in \mathbb{R})$ has the same law as $\mathbb{S}$.

It is well known that the Lévy measure Π of $\mathbb{S}$ has the form

$$\Pi(dx) = \left(c^+\mathbf{1}_{\{x>0\}} + c^-\mathbf{1}_{\{x<0\}}\right)\frac{dx}{|x|^{\alpha+1}} \tag{1}$$

where c^+ and c^- are two non-negative constants such that $c^+ + c^- > 0$. In particular, the process $(\mathbb{S}_x\,,\, x \geq 0)$ has no positive jumps (resp. no negative jumps) if and only if $c^+ = 0$ (resp. $c^- = 0$). Given $\delta > 0$, we consider a diffusion X in the random potential

$$\mathbb{V}_x = \mathbb{S}_x - \delta x.$$

Since the index α of the stable process $\mathbb{S}$ is larger than 1, we have $\mathbf{E}[\mathbb{V}_x] = -\delta x$, and therefore

$$\lim_{x\to+\infty} \mathbb{V}_x = -\infty \quad \text{and} \quad \lim_{x\to-\infty} \mathbb{V}_x = +\infty \quad \text{almost surely.}$$

This two facts easily imply that X is transient towards $+\infty$ (see the beginning of Section 2.1). We have already mentioned that, when $\mathbb{S}$ has no positive jumps (*i.e.* $c^+ = 0$), the rate of transience of X is given in [15] and X_t has polynomial growth. Thus, we here assume that $\mathbb{S}$ possesses positive jumps.

Theorem 1. *Assume that $c^+ > 0$, then*

$$\frac{X_t}{\log^\alpha t} \xrightarrow[t\to\infty]{\text{law}} \mathcal{E}\left(\frac{c^+}{\alpha}\right),$$

where $\mathcal{E}(c^+/\alpha)$ denotes an exponential law with parameter c^+/α. This result also holds with $\sup_{s\leq t} X_s$ or $\inf_{s\geq t} X_s$ in place of X_t.

The asymptotic behaviour of X is in this case very different from the one observed when $\mathbb{V}$ is a drifted Brownian motion. Here, the rate of growth is very slow: it is the same as in the recurrent setting. We also note that neither the rate of growth nor the limiting law depend on the value of the drift parameter δ.

Theorem 1 has a simple heuristic explanation: the "localisation phenomenon" for the diffusion X tells us that the time needed to reach a positive level x is approximatively exponentially proportional to the biggest ascending barrier of $\mathbb{V}$ on the interval $[0, x]$. In the case of a Brownian potential, or more generally a spectrally negative Lévy potential, the addition of a negative drift somehow "kills" the ascending barriers, thus accelerating the diffusion and leading to a polynomial rate of transience. However, in our setting, the biggest ascending barrier on $[0, x]$ of the stable process $\mathbb{S}$ is of the same order as its biggest jump on this interval. Since the addition of a drift has no influence on the jumps of the potential process, the time needed to reach level x still remains of the same order as in the recurrent case (*i.e.* when the drift is zero) and yields a logarithmic rate of transience.

2 Proof of the theorem.

2.1 Representation of X and of its hitting times.

In the remainder of this paper, we indifferently use the notation $\mathbb{V}_x$ or $\mathbb{V}(x)$. Let us first recall the classical representation of the diffusion X in the random potential $\mathbb{V}$ from a Brownian motion through a random change of scale and a random change of time (see [2] or [12] for details). Let $(B_t,\ t \geq 0)$ denote a standard Brownian motion independent of $\mathbb{V}$ and let σ stand for its hitting times:

$$\sigma(x) \stackrel{\text{def}}{=} \inf(t \geq 0\,,\ B_t = x).$$

Define the scale function of the diffusion X,

$$\mathbb{A}(x) \stackrel{\text{def}}{=} \int_0^x e^{\mathbb{V}_y} dy \qquad \text{for } x \in \mathbb{R}. \tag{2}$$

Since $\lim_{x\to+\infty} \mathbb{V}_x/x = -\delta$ and $\lim_{x\to-\infty} \mathbb{V}_x/x = \delta$ almost surely, it is clear that

$$\mathbb{A}(\infty) = \lim_{x\to+\infty} \mathbb{A}(x) < \infty \qquad \text{and} \qquad \lim_{x\to-\infty} \mathbb{A}(x) = -\infty \qquad \text{almost surely.}$$

Let $\mathbb{A}^{-1} : (-\infty, \mathbb{A}(\infty)) \mapsto \mathbb{R}$ denote the inverse of $\mathbb{A}$ and define

$$\mathbb{T}(t) \stackrel{\text{def}}{=} \int_0^t e^{-2\mathbb{V}(\mathbb{A}^{-1}(B_s))} ds \qquad \text{for } 0 \leq t < \sigma(\mathbb{A}(\infty)).$$

Similarly, let $\mathbb{T}^{-1}$ denote the inverse of $\mathbb{T}$. According to Brox [2] (see also [12]), the diffusion X in the random potential $\mathbb{V}$ may be represented in the form

$$X_t = \mathbb{A}^{-1}\Big(B_{\mathbb{T}^{-1}(t)}\Big). \tag{3}$$

It is now clear that, under our assumptions, the diffusion X is transient toward $+\infty$. We will study X via its hitting times H defined by

$$H(r) \stackrel{\text{def}}{=} \inf(t \geq 0\,,\ X_t = r) \qquad \text{for } r \geq 0.$$

Let $(L(t,x),\ t \geq 0,\ x \in \mathbb{R})$ stand for the bi-continuous version of the local time process of B. In view of (3), we can write

$$H(r) = \mathbb{T}\left(\sigma(\mathbb{A}(r))\right) = \int_0^{\sigma(\mathbb{A}(r))} e^{-2\mathbb{V}(\mathbb{A}^{-1}(B_s))} ds = \int_{-\infty}^{\mathbb{A}(r)} e^{-2\mathbb{V}(\mathbb{A}^{-1}(x))} L(\sigma(\mathbb{A}(r)), x) dx.$$

Making use of the change of variable $x = \mathbb{A}(y)$, we get

$$H(r) = \int_{-\infty}^{r} e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)), \mathbb{A}(y)) dy = I_1(r) + I_2(r) \tag{4}$$

where

$$\begin{aligned} I_1(r) &\stackrel{\text{def}}{=} \int_0^r e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)), \mathbb{A}(y)) dy, \\ I_2(r) &\stackrel{\text{def}}{=} \int_{-\infty}^0 e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)), \mathbb{A}(y)) dy. \end{aligned}$$

2.2 Proof of Theorem 1.

Given a càdlàg process $(Z_t\ ,\ t \geq 0)$, we denote by $\Delta_t Z = Z_t - Z_{t-}$ the size of the jump at time t. We also use the notation $Z_t^\natural$ to denote the largest positive jump of Z before time t,

$$Z_t^\natural \stackrel{\text{def}}{=} \sup_{0\leq s\leq t} \Delta_s Z.$$

Let $Z_t^\#$ stand for the largest ascending barrier on $[0,t]$, namely:

$$Z_t^\# \stackrel{\text{def}}{=} \sup_{0\leq x\leq y\leq t} (Z_y - Z_x).$$

We also define the functionals:

$$\overline{Z}_t \stackrel{\text{def}}{=} \sup_{s\in[0,t]} Z_s \qquad \text{(running unilateral maximum)}$$

$$\underline{Z}_t \stackrel{\text{def}}{=} \inf_{s\in[0,t]} Z_s \qquad \text{(running unilateral minimum)}$$

$$Z_t^* \stackrel{\text{def}}{=} \sup_{s\in[0,t]} |Z_s| \qquad \text{(running bilateral supremum)}$$

We start with a simple lemma concerning the fluctuations of the potential process.

Lemma 1. *There exist two constants $c_1, c_2 > 0$ such that for all $a, x > 0$*

$$\mathbf{P}\{\mathbb{V}_x^\# \leq a\} \leq e^{-c_1 \frac{x}{a^\alpha}}, \tag{5}$$

and whenever $\frac{a}{x}$ is sufficiently large,

$$\mathbf{P}\{\mathbb{V}_x^* > a\} \leq c_2 \frac{x}{a^\alpha}. \tag{6}$$

Proof. Recall that $\mathbb{V}_x = \mathbb{S}_x - \delta x$. In view of the form of the density of the Lévy measure of $\mathbb{S}$ given in (1), we get

$$\mathbf{P}\{\mathbb{V}_x^\# \leq a\} \leq \mathbf{P}\{\mathbb{V}_x^\natural \leq a\} = \exp\left(-x\int_a^\infty \frac{c^+}{y^{\alpha+1}}dy\right) = \exp\left(-\frac{c^+}{\alpha}\frac{x}{a^\alpha}\right).$$

This yields (5). From the scaling property of the stable process $\mathbb{S}$, we also have

$$\mathbf{P}\{\mathbb{V}_x^* > a\} = \mathbf{P}\left\{x^{\frac{1}{\alpha}} \sup_{t\in[0,1]} |\mathbb{S}_t - \delta x^{1-\frac{1}{\alpha}} t| > a\right\} \leq \mathbf{P}\left\{\mathbb{S}_1^* > \frac{a}{x^{\frac{1}{\alpha}}} - \delta x^{1-\frac{1}{\alpha}}\right\}.$$

Notice further that $a/x^{1/\alpha} - \delta x^{1-1/\alpha} > a/(2x^{1/\alpha})$ whenever a/x is large enough. Therefore, making use of a classical estimate concerning the tail distribution of the stable process $\mathbb{S}$ (*c.f.* Proposition 4, p221 of [1]), we find that

$$\mathbf{P}\{\mathbb{V}_x^* > a\} \leq \mathbf{P}\left\{\mathbb{S}_1^* > \frac{a}{2x^{\frac{1}{\alpha}}}\right\} \leq \mathbf{P}\left\{\overline{\mathbb{S}}_1 > \frac{a}{2x^{\frac{1}{\alpha}}}\right\} + \mathbf{P}\left\{\underline{\mathbb{S}}_1 < -\frac{a}{2x^{\frac{1}{\alpha}}}\right\} \leq c_2 \frac{x}{a^\alpha}.$$

□

Proposition 1. *There exists a constant $c_3 > 0$ such that, for all r sufficiently large and all $x \geq 0$,*

$$\mathbf{P}\{\mathbb{V}_r^{\#} \geq x+\log^4 r\}-c_3 e^{-\log^2 r} \leq \mathbf{P}\{\log I_1(r) \geq x\} \leq \mathbf{P}\{\mathbb{V}_r^{\#} \geq x-\log^4 r\}+c_3 e^{-\log^2 r}.$$

Proof. This estimate was first proved by Hu and Shi (see Lemma 4.1 of [5]) when the potential process is close to a standard Brownian motion. A similar result is given in Proposition 3.2 of [14] when $\mathbb{V}$ is a random walk in the domain of attraction of a stable law. As explained by Shi [12], the key idea is the combined use of Ray-Knight's Theorem and Laplace's method. However, in our setting, additional difficulties appear since the potential process is neither flat on integer interval nor continuous. We shall therefore give a complete proof but one can still look in [5] and [14] for additional details. Recall that

$$I_1(r) = \int_0^r e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)), \mathbb{A}(y)) dy,$$

where L is the local time of the Brownian motion B (independent of $\mathbb{V}$). Let $(U(t),\ t \geq 0)$ denote a two-dimensional squared Bessel process starting from zero, also independent of $\mathbb{V}$. According to the first Ray-Knight Theorem (*c.f.* Theorem 2.2 p455 of [10]), for any $x > 0$ the process $(L(\sigma(x), x-y),\ 0 \leq y \leq x)$ has the same law as $(U(y),\ 0 \leq y \leq x)$. Therefore, making use of the scaling property of the Brownian motion and the independence of $\mathbb{V}$ and B, for each fixed $r > 0$, the random variable $I_1(r)$ has the same law as

$$\widetilde{I}_1(r) \stackrel{\text{def}}{=} \mathbb{A}(r) \int_0^r e^{-\mathbb{V}_y} U\left(\frac{\mathbb{A}(r)-\mathbb{A}(y)}{\mathbb{A}(r)}\right) dy.$$

We simply need to prove the proposition for $\widetilde{I}_1$ instead of I_1. In the rest of the proof, we assume that r is very large.

Proof of the upper bound. Define the event

$$\mathcal{E}_1 \stackrel{\text{def}}{=} \left\{ \sup_{t\in(0,1]} \frac{U(t)}{t\log\left(\frac{8}{t}\right)} \leq r \right\}.$$

According to Lemma 6.1 of [5], $\mathbf{P}\{\mathcal{E}_1^c\} \leq c_4 e^{-r/2}$ for some constant $c_4 > 0$. On $\mathcal{E}_1$, we have

$$\begin{aligned} \widetilde{I}_1(r) &\leq r \int_0^r e^{-\mathbb{V}_y} (\mathbb{A}(r)-\mathbb{A}(y)) \log\left(\frac{8\mathbb{A}(r)}{\mathbb{A}(r)-\mathbb{A}(y)}\right) dy \\ &= r \int_0^r \left(\int_y^r e^{\mathbb{V}_z-\mathbb{V}_y} dz\right) \log\left(\frac{8\mathbb{A}(r)}{\mathbb{A}(r)-\mathbb{A}(y)}\right) dy \\ &\leq r^2 e^{\mathbb{V}_r^{\#}} \int_0^r \log\left(\frac{8\mathbb{A}(r)}{\mathbb{A}(r)-\mathbb{A}(y)}\right) dy. \end{aligned}$$

Notice also that $\mathbb{A}(r) = \int_0^r e^{\mathbb{V}_z} dz \le r e^{\overline{\mathbb{V}}_r}$ and similarly $\mathbb{A}(r) - \mathbb{A}(y) \ge (r-y)e^{\underline{\mathbb{V}}_r}$. Therefore

$$\begin{aligned}\int_0^r \log\left(\frac{8\mathbb{A}(r)}{\mathbb{A}(r)-\mathbb{A}(y)}\right) dy &\le r(\overline{\mathbb{V}}_r - \underline{\mathbb{V}}_r) + \int_0^r \log\left(\frac{8r}{r-y}\right) dy \\ &= r(\overline{\mathbb{V}}_r - \underline{\mathbb{V}}_r + 1 + \log 8).\end{aligned}$$

Define the set $\mathcal{E}_2 \stackrel{\text{def}}{=} \{\overline{\mathbb{V}}_r - \underline{\mathbb{V}}_r \le e^{\log^3 r}\}$. In view of Lemma 1,

$$\mathbf{P}\{\mathcal{E}_2^c\} \le \mathbf{P}\left\{\mathbb{V}_r^* > \frac{1}{2}e^{\log^3 r}\right\} \le e^{-\log^2 r}.$$

Therefore, $\mathbf{P}\{(\mathcal{E}_1 \cap \mathcal{E}_2)^c\} \le 2e^{-\log^2 r}$ and on $\mathcal{E}_1 \cap \mathcal{E}_2$,

$$\widetilde{I}_1(r) \le r^3(e^{\log^3 r} + 1 + \log 8)e^{\mathbb{V}_r^\#} \le e^{\log^4 r + \mathbb{V}_r^\#}.$$

This completes the proof of the upper bound.

Proof of the lower bound. Define by induction

$$\begin{cases} \gamma_0 \stackrel{\text{def}}{=} 0, \\ \gamma_{k+1} \stackrel{\text{def}}{=} \inf(t > \gamma_n,\ |\mathbb{V}_t - \mathbb{V}_{\gamma_k}| \ge 1). \end{cases}$$

The sequence $(\gamma_{k+1} - \gamma_k,\ k \ge 0)$ is i.i.d. and distributed as $\gamma_1 = \inf\{t > 0 : |\mathbb{V}_t| \ge 1\}$. We denote by $\lfloor x \rfloor$ the integer part of x. We also use the notation $\epsilon \stackrel{\text{def}}{=} e^{-\log^3 r}$. Consider the following events

$$\begin{aligned}\mathcal{E}_3 &\stackrel{\text{def}}{=} \left\{\gamma_{\lfloor r^2 \rfloor} > r\right\}, \\ \mathcal{E}_4 &\stackrel{\text{def}}{=} \{\gamma_k - \gamma_{k-1} \ge 2\epsilon \ \text{ for all } k = 1, 2 \ldots, \lfloor r^2 \rfloor\}.\end{aligned}$$

With the help of Markov's inequality, we get

$$\mathbf{P}\left\{\mathcal{E}_3^c\right\} = \mathbf{P}\left\{e^{-\gamma_{\lfloor r^2 \rfloor}} \ge e^{-r}\right\} \le e^r \mathbf{E}\left[e^{-\gamma_{\lfloor r^2 \rfloor}}\right] = e^r \mathbf{E}\left[e^{-\gamma_1}\right]^{\lfloor r^2 \rfloor} \le e^{-r},$$

where we used that r is very large and that $\mathbf{E}\left[e^{-\gamma_1}\right] < 1$ for the last inequality (because γ_1 is non-negative and not identically zero). We also have

$$\begin{aligned}\mathbf{P}\{\mathcal{E}_4^c\} \le \sum_{k=1}^{\lfloor r^2 \rfloor} \mathbf{P}\{\gamma_k - \gamma_{k-1} < 2\epsilon\} &\le \lfloor r^2 \rfloor \mathbf{P}\{\gamma_1 < 2\epsilon\} \\ &\le \lfloor r^2 \rfloor \mathbf{P}\{\mathbb{V}_{2\epsilon}^* \ge 1\} \\ &\le e^{-\log^2 r},\end{aligned}$$

where we used Lemma 1 for the last inequality. Define also

$$\mathcal{E}_5 \stackrel{\text{def}}{=} \{|\mathbb{V}_x - \mathbb{V}_r| < 1 \ \text{ for all } x \in [r - 2\epsilon, r]\}.$$

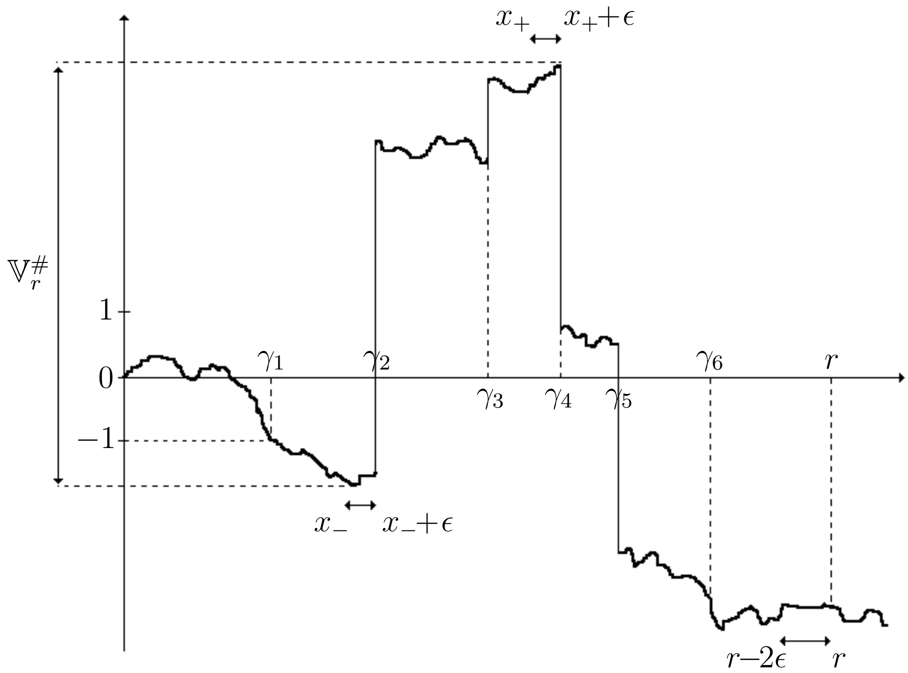

Figure 1: Sample path of $\mathbb{V}$ on $\mathcal{E}_6$.

From time reversal, the processes $(\mathbb{V}_t,\ 0 \le t \le 2\epsilon)$ and $(\mathbb{V}_r - \mathbb{V}_{(r-t)^-}, 0 \le t \le 2\epsilon)$ have the same law. Thus,

$$\mathbf{P}\{\mathcal{E}_5^c\} \le \mathbf{P}\{\mathbb{V}_{2\epsilon}^* \ge 1\} \le e^{-\log^2 r}.$$

Setting $\mathcal{E}_6 \stackrel{\text{def}}{=} \mathcal{E}_3 \cap \mathcal{E}_4 \cap \mathcal{E}_5$, we get $\mathbf{P}\{\mathcal{E}_6^c\} \le 3e^{-\log^2 r}$. Moreover, it is easy to check (see figure 1) that on $\mathcal{E}_6$, we can always find x_-, x_+ such that:

$$\begin{cases} 0 \le x_- \le x_+ \le r - 2\epsilon, \\ \text{for any } a \in [x_-, x_- + \epsilon],\ |\mathbb{V}_{x_-} - \mathbb{V}_a| \le 2, \\ \text{for any } b \in [x_+, x_+ + \epsilon],\ |\mathbb{V}_{x_+} - \mathbb{V}_b| \le 2, \\ \mathbb{V}_{x_+} - \mathbb{V}_{x_-} \ge \mathbb{V}_r^\# - 4. \end{cases}$$

Let us also define

$$\begin{aligned} \mathcal{E}_7 &\stackrel{\text{def}}{=} \mathcal{E}_6 \cap \left\{ \inf_{y \in [x_-, x_- + \epsilon]} U\left(\frac{\mathbb{A}(r) - \mathbb{A}(y)}{\mathbb{A}(r)}\right) \ge \frac{\mathbb{A}(r) - \mathbb{A}(x_-)}{\mathbb{A}(r)} e^{-2\log^2 r} \right\}, \\ \mathcal{E}_8 &\stackrel{\text{def}}{=} \left\{ \mathbb{V}_r^\# \ge 3\log^2 r \right\}. \end{aligned}$$

We finally set $\mathcal{E}_9 \stackrel{\text{def}}{=} \mathcal{E}_7 \cap \mathcal{E}_8$. Then on $\mathcal{E}_9$, we have, for all r large enough,

$$\begin{aligned}
\widetilde{I}_1(r) &\geq \mathbb{A}(r) \int_{x_-}^{x_-+\epsilon} e^{-\mathbb{V}_y} U\left(\frac{\mathbb{A}(r)-\mathbb{A}(y)}{\mathbb{A}(r)}\right) dy \\
&\geq e^{-\mathbb{V}_{x_-}-2-2\log^2 r} \int_{x_-}^{x_-+\epsilon} (\mathbb{A}(r)-\mathbb{A}(x_-)) dy \\
&= e^{-\mathbb{V}_{x_-}-2-2\log^2 r - \log^3 r} \int_{x_-}^{r} e^{\mathbb{V}_y} dy \\
&\geq e^{-\mathbb{V}_{x_-}-2-2\log^2 r - \log^3 r} \int_{x_+}^{x_++\epsilon} e^{\mathbb{V}_y} dy \\
&\geq e^{\mathbb{V}_{x_+}-\mathbb{V}_{x_-}-4-2\log^2 r - 2\log^3 r} \\
&\geq e^{\mathbb{V}_r^{\#}-\log^4 r}.
\end{aligned}$$

This proves the lower bound on $\mathcal{E}_9$. It simply remains to show that $\mathbf{P}\{\mathcal{E}_9^c\} \leq c_5 e^{-\log^2 r}$. According to Lemma 6.1 of [5], for any $0 < a < b$ and any $\eta > 0$, we have

$$\mathbf{P}\left\{\inf_{a<t<b} U(t) \leq \eta b\right\} \leq 2\sqrt{\eta} + 2\exp\left(-\frac{\eta}{2(1-a/b)}\right).$$

Therefore, making use of the independence of $\mathbb{V}$ and U, we find

$$\begin{aligned}
\mathbf{P}\{\mathcal{E}_9^c\} &\leq \mathbf{P}\{\mathcal{E}_6^c\} + \mathbf{P}\{\mathcal{E}_8^c\} + \mathbf{P}\{\mathcal{E}_7^c \cap \mathcal{E}_6 \cap \mathcal{E}_8\} \\
&\leq \mathbf{P}\{\mathcal{E}_6^c\} + \mathbf{P}\{\mathcal{E}_8^c\} + 2e^{-\log^2 r} + 2\mathbf{E}\left[e^{-\frac{1}{2}\mathbb{J}(r) e^{-2\log^2 r}} \mathbf{1}_{\mathcal{E}_6\cap\mathcal{E}_8}\right],
\end{aligned}$$

where

$$\mathbb{J}(r) \stackrel{\text{def}}{=} \frac{\mathbb{A}(r)-\mathbb{A}(x_-)}{\mathbb{A}(x_-+\epsilon)-\mathbb{A}(x_-)}.$$

We have already proved that $\mathbf{P}\{\mathcal{E}_6^c\} \leq 3e^{-\log^2 r}$. Using Lemma 1, we also check that $\mathbf{P}\{\mathcal{E}_8^c\} \leq e^{-\log^2 r}$. Thus, it remains to show that

$$\mathbf{E}\left[e^{-\frac{1}{2}\mathbb{J}(r) e^{-2\log^2 r}} \mathbf{1}_{\mathcal{E}_6\cap\mathcal{E}_8}\right] \leq c_6 e^{-\log^2 r}. \tag{7}$$

Notice that, on $\mathcal{E}_6$,

$$\mathbb{A}(r) - \mathbb{A}(x_-) = \int_{x_-}^{r} e^{\mathbb{V}_y} dy \geq \int_{x_+}^{x_++\epsilon} e^{\mathbb{V}_y} dy \geq e^{\log^3 r + \mathbb{V}_{x_+} - 2},$$

and also

$$\mathbb{A}(x_-+\epsilon) - \mathbb{A}(x_-) = \int_{x_-}^{x_-+\epsilon} e^{\mathbb{V}_y} dy \leq e^{\log^3 r + \mathbb{V}_{x_-} + 2}.$$

Therefore, on $\mathcal{E}_6 \cap \mathcal{E}_8$,

$$\mathbb{J}(r) \geq e^{\mathbb{V}_{x_+}-\mathbb{V}_{x_-}-4} \geq e^{\mathbb{V}_r^{\#}-8} \geq e^{\mathbb{V}_r^{\natural}-8} \geq e^{3\log^2 r - 8}$$

which clearly yields (7) and the proof of the proposition is complete. □

Lemma 2. *We have*

$$\frac{\mathbb{V}^{\#}_r}{r^{1/\alpha}} \xrightarrow[r\to\infty]{law} \mathbb{S}^{\natural}_1.$$

Proof. Let $f:[0,1]\mapsto\mathbb{R}$ be a deterministic càdlàg function. For $\lambda\geq 0$, define

$$f_\lambda(x) \stackrel{\text{def}}{=} f(x)-\lambda x.$$

We first show that

$$\lim_{\lambda\to\infty} f^{\#}_\lambda(1) = f^{\natural}(1). \tag{8}$$

It is clear that $f^{\natural}(1)=f^{\natural}_\lambda(1)\leq f^{\#}_\lambda(1)$ for any $\lambda>0$. Thus, we simply need to prove that $\limsup f^{\#}_\lambda(1)\leq f^{\natural}(1)$. Let $\eta>0$ and set

$$\begin{aligned} A(\eta,\lambda) &\stackrel{\text{def}}{=} \sup\left\{f_\lambda(y)-f_\lambda(x) : 0\leq x\leq y\leq 1 \text{ and } y-x\leq\eta\right\}, \\ B(\eta,\lambda) &\stackrel{\text{def}}{=} \sup\left\{f_\lambda(y)-f_\lambda(x) : 0\leq x\leq y\leq 1 \text{ and } y-x>\eta\right\}, \end{aligned}$$

so that

$$f^{\#}_\lambda(1) = \max\{A(\eta,\lambda),B(\eta,\lambda)\}. \tag{9}$$

Notice that $A(\eta,\lambda)\leq A(\eta)$ where

$$A(\eta)\stackrel{\text{def}}{=} A(\eta,0) = \sup\left\{f(y)-f(x) : 0\leq x\leq y\leq 1 \text{ and } y-x\leq\eta\right\}.$$

Since f is càdlàg, we have $\lim_{\eta\to 0} A(\eta) = f^{\natural}(1)$. Thus, for any $\varepsilon>0$, we can find $\eta_0>0$ small enough such that

$$\limsup_{\lambda\to\infty} A(\eta_0,\lambda)\leq f^{\natural}(1)+\varepsilon. \tag{10}$$

Notice also that

$$\begin{aligned} B(\eta_0,\lambda) &\leq \sup\left\{f(y)-f(x)-\eta_0\lambda : 0\leq x\leq y\leq 1 \text{ and } y-x>\eta_0\right\} \\ &\leq f^{\#}(1)-\eta_0\lambda \end{aligned}$$

which implies

$$\lim_{\lambda\to\infty} B(\eta_0,\lambda) = -\infty. \tag{11}$$

The combination of (9), (10) and (11) yields (8). Making use of the scaling property of the stable process $\mathbb{S}$, for any fixed $r>0$,

$$(\mathbb{V}_y\ ,\ 0\leq y\leq r) \stackrel{\text{law}}{=} (r^{1/\alpha}\mathbb{S}_y - \delta r y\ ,\ 0\leq y\leq 1).$$

Therefore, setting $\mathbb{R}(z) = (\mathbb{S}_{\cdot} - z\cdot)^{\#}_1$, we get the equality in law:

$$\frac{\mathbb{V}^{\#}_r}{r^{1/\alpha}} \stackrel{\text{law}}{=} \mathbb{R}(\delta r^{1-1/\alpha}). \tag{12}$$

Making use of (8), we see that $\mathbb{R}(z)$ converges almost surely towards $\mathbb{S}^{\natural}_1$ as z goes to infinity. Since $\alpha>1$ and $\delta>0$, we also have $\delta r^{1-1/\alpha}\to\infty$ as r goes to infinity and we conclude from (12) that

$$\frac{\mathbb{V}^{\#}_r}{r^{1/\alpha}} \xrightarrow[r\to\infty]{\text{law}} \mathbb{S}^{\natural}_1.$$

□

Proof of Theorem 1. Recall that the random variable $\mathbb{S}_1^\natural$ denotes the largest positive jump of $\mathbb{S}$ over the interval $[0,1]$. In view of the Lévy measure of $\mathbb{S}$ given by (1), this random variable has a continuous density. Thus, on the one hand, the combination of Proposition 1 and Lemma 2 readily shows that

$$\frac{\log(I_1(r))}{r^{1/\alpha}} \xrightarrow[r\to\infty]{\text{law}} \mathbb{S}_1^\natural. \tag{13}$$

On the other hand, the random variables $\mathbb{A}(\infty) = \lim_{x\to\infty}\mathbb{A}(x)$ and $\int_{-\infty}^0 e^{-\mathbb{V}_y}dy$ have the same law. We have already noticed that these random variables are almost surely finite. Since the function $L(t,\cdot)$ is, for any fixed t, continuous with compact support, we get

$$I_2(r) = \int_{-\infty}^0 e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)),\mathbb{A}(y))dy \le \sup_{z\in(-\infty,0]} L(\sigma(\mathbb{A}(\infty)),z)\int_{-\infty}^0 e^{-\mathbb{V}_y}dy < \infty.$$

Therefore,

$$\sup_{r\ge 0} I_2(r) < \infty \quad \text{almost surely.} \tag{14}$$

Combining (4), (13) and (14), we deduce that

$$\frac{\log(H(r))}{r^{1/\alpha}} \xrightarrow[r\to\infty]{\text{law}} \mathbb{S}_1^\natural$$

which, from the definition of the hitting times H, yields

$$\frac{\sup_{s\le t} X_s}{\log^\alpha t} \xrightarrow[t\to\infty]{\text{law}} \left(\frac{1}{\mathbb{S}_1^\natural}\right)^\alpha.$$

Moreover, according to the density of the Lévy measure of $\mathbb{S}$, we have

$$\mathbf{P}\{\mathbb{S}_1^\natural \le x\} = \exp\left(-\int_x^\infty \frac{c^+}{y^{\alpha+1}}dy\right) = \exp\left(-\frac{c^+}{\alpha x^\alpha}\right).$$

Therefore, the random variable $(1/\mathbb{S}_1^\natural)^\alpha$ has an exponential distribution with parameter c^+/α so the proof of the theorem for $\sup_{s\le t} X_s$ is complete. We finally use the classical argument given by Kawazu and Tanaka, p201 [7] to obtain the corresponding results for X_t and $\inf_{s\ge t} X_s$. □

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