Remarks - September 2018

R. Sena-Dias (I) - Complete extremal Kähler metrics on non-compact manifolds: an (incomplete) overview.

Canonical Kähler metrics on non-cpct mnfds

I. Ricci-flat Kähler

1). Complement of divisors

Yau's thm: (M, ω, J) cpct Kähler mnfd, $\Omega \in c_1(M)$ 2-form. Then $\exists\, \omega'$ Kähler metric in $[\omega]$ s.th. $Ric(\omega') = \Omega$.

Corollary: (M, ω, J) cpct Kähler, $c_1(M) = 0$. $\exists\, \omega'$ Kähler in $[\omega]$ s.th. $Ric(\omega') = 0$.

. Tian-Yau '90: (M, ω, J) cpct Kähler, $D \subset M$ divisor, smooth, ample, $\Omega \in c_1(K_X \otimes D)$. Then $\exists\, \omega'$ Kähler metric on $M \setminus D$, cplte, s. th. $Ric(\omega'|_{M\setminus D}) = \Omega|_{M \setminus D}$.

. Corollary: (M, ω, J) cpct Kähler, D anti-canonical ample and smooth. Then $\exists\, \omega'$ cplete, Kähler, Ricci-flat.

↳ rmks: 1) Monge-Ampère techniques;
2) D could be mildly singular

3) asymptotic behaviour.

Q (Yau: '78 ICM): are all cplte RFK arising this way?

→ No - counter-exple (Anderson - Kronheimer - Le Brun)

2- Resolution of singularities.

M variety - X is a resolution of M if it is smooth, and if $\exists$ $X \dashrightarrow M$ birat^al map which is proper.

X is minimal if all other resolutions factor through X.

↳ ex: . $\Gamma \subset U(m)$ finite; Γ acts freely on $\mathbb{C}^m \setminus \{0\}$, and $\mathbb{C}^m/\Gamma$ has an isolated singularity. X_Γ is the min. resol° of $\mathbb{C}^m/\Gamma$

. $\tau \to \mathbb{CP}^{m-1}$ taut^al bdle $\quad \tau^2 \dashrightarrow \mathbb{C}^m/\mathbb{Z}_2$

$$([z], z \otimes z) \to \pm z, \quad z \in \mathbb{C}^m$$

ALE:

. def°: (X^n, g) Riem^an metric is ALE if

. $\Gamma \subset O(n)$ finite

. $K \subset X$, $B \subset \mathbb{R}^n$

. $\pi : X \setminus K \to \mathbb{R}^n \setminus B/\Gamma$ covering

. $|\nabla^k (\pi_* g - g_{euc})| \leq C r^{-n-k}$

Kähler setting: (X^{2m}, ω, J) Kähler, $\Gamma \subset U(m)$, π holo^ic

a) $\dim_{\mathbb{C}} 2$, ALE, rfk

Eguchi - Hanson, Gibbons - Hawking, Kronheimer

Thm (Kronheimer '89):

$\Gamma \subset SU(2)$ finite ($\leadsto$ acts freely on $\mathbb{C}^2 \setminus \{0\}$)
Then X_Γ carries an family ALE RFK.
If simply-connected X^4 admits an RFK ALE metric, then it is isometric to one of the constructed.

Γ's can be cyclic $\leadsto$ metrics have S^1-symm.
dihedral
3 more

b) dim > 2 ALE RFK

Calabi: explicit rfk on $\mathbb{C}^m \to \mathbb{CP}^{m-1}$

Thm (Joyce '01):

$\Gamma \subset SU(m)$ finite, acting freely on $\mathbb{C}^m \setminus \{0\}$ s. th. $\mathbb{C}^m/\Gamma$ has isolated sing at 0.
Then $\exists$ Kähler ALE rfk metric on X_Γ, which is unique in its Kähler class

1) Joyce wanted to use such metrics special holonomy metrics

2) J. claims explicit construc° are rare

2(3) dim 2 not ALE

Gibbons - Hawking / LeBrun : rfk metric on X_Γ (Γ cyclic) not ALE -

"ALF"

def°: (X^4, g) Riem^{an} is said to be ALF

if : - $\exists\ \Gamma \subset O(3)$ finite
$O(2)$, $O(1)$

- $K \subset X$ cpct . $B \subset \mathbb{R}^3_{\mathbb{R}^2\ \mathbb{R}}$

- $\pi : X \setminus K \to \frac{\mathbb{R}^3 \setminus B}{\Gamma}$ circle bdle
$\mathbb{T}^2$, S^1-bdle over $\mathbb{T}^2$

with a connec° η s. th.

$$|\nabla^k (g - \pi^* g_{euc} - \eta \otimes \eta)| \leq C r^{-4-k}$$

blue: ALG red: ALH

Auvray construc° of ALF metrics on X_Γ, Γ dihedral

Cherkis - Kapustin conjectured all RFK cplte are ALE, F, G, H

Hein counterexample with vol$(B(x,r)) \sim r^{4/3}$, $|\text{Riem}| \sim r^{-2}$

Thm (Chen - Chen) :

X cplte RFK s. th. $|\text{Riem}| \lesssim r^{-2-\varepsilon}$
then it is ALE, F, G or H.

Classific° within each class

ALF case due to Minerbe -

B- Scalar flat case

1). Burns, LeBrun constructed such metrics on $O(-p)$ that are ALE + extra exples by L. around 91

. Joyce '95 : ansatz for RFK metrics on open sets in $\mathbb{C}^2$ (toric)

. Thm (Calderbank - Singer '04):

$\Gamma \subset U(2)$ cyclic -

$\exists$ SFK ALE metrics on X_Γ

SFK : scalar-flat Kähler

↳ rmk : 1) globalize Joyce

2) toric metrics

. Thm (Lock - Viaclovsky '10):

$\Gamma \subset U(2)$ finite, acts freely on $\mathbb{C}^2 \setminus \{0\}$, $\mathbb{C}^2/\Gamma$ has isolated sing^{ty} -

Then : $\exists$ SFK ALE metrics on X_Γ -

2) examples of non ALE SFK metrics

Abreu - S. : $\Gamma \subset U(2)$ cyclic

then X_Γ admits a 2-param family of SFK metrics which are not ALE - most of them are not RFK

(ii) . Toric scalar-flat Kähler metrics .

I. Motivation

. Thm (Donaldson) :

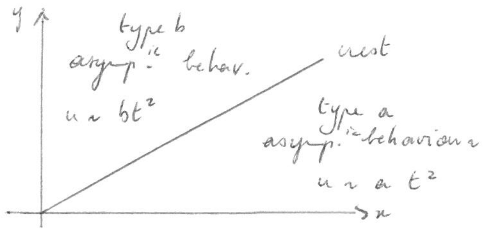
X^4 cpct toric mfd admits a cscK metric iff it is "stable" .

This is proved using cont^{ty} method . specific to toric - What happens to the method in the unstable case ?

"Conjecture" (D.) : in the unst. case, there is a

Sena-Dias - (5)

subsqce of certain rescalings of metrics in the continuity method that converges to "toric" cplte sfk metrics which have "crest like" asymptotic behaviour



$t = x+y$ Energy?

<u>Goal:</u> find such metrics.

II - <u>Toric geometry</u>.

<u>def°</u>: (X, ω, J) is said to be <u>toric Kähler</u> if it admits an effective holo-ic Ham-an $\mathbb{T}^n$ ac°, whose moment map is proper

to make the non-cpt case work!

Image of the moment map $P = \{x \in \mathbb{R}^n, x \cdot v_i - \lambda_i > 0\}$

$0 < \mathrm{Fix}(\mathbb{T}^n) < +\infty$

X is an "unbdd toric mnfd" if: P is unbdd.

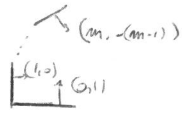
X^4 is "strictly unbdd" if: P is unbdd, 2 unbdd edges not ∥

<u>Examples</u>: 1. $\mathbb{C}^2 \circlearrowleft S^1 \times S^1$. $\mu(z_1, z_2) = (|z_1|^2, |z_2|^2)$

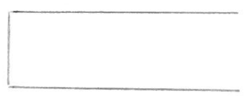
2. $\Gamma \subset U(2)$ finite, acts freely on $\mathbb{C}^2 \setminus \{0\}$ s.th. $\mathbb{C}^2/\Gamma$ has an isolated sing^{ty}.

X_Γ minimal resol° of $\mathbb{C}^2/\Gamma$ - strictly unbdd toric

$\Gamma = \mathbb{Z}_m \rightsquigarrow A_{m-1}$ sing^{ty}



3- $S^2 \times \mathbb{C} \circlearrowleft S^1 \times S^1$



not strictly unbdd

. Thm (Abreu - S, S.)

X^2 strictly unbded toric surface then it admits a 2-parameter family of complete SFK metrics which are not ALE, and most of these are not RFK, and their energies are finite.

Thm (Weber):

Most of the above metrics are not "ALF".

Metric toric geometry

Guillemin : (X, ω, J) toric Kähler.
there is a "canonical" cplx str. J_G

Abreu : $\left\{ \begin{array}{l} J \text{ toric on a} \\ \text{given mnfd s.th.} \\ J \text{ biholo to } J_G \end{array} \right\} \xleftarrow{*} \left\{ \begin{array}{l} u : P \to \mathbb{R} \\ u \text{ cvx} \\ u - u_G \text{ is } C^\infty(\bar{P}) \end{array} \right\}$

where u_G is a given func° depending only on P:

$u_G = \frac{1}{2} \sum_i l_i \log l_i \quad , \quad l_i(x) = x \cdot v_i - \lambda_i$

A toric K. mnfd admits an open dense where $\mathbb{T}^n$ acts freely

$\overset{\circ}{X} \cong P \times \mathbb{T}^n$

(x, θ) "ac°-angle coordinates"

$\underset{g}{g_u} = u_{ij}\, dx^i \otimes dx^j + u^{ij}\, d\theta_i \otimes d\theta_j \xleftarrow{*} u$

$*$, Legendre dual of the K. pot^{ial}.

Abreu : $\mathrm{scal}(g_u) = -\frac{\partial^2 u^{ij}}{\partial x_i \partial x_j}$

Construction

Joyce ansatz (translated by Donaldson into toric):

$\xi : \mathcal{U} \subset \mathbb{H} = \{ (H, r),\ r > 0 \} \to \mathbb{R}^2$

· $\det \left(\frac{\partial \xi}{\partial (H, r)} \right) > 0$

· $\frac{\partial^2 \xi}{\partial H^2} + \frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r} \frac{\partial \xi}{\partial r} = 0$

(PDE for axi-symm$^{\underline{ic}}$ harm$^{\underline{ic}}$ func$^{\circ}$).

$u : V \subset \mathbb{R}^2 \to \mathbb{R}$ convex,

$\frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = 0$

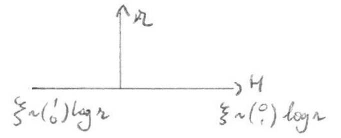
Global version of Joyce

Need ξ such that:

1) $\det \left[\frac{\partial \xi}{\partial (H, r)} \right] > 0$;
2) bdry bhv of ξ needs to transform into the right bdry bhv for u;
3) bhv of ξ at ∞ to correspond to a cplte metric

↳ ex: · $\xi = \left(\log(-H + \sqrt{H^2 + r^2}),\ \log(H + \sqrt{H^2 + r^2}) \right)$

then: · $\det \left(\frac{\partial \xi}{\partial (H, r)} \right) > 0$

· 

$u = x \log x + y \log y = u_{euc}$ on $V = (\mathbb{R}^{+*})^2$

Linear comb$^{\circ}$ of such ξ give sfk metrics - these are ALE (Calderbank-Singer '04)

Donaldson's idea: can add $H\nu$ to ξ to get another sol$^{\circ}$

$\xi + H\nu : \mathbb{H} \to \mathbb{R}^2$

will yield an appropriate metric
For $\nu \in$ cone determined by ν_1 and ν_d,
this will give the metrics in the thm.

$\xi + H\nu$ - $\det \frac{\partial^2(\xi + H\nu)}{\partial(\theta, r)} > 0$

- bdry cond$^\circ$ on ξ unchanged by adding $H\nu$

- the metric ass. to $\xi + H\nu$ needs ν to be in the above cone to be cplte.