Hierarchical Bayesian Modelling for univariate geo-referenced spatial data (BCG chap 5)

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9th May 2006

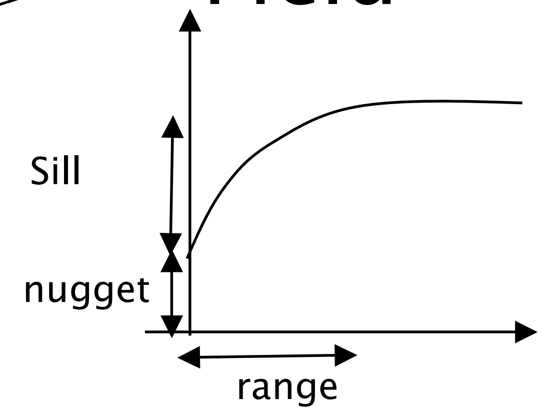
The Univariate stationary Gaussian isotropic case

Mean effect

Spatial Field

$$Y(s) = \mu(s) + \nu(s)$$

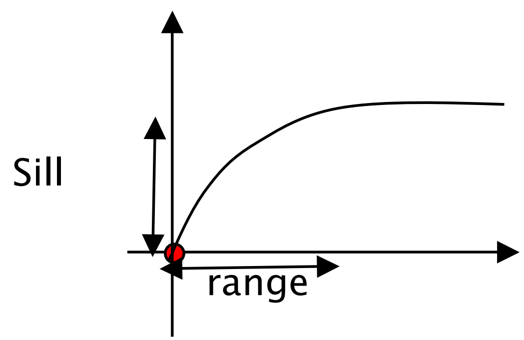
$$\mu(s) = X(s)\beta$$



Latent structured phenomenon

$$Y(s) = \mu(s) + Z(s) + \varepsilon(s)$$

« Pure » Error iid noise

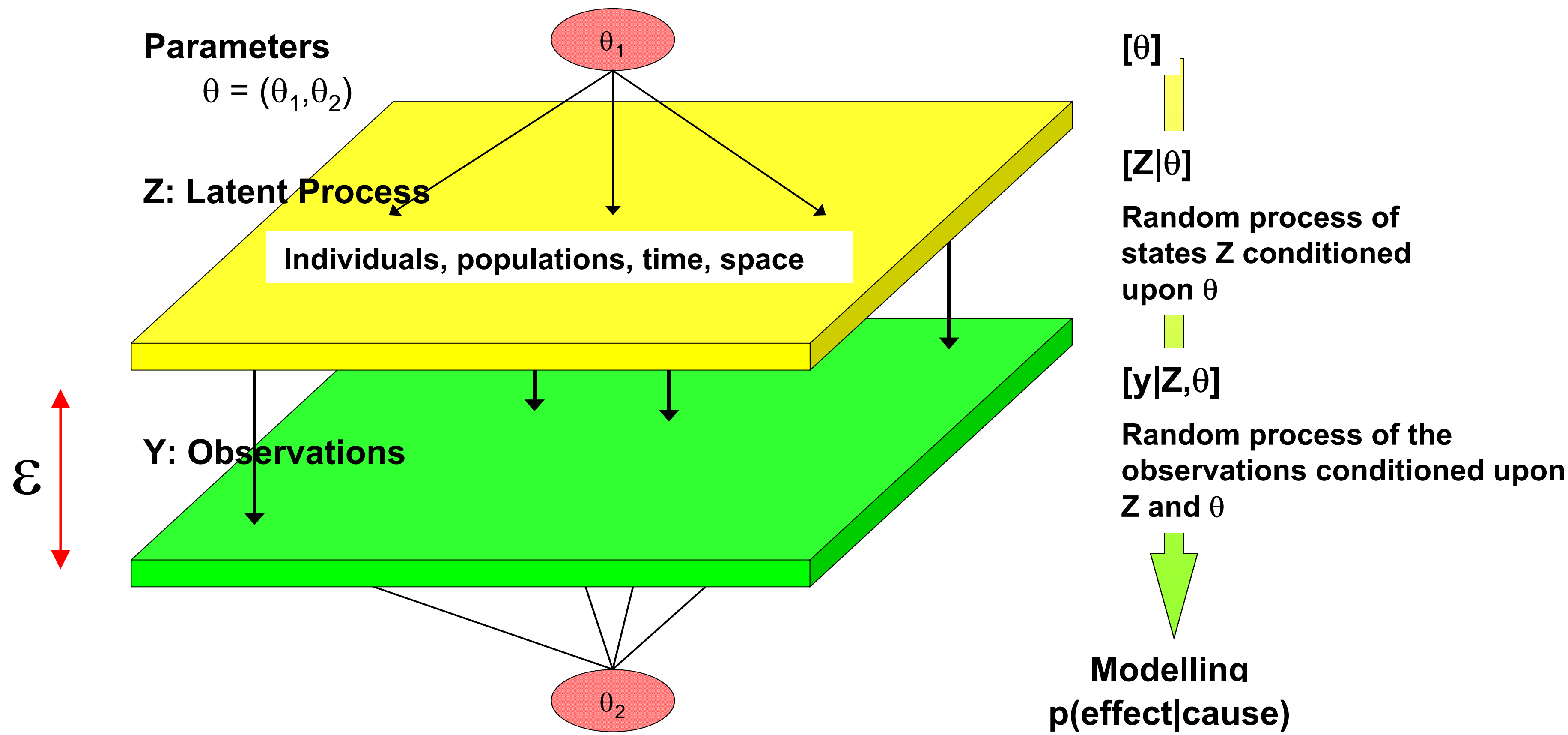


Given (μ,Z) the Y(s)'s are cond't independent

Conditionnal (hierarchical) modelling strategy in HBM

- **Modelling : Capacity to accommodate complexity**

Parameters, latent states process and observation model can be modelled independently



Conditionnal (hierarchical) modelling strategy in HBM

- **Modelling : Capacity to accommodate complexity**

Each term [Z|θ] and [y|Z,θ] can be very complex if seen « globally », but is constructed from simple local interactions

« How does this global model works ? » → $\sum_{\text{component } i}$ { *« How does this component works, conditioned upon the elements that directly affect it ? »* }

Global (complex) model → $\sum_{i}$ Local (simple) models

→ **A natural framework to formalize knowledge**

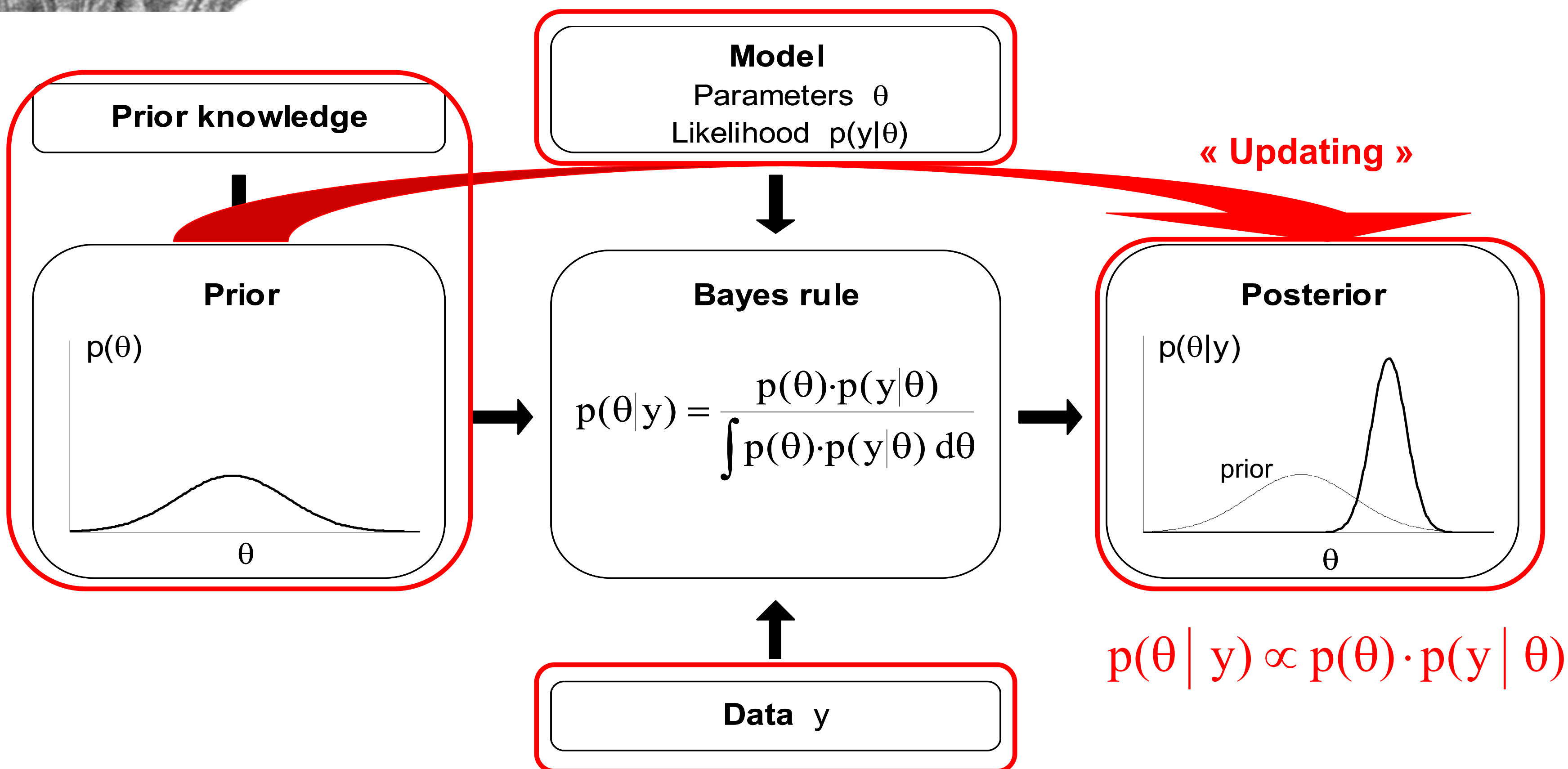
→ **Enhances dialog between modeler and practitionner**

The Bayes rule (*Bayes theorem*)

Sir. Reverend Thomas Bayes (1702-1761)

Source : Brooks, S.P., 2003. Bayesian computation : a statistical revolution. Phil. Trans. R. Soc. Lond. A – 361: 2681-2697.

« Information processor »

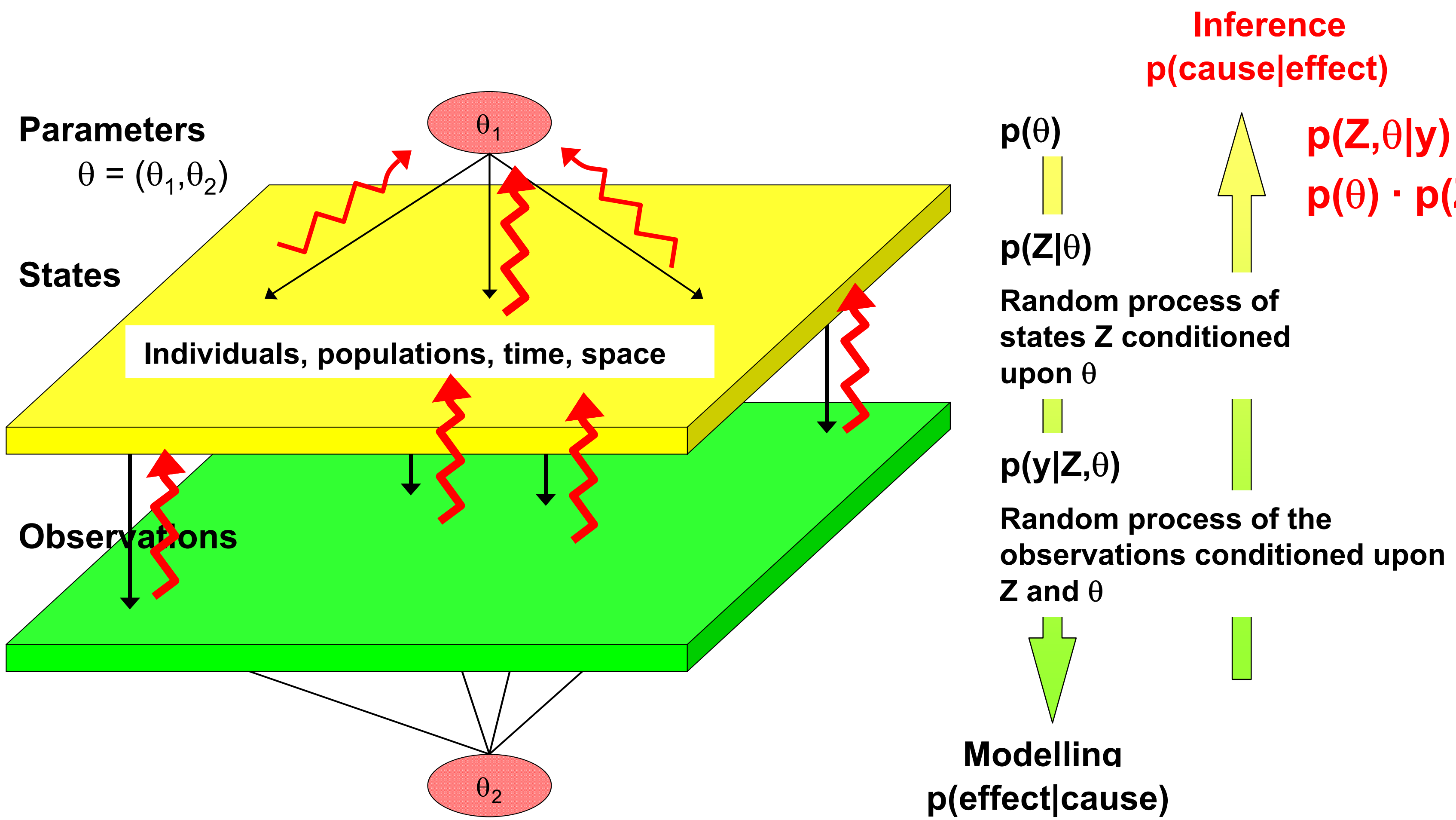


Conditionnal (hierarchical) modelling strategy in HBM
Take advantage of conditional independence

- **Inferences**

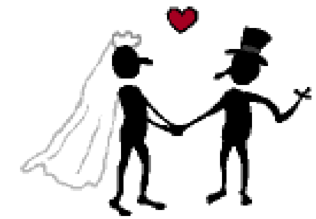
The joint posterior pdf splits into 3 components

$\Rightarrow$ **Easy to switch from "Modelling" to "Inferences"**



$$p(Z,\theta|y) \propto p(\theta) \cdot p(Z|\theta) \cdot p(y|Z,\theta)$$

MCMC and bayesian inference go along well



1) Draw N replicates from [θ|y]

2) use θ sample to get « empirical » estimates : mean, variance, percentiles from θ.

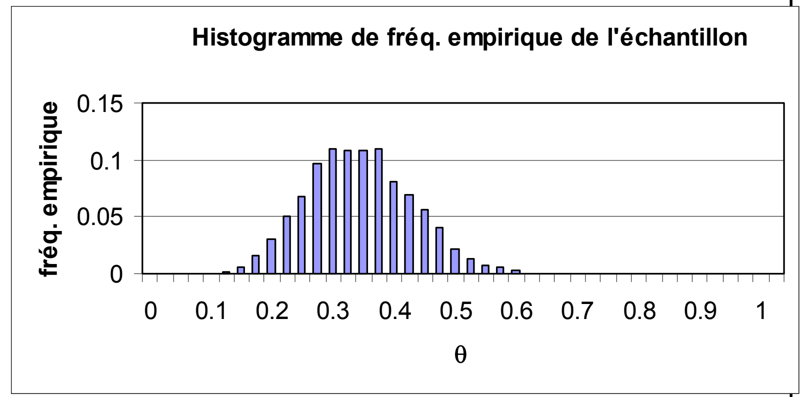
Unknown pdf [θ|y]

⇩

sample from [θ|y] :

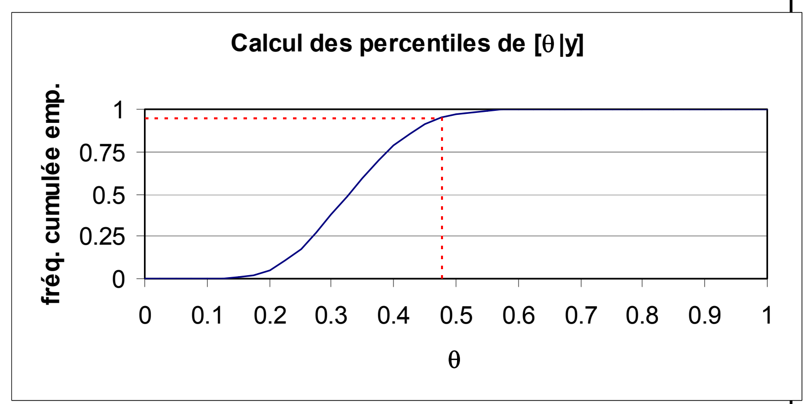
$(\theta^{(1)}, \theta^{(2)}, \ldots , \theta^{(N)})$

⇨



$$\text{Moyenne}: \bar{\theta} = \frac{1}{N}\sum_{i=1}^{N}\theta^{(i)}$$

$$\text{Variance}: \frac{1}{\text{N}}\sum_{i=1}^{N}\left(\theta^{(i)} - \bar{\theta}\right)^2$$



MCMC without tears : WinBUGS

Bayesian inference

Prior (known) upon θ

Model likelihood (known)

Posterior of θ

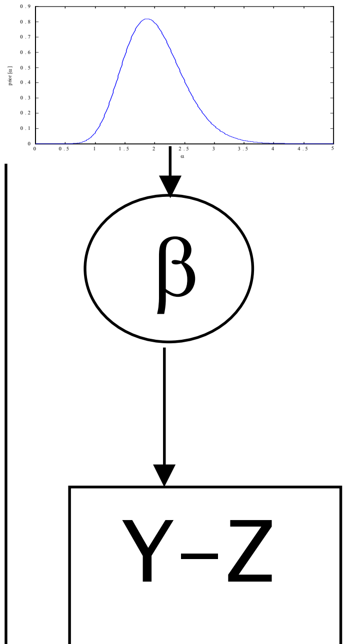
$$[\theta, Z|y] = \frac{[\theta][Z|\theta][y|\theta, Z]}{\iint_{\Theta, Z} [\theta, Z][y|\theta, Z] \, d\theta \, dZ}$$

Do not depend from θ and often untractable analytically. Can be considered as a constant

$$[\theta|y] = k \cdot [\theta][y|\theta]$$

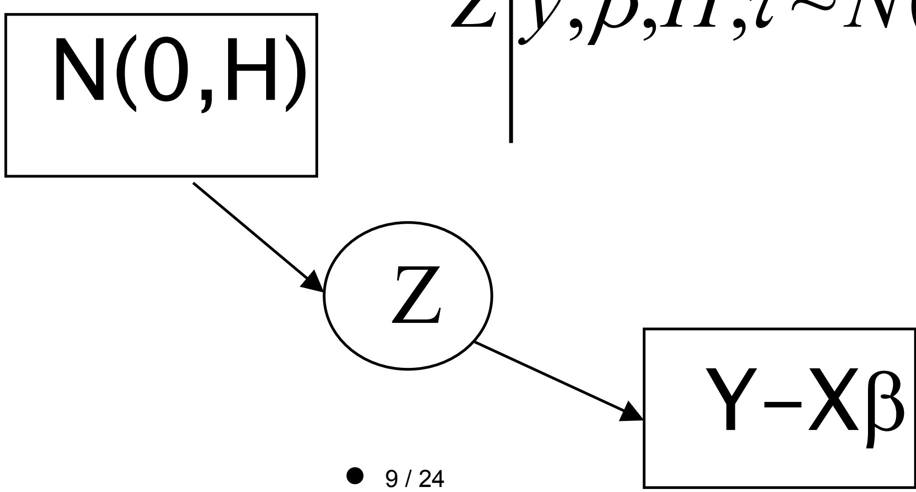
MCMC can help by providing « samples » from [θ|y]

Take advantage of conjugacy for bayesian Inference



$$\beta \mid y, Z, \tau \sim N\left((X'X)^{-1} X'(Y-Z), \left(\frac{X'X}{(\tau)^2} \right)^{-1} \right)$$

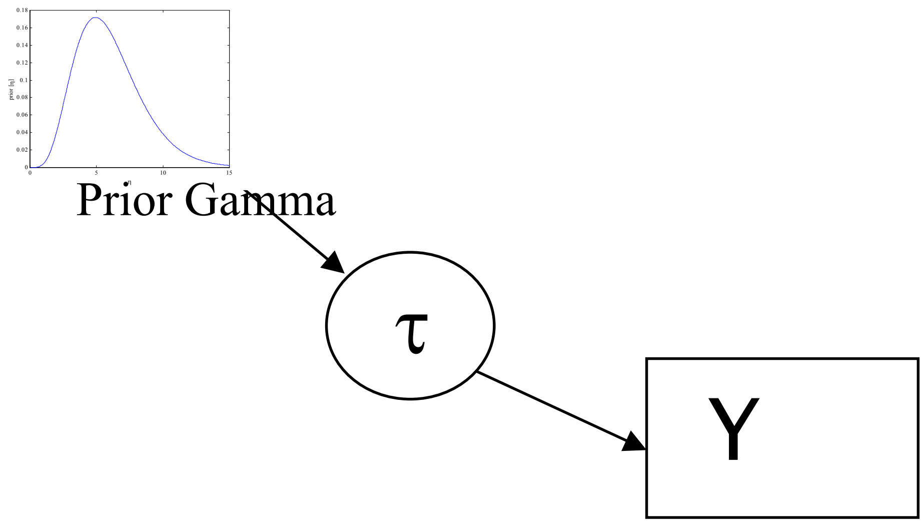
$$Z \mid y, \beta, H, \tau \sim N\left(\left(H + \frac{I}{(\tau)^2} \right)^{-1} \left(\frac{Y - X\beta}{(\tau)^2} \right), \left(H + \frac{I}{(\tau)^2} \right)^{-1} \right)$$



H(ϕ,κ,σ): cov matrix for Z

τ : nugget variance

Take advantage of conjugacy for bayesian Inference (cont'd)



H(φ,κ,σ): cov matrix for Z

τ : nugget variance

$$\frac{1}{(\tau)^2} \Big| y, Z, \beta \sim Gamma\,(\bullet,\bullet)$$

$$\frac{1}{(\sigma)^2} \Big| Z, \beta, C(\phi,\kappa) \sim Gamma\,(\bullet,\bullet)$$

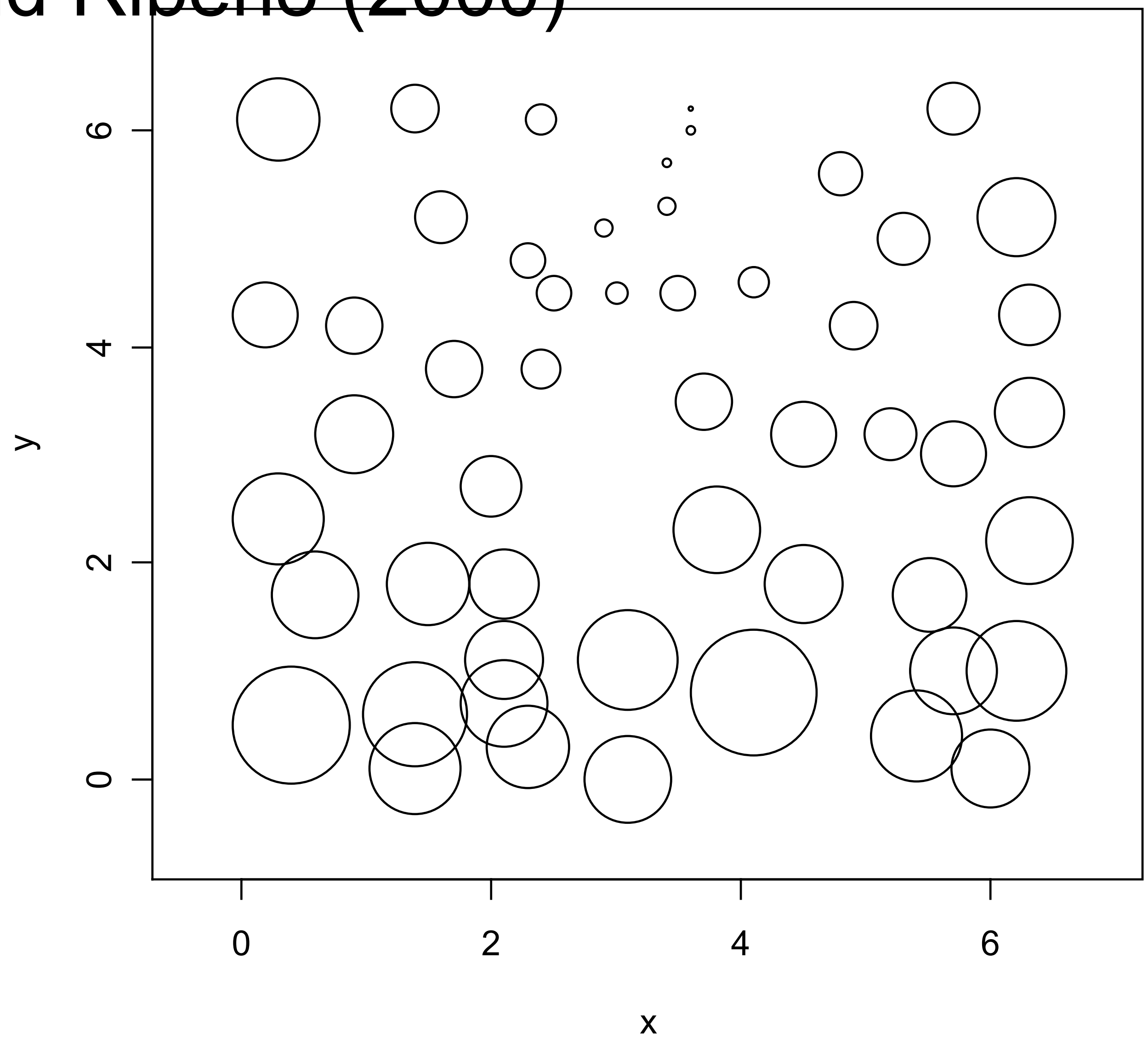
Can't take advantage of conjugacy for ϕ,κ!!!

$H(\phi,\kappa,\sigma)$: cov matrix for Z

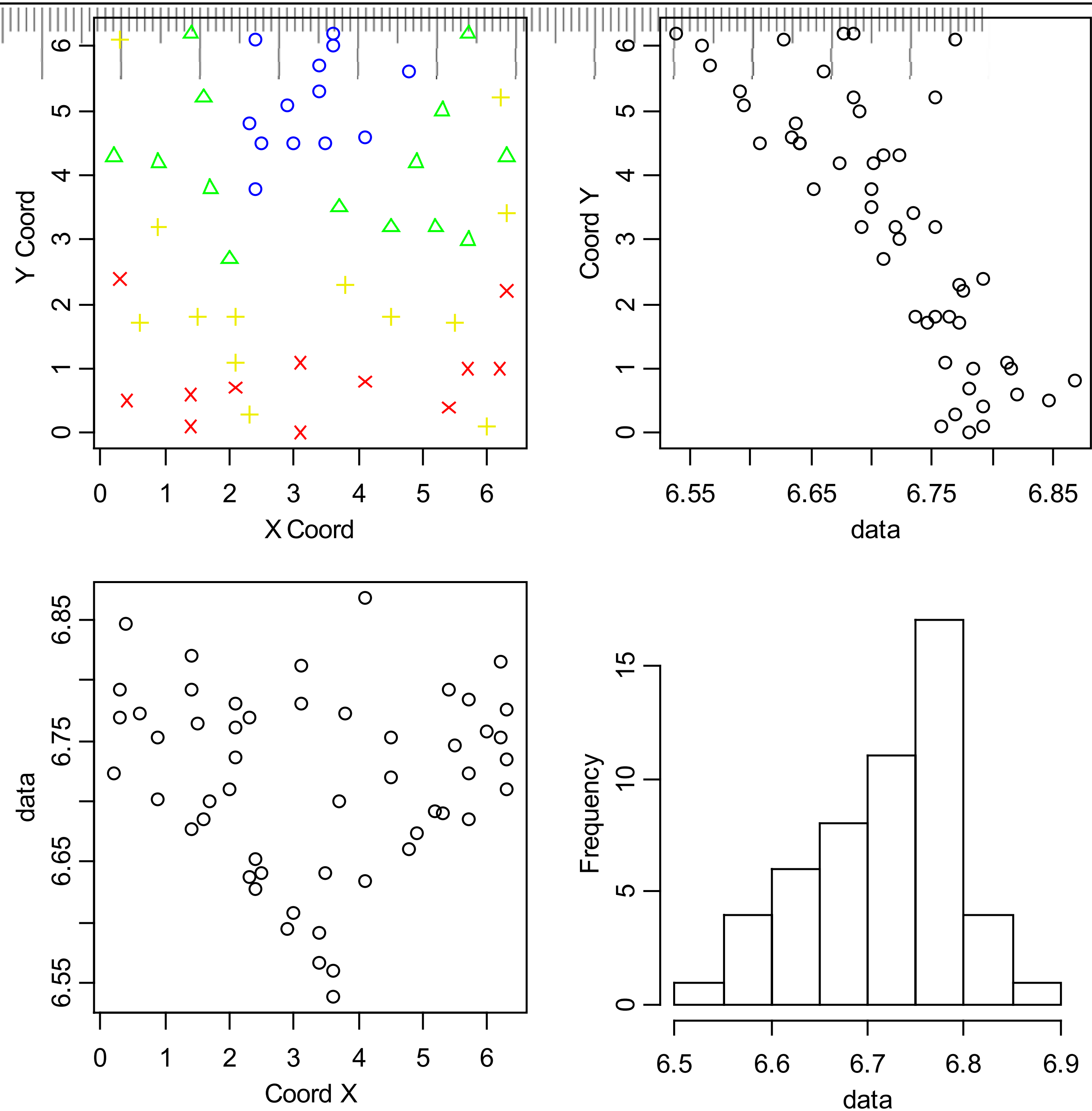
$H(\phi,\kappa,\sigma)=\sigma^2 C(\phi,\kappa)$

$$\left[\phi,\kappa \middle| \sigma, Z\right] = k \times \left[\phi,\kappa\right] \times \frac{\exp(-Z' C^{-1}(\phi,\kappa) Z / 2(\sigma)^2)}{\sqrt{\det(C(\phi,\kappa))}}$$

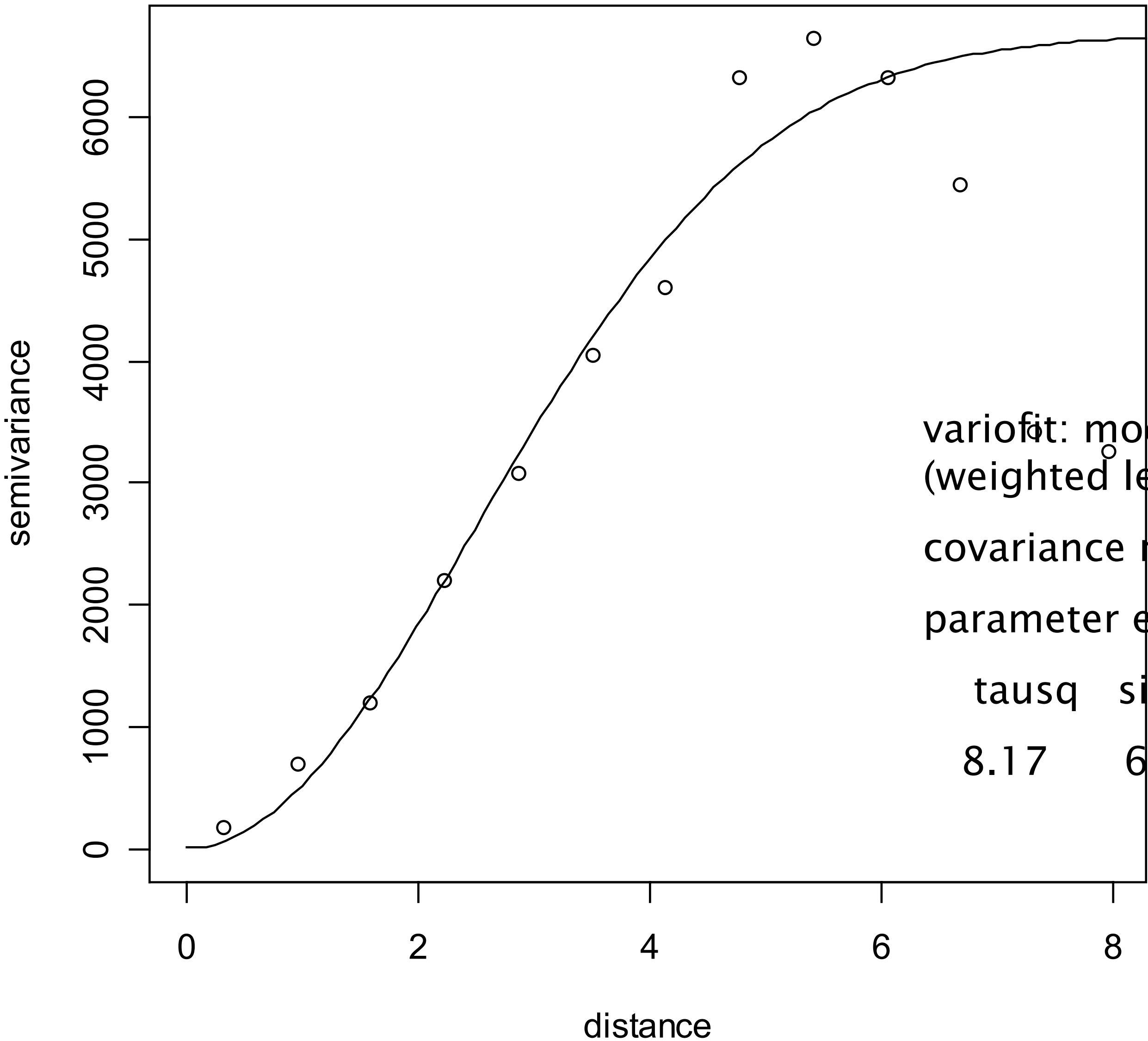
Elevation data
From Diggle and Riberio (2000)



Data description



Traditional Inference



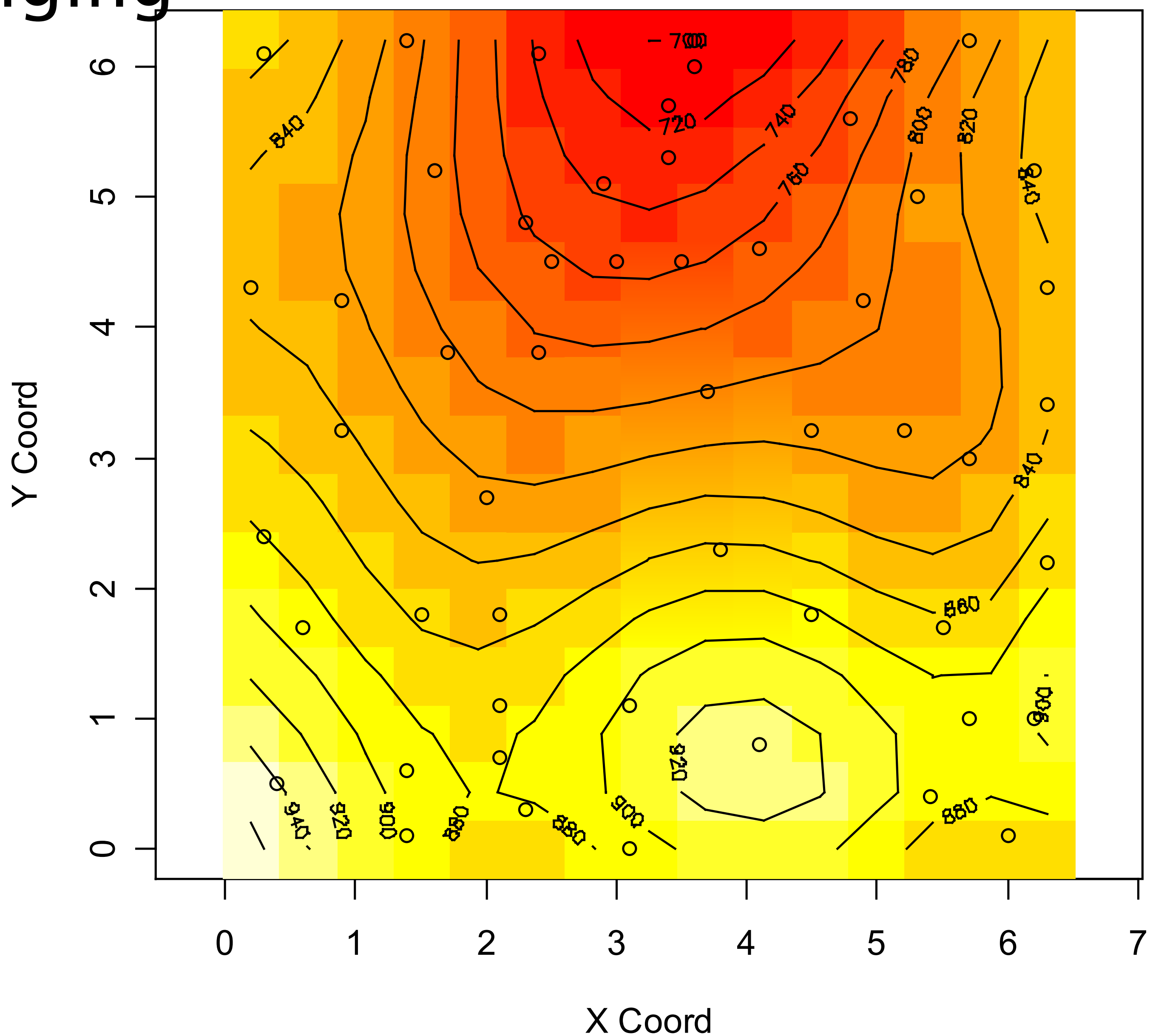
variofit: model parameters estimated by WLS (weighted least squares):

covariance model is: powered.exponential

parameter estimates:

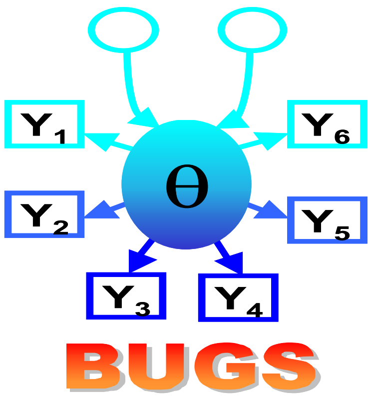
tausq	sigmasq	phi	kappa
8.17	6665	3.5	2

Frequentist kriging



Using GeoR

Friendly softwares, e.g. WinBUGS®



"Bayesian inference Using Gibbs Sampler"

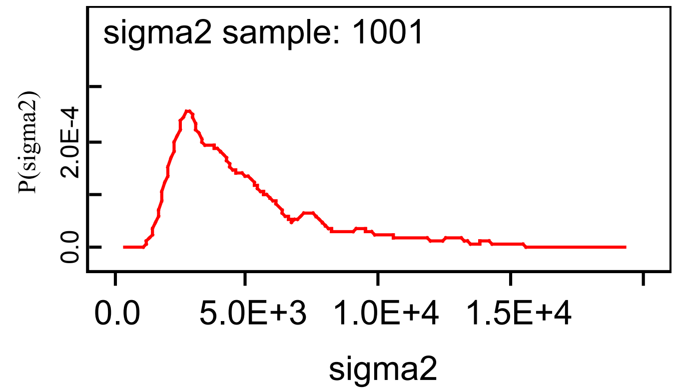
Medical Research Council, Biostatistics Unit, Cambridge, UK

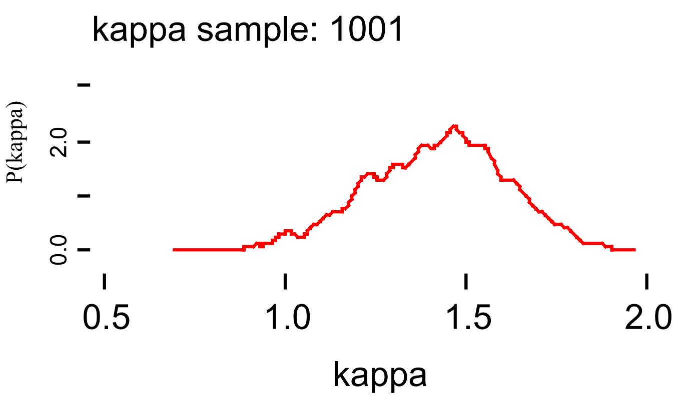
Imperial College, London, UK

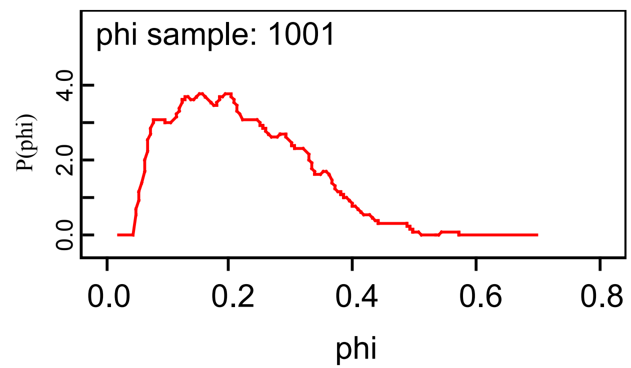
- **Bayesian statistical modeling**
- **Very popular**
- **Sampling based methods using Gibbs sampling (eventually hybrid)**
- **Models can be described graphically**
- **Freeware, available at : www.mrc-bsu.cam.ac.uk/bugs/**
- **Extensions :** { **Spatial models : GeoBUGS®**
Convergence : Coda®, Boa® (R packages)
- **Coupling with R**

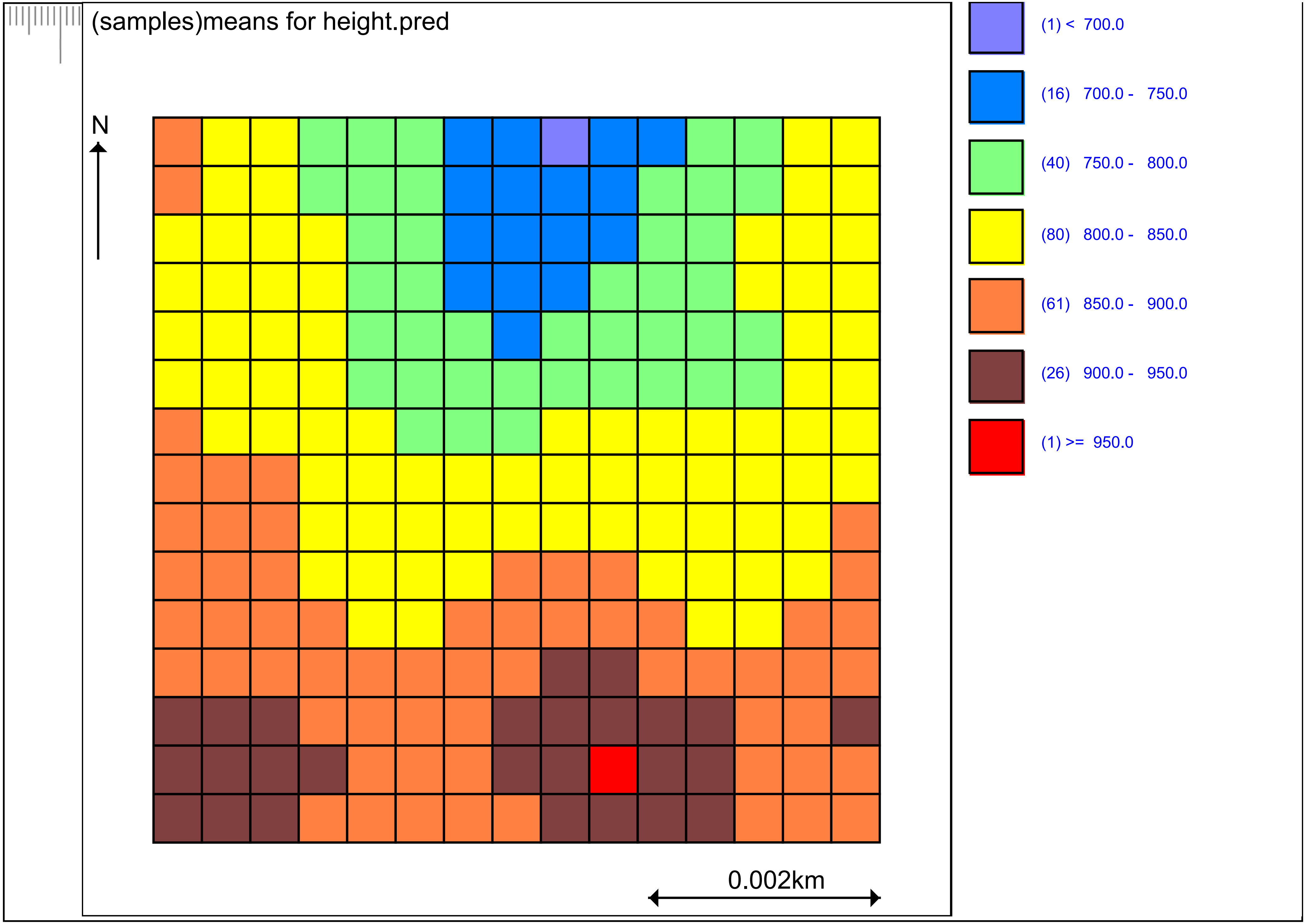
Bayesian Kriging

	Mean	Var
kappa	1.421	0.1934
phi	0.2178	0.102
sigma2	5145.0	3003.0








(samples)means for height.pred
N
0.002km
(1) < 700.0
(16) 700.0 - 750.0
(40) 750.0 - 800.0
(80) 800.0 - 850.0
(61) 850.0 - 900.0
(26) 900.0 - 950.0
(1) >= 950.0

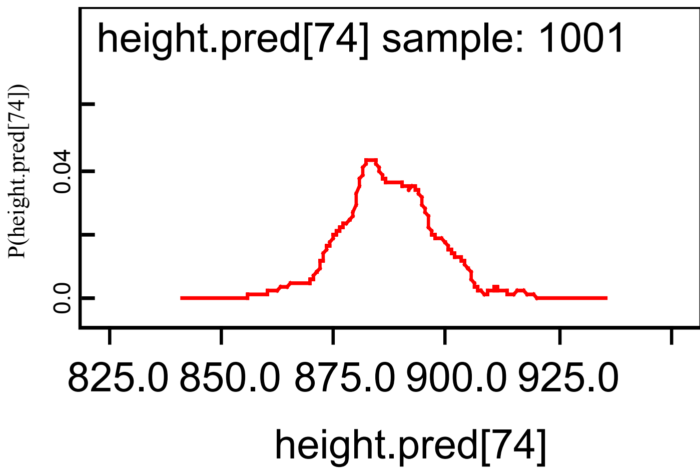
Bayes est=887.2±2*10.8

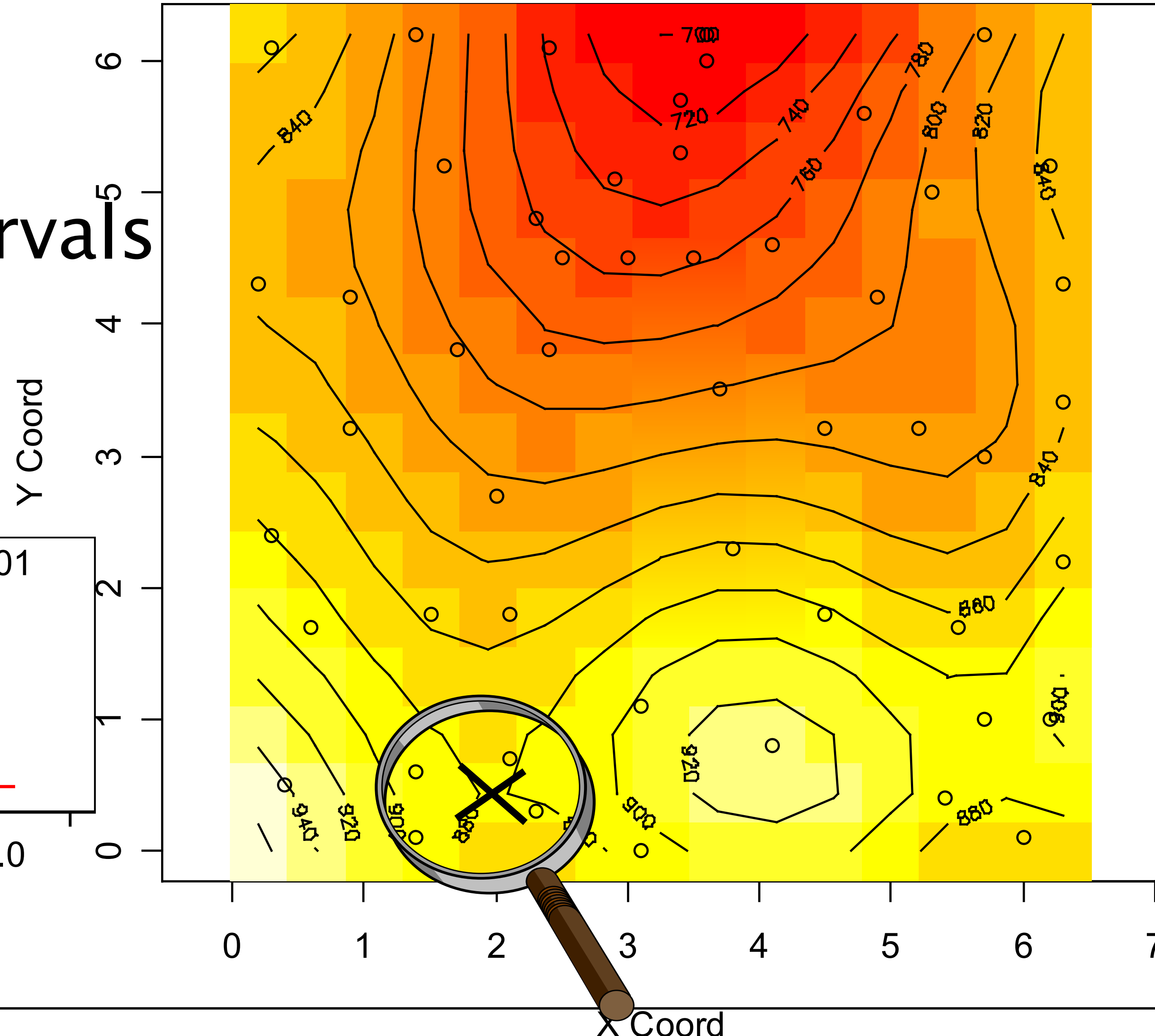
Freq. est=878.7±2* 3.2

$$[y_{new}|y]=\int_{\Theta}[y_{new}|y,\theta][\theta|y]d\theta$$

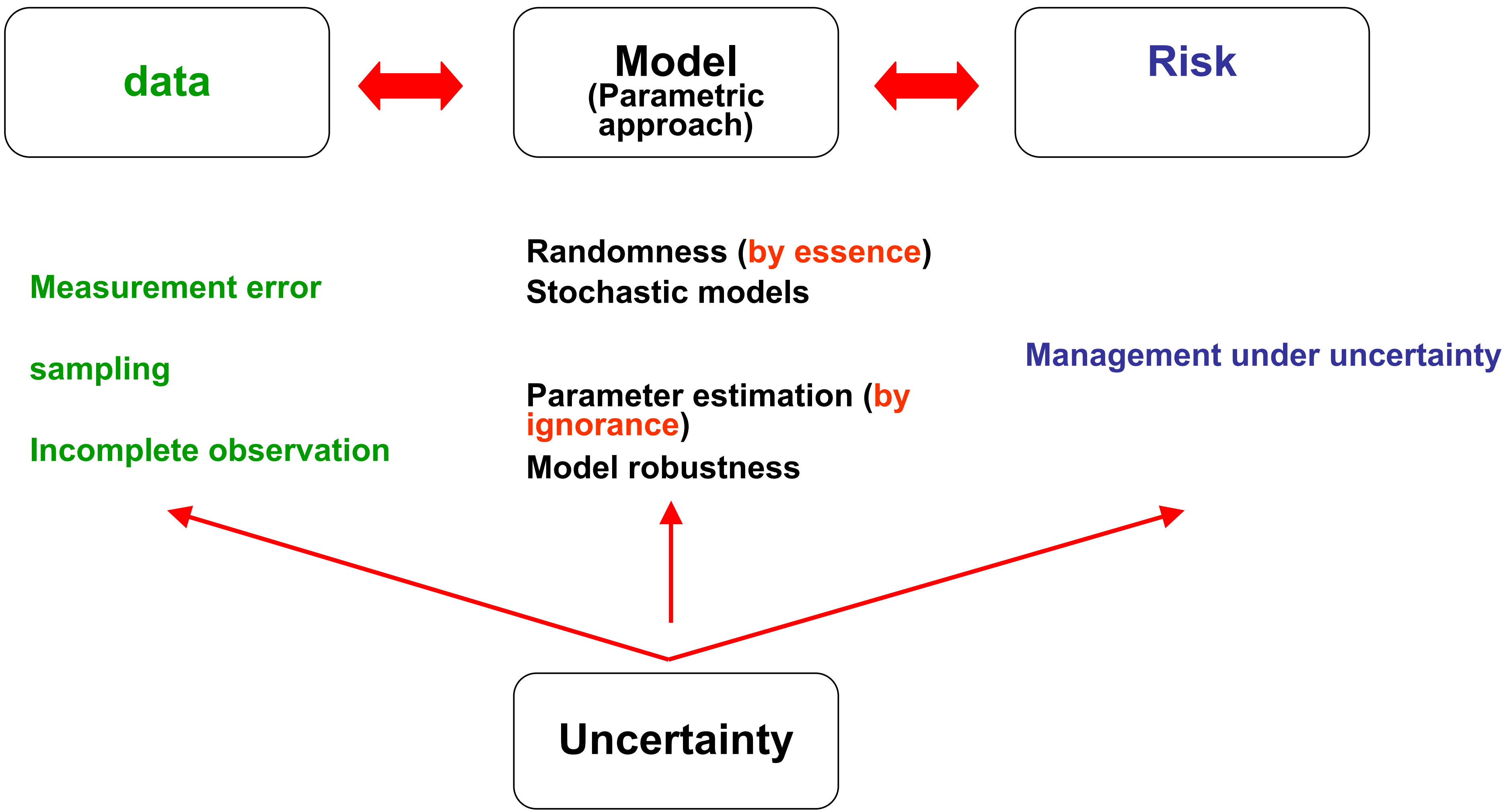
$$\theta=(\beta,\kappa,\phi,\sigma)$$

Confidence intervals widen !



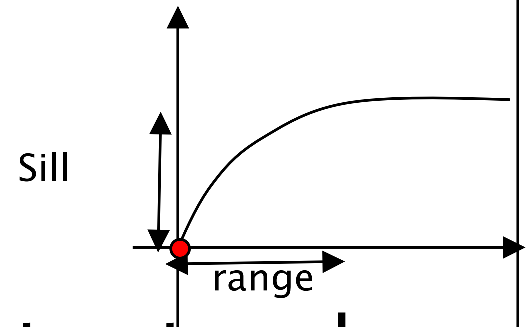


Probabilizing uncertainty : to bet or not to bet



The Univariate stationary Non-Gaussian isotropic case

$$\mu(s)=X(s)\beta$$ ← Mean effect



$$Z(s)\sim N(0,H(\phi,\kappa,\sigma))$$ ← Latent structured phenomenon

Exemples for *Process model* : Gauss+Matern, powered exp…

$$Y(s)\big|Z,\theta,X\sim\ell(Z,\theta,X)$$

~~$Y(s)=\mu(s)+Z(s)+\varepsilon(s)$~~

Exemples for *data model* : Bernoulli, Poisson…

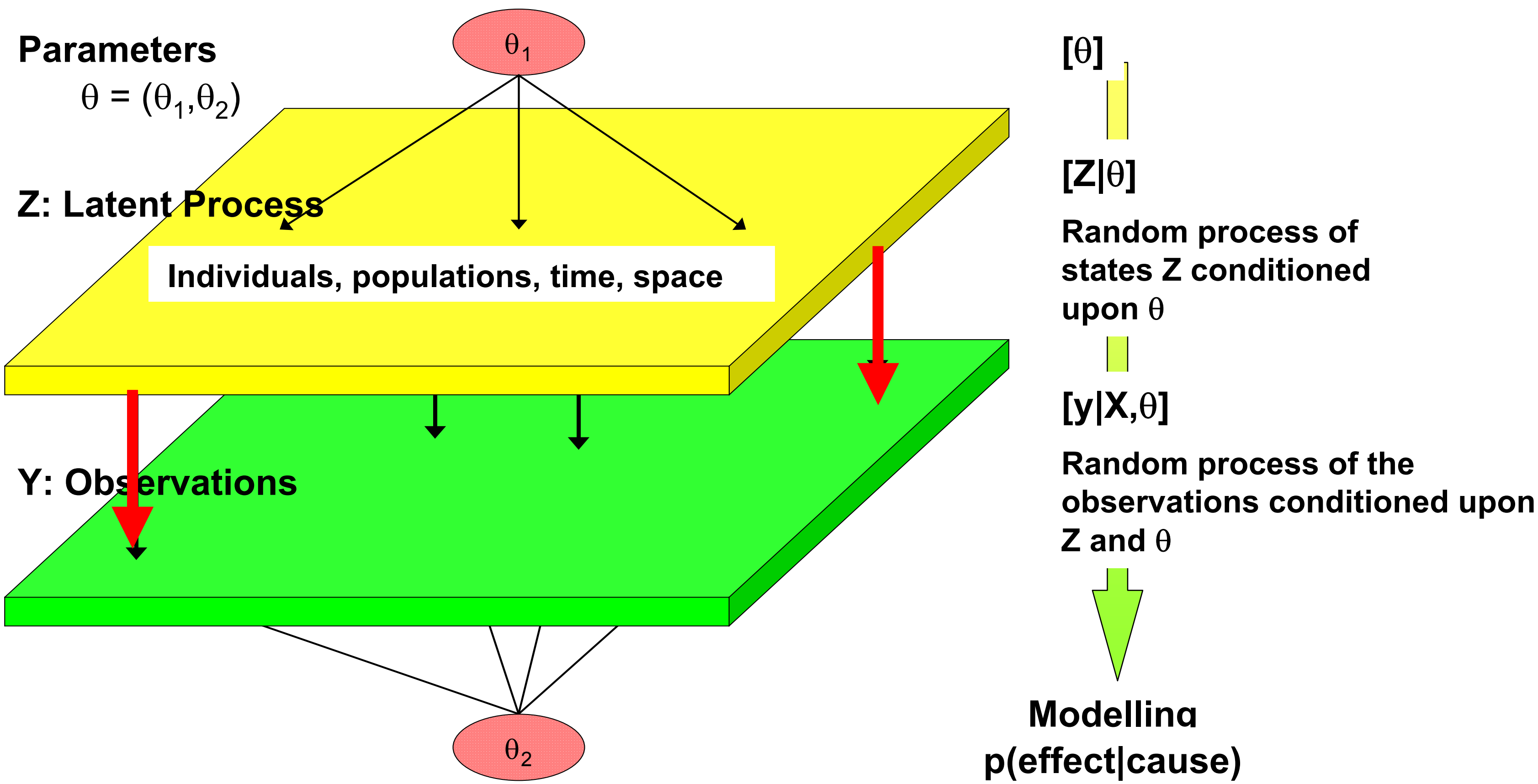
$$Y(s)\big|Z,\theta,X\sim dBern(p(s));\log\left(\frac{p(s)}{1-p(s)}\right)=X\beta+Z$$

Given (μ,Z) the Y(s)'s are cond't independent

Conditionnal (hierarchical) modelling strategy in HBM

- **Modelling : Capacity to accommodate complexity**

Parameters, states process and observations process can be modelled independently



What ‘s next in chapter 5?

How to describe non-stationarity (5.3)

Play it again with areal data (lattice)…

THANKS!