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KINETIC THEORY OF PLASMAS: TRANSLATIONAL ENERGY

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In the present study, we derive from kinetic theory a unified fluid model for multicomponent plasmas by accounting for the electromagnetic field influence. We deal with a possible thermal nonequilibrium of the translational energy of the particles, neglecting their internal energy and reactive collisions. Given the strong disparity of mass between the electrons and heavy particles, such as molecules, atoms, and ions, we conduct a dimensional analysis of the Boltzmann equation and introduce a scaling based on a multiscale perturbation parameter equal to the square root of the ratio of the electron mass to a characteristic heavy-particle mass. We then generalize the Chapman–Enskog method, emphasizing the role of the perturbation parameter on the collisional operator, the streaming operator, and the collisional invariants of the Boltzmann equation. The system is examined at successive orders of approximation, each corresponding to a physical timescale. At the highest approximation order investigated, the multicomponent Navier–Stokes regime is reached for the heavy particles and is coupled to first-order drift-diffusion equations for the electrons. The transport coefficients are then calculated in terms of bracket operators whose mathematical structure allows for positivity properties to be determined and Onsager's reciprocal relations to hold. The transport coefficients exhibit an anisotropic behavior when the magnetic field is strong enough. We also give a

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complete description of the Kolesnikov effect, i.e. the crossed contributions to the mass and energy transport fluxes coupling the electrons and heavy particles. Finally, the first and second laws of thermodynamics are proved to be satisfied by deriving a total energy equation and an entropy equation. Moreover, the purely convective system of equations is shown to be hyperbolic.

Keywords: Kinetic theory; plasmas in thermal nonequilibrium; conservation equations; multicomponent transport properties.

AMS Subject Classification: 82C40, 76X05, 41A60

1. Introduction

Plasmas are ionized gas mixtures, either magnetized or not, that have many practical applications. For instance, lightning is a well-known natural plasma and has been studied for many years.[4] A second application is encountered in hypersonic flows; when a spacecraft enters into a planetary atmosphere at hypervelocity, the gas temperature and pressure strongly rise through a shock wave, consequently, dissociation and ionization of the gas particles occur in the shock layer. Atmospheric entry plasmas are reproduced in dedicated wind-tunnels such as plasmatrons, arc-jet facilities, and shock-tubes.[45,49,53] A third example was found about two decades ago, when large-scale electrical discharges were discovered in the mesosphere and lower ionosphere above large thunderstorms; these plasmas are now commonly referred to as sprites.[8,46] Fourth, discharges at atmospheric pressure have received renewed attention in recent years due to their ability to enhance the reactivity of a variety of gas flows for applications ranging from surface treatment to flame stabilization and ignition (see Refs. 44, 48, 51 and 54 and references cited therein). Fifth, Hall thrusters are being developed to replace chemical systems for many on-orbit propulsion tasks on communications and exploration spacecraft.[3,9] Finally, two important applications of magnetized plasmas are laboratory thermonuclear fusion[6,50] and the magnetic reconnection phenomenon in astrophysics.[56]

Depending on the magnitude of the ratio of the reference particle mean free path to the system characteristic length (Knudsen number), two different approaches are generally followed to describe the transport of mass, momentum, and energy in a plasma[5]: either a particle approach at high values of the Knudsen number (solution to the Boltzmann equation using Monte Carlo methods), or a fluid approach at low values (solution to macroscopic conservation equations by means of computational fluid dynamics methods). In this work, we study plasmas that can be described by a fluid approach, which encompasses most of the above-mentioned applications. In this case, kinetic theory can be used to obtain the governing conservation equations and transport fluxes. Hence, closure of the problem is realized at the microscopic level by determining from experimental measurements either the potentials of interaction between the gas particles or the cross-sections, allowing for the transport coefficients to be computed.

A complete model of plasmas should allow for the following physical phenomena to be described

- Thermal nonequilibrium of the translational energy,
- Influence of the electromagnetic field,
- Occurrence of reactive collisions,
- Excitation of internal degrees of freedom.

So far, no such unified model has been derived by means of kinetic theory. Besides, a derivation of the mathematical structure of the conservation equations also appears to be crucial in the design of the associated numerical methods. In the present study, we investigate based on our previous work the thermal nonequilibrium of the translational energy[40] and the influence of the magnetic field.[28] We generalize the Chapman–Enskog method within the context of a dimensional analysis of the Boltzmann equation, emphasizing the role of a multiscale perturbation parameter on the collisional operator, the streaming operator, and the collisional invariants of the Boltzmann equation. Then, we obtain macroscopic equations eventually leading to a sound entropy structure. Moreover, the purely convective system of equations is shown to be hyperbolic. Let us now describe in more detail how these issues are currently addressed in the literature.

First, a multiscale analysis is essential to solve the Boltzmann equation governing the velocity distribution functions. We recall that a fluid can be described in the continuum limit provided that the Knudsen number is small. In the case of plasmas, a thermal nonequilibrium may occur between the velocity distribution functions of the electrons and heavy particles (atoms, molecules, and ions), given the strong disparity of mass between both types of species. The square root of the ratio of the electron mass to a characteristic heavy-particle mass represents an additional small parameter to be accounted for in the derivation of an asymptotic solution to the Boltzmann equation. The literature abounds with expressions for the scaling for the perturbative solution method. For instance, significant results are given in Refs. 15, 19, 24, 35 and 57. Petit and Darrozes[47] have suggested that the only sound scaling is obtained by means of a dimensional analysis of the Boltzmann equation. Moreover, they have deduced that the Knudsen number is proportional to the square root of the electron heavy-particle mass ratio. Subsequently, Degond and Lucquin[20,21] have established a formal theory of epochal relaxation based on such a scaling. In their study, the mean velocity of the electrons was shown to vanish in an inertial reference frame. Moreover, the heavy-particle diffusive fluxes were scarcely dealt with since their work is restricted to a single type of heavy particles, and thus no multicomponent diffusion was to be found. In such a simplified context, the details of the interaction between the heavy particles and electrons degenerate and the positivity of the entropy production is straightforward. We will establish a theory based on a multiscale analysis for multicomponent plasmas (which includes the single heavy-particle case) where the mean electron velocity is the mean heavy-particle velocity in the absence of external forces. As an alternative, Magin and Degrez[40] have also proposed a model for multicomponent plasmas in a hydrodynamic velocity frame. They have applied a multiscale analysis to the derivation of the multicomponent transport fluxes and coefficients. However, since the hydrodynamic velocity is used to define the reference frame instead of the mean heavy-particle velocity, the Chapman–Enskog method requires additional low order terms in the integral equation for the electron perturbation function to ensure mass conservation. Finally, we also desire that the development of thermal equilibrium models shall always be obtained as a particular case of the general theory.

Second, the magnetic field induces anisotropic transport fluxes when the electron collision frequency is lower than the electron cyclotron frequency of gyration around the magnetic lines. Braginskii[10] has studied the case of fully ionized plasmas composed of one single ion species. Recently, Bobrova *et al.* have generalized the previous work to multicomponent plasmas. However, the scaling used in these studies does not comply with a dimensional analysis of the Boltzmann equation. Lucquin[37,38] has investigated magnetized plasmas in this framework. Nevertheless, the same limitation is found for the diffusive fluxes as in Refs. 20 and 21. Finally, Giovangigli and Graille[28] have studied the Enskog expansion of magnetized plasmas and obtained macroscopic equations together with expressions for transport fluxes and coefficients. Unfortunately, the difference of mass between the electrons and heavy particles is not accounted for in their work.

Third, plasmas are strongly reactive gas mixtures. The kinetic mechanism comprises numerous reactions[12]: dissociation of molecules by electron and heavy-particle impact, three-body recombination, ionization by electron and heavy-particle impact, associative ionization, dissociative recombination, radical reactions, charge exchange . . . Giovangigli and Massot[29] have derived a formal theory of chemically reacting flows for the case of neutral gases. Subsequently, Giovangigli and Graille[28] have studied the case of ionized gases. We recall that their scaling does not take into account the mass disparity between electrons and heavy particles. Besides, in chemical equilibrium situations, a long-standing theoretical debate in the literature deals with nonuniqueness of the two-temperature Saha equation for quasi-equilibrium composition. Recently, Giordano and Capitelli[30] have emphasized the importance of the physical constraints imposed on the system by using a thermodynamic approach. A derivation based on kinetic theory should further improve the understanding of the problem. Choquet *et al.*[16,17] have already studied the case of ionization reactions by electron impact.

Fourth, molecules rotate and vibrate, and moreover, the electronic energy levels of atoms and molecules can be excited. Generally, the rotational energy mode is considered to be fully excited above a few Kelvins. In a plasma environment, the vibrational and electronic energy modes are also significantly excited. The detailed treatment of the internal degrees of freedom is however beyond the scope of the present work where we will only tackle the translational energy in the context of thermal nonequilibrium. The reader is thus referred to the specialized literature for the treatment of the internal energy.[11,39,43]

Fifth, the development of numerical methods to solve conservation equations relies on the identification of their intrinsic mathematical structure. For instance, the system of conservation equations of mass, momentum, and energy is known to be nonconservative for thermal nonequilibrium ionized gases. Therefore, this formulation is not suitable for numerical approximations of discontinuous solutions. Coquel and Marmignon[18] have derived a well-posed conservative formulation based on a phenomenological approach. Nevertheless, their derivation is not consistent with a scaling based on a dimensional analysis. We will show that kinetic theory, based on first principles, naturally allows for an adequate mathematical structure to be obtained, as opposed to the phenomenological approach.

In this work, we propose to derive the multicomponent plasma conservation equations of mass, momentum, and energy, as well as the expressions for the associated multicomponent transport fluxes and coefficients. The multicomponent Navier–Stokes regime is reached for the heavy particles and is coupled to first-order drift-diffusion equations for the electrons. We deal here with first-order equations for electrons, thus one order beyond the expansion commonly investigated in the literature. The derivation relies on kinetic theory and is based on the ansatz that the particles of the plasma are inert and only possess translational degrees of freedom. The electromagnetic field influence is accounted for. In Sec. 2, we express the Boltzmann equation in a noninertial reference frame. We show that the mean heavy-particle velocity is a suitable choice for the reference frame velocity. This step is essential to establish a formalism where the electrons follow the bulk movement of the plasma. Then, we define the reference quantities of the system in order to derive the scaling of the Boltzmann equation from a dimensional analysis. The multiscale aspect occurs in both the streaming operator and collision operator of the Boltzmann equation. Consequently, Sec. 3 is devoted to the scaling of the partial collision operators between unlike particles. We determine the collisional invariants associated with respectively the electrons and the heavy particles. In Sec. 4, we use a Chapman–Enskog method to derive macroscopic conservation equations. The system is examined at successive orders of approximation, each corresponding to a physical timescale. For that purpose, scalar products and linearized collision operators are introduced. The global expressions for the macroscopic fluid equations are then provided up to Navier–Stokes equations for the heavy particles and first-order drift-diffusion equations for the electrons. We also prove that our choice of reference frame is essential in order to reach this expansion level. In Sec. 5, we establish the formal existence and uniqueness of a solution to the Boltzmann equation. The multicomponent transport coefficients are then calculated in terms of bracket operators whose mathematical structure allows for the sign of the transport coefficients to be determined, including for the Kolesnikov effect, or the crossed contributions to the mass and energy transport fluxes coupling the electrons and heavy particles. The explicit expressions for the transport coefficients can be obtained by means of a Galerkin spectral method[14]; this is not treated in the present study. Finally, in Sec. 6, the first and second laws of thermodynamics are proved to be satisfied by deriving a total energy equation and an entropy equation. Moreover, Onsager's reciprocal relations hold between the transport coefficients. Then, we identify, from a fluid standpoint, the mathematical structure of the purely convective system of macroscopic equations. Hence, we demonstrate that kinetic theory can be used as a guideline in the derivation of the macroscopic conservation equations as well as in the design of the associated numerical methods.

Beyond the obvious interest from the point of view of applications and design of numerical schemes, the present study also yields a formal kinetic theory of mixtures of separate masses, where the light species obey a scaling of the Boltzmann equation characteristic of neutral gases in the low Mach number limit (yielding parabolic macroscopic equations) whereas the heavy species obey a scaling characteristic of neutral gases in the compressible gas dynamics regime (yielding hyperbolic macroscopic equations). The original treatment of the purely parabolic and hyperbolic scalings was first provided by Bardos *et al.*[2] These scalings, quite standard, can be used for various asymptotics such as the Vlasov–Navier–Stokes equations in different regimes investigated by Goudon *et al.*[32,33] A rigorous derivation of a set of macroscopic equations in the situation where the hyperbolic and parabolic scalings are entangled in the same problem is an original result obtained in the present work.

2. Boltzmann's Equation

2.1. *Assumptions*

(1) The plasma is described by the kinetic theory of gases based on classical mechanics, provided that: (a) The mean distance between the gas particles $1/(n^0)^{1/3}$ is larger than the thermal de Broglie wavelength, where n^0 is a reference number density,[34] (b) The square of the ratio of the electron thermal speed V_{e}^0 to the speed of light is small.
(2) Reactive collisions and particle internal energy are not accounted for.
(3) The particle interactions are modeled as binary encounters by means of a Boltzmann collision operator, provided that: (a) The gas is sufficiently dilute, i.e. the mean distance between the gas particles $1/(n^0)^{1/3}$ is larger than the particle interaction distance $(\sigma^0)^{1/2}$, where σ^0 is a reference differential cross-section common to all species, (b) The plasma parameter, quantity proportional to the number of electrons in a sphere of radius equal to the Debye length, is supposed to be large. Hence, multiple charged particle interactions are treated as equivalent binary collisions by means of a Coulomb potential screened at the Debye length.[1,23]
(4) The plasma is composed of electrons and a multicomponent mixture of heavy particles (atoms, molecules, and ions). The ratio of the electron mass m_{e}^0 to a

characteristic heavy-particle mass m_h^0 is such that the nondimensional number $\varepsilon = \sqrt{m_e^0/m_h^0}$ is small.

(5) The pseudo-Mach number, defined as a reference hydrodynamic velocity divided by the heavy-particle thermal speed, $M_h = v^0/V_h^0$, is supposed to be of order one.

(6) The macroscopic timescale t^0 is assumed to be comparable with the heavy-particle kinetic timescale t_h^0 divided by ε. The macroscopic length scale is based on a reference convective length $L^0 = v^0 t^0$.

(7) The reference electrical and thermal energies of the system are of the same order of magnitude.

The mean free path l^0 and macroscopic length scale L^0 allow for the Knudsen number to be defined $Kn = l^0/L^0$. It will be shown that this quantity is small, provided that assumptions (4)–(6) are satisfied. Therefore, a continuum description of the system is deemed to be possible.

2.2. *Inertial reference frame*

The choice of a proper reference frame will prove to be essential in the following multiscale analysis. Two such frames are commonly used in the literature. Degond and Lucquin[20,21] work in the inertial reference frame, as do Ferziger and Kaper.[26] The second reference frame is presented in the following section. Considering assumptions (1)–(3), the temporal evolution of the velocity distribution function $f_i^\star$ of the plasma particles i is governed in the phase space $(\mathbf{x}^\star, \mathbf{c}_i^\star)$ by the Boltzmann equation[13,26]

$$\mathscr{D}_i^\star(f_i^\star) = \sum_{j\in \mathrm{S}} \mathcal{J}_{ij}^\star(f_i^\star, f_j^\star), \quad i \in \mathrm{S}, \tag{2.1}$$

where symbol S is the set of indices of the gas species. Dimensional quantities are denoted by the superscript *. The streaming operator reads

$$\mathscr{D}_i^\star(f_i^\star) = \partial_{t^\star} f_i^\star + \mathbf{c}_i^\star \cdot \boldsymbol{\partial}_{\mathbf{x}^\star} f_i^\star + \frac{q_i^\star}{m_i^\star}(\mathbf{E}^\star + \mathbf{c}_i^\star \wedge \mathbf{B}^\star) \cdot \boldsymbol{\partial}_{\mathbf{c}_i^\star} f_i^\star, \quad i \in \mathrm{S}, \tag{2.2}$$

in an inertial reference frame. Symbol $t^\star$ stands for time, $\mathbf{E}^\star$, the electric field, $\mathbf{B}^\star$, the magnetic field, $m_i^\star$, the mass of the particle i, and $q_i^\star$, its charge. The partial collision operator of particle j impacting on particle i reads

$$\mathcal{J}_{ij}^\star(f_i^\star, f_j^\star) = \int (f_i^{\star\prime} f_j^{\star\prime} - f_i^\star f_j^\star)|\mathbf{c}_i^\star - \mathbf{c}_j^\star| \sigma_{ij}^\star \mathrm{d}\boldsymbol{\omega} \mathrm{d}\mathbf{c}_j^\star, \quad i, j \in \mathrm{S}. \tag{2.3}$$

After collision, quantities are denoted by the superscript $'$. The differential cross-section $\sigma_{ij}^\star = \sigma_{ij}^\star[\mu_{ij}^\star|\mathbf{c}_i^\star - \mathbf{c}_j^\star|^2/(\mathrm{k_B}T^0), \boldsymbol{\omega}\cdot\mathbf{e}]$ depends on the relative kinetic energy of the colliding particles and the cosine of the angle between the unit vectors of relative velocities $\boldsymbol{\omega} = (\mathbf{c}_i^{\star\prime} - \mathbf{c}_j^{\star\prime})/|\mathbf{c}_i^{\star\prime} - \mathbf{c}_j^{\star\prime}|$ and $\mathbf{e} = (\mathbf{c}_i^\star - \mathbf{c}_j^\star)/|\mathbf{c}_i^\star - \mathbf{c}_j^\star|$. Quantity $\mu_{ij}^\star = m_i^\star m_j^\star/(m_i^\star + m_j^\star)$ is the reduced mass of the particle pair, T^0, a reference temperature, and k_B, Boltzmann's constant. Therefore, the differential cross-sections are symmetric with respect to their indices $i, j \in S$, i.e. $\sigma^\star_{ij} = \sigma^\star_{ji}$. The collision operator reads in a compact form

$$\mathcal{J}^\star_i = \sum_{j \in S} \mathcal{J}^\star_{ij}(f^\star_i, f^\star_j), \quad i \in S. \tag{2.4}$$

2.3. *Noninertial reference frame*

Sutton and Sherman,[52] as Chapman and Cowling,[14] have proposed a noninertial reference frame based on the hydrodynamic velocity

$$\rho^\star \mathbf{v}^\star = \sum_{j \in S} \int m^\star_j \mathbf{c}^\star_j f^\star_j \mathrm{d}\mathbf{c}^\star_j, \tag{2.5}$$

where the mixture mass density is defined as $\rho^\star = \sum_{j \in S} \rho^\star_j$. Symbol $\rho^\star_i = n^\star_i m^\star_i$ stands for the partial mass density, and $n^\star_i = \int f^\star_i \mathrm{d}\mathbf{c}^\star_i$, the partial number density. It is a convenient choice since it is the reference frame associated with the definition of the peculiar velocities

$$\mathbf{C}^{\mathbf{v}\star}_i = \mathbf{c}^\star_i - \mathbf{v}^\star, \quad i \in S, \tag{2.6}$$

induced from the usual momentum constraint. We infer from definition (2.5) that the global diffusion flux vanishes

$$\sum_{j \in S} \int m^\star_j \mathbf{C}^{\mathbf{v}\star}_j f^\star_j \mathrm{d}\mathbf{c}^\star_j = 0, \tag{2.7}$$

that is, the standard momentum constraint.

Given the strong disparity of mass between the electrons and heavy particles, a frame linked with the heavy particles appears to be a rather more natural choice for plasmas, as fully justified in the following detailed multiscale analysis. Thus, we define the mean electron velocity and mean heavy-particle velocity

$$\rho^\star_e \mathbf{v}^\star_e = \int m^\star_e \mathbf{c}^\star_e f^\star_e \mathrm{d}\mathbf{c}^\star_e, \quad \rho^\star_h \mathbf{v}^\star_h = \sum_{j \in H} \int m^\star_j \mathbf{c}^\star_j f^\star_j \mathrm{d}\mathbf{c}^\star_j, \tag{2.8}$$

where the heavy-particle mass density reads $\rho^\star_h = \sum_{j \in H} \rho^\star_j$. Symbol H stands for the set of indices of heavy particles. In this $\mathbf{v}^\star_h$ frame, the free electrons interact with heavy particles whose global movement is frozen in space. A similar viewpoint is commonly adopted in the quantum theory of molecules when the Born–Oppenheimer approximation is used to study the motion of the bound electrons about the nuclei.[7] Based on the following definition of peculiar velocities

$$\mathbf{C}^\star_i = \mathbf{c}^\star_i - \mathbf{v}^\star_h, \quad i \in S, \tag{2.9}$$

the heavy-particle diffusion flux vanishes

$$\sum_{j \in H} \int m^\star_j \mathbf{C}^\star_j f^\star_j \mathrm{d}\mathbf{c}^\star_j = 0. \tag{2.10}$$

For now, we defer the choice of the reference velocity. We use the symbol $\mathbf{u}^\star$ to define the peculiar velocities $\mathbf{C}_i^{\mathbf{u}\star} = \mathbf{c}_i^\star - \mathbf{u}^\star$, $i \in \mathrm{S}$. Then, the Boltzmann equation is expressed in a frame moving at $\mathbf{u}^\star$ velocity by means of this change of variables. Hence, the streaming operator (2.2) is transformed into the expression

$$\mathscr{D}_i^\star(f_i^\star) = \partial_{t^\star} f_i^\star + (\mathbf{C}_i^{\mathbf{u}\star} + \mathbf{u}^\star) \cdot \boldsymbol{\partial}_{\mathbf{x}^\star} f_i^\star + \frac{q_i^\star}{m_i^\star}[\mathbf{E}^\star + (\mathbf{C}_i^{\mathbf{u}\star} + \mathbf{u}^\star) \wedge a\mathbf{B}^\star] \cdot \boldsymbol{\partial}_{\mathbf{C}_i^{\mathbf{u}\star}} f_i^\star$$
$$- \frac{\mathrm{D}\mathbf{u}^\star}{\mathrm{D}t^\star} \cdot \boldsymbol{\partial}_{\mathbf{C}_i^{\mathbf{u}\star}} f_i^\star - (\boldsymbol{\partial}_{\mathbf{C}_i^{\mathbf{u}\star}} f_i^\star \otimes \mathbf{C}_i^{\mathbf{u}\star}) : \boldsymbol{\partial}_{\mathbf{x}^\star} \mathbf{u}^\star, \tag{2.11}$$

where $\mathrm{D}/\mathrm{D}t^\star = \partial_{t^\star} + \mathbf{u}^\star \cdot \boldsymbol{\partial}_{\mathbf{x}^\star}$. The partial collision operator (2.3) is found to be

$$\mathcal{J}_{ij}^\star\left(f_i^\star, f_j^\star\right) = \int (f_i'^\star f_j'^\star - f_i^\star f_j^\star)|\mathbf{C}_i^{\mathbf{u}\star} - \mathbf{C}_j^{\mathbf{u}\star}|\sigma_{ij}^\star \mathrm{d}\boldsymbol{\omega}\mathrm{d}\mathbf{C}_j^{\mathbf{u}\star}, \quad i, j \in \mathrm{S}. \tag{2.12}$$

In a noninertial reference frame, the velocity distribution function $f_i^\star$, the differential cross-section $\sigma_{ij}^\star = \sigma_{ij}^\star[\mu_{ij}^\star|\mathbf{C}_i^{\mathbf{u}\star} - \mathbf{C}_j^{\mathbf{u}\star}|^2/(\mathrm{k_B}T^0),\ \boldsymbol{\omega}\cdot\mathbf{e}]$, as well as both the unit vectors $\boldsymbol{\omega} = (\mathbf{C}_i^{\mathbf{u}\prime\star} - \mathbf{C}_j^{\mathbf{u}\prime\star})/|\mathbf{C}_i^{\mathbf{u}\prime\star} - \mathbf{C}_j^{\mathbf{u}\prime\star}|$ and $\mathbf{e} = (\mathbf{C}_i^{\mathbf{u}\star} - \mathbf{C}_j^{\mathbf{u}\star})/|\mathbf{C}_i^{\mathbf{u}\star} - \mathbf{C}_j^{\mathbf{u}\star}|$ depend on the peculiar velocities. For simplicity, the notation is the same as for the inertial reference frame, where the previous quantities depend on the absolute velocities.

2.4. *Collisional invariants*

We now define collisional invariants in a reference frame moving at velocity $\mathbf{u}$.

Definition 2.1. The space of scalar collisional invariants $\mathcal{I}^{\mathbf{u}\star}$ is spanned by the following elements

$$\begin{cases} \psi^{\mathbf{u},j\star} = (m_i^\star \delta_{ij})_{i\in\mathrm{S}}, & j \in \mathrm{S}, \\ \psi^{\mathbf{u},n^\mathrm{S}+\nu\star} = (m_i^\star C_{i\nu}^{\mathbf{u}\star})_{i\in\mathrm{S}}, & \nu \in \{1,2,3\}, \\ \psi^{\mathbf{u},n^\mathrm{S}+4\star} = \left(\dfrac{1}{2} m_i^\star \mathbf{C}_i^{\mathbf{u}\star}\cdot\mathbf{C}_i^{\mathbf{u}\star}\right)_{i\in\mathrm{S}}, & \end{cases} \tag{2.13}$$

where symbol n^S denotes the cardinality of the set of species S.

We introduce the scalar product

$$\langle\!\langle \xi^\star, \zeta^\star \rangle\!\rangle^{\mathbf{u}\star} = \sum_{j\in\mathrm{S}} \int \xi_j^\star \odot \bar{\zeta}_j^{\,\star} \mathrm{d}\mathbf{C}_j^{\mathbf{u}\star}, \tag{2.14}$$

for families $\xi^\star = (\xi_i^\star)_{i\in\mathrm{S}}$ and $\zeta^\star = (\zeta_i^\star)_{i\in\mathrm{S}}$. The symbol $\odot$ stands for the fully contracted product in space, and the symbol $^-$ for the conjugate transpose operation. The collision operator $\mathcal{J}^\star = (\mathcal{J}_i^\star)_{i\in\mathrm{S}}$ defined in Eq. (2.4) obeys the following property.

Property 2.1. The collision operator $\mathcal{J}^\star$ is orthogonal to the space of collisional invariants $\mathcal{I}^{\mathbf{u}\star}$, i.e. $\langle\!\langle \psi^{\mathbf{u},l\star}, \mathcal{J}^\star \rangle\!\rangle^{\mathbf{u}\star} = 0$, for all $l \in \{1, \ldots, n^\mathrm{S} + 4\}$.

Proof. The projection of the collision operator $\mathcal{J}^\star$ onto $\psi^{\mathbf{u},l\star}$, $l \in \{1,\ldots,n^{\mathrm{S}}+4\}$, is shown to be

$$\frac{1}{4}\sum_{i,j\in\mathrm{S}}\int(f_i^{\star\prime}f_j^{\star\prime}-f_i^\star f_j^\star)(\psi_i^{\mathbf{u},l\star}+\psi_j^{\mathbf{u},l\star}-\hat{\psi}_i^{\mathbf{u},l'\star}-\hat{\psi}_j^{\mathbf{u},l'\star})$$
$$\times|\mathbf{C}_i^{\mathbf{u}\star}-\mathbf{C}_j^{\mathbf{u}\star}|\sigma_{ij}^\star \mathrm{d}\boldsymbol{\omega}\mathrm{d}\mathbf{C}_i^{\mathbf{u}\star}\mathrm{d}\mathbf{C}_j^{\mathbf{u}\star},$$

see for instance Chapman and Cowling.[14] This expression vanishes for all $l \in \{1,\ldots,n^{\mathrm{S}}+4\}$. □

Finally, the macroscopic properties can be expressed by means of the scalar product of the distribution functions and the collisional invariants

$$\begin{cases}\langle\langle f^\star,\psi^{i\star}\rangle\rangle^{\mathbf{u}*}=\rho_i^\star, & i\in\mathrm{S},\\ \langle\langle f^\star,\psi^{n^{\mathrm{S}}+\nu\star}\rangle\rangle^{\mathbf{u}*}=\rho^\star(v_\nu^\star-u_\nu^\star), & \nu\in\{1,2,3\},\\ \langle\langle f^\star,\psi^{n^{\mathrm{S}}+4\star}\rangle\rangle^{\mathbf{u}\star}=\rho^\star e^{\mathbf{v}\star}+\dfrac{1}{2}\rho^\star(\mathbf{v}^\star-\mathbf{u}^\star)\cdot(\mathbf{v}^\star-\mathbf{u}^\star), & \end{cases}$$

where quantity $e^{\mathbf{v}\star}$ stands for the gas thermal energy per unit mass in the hydrodynamic velocity frame.

2.5. *Dimensional analysis*

A sound scaling of the Boltzmann equation is deduced from a dimensional analysis inspired by Petit and Darrozes.[47] First, reference quantities are introduced in Table 1. The characteristic temperature, number density, differential cross-section, mean free path, macroscopic timescale, hydrodynamic velocity, macroscopic length,

Table 1. Reference quantities.

	Common to all species	
Temperature	T^0	
Number density	n^0	
Differential cross-section	σ^0	
Mean free path	l^0	
Macroscopic timescale	t^0	
Hydrodynamic velocity	v^0	
Macroscopic length	L^0	
Electric field	E^0	
Magnetic field	B^0	
	Electrons	Heavy particles
Mass	m_{e}^0	m_h^0
Thermal speed	V_{e}^0	V_h^0
Kinetic timescale	t_{e}^0	t_h^0

and electric and magnetic fields are assumed to be common to all species. Reference dimensional quantities are denoted by the superscript "0". The nondimensional number

$$\varepsilon = \sqrt{\frac{m_{\mathrm{e}}^0}{m_h^0}} \tag{2.15}$$

quantifies the ratio of the electron mass to a reference heavy-particle mass. According to assumption (4), the value of ε is small. Consequently, electrons exhibit a larger thermal speed than that of heavy particles

$$V_{\mathrm{e}}^0 = \sqrt{\frac{\mathrm{k_B}T^0}{m_{\mathrm{e}}^0}}, \quad V_h^0 = \sqrt{\frac{\mathrm{k_B}T^0}{m_h^0}} = \varepsilon V_{\mathrm{e}}^0. \tag{2.16}$$

Moreover, the electron and heavy-particle temperatures may be distinct, provided that Eq. (2.16) does not fail to describe the order of magnitude of the thermal speeds. The differential cross-sections are of the same order of magnitude σ^0. Hence, the characteristic mean free path $l^0 = 1/(n^0\sigma^0)$ is found to be identical for all species. As a result, the kinetic timescale, or relaxation time of a distribution function towards its respective quasi-equilibrium state, is lower for electrons than for heavy particles

$$t_{\mathrm{e}}^0 = \frac{l^0}{V_{\mathrm{e}}^0}, \quad t_h^0 = \frac{l^0}{V_h^0} = \frac{t_{\mathrm{e}}^0}{\varepsilon}. \tag{2.17}$$

Assumption (6) states that the macroscopic timescale reads

$$t^0 = \frac{t_h^0}{\varepsilon}. \tag{2.18}$$

It is shown in Sec. 4 that this quantity corresponds to the average time during which electrons and heavy particles exchange their energy through encounters. In addition, the macroscopic temporal and spatial scales are linked by the expression

$$L^0 = v^0 t^0, \tag{2.19}$$

where the hydrodynamic velocity is determined by the pseudo-Mach number $M_h = v^0/V_h^0$. Given assumption (5), the pseudo-Mach number is of order one. Hence, the Knudsen number

$$Kn = \frac{l^0}{L^0} = \frac{\varepsilon}{M_h}, \tag{2.20}$$

is small, due to our choice of macroscopic and temporal scales, leading to a continuum description of the gas. Finally, following assumption (7), the reference electric field is such that

$$q^0 E^0 L^0 = \mathrm{k_B}T^0. \tag{2.21}$$

The intensity of the magnetic field is governed by the Hall numbers of the electrons and heavy particles

$$\beta_{\mathrm{e}} = \frac{q^0 B^0}{m_{\mathrm{e}}^0} t_{\mathrm{e}}^0 = \varepsilon^{1-b}, \quad \beta_h = \frac{q^0 B^0}{m_h^0} t_h^0 = \varepsilon \beta_{\mathrm{e}}, \tag{2.22}$$

defined as the Larmor frequencies, $q^0 B^0/m_{\mathrm{e}}$ for the electrons and $q^0 B^0/m_h^0$ for the heavy particles, multiplied by their corresponding kinetic timescale. The magnetic field is assumed to be proportional to a power of ε by means of an integer $b \leq 1$. The physical interpretation of the b parameter appears in Sec. 5.5.

The dimensional analysis can be summarized as follows: (a) Two spatial scales were introduced, one spatial scale at the microscopic level and one spatial scale at the macroscopic level, they are related by Eq. (2.20); (b) Whereas three temporal scales were defined in Eq. (2.17), two timescales at the microscopic level, respectively for the electrons and for the heavy particles, and one timescale at the macroscopic level, given in Eq. (2.18), common to all species.

Nondimensional variables are based on the reference quantities. They are denoted by dropping the superscript *. In particular, one has the following expressions for the particle velocities

$$\mathbf{c}_{\mathrm{e}}^\star = V_{\mathrm{e}}^0 \mathbf{c}_{\mathrm{e}}, \quad \mathbf{c}_i^\star = V_h^0 \mathbf{c}_i, \ \ i \in \mathrm{H}. \tag{2.23}$$

The reference hydrodynamic velocity, mean electron velocity, and mean heavy-particle velocity are equal to v^0. The hydrodynamic velocity defined in Eq. (2.5) is

$$(\rho_h + \varepsilon^2 \rho_{\mathrm{e}})\mathbf{v} = \rho_h \mathbf{v}_h + \varepsilon^2 \rho_{\mathrm{e}} \mathbf{v}_{\mathrm{e}}, \tag{2.24}$$

in terms of nondimensional variables, whereas the mean electron and heavy-particle velocities given in Eq. (2.8) read

$$\rho_{\mathrm{e}} M_h \mathbf{v}_{\mathrm{e}} = \frac{1}{\varepsilon} \int \mathbf{c}_{\mathrm{e}} f_{\mathrm{e}} \, \mathrm{d}\mathbf{c}_{\mathrm{e}}, \quad \rho_h M_h \mathbf{v}_h = \sum_{j \in \mathrm{H}} \int m_j \mathbf{c}_j f_j \, \mathrm{d}\mathbf{c}_j. \tag{2.25}$$

The peculiar velocities are given by the relations

$$\mathbf{C}_{\mathrm{e}}^{\mathbf{u}} = \mathbf{c}_{\mathrm{e}} - \varepsilon M_h \mathbf{u}, \quad \mathbf{C}_i^{\mathbf{u}} = \mathbf{c}_i - M_h \mathbf{u}, \ \ i \in \mathrm{H}. \tag{2.26}$$

Usually, they are associated with the momentum constraints of the mixture, so that the natural choice is $\mathbf{u} = \mathbf{v}$. In such a case, we get the following relation

$$\sum_{j \in \mathrm{H}} \int m_j \mathbf{C}_j^{\mathbf{v}} f_j \, \mathrm{d}\mathbf{C}_j^{\mathbf{v}} + \varepsilon \int \mathbf{C}_{\mathrm{e}}^{\mathbf{v}} f_{\mathrm{e}} \, \mathrm{d}\mathbf{C}_{\mathrm{e}}^{\mathbf{v}} = 0.$$

However, the hydrodynamic velocity of the mixture, electrons included, can also be expanded in the ε parameter and thus receives contributions at various ε orders in the Chapman–Enskog method. Since the reference frame should not depend on the expansion order, we could mimic the approach of Lucquin and Degond[20,21,37] and take $\mathbf{u} = 0$, which means working in the inertial reference frame. However, we follow a different path, not only by choosing the mean heavy-particle velocity as reference velocity, $\mathbf{u} = \mathbf{v}_h$, but also by defining the peculiar velocities based on this quantity, as opposed to Petit and Darrozes.[47] The rationale for such a choice is threefold: (a) The mean heavy-particle velocity $\mathbf{v}_h$ does not depend on ε while still being a perturbation of the hydrodynamic velocity $\mathbf{v}$ of the complete mixture up to second order in ε

$$(\rho_h + \varepsilon^2 \rho_\mathrm{e}) M_h (\mathbf{v} - \mathbf{v}_h) = \varepsilon \int \mathbf{C}_\mathrm{e}^{\mathbf{v}_h} f_\mathrm{e} \, \mathrm{d}\mathbf{C}_\mathrm{e}^{\mathbf{v}_h}, \tag{2.27}$$

since quantity $\int \mathbf{C}_\mathrm{e}^{\mathbf{v}_h} f_\mathrm{e} \, \mathrm{d}\mathbf{C}_\mathrm{e}^{\mathbf{v}_h}$ taken with f_e as a perturbation of a Maxwell–Boltzmann distribution will be of $\mathcal{O}(\varepsilon)$ in the Chapman–Enskog expansion presented in Sec. 4; (b) It will prove to be the natural reference frame in which the heavy particles thermalize in the context of the proposed multiscale analysis; (c) It will also prove to be the only available choice for electron thermalization and successive order solutions, thus making the proposed change of reference frame optimal and leading to a rigorous formalism as well as a simplified algebra. In the following, since there is no ambiguity, we will drop the $\mathbf{v}_h$ superscript in the use of the peculiar velocities $\mathbf{C}_\mathrm{e}^{\mathbf{v}_h}$ and $\mathbf{C}_i^{\mathbf{v}_h}$, $i \in \mathrm{H}$.

Consequently, the heavy-particle diffusion flux vanishes, as shown in Eq. (2.10)

$$\sum_{j \in \mathrm{H}} \int m_j \mathbf{C}_j f_j \, \mathrm{d}\mathbf{C}_j = 0. \tag{2.28}$$

We investigate the system at the macroscopic time $t^\star = t^0 t$ and macroscopic length $\mathbf{x}^\star = L^0 \mathbf{x}$. Thus, the Boltzmann equation (2.1) can be expressed, in nondimensional form, respectively for the electrons and heavy particles, as

$$\begin{aligned} &\partial_t f_\mathrm{e} + \frac{1}{\varepsilon M_h} (\mathbf{C}_\mathrm{e} + \varepsilon M_h \mathbf{v}_h) \cdot \boldsymbol{\partial}_\mathbf{x} f_\mathrm{e} + \varepsilon^{-(1+b)} q_\mathrm{e} \big[(\mathbf{C}_\mathrm{e} + \varepsilon M_h \mathbf{v}_h) \wedge \mathbf{B} \big] \cdot \boldsymbol{\partial}_{\mathbf{C}_\mathrm{e}} f_\mathrm{e} \\ &\quad + \left(\frac{1}{\varepsilon M_h} q_\mathrm{e} \mathbf{E} - \varepsilon M_h \frac{\mathrm{D}\mathbf{v}_h}{\mathrm{D}t} \right) \cdot \boldsymbol{\partial}_{\mathbf{C}_\mathrm{e}} f_\mathrm{e} - (\boldsymbol{\partial}_{\mathbf{C}_\mathrm{e}} f_\mathrm{e} \otimes \mathbf{C}_\mathrm{e}) {:} \boldsymbol{\partial}_\mathbf{x} \mathbf{v}_h = \frac{1}{\varepsilon^2} \mathcal{J}_\mathrm{e}, \end{aligned} \tag{2.29}$$

$$\begin{aligned} &\partial_t f_i + \frac{1}{M_h} (\mathbf{C}_i + M_h \mathbf{v}_h) \cdot \boldsymbol{\partial}_\mathbf{x} f_i + \varepsilon^{1-b} \frac{q_i}{m_i} [(\mathbf{C}_i + M_h \mathbf{v}_h) \wedge \mathbf{B}] \cdot \boldsymbol{\partial}_{\mathbf{C}_i} f_i \\ &\quad + \left(\frac{1}{M_h} \frac{q_i}{m_i} \mathbf{E} - M_h \frac{\mathrm{D}\mathbf{v}_h}{\mathrm{D}t} \right) \cdot \boldsymbol{\partial}_{\mathbf{C}_i} f_i - (\boldsymbol{\partial}_{\mathbf{C}_i} f_i \otimes \mathbf{C}_i) {:} \boldsymbol{\partial}_\mathbf{x} \mathbf{v}_h = \frac{1}{\varepsilon} \mathcal{J}_i, \quad i \in \mathrm{H}, \end{aligned} \tag{2.30}$$

where the collision operators read

$$\mathcal{J}_\mathrm{e} = \mathcal{J}_\mathrm{ee}(f_\mathrm{e}, f_\mathrm{e}) + \sum_{j \in \mathrm{H}} \mathcal{J}_{\mathrm{e}j}(f_\mathrm{e}, f_j), \tag{2.31}$$

$$\mathcal{J}_i = \frac{1}{\varepsilon} \mathcal{J}_{i\mathrm{e}}(f_i, f_\mathrm{e}) + \sum_{j \in \mathrm{H}} \mathcal{J}_{ij}(f_i, f_j), \quad i \in \mathrm{H}. \tag{2.32}$$

Let us emphasize that Eq. (2.29) for the electrons exhibits a similar scaling as that of the kinetic equation for neutral gases in the low Mach number limit (yielding parabolic macroscopic equations), whereas the scaling of Eq. (2.30) for the heavy particles is typical of that of the kinetic equation for neutral gases in the compressible gas dynamics regime (yielding hyperbolic macroscopic equations).[2] Therefore, the coupled system of kinetic Eqs. (2.29) and (2.30) combines the usual scalings and the mathematical structure of the resulting system of macroscopic equations has to be identified.

The collisional invariants (2.13) depend on the mass ratio as well, as shown in their following nondimensional form.

Definition 2.2. The space of scalar collisional invariants $\mathcal{I}$ is spanned by the following elements $\psi^l_\varepsilon = (\psi^l_{\mathrm{e}}, \psi^l_h)$, $l \in \{1, \ldots, n^{\mathrm{S}} + 4\}$, with

$$\begin{cases} \psi^j_{\mathrm{e}} = \varepsilon^2 \delta_{\mathrm{e}j}, & \psi^j_h = (m_i \delta_{ij})_{i\in\mathrm{H}}, & j \in \mathrm{S}, \\ \psi^{n^{\mathrm{S}}+\nu}_{\mathrm{e}} = \varepsilon C_{\mathrm{e}\nu}, & \psi^{n^{\mathrm{S}}+\nu}_h = (m_i C_{i\nu})_{i\in\mathrm{H}}, & \nu \in \{1,2,3\}, \\ \psi^{n^{\mathrm{S}}+4}_{\mathrm{e}} = \dfrac{1}{2}\mathbf{C}_{\mathrm{e}} \cdot \mathbf{C}_{\mathrm{e}}, & \psi^{n^{\mathrm{S}}+4}_h = \left(\dfrac{1}{2} m_i \mathbf{C}_i \cdot \mathbf{C}_i\right)_{i\in\mathrm{H}}. & \end{cases} \tag{2.33}$$

It is worth noticing the influence of the hierarchy of scales: whereas the scaling does not introduce any structural change in the mass and energy collisional invariants, the electron contribution disappears from the momentum collisional invariant vector in the limit of ε tending to zero. A similar behavior can be observed for the total mass; however, the single species collisional invariants are not affected.

For a family $\xi = (\xi_i)_{i\in\mathrm{S}}$, we introduce two separate contributions: ξ_{e}, concerning the electrons, and $\xi_h = (\xi_i)_{i\in\mathrm{H}}$, concerning the heavy particles. Hence, the scalar product between the families $\xi = (\xi_i)_{i\in\mathrm{S}}$ and $\zeta = (\zeta_i)_{i\in\mathrm{S}}$ defined in Eq. (2.14) is decomposed into a sum of partial scalar products with different scales

$$\langle\!\langle \xi, \zeta \rangle\!\rangle = \langle\!\langle \xi_{\mathrm{e}}, \zeta_{\mathrm{e}} \rangle\!\rangle_{\mathrm{e}} + \varepsilon^3 \langle\!\langle \xi_h, \zeta_h \rangle\!\rangle_h, \tag{2.34}$$

given by the expressions

$$\langle\!\langle \xi_{\mathrm{e}}, \zeta_{\mathrm{e}} \rangle\!\rangle_{\mathrm{e}} = \int \xi_{\mathrm{e}} \odot \bar{\zeta}_{\mathrm{e}} \mathrm{d}\mathbf{C}_{\mathrm{e}}, \quad \langle\!\langle \xi_h, \zeta_h \rangle\!\rangle_h = \sum_{j\in\mathrm{H}} \int \xi_j \odot \bar{\zeta}_j \mathrm{d}\mathbf{C}_j. \tag{2.35}$$

Finally, we introduce the collision operator $\mathcal{J}_\varepsilon = \left(\varepsilon \mathcal{J}_{\mathrm{e}}, \frac{1}{\varepsilon}\mathcal{J}_h\right)$, where Eq. (2.29) has been multiplied by a factor ε^3 corresponding to a scaling of the two Boltzmann equations coherent with the definition of the scalar product. Then, we derive the following property.

Property 2.2. The collision operator $\mathcal{J}_\varepsilon$ is orthogonal to the space of collisional invariants $\mathcal{I}$, i.e. $\langle\!\langle \psi^l_\varepsilon, \mathcal{J}_\varepsilon \rangle\!\rangle = 0$, for all $l \in \{1, \ldots, n^{\mathrm{S}} + 4\}$. Furthermore, the three types of pairwise interaction terms in $\langle\!\langle \psi_\varepsilon^l, \mathfrak{J}_\varepsilon \rangle\!\rangle$ separately vanish, i.e.

$$\langle\!\langle \psi_{\mathrm{e}}^l, \mathfrak{J}_{\mathrm{ee}} \rangle\!\rangle_{\mathrm{e}} = 0, \tag{2.36}$$

$$\sum_{j\in\mathrm{H}} \langle\!\langle \psi_{\mathrm{e}}^l, \mathfrak{J}_{\mathrm{e}j} \rangle\!\rangle_{\mathrm{e}} + \langle\!\langle \psi_h^l, \mathfrak{J}_{h\mathrm{e}} \rangle\!\rangle_h = 0, \tag{2.37}$$

$$\sum_{j\in\mathrm{H}} \langle\!\langle \psi_h^l, \mathfrak{J}_{hj} \rangle\!\rangle_h = 0, \tag{2.38}$$

respectively for the electron, electron heavy-particle, and heavy-particle interactions.

Proof. The projection of the collision operator $\mathfrak{J}_\varepsilon$ onto ψ_ε^l, $l \in \{1, \ldots, n^{\mathrm{S}} + 4\}$, is given by the expression

$$\langle\!\langle \psi_\varepsilon^l, \mathfrak{J}_\varepsilon \rangle\!\rangle = \varepsilon \langle\!\langle \psi_{\mathrm{e}}^l, \mathfrak{J}_{\mathrm{ee}} \rangle\!\rangle_{\mathrm{e}} + \varepsilon \sum_{j\in\mathrm{H}} \langle\!\langle \psi_{\mathrm{e}}^l, \mathfrak{J}_{\mathrm{e}j} \rangle\!\rangle_{\mathrm{e}} + \varepsilon \langle\!\langle \psi_h^l, \mathfrak{J}_{h\mathrm{e}} \rangle\!\rangle_h$$
$$+ \varepsilon^2 \sum_{j\in\mathrm{H}} \langle\!\langle \psi_h^l, \mathfrak{J}_{hj} \rangle\!\rangle_h.$$

The terms of this sum are examined by interaction pairs

$$\begin{aligned}
\langle\!\langle \psi_{\mathrm{e}}^l, \mathfrak{J}_{\mathrm{ee}} \rangle\!\rangle_{\mathrm{e}} &= \frac{1}{4} \int (f_{\mathrm{e}}' f_{\mathrm{e}1}' - f_{\mathrm{e}} f_{\mathrm{e}1})(\psi_{\mathrm{e}}^l + \psi_{\mathrm{e}1}^l - \psi_{\mathrm{e}}^{l\prime} - \psi_{\mathrm{e}1}^{l\prime}) \\
&\quad \times |\mathbf{C}_{\mathrm{e}} - \mathbf{C}_{\mathrm{e}1}| \sigma_{\mathrm{ee}1} \mathrm{d}\boldsymbol{\omega} \mathrm{d}\mathbf{C}_{\mathrm{e}} \mathrm{d}\mathbf{C}_{\mathrm{e}1}, \\
\sum_{j\in\mathrm{H}} \langle\!\langle \psi_{\mathrm{e}}^l, \mathfrak{J}_{\mathrm{e}j} \rangle\!\rangle_{\mathrm{e}} + \langle\!\langle \psi_h^l, \mathfrak{J}_{h\mathrm{e}} \rangle\!\rangle_h &= \frac{1}{2} \sum_{j\in\mathrm{H}} \int (f_{\mathrm{e}}' f_j' - f_{\mathrm{e}} f_j)(\psi_{\mathrm{e}}^l + \psi_j^l - \psi_{\mathrm{e}}^{l\prime} - \psi_j^{l\prime}) \\
&\quad \times |\mathbf{C}_{\mathrm{e}} - \varepsilon \mathbf{C}_j| \sigma_{\mathrm{e}j} \mathrm{d}\boldsymbol{\omega} \mathrm{d}\mathbf{C}_{\mathrm{e}} \mathrm{d}\mathbf{C}_j, \\
\sum_{j\in\mathrm{H}} \langle\!\langle \psi_h^l, \mathfrak{J}_{hj} \rangle\!\rangle_h &= \frac{1}{4} \sum_{i,j\in\mathrm{H}} \int \left(f_i' f_j' - f_i f_j\right) \left(\psi_i^l + \psi_j^l - \psi_i^{l\prime} - \psi_j^{l\prime}\right) \\
&\quad \times |\mathbf{C}_i - \mathbf{C}_j| \sigma_{ij} \mathrm{d}\boldsymbol{\omega} \mathrm{d}\mathbf{C}_i \mathrm{d}\mathbf{C}_j.
\end{aligned}$$

These expressions vanish and thus the sum $\langle\!\langle \psi_\varepsilon^l, \mathfrak{J}_\varepsilon \rangle\!\rangle = 0$. □

The multiscale analysis occurs at three levels: (a) In the kinetic equations (2.29) and (2.30); (b) In the collisional invariants (2.33) and thus in the conservation of the associated macroscopic quantities; (c) In the collision operators. Encounters between particles of the same type are dealt with as usual,[14] whereas the collision operators between unlike particles (electron heavy-particle interactions) depend on the ε parameter via their relative kinetic energy and velocity, and the vectors $\boldsymbol{\omega}$ and $\mathbf{e}$. The scaling of these operators is investigated in the following section.

3. Preliminary Results

3.1. *Electron heavy-particle collision dynamics*

The study of the electron heavy-particle collision dynamics yields the dependence of the peculiar velocities on the ε parameter. First, we express momentum conservation in terms of the peculiar velocities in the heavy-particle reference frame. Considering a collision of a heavy species, $i \in \mathrm{H}$, against an electron, the peculiar velocities after collision $\mathbf{C}_i'$ and $\mathbf{C}_\mathrm{e}'$ are related to their counterpart before collision $\mathbf{C}_i$ and $\mathbf{C}_\mathrm{e}$

$$\begin{aligned} \mathbf{C}_i' &= \frac{\varepsilon}{m_i+\varepsilon^2}\mathbf{C}_\mathrm{e} + \frac{m_i}{m_i+\varepsilon^2}\mathbf{C}_i + s\frac{\varepsilon}{m_i+\varepsilon^2}|\varepsilon\mathbf{C}_i - \mathbf{C}_\mathrm{e}|\omega, \quad i \in H, \\ \mathbf{C}_\mathrm{e}' &= \frac{\varepsilon^2}{m_i+\varepsilon^2}\mathbf{C}_\mathrm{e} + \frac{\varepsilon m_i}{m_i+\varepsilon^2}\mathbf{C}_i - s\frac{m_i}{m_i+\varepsilon^2}|\varepsilon\mathbf{C}_i - \mathbf{C}_\mathrm{e}|\omega, \end{aligned} \tag{3.1}$$

provided that the mean heavy-particle velocity is not modified by this single collision event. The direction of the relative velocities after collision is defined in their center of mass by

$$\boldsymbol{\omega} = s\frac{\varepsilon\mathbf{C}_i' - \mathbf{C}_\mathrm{e}'}{|\varepsilon\mathbf{C}_i' - \mathbf{C}_\mathrm{e}'|}.$$

Symbol s stands for an integer either equal to $+1$ for the collision operator $\mathcal{J}_{i\mathrm{e}}$, $i \in \mathrm{H}$, or -1 for $\mathcal{J}_{\mathrm{e}i}$, $i \in \mathrm{H}$. This notation is consistent with the definition of $\boldsymbol{\omega}$ in Eq. (2.12). We are now able to expand the crossed-collision operators.

3.2. *Expansion of the collision operator* $\mathcal{J}_{i\mathrm{e}}$

Dimensional analysis yields the following expression for the nondimensional collision operator $\mathcal{J}_{i\mathrm{e}}$, $i \in \mathrm{H}$,

$$\begin{aligned} \mathcal{J}_{i\mathrm{e}}(f_i, f_\mathrm{e})(\mathbf{C}_i) = \int \sigma_{ie}\left(|\boldsymbol{\gamma}_\mathrm{e}|^2, \boldsymbol{\omega}\cdot\frac{\boldsymbol{\gamma}_\mathrm{e}}{|\boldsymbol{\gamma}_\mathrm{e}|}\right)|\varepsilon\mathbf{C}_i - \mathbf{C}_\mathrm{e}| \\ \times [f_i(\mathbf{C}_i')f_\mathrm{e}(\mathbf{C}_\mathrm{e}') - f_i(\mathbf{C}_i)f_\mathrm{e}(\mathbf{C}_\mathrm{e})]\mathrm{d}\boldsymbol{\omega}\mathrm{d}\mathbf{C}_\mathrm{e}, \end{aligned} \tag{3.2}$$

where the relative kinetic energy and the vector $\mathbf{e}$ are expressed by means of the vector $\boldsymbol{\gamma}_\mathrm{e} = s(\varepsilon\mathbf{C}_i - \mathbf{C}_\mathrm{e})/(1+\varepsilon^2/m_i)^{1/2}$.

We then introduce the generalized momentum cross-section[14] in a thermal nonequilibrium context[42]

$$Q_{i\mathrm{e}}^{(l)}(|\boldsymbol{\gamma}_\mathrm{e}|^2) = 2\pi\int_0^\pi \sigma_{i\mathrm{e}}(|\boldsymbol{\gamma}_\mathrm{e}|^2, \cos\theta)(1-\cos^l\theta)\sin\theta\mathrm{d}\theta, \quad i \in \mathrm{H}, \quad l \in \mathbb{N}_0, \tag{3.3}$$

where symbol θ stands for the angle between the vectors $\boldsymbol{\omega}$ and $\mathbf{e}$. For $l = 1$, this cross-section represents the average momentum transferred in encounters from electrons to i heavy particles for a given value of the relative kinetic energy.

Theorem 3.1. *The collision operator* $\mathcal{J}_{i\mathrm{e}}$, $i \in \mathrm{H}$, *can be expanded in the form*

$$\mathcal{J}_{i\mathrm{e}}(f_i, f_\mathrm{e})(\mathbf{C}_i) = \varepsilon \mathcal{J}^1_{i\mathrm{e}}(f_i, f_\mathrm{e})(\mathbf{C}_i) + \varepsilon^2 \mathcal{J}^2_{i\mathrm{e}}(f_i, f_\mathrm{e})(\mathbf{C}_i) + \varepsilon^3 \mathcal{J}^3_{i\mathrm{e}}(f_i, f_\mathrm{e})(\mathbf{C}_i) + \mathcal{O}(\varepsilon^4). \tag{3.4}$$

The zeroth-order collision operator $\mathcal{J}^0_{i\mathrm{e}}(f_i, f_\mathrm{e})(\mathbf{C}_i)$, $i \in \mathrm{H}$, *vanishes. The first-order term* $\mathcal{J}^1_{i\mathrm{e}}$, $i \in \mathrm{H}$, *reads*

$$\mathcal{J}^1_{i\mathrm{e}}(f_i, f_\mathrm{e})(\mathbf{C}_i) = -\frac{1}{m_i}\partial_{\mathbf{C}_i} f_i(\mathbf{C}_i) \cdot \int Q^{(1)}_{i\mathrm{e}}(|\boldsymbol{\gamma}_\mathrm{e}|^2)|\boldsymbol{\gamma}_\mathrm{e}|\boldsymbol{\gamma}_\mathrm{e} f_\mathrm{e}(\boldsymbol{\gamma}_\mathrm{e})\mathrm{d}\boldsymbol{\gamma}_\mathrm{e}, \quad i \in \mathrm{H}. \tag{3.5}$$

The second-order term $\mathcal{J}^2_{i\mathrm{e}}$, $i \in \mathrm{H}$, *is found to be*

$$\begin{aligned}\mathcal{J}^2_{i\mathrm{e}}(f_i, f_\mathrm{e})(\mathbf{C}_i) = &-\frac{1}{m_i}\partial_{\mathbf{C}_i}(f_i(\mathbf{C}_i)\mathbf{C}_i) : \int Q^{(1)}_{i\mathrm{e}}(|\boldsymbol{\gamma}_\mathrm{e}|^2)|\boldsymbol{\gamma}_\mathrm{e}|\partial_{\mathbf{C}_\mathrm{e}} f_\mathrm{e}(\boldsymbol{\gamma}_\mathrm{e}) \otimes \boldsymbol{\gamma}_\mathrm{e}\mathrm{d}\boldsymbol{\gamma}_\mathrm{e}\\
&+\frac{1}{4m_i^2}\partial^2_{\mathbf{C}_i\mathbf{C}_i} f_i(\mathbf{C}_i) : \int Q^{(2)}_{i\mathrm{e}}(|\boldsymbol{\gamma}_\mathrm{e}|^2)|\boldsymbol{\gamma}_\mathrm{e}|(|\boldsymbol{\gamma}_\mathrm{e}|^2\mathbb{I} - 3\boldsymbol{\gamma}_\mathrm{e}\otimes\boldsymbol{\gamma}_\mathrm{e}) f_\mathrm{e}(\boldsymbol{\gamma}_\mathrm{e})\mathrm{d}\boldsymbol{\gamma}_\mathrm{e}\\
&+\frac{1}{m_i^2}\partial^2_{\mathbf{C}_i\mathbf{C}_i} f_i(\mathbf{C}_i) : \int Q^{(1)}_{i\mathrm{e}}(|\boldsymbol{\gamma}_\mathrm{e}|^2)|\boldsymbol{\gamma}_\mathrm{e}|\boldsymbol{\gamma}_\mathrm{e}\otimes\boldsymbol{\gamma}_\mathrm{e} f_\mathrm{e}(\boldsymbol{\gamma}_\mathrm{e})\mathrm{d}\boldsymbol{\gamma}_\mathrm{e}.\end{aligned} \tag{3.6}$$

Finally, the third-order term $\mathcal{J}^3_{i\mathrm{e}}$, $i \in \mathrm{H}$, *is given by*

$$\begin{aligned}&\mathcal{J}^3_{i\mathrm{e}}(f_i, f_\mathrm{e})(\mathbf{C}_i)\\
&\quad= \frac{1}{m_i}\partial_{\mathbf{C}_i}\left(\frac{1}{2} f_i(\mathbf{C}_i)\mathbf{C}_i\otimes\mathbf{C}_i\right) \odot \int Q^{(1)}_{i\mathrm{e}}(|\boldsymbol{\gamma}_\mathrm{e}|^2)|\boldsymbol{\gamma}_\mathrm{e}|\partial^2_{\mathbf{C}_\mathrm{e}\mathbf{C}_\mathrm{e}} f_\mathrm{e}(\boldsymbol{\gamma}_\mathrm{e}) \otimes \boldsymbol{\gamma}_\mathrm{e}\mathrm{d}\boldsymbol{\gamma}_\mathrm{e}\\
&\qquad+\frac{1}{2m_i^2}\partial_{\mathbf{C}_i} f_i(\mathbf{C}_i) \cdot \int Q^{(1)}_{i\mathrm{e}}(|\boldsymbol{\gamma}_\mathrm{e}|^2)|\boldsymbol{\gamma}_\mathrm{e}|\boldsymbol{\gamma}_\mathrm{e}\otimes\boldsymbol{\gamma}_\mathrm{e} \cdot \partial_{\mathbf{C}_\mathrm{e}} f_\mathrm{e}(\boldsymbol{\gamma}_\mathrm{e})\mathrm{d}\boldsymbol{\gamma}_\mathrm{e}\\
&\qquad+\frac{1}{m_i^2}\partial_{\mathbf{C}_i}(\partial_{\mathbf{C}_i} f_i(\mathbf{C}_i)\otimes\mathbf{C}_i) \odot \int Q^{(1)}_{i\mathrm{e}}(|\boldsymbol{\gamma}_\mathrm{e}|^2)|\boldsymbol{\gamma}_\mathrm{e}|\boldsymbol{\gamma}_\mathrm{e}\otimes\boldsymbol{\gamma}_\mathrm{e}\otimes\partial_{\mathbf{C}_\mathrm{e}} f_\mathrm{e}(\boldsymbol{\gamma}_\mathrm{e})\mathrm{d}\boldsymbol{\gamma}_\mathrm{e}\\
&\qquad+\frac{1}{4m_i^2}\partial_{\mathbf{C}_i}(\partial_{\mathbf{C}_i} f_i(\mathbf{C}_i)\otimes\mathbf{C}_i) \odot \int Q^{(2)}_{i\mathrm{e}}(|\boldsymbol{\gamma}_\mathrm{e}|^2)|\boldsymbol{\gamma}_\mathrm{e}|(|\boldsymbol{\gamma}_\mathrm{e}|^2\mathbb{I}\\
&\qquad-3\boldsymbol{\gamma}_\mathrm{e}\otimes\boldsymbol{\gamma}_\mathrm{e})\otimes\partial_{\mathbf{C}_\mathrm{e}} f_\mathrm{e}(\boldsymbol{\gamma}_\mathrm{e})\mathrm{d}\boldsymbol{\gamma}_\mathrm{e}\\
&\qquad-\frac{1}{4m_i^2}\partial^3_{\mathbf{C}_i\mathbf{C}_i\mathbf{C}_i} f_i(\mathbf{C}_i) \odot \int Q^{(1)}_{i\mathrm{e}}(|\boldsymbol{\gamma}_\mathrm{e}|^2)|\boldsymbol{\gamma}_\mathrm{e}|(|\boldsymbol{\gamma}_\mathrm{e}|^2\mathbb{I} + \boldsymbol{\gamma}_\mathrm{e}\otimes\boldsymbol{\gamma}_\mathrm{e})\otimes\boldsymbol{\gamma}_\mathrm{e} f_\mathrm{e}(\boldsymbol{\gamma}_\mathrm{e})\mathrm{d}\boldsymbol{\gamma}_\mathrm{e}\\
&\qquad-\frac{1}{4m_i^2}\partial^3_{\mathbf{C}_i\mathbf{C}_i\mathbf{C}_i} f_i(\mathbf{C}_i) \odot \int Q^{(2)}_{i\mathrm{e}}(|\boldsymbol{\gamma}_\mathrm{e}|^2)|\boldsymbol{\gamma}_\mathrm{e}|(|\boldsymbol{\gamma}_\mathrm{e}|^2\mathbb{I} - 3\boldsymbol{\gamma}_\mathrm{e}\otimes\boldsymbol{\gamma}_\mathrm{e})\otimes\boldsymbol{\gamma}_\mathrm{e} f_\mathrm{e}(\boldsymbol{\gamma}_\mathrm{e})\mathrm{d}\boldsymbol{\gamma}_\mathrm{e}\\
&\qquad+\frac{1}{12m_i^2}\partial^3_{\mathbf{C}_i\mathbf{C}_i\mathbf{C}_i} f_i(\mathbf{C}_i) \odot \int Q^{(3)}_{i\mathrm{e}}(|\boldsymbol{\gamma}_\mathrm{e}|^2)|\boldsymbol{\gamma}_\mathrm{e}|(3|\boldsymbol{\gamma}_\mathrm{e}|^2\mathbb{I} - 5\boldsymbol{\gamma}_\mathrm{e}\otimes\boldsymbol{\gamma}_\mathrm{e})\otimes\boldsymbol{\gamma}_\mathrm{e} f_\mathrm{e}(\boldsymbol{\gamma}_\mathrm{e})\mathrm{d}\boldsymbol{\gamma}_\mathrm{e}\\
&\qquad+\frac{3}{2m_i}\mathcal{J}^1_{i\mathrm{e}}(f_i, f_\mathrm{e})(\mathbf{C}_i).\end{aligned} \tag{3.7}$$

Proof. The change of variable $\mathrm{d}\mathbf{C}_{\mathrm{e}} = -(1+\varepsilon^2/m_i)^{3/2}\mathrm{d}\boldsymbol{\gamma}_{\mathrm{e}}$ allows for the differential cross-section dependence on ε to be eliminated

$$\mathcal{J}_{i\mathrm{e}}(f_i, f_{\mathrm{e}})(\mathbf{C}_i) = \int \sigma_{i\mathrm{e}}\left(|\boldsymbol{\gamma}_{\mathrm{e}}|^2, \boldsymbol{\omega}\cdot\frac{\boldsymbol{\gamma}_{\mathrm{e}}}{|\boldsymbol{\gamma}_{\mathrm{e}}|}\right)|\boldsymbol{\gamma}_{\mathrm{e}}|(1+\varepsilon^2/m_i)^2$$
$$\times [f_i(\mathbf{C}_i')f_{\mathrm{e}}(\mathbf{C}_{\mathrm{e}}') - f_i(\mathbf{C}_i)f_{\mathrm{e}}(\mathbf{C}_{\mathrm{e}})]\mathrm{d}\boldsymbol{\omega}\mathrm{d}\boldsymbol{\gamma}_{\mathrm{e}}, \quad i \in \mathrm{H}.$$

Then, the peculiar velocities are expanded in a power series of ε

$$\mathbf{C}_i' = \mathbf{C}_i + \varepsilon\frac{1}{m_i}\mathbf{a}_1 - \varepsilon^3\frac{1}{2m_i^2}\mathbf{a}_1 + \mathcal{O}(\varepsilon^4), \quad \mathbf{a}_1 = -\boldsymbol{\gamma}_{\mathrm{e}} + |\boldsymbol{\gamma}_{\mathrm{e}}|\boldsymbol{\omega}, \;\; i \in \mathrm{H},$$

$$\mathbf{C}_{\mathrm{e}}' = -|\boldsymbol{\gamma}_{\mathrm{e}}|\boldsymbol{\omega} + \varepsilon\mathbf{C}_i + \varepsilon^2\frac{1}{m_i}\mathbf{a}_2 + \mathcal{O}(\varepsilon^4), \quad \mathbf{a}_2 = -\boldsymbol{\gamma}_{\mathrm{e}} + \frac{1}{2}|\boldsymbol{\gamma}_{\mathrm{e}}|\boldsymbol{\omega},$$

$$\mathbf{C}_{\mathrm{e}} = -\boldsymbol{\gamma}_{\mathrm{e}} + \varepsilon\mathbf{C}_i - \varepsilon^2\frac{1}{2m_i}\boldsymbol{\gamma}_{\mathrm{e}} + \mathcal{O}(\varepsilon^4).$$

Hence, the distribution functions are found to be

$$f_i(\mathbf{C}_i') = f_i(\mathbf{C}_i) + \varepsilon\frac{1}{m_i}\partial_{\mathbf{C}_i}f_i(\mathbf{C}_i)\cdot\mathbf{a}_1 + \varepsilon^2\frac{1}{2m_i^2}\partial^2_{\mathbf{C}_i\mathbf{C}_i}f_i(\mathbf{C}_i):(\mathbf{a}_1\otimes\mathbf{a}_1)$$
$$+\varepsilon^3\frac{1}{6m_i^3}\partial^3_{\mathbf{C}_i\mathbf{C}_i\mathbf{C}_i}f_i(\mathbf{C}_i)\odot(\mathbf{a}_1\otimes\mathbf{a}_1\otimes\mathbf{a}_1)$$
$$-\varepsilon^3\frac{1}{2m_i^2}\partial_{\mathbf{C}_i}f_i(\mathbf{C}_i)\cdot\mathbf{a}_1 + \mathcal{O}(\varepsilon^4), \quad i \in \mathrm{H},$$

$$f_{\mathrm{e}}(\mathbf{C}_{\mathrm{e}}') = f_{\mathrm{e}}(-|\boldsymbol{\gamma}_{\mathrm{e}}|\boldsymbol{\omega}) + \varepsilon\partial_{\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}(-|\boldsymbol{\gamma}_{\mathrm{e}}|\boldsymbol{\omega})\cdot\mathbf{C}_i + \varepsilon^2\frac{1}{2}\partial^2_{\mathbf{C}_{\mathrm{e}}\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}(-|\boldsymbol{\gamma}_{\mathrm{e}}|\boldsymbol{\omega}):(\mathbf{C}_i\otimes\mathbf{C}_i)$$
$$+\varepsilon^2\frac{1}{m_i}\partial_{\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}(-|\boldsymbol{\gamma}_{\mathrm{e}}|\boldsymbol{\omega})\cdot\mathbf{a}_2 + \varepsilon^3\frac{1}{6}\partial^3_{\mathbf{C}_{\mathrm{e}}\mathbf{C}_{\mathrm{e}}\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}(-|\boldsymbol{\gamma}_{\mathrm{e}}|\boldsymbol{\omega})\odot(\mathbf{C}_i\otimes\mathbf{C}_i\otimes\mathbf{C}_i)$$
$$+\varepsilon^3\frac{1}{m_i}\partial^2_{\mathbf{C}_{\mathrm{e}}\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}(-|\boldsymbol{\gamma}_{\mathrm{e}}|\boldsymbol{\omega}):(\mathbf{C}_i\otimes\mathbf{a}_2) + \mathcal{O}(\varepsilon^4),$$

$$f_{\mathrm{e}}(\mathbf{C}_{\mathrm{e}}) = f_{\mathrm{e}}(-\boldsymbol{\gamma}_{\mathrm{e}}) + \varepsilon\partial_{\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}(-\boldsymbol{\gamma}_{\mathrm{e}})\cdot\mathbf{C}_i + \varepsilon^2\frac{1}{2}\partial^2_{\mathbf{C}_{\mathrm{e}}\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}(-\boldsymbol{\gamma}_{\mathrm{e}}):\mathbf{C}_i\otimes\mathbf{C}_i$$
$$-\varepsilon^2\frac{1}{2m_i}\partial_{\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}(-\boldsymbol{\gamma}_{\mathrm{e}})\cdot\boldsymbol{\gamma}_{\mathrm{e}} + \varepsilon^3\frac{1}{6}\partial^3_{\mathbf{C}_{\mathrm{e}}\mathbf{C}_{\mathrm{e}}\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}(-\boldsymbol{\gamma}_{\mathrm{e}})\odot\mathbf{C}_i\otimes\mathbf{C}_i\otimes\mathbf{C}_i$$
$$-\varepsilon^3\frac{1}{2m_i}\partial^2_{\mathbf{C}_{\mathrm{e}}\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}(-\boldsymbol{\gamma}_{\mathrm{e}}):\mathbf{C}_i\otimes\boldsymbol{\gamma}_{\mathrm{e}} + \mathcal{O}(\varepsilon^4).$$

Combining these equations, the zeroth-order term $\mathcal{J}^0_{i\mathrm{e}}$, $i \in \mathrm{H}$, is thus given by

$$\mathcal{J}^0_{i\mathrm{e}}(f_i, f_{\mathrm{e}})(\mathbf{C}_i) = f_i(\mathbf{C}_i)\int \sigma_{i\mathrm{e}}(|\boldsymbol{\gamma}_{\mathrm{e}}|^2, \boldsymbol{\omega}\cdot\mathbf{e})|\boldsymbol{\gamma}_{\mathrm{e}}|^3$$
$$\times [f_{\mathrm{e}}(|\boldsymbol{\gamma}_{\mathrm{e}}|\boldsymbol{\omega}) - f_{\mathrm{e}}(|\boldsymbol{\gamma}_{\mathrm{e}}|\mathbf{e})]\mathrm{d}\boldsymbol{\omega}\mathrm{d}\mathbf{e}\mathrm{d}|\boldsymbol{\gamma}_{\mathrm{e}}|.$$

Interchanging $\mathbf{e}$ and $\boldsymbol{\omega}$, the integral is shown to vanish. Then, Eqs. (3.5)–(3.7) are obtained after some lengthy calculation. □

Theorem 3.1 admits three corollaries.

Corollary 3.1. *The first-order collision operator* $\mathfrak{J}^1_{i\mathrm{e}}(f_i, f_\mathrm{e})$, $i \in \mathrm{H}$, *vanishes when* f_e *is an isotropic function of the velocity* $\mathbf{C}_\mathrm{e}$.

Proof. Expression (3.5) immediately yields that the integrand is an odd function of $\boldsymbol{\gamma}_\mathrm{e}$ if f_e is isotropic in the heavy-particle reference frame, so that the first-order collision operator vanishes. □

A collision frequency is defined as a Maxwell–Boltzmann averaged momentum cross-section

$$\nu_{i\mathrm{e}} = \frac{1}{T_\mathrm{e}} \int Q^{(1)}_{i\mathrm{e}}(|\boldsymbol{\gamma}_\mathrm{e}|^2)|\boldsymbol{\gamma}_\mathrm{e}|^3 f^0_\mathrm{e}(\boldsymbol{\gamma}_\mathrm{e})\mathrm{d}\boldsymbol{\gamma}_\mathrm{e}, \quad i \in \mathrm{H},$$

where $f^0_\mathrm{e}(\boldsymbol{\gamma}_\mathrm{e}) = n_\mathrm{e} \exp[-\boldsymbol{\gamma}_\mathrm{e} \cdot \boldsymbol{\gamma}_\mathrm{e}/(2T_\mathrm{e})]/(2\pi T_\mathrm{e})^{3/2}$.

Corollary 3.2. *Considering the function* $f^0_\mathrm{e} = n_\mathrm{e} \exp[-\mathbf{C}_\mathrm{e} \cdot \mathbf{C}_\mathrm{e}/(2T_\mathrm{e})]/(2\pi T_\mathrm{e})^{3/2}$, *the second-order collision operator reads*

$$\mathfrak{J}^2_{i\mathrm{e}}(f_i, f^0_\mathrm{e})(\mathbf{C}_i) = \frac{\nu_{i\mathrm{e}}}{3m_i}\left(\boldsymbol{\partial}_{\mathbf{C}_i} \cdot (f_i \mathbf{C}_i) + \frac{T_\mathrm{e}}{m_i}\Delta_{\mathbf{C}_i} f_i\right), \quad i \in \mathrm{H}. \tag{3.8}$$

Proof. A direct calculation of $\mathfrak{J}^2_{i\mathrm{e}}(f_i, f^0_\mathrm{e})(\mathbf{C}_i)$ given in (3.6) immediately yields expression (3.8) if $f^0_\mathrm{e} = n_\mathrm{e} \exp[-\mathbf{C}_\mathrm{e} \cdot \mathbf{C}_\mathrm{e}/(2T_\mathrm{e})]/(2\pi T_\mathrm{e})^{3/2}$. □

Corollary 3.3. *The third-order collision operator* $\mathfrak{J}^3_{i\mathrm{e}}(f_i, f_\mathrm{e})$, $i \in \mathrm{H}$, *vanishes when* f_e *is an isotropic function of the velocity* $\mathbf{C}_\mathrm{e}$.

Proof. Expression (3.7) immediately yields that the integrand is an odd function of $\boldsymbol{\gamma}_\mathrm{e}$ if f_e is isotropic in the heavy-particle reference frame, so that the third-order collision operator vanishes. □

3.3. *Expansion of the collision operator* $\mathfrak{J}_{\mathrm{e}i}$

Dimensional analysis yields the following expression for the nondimensional collision operator $\mathfrak{J}_{\mathrm{e}i}$, $i \in \mathrm{H}$,

$$\mathfrak{J}_{\mathrm{e}i}(f_\mathrm{e}, f_i)(\mathbf{C}_\mathrm{e}) = \int \sigma_{\mathrm{e}i}\left(\frac{m_i|\mathbf{C}_\mathrm{e} - \varepsilon\mathbf{C}_i|^2}{m_i + \varepsilon^2}, \boldsymbol{\omega} \cdot \mathbf{e}\right)|\mathbf{C}_\mathrm{e} - \varepsilon\mathbf{C}_i|$$
$$\times [f_\mathrm{e}(\mathbf{C}'_\mathrm{e})f_i(\mathbf{C}'_i) - f_\mathrm{e}(\mathbf{C}_\mathrm{e})f_i(\mathbf{C}_i)]\mathrm{d}\boldsymbol{\omega}\mathrm{d}\mathbf{C}_i. \tag{3.9}$$

The original set of variables $\{\mathbf{C}_\mathrm{e}, \mathbf{C}_i, \boldsymbol{\omega}\}$ is retained. We introduce the momentum cross-section

$$Q_{\mathrm{e}i}^{(1)}(|\mathbf{C}_\mathrm{e}|^2) = 2\pi \int_0^\pi \sigma_{\mathrm{e}i}(|\mathbf{C}_\mathrm{e}|^2, \cos\theta)(1-\cos\theta)\sin\theta \mathrm{d}\theta, \quad i \in \mathrm{H}, \tag{3.10}$$

representing the average momentum transferred in encounters from i heavy particles to electrons. It is equal to the cross-section $Q_{i\mathrm{e}}^{(1)}$.

Theorem 3.2. *The collision operator* $\mathcal{J}_{\mathrm{e}i}$, $i \in \mathrm{H}$, *can be expanded in the form*

$$\begin{aligned}\mathcal{J}_{\mathrm{e}i}(f_\mathrm{e}, f_i)(\mathbf{C}_\mathrm{e}) = \mathcal{J}_{\mathrm{e}i}^0(f_\mathrm{e}, f_i)(\mathbf{C}_\mathrm{e}) + \varepsilon \mathcal{J}_{\mathrm{e}i}^1(f_\mathrm{e}, f_i)(\mathbf{C}_\mathrm{e}) + \varepsilon^2 \mathcal{J}_{\mathrm{e}i}^2(f_\mathrm{e}, f_i)(\mathbf{C}_\mathrm{e}) \\ + \varepsilon^3 \mathcal{J}_{\mathrm{e}i}^3(f_\mathrm{e}, f_i)(\mathbf{C}_\mathrm{e}) + \mathcal{O}(\varepsilon^4).\end{aligned} \tag{3.11}$$

The zeroth-order term $\mathcal{J}_{\mathrm{e}i}^0$, $i \in \mathrm{H}$, *is given by the expression*

$$\begin{aligned}\mathcal{J}_{\mathrm{e}i}^0(f_\mathrm{e}, f_i)(\mathbf{C}_\mathrm{e}) = \int f_i(\mathbf{C}_i)\mathrm{d}\mathbf{C}_i \int \sigma_{\mathrm{e}i}\left(|\mathbf{C}_\mathrm{e}|^2, \boldsymbol{\omega}\cdot\frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\right) \\ \times |\mathbf{C}_\mathrm{e}|[f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega}) - f_\mathrm{e}(\mathbf{C}_\mathrm{e})]\mathrm{d}\boldsymbol{\omega}.\end{aligned} \tag{3.12}$$

The first-order term $\mathcal{J}_{i\mathrm{e}}^1$, $i \in \mathrm{H}$, *reads*

$$\begin{aligned}\mathcal{J}_{\mathrm{e}i}^1(f_\mathrm{e}, f_i)(\mathbf{C}_\mathrm{e}) = \left(\int f_i(\mathbf{C}_i)\mathbf{C}_i\mathrm{d}\mathbf{C}_i\right) \\ \cdot\left\{\partial_{\mathbf{C}_\mathrm{e}} \int \sigma_{\mathrm{e}i}\left(|\mathbf{C}_\mathrm{e}|^2, \frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\cdot\boldsymbol{\omega}\right)[f_\mathrm{e}(\mathbf{C}_\mathrm{e}) - f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega})]|\mathbf{C}_\mathrm{e}|\mathrm{d}\boldsymbol{\omega}\right. \\ \left. + \int \sigma_{\mathrm{e}i}\left(|\mathbf{C}_\mathrm{e}|^2, \frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\cdot\boldsymbol{\omega}\right)|\mathbf{C}_\mathrm{e}|[\partial_{\mathbf{C}_\mathrm{e}} f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega}) - \partial_{\mathbf{C}_\mathrm{e}} f_\mathrm{e}(\mathbf{C}_\mathrm{e})]\mathrm{d}\boldsymbol{\omega}\right\}.\end{aligned} \tag{3.13}$$

The second-order term $\mathcal{J}_{i\mathrm{e}}^2$, $i \in \mathrm{H}$, *is found to be*

$$\begin{aligned}\mathcal{J}_{\mathrm{e}i}^2(f_\mathrm{e}, f_i)(\mathbf{C}_\mathrm{e}) = \frac{1}{m_i} K_{\mathrm{e}i}^{2,1}(\mathbf{C}_\mathrm{e}) \int f_i(\mathbf{C}_i)\mathrm{d}\mathbf{C}_i \\ + \frac{1}{2}\mathbf{K}_{\mathrm{e}i}^{2,2}(\mathbf{C}_\mathrm{e}) : \int f_i(\mathbf{C}_i)\mathbf{C}_i \otimes \mathbf{C}_i\mathrm{d}\mathbf{C}_i,\end{aligned} \tag{3.14}$$

with

$$\begin{aligned}K_{\mathrm{e}i}^{2,1}(\mathbf{C}_\mathrm{e}) = \partial_{\mathbf{C}_\mathrm{e}} \cdot \int \sigma_{\mathrm{e}i}\left(|\mathbf{C}_\mathrm{e}|^2, \frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\cdot\boldsymbol{\omega}\right)(\mathbf{C}_\mathrm{e} - |\mathbf{C}_\mathrm{e}|\boldsymbol{\omega})|\mathbf{C}_\mathrm{e}|f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega} \\ - |\mathbf{C}_\mathrm{e}|\mathbf{C}_\mathrm{e} \cdot \int \partial_{\mathbf{C}_\mathrm{e}}\sigma_{\mathrm{e}i}\left(|\mathbf{C}_\mathrm{e}|^2, \frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\cdot\boldsymbol{\omega}\right)[f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega}) - f_\mathrm{e}(\mathbf{C}_\mathrm{e})]\mathrm{d}\boldsymbol{\omega},\end{aligned}$$

and

$$\mathbf{K}_{\mathrm{e}i}^{2,2}(\mathbf{C}_\mathrm{e}) = \partial^2_{\mathbf{C}_\mathrm{e}\mathbf{C}_\mathrm{e}} \int \sigma_{\mathrm{e}i}\left(|\mathbf{C}_\mathrm{e}|^2, \frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\cdot\boldsymbol{\omega}\right)|\mathbf{C}_\mathrm{e}|[f_\mathrm{e}(\mathbf{C}_\mathrm{e}|\boldsymbol{\omega}) - f_\mathrm{e}(\mathbf{C}_\mathrm{e})]\mathrm{d}\boldsymbol{\omega}$$
$$+ 2\int \partial_{\mathbf{C}_\mathrm{e}}\left(\sigma_{\mathrm{e}i}\left(|\mathbf{C}_\mathrm{e}|^2, \frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\cdot\boldsymbol{\omega}\right)|\mathbf{C}_\mathrm{e}|\right)\otimes[\partial_{\mathbf{C}_\mathrm{e}} f_\mathrm{e}(\mathbf{C}_\mathrm{e}) - \partial_{\mathbf{C}_\mathrm{e}} f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega})]\mathrm{d}\boldsymbol{\omega}$$
$$+ |\mathbf{C}_\mathrm{e}|\int \sigma_{\mathrm{e}i}\left(|\mathbf{C}_\mathrm{e}|^2, \frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\cdot\boldsymbol{\omega}\right)[\partial^2_{\mathbf{C}_\mathrm{e}\mathbf{C}_\mathrm{e}} f_\mathrm{e}(\mathbf{C}_\mathrm{e}) - \partial^2_{\mathbf{C}_\mathrm{e}\mathbf{C}_\mathrm{e}} f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega})]\mathrm{d}\boldsymbol{\omega}$$
$$+ 2|\mathbf{C}_\mathrm{e}|\int \sigma_{\mathrm{e}i}\left(|\mathbf{C}_\mathrm{e}|^2, \frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\cdot\boldsymbol{\omega}\right)\frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\otimes\boldsymbol{\omega}\,\partial^2_{\mathbf{C}_\mathrm{e}\mathbf{C}_\mathrm{e}} f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega}.$$

Proof. The relative velocity and peculiar velocities after collision are expanded in a power series of ε. For $i \in \mathrm{H}$, we have

$$|\mathbf{C}_\mathrm{e} - \varepsilon\mathbf{C}_i| = |\mathbf{C}_\mathrm{e}| - \varepsilon\frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\cdot\mathbf{C}_i + \varepsilon^2 b_1 + \mathcal{O}(\varepsilon^3),$$
$$\mathbf{C}_i' = \mathbf{C}_i + \varepsilon\frac{1}{m_i}\mathbf{a}_4 - \varepsilon^2\frac{1}{m_i}\mathbf{a}_5 + \mathcal{O}(\varepsilon^3),$$
$$\mathbf{C}_\mathrm{e}' = |\mathbf{C}_\mathrm{e}|\boldsymbol{\omega} + \varepsilon\mathbf{a}_5 + \varepsilon^2\left(\frac{1}{m_i}\mathbf{a}_4 + \mathbf{a}_6\right) + \mathcal{O}(\varepsilon^3),$$

with $b_1 = \frac{1}{2|\mathbf{C}_\mathrm{e}|}[|\mathbf{C}_i|^2 - \big(\frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\cdot\mathbf{C}_i\big)^2]$, $\mathbf{a}_4 = \mathbf{C}_\mathrm{e} - |\mathbf{C}_\mathrm{e}|\boldsymbol{\omega}$, $\mathbf{a}_5 = \mathbf{C}_i - \frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\cdot\mathbf{C}_i\boldsymbol{\omega}$, $\mathbf{a}_6 = b_1\boldsymbol{\omega}$. Hence, the distribution functions are found to be

$$f_i(\mathbf{C}_i') = f_i(\mathbf{C}_i) + \varepsilon\frac{1}{m_i}\partial_{\mathbf{C}_i} f_i(\mathbf{C}_i)\cdot\mathbf{a}_4 + \varepsilon^2\frac{1}{2m_i^2}\partial^2_{\mathbf{C}_i\mathbf{C}_i} f_i(\mathbf{C}_i) : (\mathbf{a}_4\otimes\mathbf{a}_4)$$
$$- \varepsilon^2\frac{1}{2m_i}\partial_{\mathbf{C}_i} f_i(\mathbf{C}_i)\cdot\mathbf{a}_5 + \mathcal{O}(\varepsilon^3), \quad i \in \mathrm{H},$$
$$f_\mathrm{e}(\mathbf{C}_\mathrm{e}') = f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega}) + \varepsilon\partial_{\mathbf{C}_\mathrm{e}} f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega})\cdot\mathbf{a}_5 + \frac{1}{2}\varepsilon^2\partial^2_{\mathbf{C}_\mathrm{e}\mathbf{C}_\mathrm{e}} f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega}) : (\mathbf{a}_5\otimes\mathbf{a}_5)$$
$$+ \varepsilon^2\partial_{\mathbf{C}_\mathrm{e}} f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega})\cdot\left(\frac{1}{m_i}\mathbf{a}_4 + \mathbf{a}_6\right) + \mathcal{O}(\varepsilon^3).$$

Combining these equations, we obtain Eqs. (3.12)–(3.14) after some lengthy calculation. □

Theorem 3.2 admits three corollaries. First, we define the rate of entropy produced at order ε^0 in collisions between electrons and i heavy particles

$$\Upsilon^0_{\mathrm{e}i} = -\int \mathfrak{J}^0_{\mathrm{e}i}(f_\mathrm{e}, f_i)(\mathbf{C}_\mathrm{e})\,\ln\left[\frac{(2\pi)^{3/2}n^0}{Q^0_\mathrm{e}} f_\mathrm{e}(\mathbf{C}_\mathrm{e})\right]\mathrm{d}\mathbf{C}_\mathrm{e}, \quad i \in \mathrm{H},$$

where $Q^0_\mathrm{e} = (2\pi m^0_\mathrm{e}\mathrm{k_B}T^0/\mathrm{h}^2_\mathrm{P})^{3/2}$ is the translational partition function of electrons. Symbol $\mathrm{h_P}$ stands for Planck's constant. The zeroth-order operator describes the relaxation of the electron population towards an isotropic distribution function in the heavy-particle reference frame.

Corollary 3.4. *The zeroth-order collision operator* $\mathcal{J}^0_{\mathrm{e}i}(f_\mathrm{e}, f_i)$, $i \in \mathrm{H}$, *vanishes when* f_e *is an isotropic function of the velocity* $\mathbf{C}_\mathrm{e}$. *Moreover, the zeroth-order entropy is non-negative, that is* $\Upsilon^0_{\mathrm{e}i} \geq 0$, $i \in \mathrm{H}$, *and the inequality is an equality if and only if* f_e *is an isotropic function of the velocity* $\mathbf{C}_\mathrm{e}$.

Proof. If f_e is an isotropic function of $\mathbf{C}_\mathrm{e}$, we have $f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega}) = f_\mathrm{e}(\mathbf{C}_\mathrm{e})$ for any $\boldsymbol{\omega}$ in the unit sphere, so that expression (3.12) implies that $\mathcal{J}^0_{\mathrm{e}i}(f_\mathrm{e}, f_i) = 0$. The zeroth-order entropy production rate reads

$$\begin{aligned}\Upsilon^0_{\mathrm{e}i} = -n_i \int \sigma_{\mathrm{e}i}\left(|\mathbf{C}_\mathrm{e}|^2, \boldsymbol{\omega}\cdot\frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\right)|\mathbf{C}_\mathrm{e}|^3[f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega}) - f_\mathrm{e}(\mathbf{C}_\mathrm{e})]\\ \times \ln\left[\frac{(2\pi)^{3/2}n^0}{Q^0_\mathrm{e}}f_\mathrm{e}(\mathbf{C}_\mathrm{e})\right]\mathrm{d}|\mathbf{C}_\mathrm{e}|\mathrm{d}\frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\mathrm{d}\boldsymbol{\omega},\end{aligned}$$

and interchanging $\frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}$ and $\boldsymbol{\omega}$,

$$\Upsilon^0_{\mathrm{e}i} = \frac{n_i}{2}\int \sigma_{\mathrm{e}i}\left(|\mathbf{C}_\mathrm{e}|^2, \boldsymbol{\omega}\cdot\frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\right)|\mathbf{C}_\mathrm{e}|^3\Omega(f_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega}), f_\mathrm{e}(\mathbf{C}_\mathrm{e}))\mathrm{d}|\mathbf{C}_\mathrm{e}|\mathrm{d}\frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\mathrm{d}\boldsymbol{\omega},$$

where $\Omega(x, y) = (x - y)\ln(x/y)$ is a non-negative function. We then obtain that $\Upsilon^0_{\mathrm{e}i}$, $i \in \mathrm{H}$, is non-negative and equal to 0 if and only if f_e is isotropic in the heavy-particle reference frame. □

Corollary 3.5. *The first-order collision operator* $\mathcal{J}^1_{\mathrm{e}i}(f_\mathrm{e}, f_i)$, $i \in \mathrm{H}$, *vanishes when* f_i *is an isotropic function of the velocity* $\mathbf{C}_i$.

Proof. Expression (3.13) immediately yields that the integrand is an odd function of $\mathbf{C}_i$, $i \in \mathrm{H}$, if f_i is isotropic in the heavy-particle reference frame, so that the first-order collision operator vanishes. □

Remark 3.1. So far, we note that the isotropy property of f_i is strongly related to our choice of reference frame. For example, such a property is not satisfied when $\mathbf{u} = 0$. Thus, the structure of the expansion of collision operators depends on the initial choice of reference frame. We will come back to this point in Sec. 4.8.

Corollary 3.6. *Considering the functions* $f^0_\mathrm{e} = n_\mathrm{e}\exp[-\mathbf{C}_\mathrm{e}\cdot\mathbf{C}_\mathrm{e}/(2T_\mathrm{e})]/(2\pi T_\mathrm{e})^{3/2}$ *and* $f^0_i = n_i m_i^{3/2}\exp[-m_i\mathbf{C}_i\cdot\mathbf{C}_i/(2T_h)]/(2\pi T_h)^{3/2}$, $i \in \mathrm{H}$, *the second-order collision operator* $\mathcal{J}^2_{\mathrm{e}i}(f^0_\mathrm{e}, f^0_i)(\mathbf{C}_\mathrm{e})$, $i \in \mathrm{H}$, *reads*

$$\begin{aligned}\mathcal{J}^2_{\mathrm{e}i}(f^0_\mathrm{e}, f^0_i)(\mathbf{C}_\mathrm{e}) = (T_h - T_\mathrm{e})\frac{n_i}{m_i}\frac{1}{T_\mathrm{e}}f^0_\mathrm{e}(\mathbf{C}_\mathrm{e})|\mathbf{C}_\mathrm{e}|\\ \times\left[\partial_{\mathbf{C}_\mathrm{e}}\cdot(Q^{(1)}_{\mathrm{e}i}(\mathbf{C}_\mathrm{e})\mathbf{C}_\mathrm{e}) + \left(1 - \frac{|\mathbf{C}_\mathrm{e}|^2}{T_\mathrm{e}}\right)Q^{(1)}_{\mathrm{e}i}(\mathbf{C}_\mathrm{e})\right].\end{aligned} \tag{3.15}$$

Proof. A direct calculation of $\mathcal{J}^2_{\mathrm{e}i}(f^0_{\mathrm{e}}, f_i)(\mathbf{C}_{\mathrm{e}})$ given in (3.14) immediately yields expression (3.15) if f^0_{e} and f^0_i are the Maxwell–Boltzmann distribution functions given in the assumptions of Corollary 3.6. □

3.4. *Electron and heavy-particle collisional invariants*

Based on the space of collisional invariants $\mathcal{I}$ defined in Eq. (2.33), we introduce two subspaces naturally associated with our choice of scaling.

Definition 3.1. The space of scalar electron collisional invariants $\mathcal{I}_{\mathrm{e}}$ is spanned by the following elements

$$\begin{cases} \hat{\psi}^1_{\mathrm{e}} = 1, \\ \hat{\psi}^2_{\mathrm{e}} = \dfrac{1}{2}\mathbf{C}_{\mathrm{e}} \cdot \mathbf{C}_{\mathrm{e}}. \end{cases} \tag{3.16}$$

Definition 3.2. The space of scalar heavy-particle collisional invariants $\mathcal{I}_h$ is spanned by the following elements

$$\begin{cases} \hat{\psi}^j_h = (m_i \delta_{ij})_{i\in\mathrm{H}}, & j \in \mathrm{H}, \\ \hat{\psi}^{n^{\mathrm{H}}+\nu}_h = (m_i C_{i\nu})_{i\in\mathrm{H}}, & \nu \in \{1,2,3\}, \\ \hat{\psi}^{n^{\mathrm{H}}+4}_h = \left(\dfrac{1}{2} m_i \mathbf{C}_i \cdot \mathbf{C}_i\right)_{i\in\mathrm{H}}, \end{cases} \tag{3.17}$$

where symbol n^{H} denotes the cardinality of the set of heavy particles H.

The decoupling of the collision invariants is clearly identified in the proposed scaling. More precisely, the definition of the electron linearized collision operator (given in Sec. 4) will involve the electron partial collision operator $\mathcal{J}_{\mathrm{ee}}$ and the mixed partial collision operators $\mathcal{J}^0_{\mathrm{e}i}$, $i \in \mathrm{H}$, satisfying the following important property.

Property 3.1. The partial collision operators $\mathcal{J}^0_{\mathrm{e}i}$, $i \in \mathrm{H}$, are orthogonal to the space of collisional invariants $\mathcal{I}_{\mathrm{e}}$, i.e. $\langle\!\langle \hat{\psi}^l_{\mathrm{e}}, \mathcal{J}^0_{\mathrm{e}i} \rangle\!\rangle_{\mathrm{e}} = 0$ for all $l \in \{1,2\}$.

Proof. The projection of the collision operator $\mathcal{J}^0_{\mathrm{e}i}$, $i \in \mathrm{H}$, onto $\hat{\psi}^l_{\mathrm{e}}$, $l \in \{1,2\}$ reads

$$\begin{aligned} \langle\!\langle \hat{\psi}^l_{\mathrm{e}}, \mathcal{J}^0_{\mathrm{e}i} \rangle\!\rangle_{\mathrm{e}} = n_i \int \sigma_{\mathrm{e}i} \left(|\mathbf{C}_{\mathrm{e}}|^2, \boldsymbol{\omega} \cdot \frac{\mathbf{C}_{\mathrm{e}}}{|\mathbf{C}_{\mathrm{e}}|} \right) |\mathbf{C}_{\mathrm{e}}|^3 \\ \times [f_{\mathrm{e}}(|\mathbf{C}_{\mathrm{e}}|\boldsymbol{\omega}) - f_{\mathrm{e}}(\mathbf{C}_{\mathrm{e}})] \hat{\psi}^l_{\mathrm{e}} \mathrm{d}|\mathbf{C}_{\mathrm{e}}| \mathrm{d}\boldsymbol{\omega} \mathrm{d}\frac{\mathbf{C}_{\mathrm{e}}}{|\mathbf{C}_{\mathrm{e}}|}. \end{aligned}$$

Interchanging $\boldsymbol{\omega}$ and $\frac{\mathbf{C}_{\mathrm{e}}}{|\mathbf{C}_{\mathrm{e}}|}$, the projection $\langle\!\langle \hat{\psi}^l_{\mathrm{e}}, \mathcal{J}^0_{\mathrm{e}i} \rangle\!\rangle_{\mathrm{e}}$ is shown to vanish for all $l \in \{1,2\}$. □

We emphasize that the partial collision operators $\mathcal{J}^0_{\mathrm{e}i}$, $i \in \mathrm{H}$, are not orthogonal for the scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathrm{e}}$ to the space spanned by the electron momentum. This is the reason why the vector $\mathbf{C}_e$ does not belong to $\mathcal{I}_e$. In contrast, the definition of the heavy-particle linearized collision operator (given in Sec. 4) only involves the heavy-particle partial collision operators $\mathcal{J}_{ij}$, $i, j \in \mathrm{H}$.

Subsequently, using the newly defined collisional invariants, the orthogonality Property 2.2 of the cross-collision operators can be rewritten

$$\sum_{j\in\mathrm{H}} \langle\!\langle \hat{\psi}_e^1, \mathcal{J}_{ej} \rangle\!\rangle_e = 0, \quad \langle\!\langle \hat{\psi}_h^i, \mathcal{J}_{he} \rangle\!\rangle_h = 0, \;\; i \in \mathrm{H}, \tag{3.18}$$

for mass conservation,

$$\varepsilon \sum_{j\in\mathrm{H}} \langle\!\langle C_{e\nu}, \mathcal{J}_{ej} \rangle\!\rangle_e + \langle\!\langle \hat{\psi}_h^{n^{\mathrm{H}}+\nu}, \mathcal{J}_{he} \rangle\!\rangle_h = 0, \quad \nu \in \{1,2,3\}, \tag{3.19}$$

for momentum conservation, and

$$\sum_{j\in\mathrm{H}} \langle\!\langle \hat{\psi}_e^2, \mathcal{J}_{ej} \rangle\!\rangle_e + \langle\!\langle \hat{\psi}_h^{n^{\mathrm{H}}+4}, \mathcal{J}_{he} \rangle\!\rangle_h = 0, \tag{3.20}$$

for energy conservation. This set of relations is essential since it corresponds to the conservation of mass, momentum, and energy in the electron heavy-particle interactions through the various orders in ε of the Chapman–Enskog expansion.

Then, the macroscopic properties are expressed as partial scalar products of the distribution functions and the new collisional invariants

$$\begin{cases} \langle\!\langle f_e, \hat{\psi}_e^1 \rangle\!\rangle_e = \rho_e, \\ \langle\!\langle f_e, \hat{\psi}_e^2 \rangle\!\rangle_e = \rho_e e_e, \end{cases} \tag{3.21}$$

and

$$\begin{cases} \langle\!\langle f_h, \hat{\psi}_h^i \rangle\!\rangle_h = \rho_i, & i \in \mathrm{H}, \\ \langle\!\langle f_h, \hat{\psi}_h^{n^{\mathrm{H}}+\nu} \rangle\!\rangle_h = 0, & \nu \in \{1,2,3\}, \\ \langle\!\langle f_h, \hat{\psi}_h^{n^{\mathrm{H}}+4} \rangle\!\rangle_h = \rho_h e_h. & \end{cases} \tag{3.22}$$

Symbol e_e stands for the electron thermal energy per unit mass and e_h, the heavy-particle thermal energy per unit mass. It is important to mention that these quantities are defined in the heavy-particle reference frame. Furthermore, the decoupling of the collisional invariants is also consistent with the expression for the macroscopic properties. In particular, because the electron momentum is not a collision invariant in the proposed asymptotic limit, the electron mass flux is not constrained in the heavy-particle reference frame.

Translational temperatures are introduced as averaged thermal energies in the heavy-particle reference frame as follows.

Definition 3.3. The electron and heavy-particle translational temperatures are given by

$$T_{\mathrm{e}} = \frac{2}{3n_{\mathrm{e}}}\langle\!\langle f_{\mathrm{e}}, \hat{\psi}_{\mathrm{e}}^{2}\rangle\!\rangle_{\mathrm{e}}, \tag{3.23}$$

$$T_{h} = \frac{2}{3n_{h}}\langle\!\langle f_{h}, \hat{\psi}_{h}^{n^{\mathrm{H}}+4}\rangle\!\rangle_{h}, \tag{3.24}$$

where the heavy-particle number density is $n_h = \sum_{j\in\mathrm{H}} n_j$.

Consequently, the energy can be rewritten

$$\langle\!\langle f_{\mathrm{e}}, \hat{\psi}_{\mathrm{e}}^{2}\rangle\!\rangle_{\mathrm{e}} = \frac{3}{2}n_{\mathrm{e}}T_{\mathrm{e}},$$

for the electrons, and

$$\langle\!\langle f_{h}, \hat{\psi}_{h}^{n^{\mathrm{H}}+4}\rangle\!\rangle_{h} = \frac{3}{2}n_{h}T_{h}$$

for the heavy particles. It will be shown in Sec. 4 that these two temperatures are generally different.

4. Chapman–Enskog Method

We employ an Enskog expansion to derive an approximate solution to the Boltzmann equations (2.29)–(2.30) by expanding the species distribution functions as

$$f_{\mathrm{e}} = f_{\mathrm{e}}^{0}(1+\varepsilon\phi_{\mathrm{e}}+\varepsilon^{2}\phi_{\mathrm{e}}^{2}+\varepsilon^{3}\phi_{\mathrm{e}}^{3}) + \mathcal{O}(\varepsilon^{4}), \tag{4.1}$$

$$f_{i} = f_{i}^{0}(1+\varepsilon\phi_{i}+\varepsilon^{2}\phi_{i}^{2}) + \mathcal{O}(\varepsilon^{3}), \quad i\in\mathrm{H}, \tag{4.2}$$

and by imposing that the zeroth-order contributions f_{e}^{0} and f_{h}^{0} yield the local macroscopic properties

$$\langle\!\langle f_{\mathrm{e}}^{0}, \hat{\psi}_{\mathrm{e}}^{l}\rangle\!\rangle_{\mathrm{e}} = \langle\!\langle f_{\mathrm{e}}, \hat{\psi}_{\mathrm{e}}^{l}\rangle\!\rangle_{\mathrm{e}}, \quad l\in\{1,2\}, \tag{4.3}$$

$$\langle\!\langle f_{h}^{0}, \hat{\psi}_{h}^{l}\rangle\!\rangle_{h} = \langle\!\langle f_{h}, \hat{\psi}_{h}^{l}\rangle\!\rangle_{h}, \quad l\in\{1,\ldots,n^{\mathrm{H}}+4\}. \tag{4.4}$$

Hence, based upon the dimensional analysis of Sec. 2.5, the electron Boltzmann equation (2.29) becomes

$$\begin{aligned}&\varepsilon^{-2}\mathscr{D}_{\mathrm{e}}^{-2}(f_{\mathrm{e}}^{0}) + \varepsilon^{-1}\mathscr{D}_{\mathrm{e}}^{-1}(f_{\mathrm{e}}^{0},\phi_{\mathrm{e}}) + \mathscr{D}_{\mathrm{e}}^{0}(f_{\mathrm{e}}^{0},\phi_{\mathrm{e}},\phi_{\mathrm{e}}^{2}) + \varepsilon\mathscr{D}_{\mathrm{e}}^{1}(f_{\mathrm{e}}^{0},\phi_{\mathrm{e}},\phi_{\mathrm{e}}^{2},\phi_{\mathrm{e}}^{3})\\ &\quad = \varepsilon^{-2}\mathcal{J}_{\mathrm{e}}^{-2} + \varepsilon^{-1}\mathcal{J}_{\mathrm{e}}^{-1} + \mathcal{J}_{\mathrm{e}}^{0} + \varepsilon\mathcal{J}_{\mathrm{e}}^{1} + \mathcal{O}(\varepsilon^{2}),\end{aligned} \tag{4.5}$$

where the electron streaming operators read at successive orders

$$\mathscr{D}_{\mathrm{e}}^{-2}(f_{\mathrm{e}}^{0}) = \delta_{b1}q_{\mathrm{e}}(\mathbf{C}_{\mathrm{e}}\wedge\mathbf{B})\cdot\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}^{0},$$

$$\mathscr{D}_{\mathrm{e}}^{-1}(f_{\mathrm{e}}^{0},\phi_{\mathrm{e}}) = \hat{\mathscr{D}}_{\mathrm{e}}^{-1}(f_{\mathrm{e}}^{0}) + q_{\mathrm{e}}(\delta_{b1}\mathbf{C}_{\mathrm{e}}\wedge\mathbf{B})\cdot\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}(f_{\mathrm{e}}^{0}\phi_{\mathrm{e}}),$$

$$
\begin{aligned}
\hat{\mathscr{D}}_{\mathrm{e}}^{-1}(f_{\mathrm{e}}^0) &= \frac{1}{M_h}\mathbf{C}_{\mathrm{e}}\cdot\boldsymbol{\partial}_{\mathbf{x}}f_{\mathrm{e}}^0 + q_{\mathrm{e}}\left(\frac{1}{M_h}\mathbf{E}' + \delta_{b0}\mathbf{C}_{\mathrm{e}}\wedge\mathbf{B}\right)\cdot\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}^0,\\
\mathscr{D}_{\mathrm{e}}^0(f_{\mathrm{e}}^0,\phi_{\mathrm{e}},\phi_{\mathrm{e}}^2) &= \hat{\mathscr{D}}_{\mathrm{e}}^0(f_{\mathrm{e}}^0,\phi_{\mathrm{e}}) + q_{\mathrm{e}}(\delta_{b1}\mathbf{C}_{\mathrm{e}}\wedge\mathbf{B})\cdot\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}}^2),\\
\hat{\mathscr{D}}_{\mathrm{e}}^0(f_{\mathrm{e}}^0,\phi_{\mathrm{e}}) &= \partial_t f_{\mathrm{e}}^0 + \frac{1}{M_h}\mathbf{C}_{\mathrm{e}}\cdot\boldsymbol{\partial}_{\mathbf{x}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}}) + \mathbf{v}_h\cdot\boldsymbol{\partial}_{\mathbf{x}}f_{\mathrm{e}}^0 - (\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}^0\otimes\mathbf{C}_{\mathrm{e}}):\boldsymbol{\partial}_{\mathbf{x}}\mathbf{v}_h\\
&\quad + q_{\mathrm{e}}(\delta_{b0}M_h\mathbf{v}_h\wedge\mathbf{B} + \delta_{b(-1)}\mathbf{C}_{\mathrm{e}}\wedge\mathbf{B})\cdot\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}^0\\
&\quad + q_{\mathrm{e}}\left(\frac{1}{M_h}\mathbf{E}' + \delta_{b0}\mathbf{C}_{\mathrm{e}}\wedge\mathbf{B}\right)\cdot\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}}),\\
\mathscr{D}_{\mathrm{e}}^1(f_{\mathrm{e}}^0,\phi_{\mathrm{e}},\phi_{\mathrm{e}}^2,\phi_{\mathrm{e}}^3) &= \hat{\mathscr{D}}_{\mathrm{e}}^1(f_{\mathrm{e}}^0,\phi_{\mathrm{e}},\phi_{\mathrm{e}}^2) + q_{\mathrm{e}}(\delta_{b1}\mathbf{C}_{\mathrm{e}}\wedge\mathbf{B})\cdot\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}}^3),\\
\hat{\mathscr{D}}_{\mathrm{e}}^1(f_{\mathrm{e}}^0,\phi_{\mathrm{e}},\phi_{\mathrm{e}}^2) &= \partial_t(f_{\mathrm{e}}^0\phi_{\mathrm{e}}) + \frac{1}{M_h}\mathbf{C}_{\mathrm{e}}\cdot\boldsymbol{\partial}_{\mathbf{x}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}}^2) + \mathbf{v}_h\cdot\boldsymbol{\partial}_{\mathbf{x}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}})\\
&\quad - M_h\frac{\mathrm{D}\mathbf{v}_h}{\mathrm{D}t}\cdot\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}^0 - (\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}})\otimes\mathbf{C}_{\mathrm{e}}):\boldsymbol{\partial}_{\mathbf{x}}\mathbf{v}_h\\
&\quad + q_{\mathrm{e}}(\delta_{b(-1)}M_h\mathbf{v}_h\wedge\mathbf{B} + \delta_{b(-2)}\mathbf{C}_{\mathrm{e}}\wedge\mathbf{B})\cdot\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}f_{\mathrm{e}}^0\\
&\quad + q_{\mathrm{e}}(\delta_{b0}M_h\mathbf{v}_h\wedge\mathbf{B} + \delta_{b(-1)}\mathbf{C}_{\mathrm{e}}\wedge\mathbf{B})\cdot\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}})\\
&\quad + q_{\mathrm{e}}\left(\frac{1}{M_h}\mathbf{E}' + \delta_{b0}\mathbf{C}_{\mathrm{e}}\wedge\mathbf{B}\right)\cdot\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}}^2),
\end{aligned}
$$

with the electric field expressed in the heavy-particle reference frame as $\mathbf{E}' = \mathbf{E} + \delta_{b1}M_h^2\mathbf{v}_h\wedge\mathbf{B}$. The electron collision operators are given by

$$
\begin{aligned}
\mathfrak{J}_{\mathrm{e}}^{-2} &= \cancel{\mathfrak{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0,f_{\mathrm{e}}^0)} + \sum_{j\in\mathrm{H}}\cancel{\mathfrak{J}_{\mathrm{e}j}^0(f_{\mathrm{e}}^0,f_j^0)},\\
\mathfrak{J}_{\mathrm{e}}^{-1} &= \mathfrak{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}},f_{\mathrm{e}}^0) + \mathfrak{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0,f_{\mathrm{e}}^0\phi_{\mathrm{e}})\\
&\quad + \sum_{j\in\mathrm{H}}\mathfrak{J}_{\mathrm{e}j}^0(f_{\mathrm{e}}^0\phi_{\mathrm{e}},f_j^0) + \cancel{\mathfrak{J}_{\mathrm{e}j}^0(f_{\mathrm{e}}^0,f_j^0\phi_j)} + \cancel{\mathfrak{J}_{\mathrm{e}j}^1(f_{\mathrm{e}}^0,f_j^0)}\\
\mathfrak{J}_{\mathrm{e}}^0 &= \mathfrak{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}}^2,f_{\mathrm{e}}^0) + \mathfrak{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}},f_{\mathrm{e}}^0\phi_{\mathrm{e}}) + \mathfrak{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0,f_{\mathrm{e}}^0\phi_{\mathrm{e}}^2)\\
&\quad + \sum_{j\in\mathrm{H}}\mathfrak{J}_{\mathrm{e}j}^0(f_{\mathrm{e}}^0\phi_{\mathrm{e}}^2,f_j^0) + \cancel{\mathfrak{J}_{\mathrm{e}j}^0(f_{\mathrm{e}}^0,f_j^0\phi_j^2)} + \hat{\mathfrak{J}}_{\mathrm{e}}^0,\\
\hat{\mathfrak{J}}_{\mathrm{e}}^0 &= \sum_{j\in\mathrm{H}}\mathfrak{J}_{\mathrm{e}j}^0(f_{\mathrm{e}}^0\phi_{\mathrm{e}},f_j^0\phi_j) + \cancel{\mathfrak{J}_{\mathrm{e}j}^1(f_{\mathrm{e}}^0\phi_{\mathrm{e}},f_j^0)} + \mathfrak{J}_{\mathrm{e}j}^1(f_{\mathrm{e}}^0,f_j^0\phi_j) + \mathfrak{J}_{\mathrm{e}j}^2(f_{\mathrm{e}}^0,f_j^0),\\
\mathfrak{J}_{\mathrm{e}}^1 &= \mathfrak{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}}^3,f_{\mathrm{e}}^0) + \mathfrak{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}}^2,f_{\mathrm{e}}^0\phi_{\mathrm{e}}) + \mathfrak{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}},f_{\mathrm{e}}^0\phi_{\mathrm{e}}^2) + \mathfrak{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0,f_{\mathrm{e}}^0\phi_{\mathrm{e}}^3)\\
&\quad + \sum_{j\in\mathrm{H}}\mathfrak{J}_{\mathrm{e}j}^0(f_{\mathrm{e}}^0\phi_{\mathrm{e}}^3,f_j^0) + \cancel{\mathfrak{J}_{\mathrm{e}j}^0(f_{\mathrm{e}}^0,f_j^0\phi_j^3)} + \hat{\mathfrak{J}}_{\mathrm{e}}^1,
\end{aligned}
$$

$$\begin{aligned}\hat{\mathfrak{J}}_{\mathrm{e}}^{1} = \sum_{j\in\mathrm{H}} \{&\mathfrak{J}_{\mathrm{e}j}^{0}(f_{\mathrm{e}}^{0}\phi_{\mathrm{e}}^{2}, f_{j}^{0}\phi_{j}) + \mathfrak{J}_{\mathrm{e}j}^{0}(f_{\mathrm{e}}^{0}\phi_{\mathrm{e}}, f_{j}^{0}\phi_{j}^{2}) + \cancel{\mathfrak{J}_{\mathrm{e}j}^{1}(f_{\mathrm{e}}^{0}\phi_{\mathrm{e}}^{2}, f_{j}^{0})}\\ &+ \mathfrak{J}_{\mathrm{e}j}^{1}(f_{\mathrm{e}}^{0}\phi_{\mathrm{e}}, f_{j}^{0}\phi_{j}) + \mathfrak{J}_{\mathrm{e}j}^{1}(f_{\mathrm{e}}^{0}, f_{j}^{0}\phi_{j}^{2}) + \mathfrak{J}_{\mathrm{e}j}^{2}(f_{\mathrm{e}}^{0}\phi_{\mathrm{e}}, f_{j}^{0})\\ &+ \mathfrak{J}_{\mathrm{e}j}^{2}(f_{\mathrm{e}}^{0}, f_{j}^{0}\phi_{j}) + \mathfrak{J}_{\mathrm{e}j}^{3}(f_{\mathrm{e}}^{0}, f_{j}^{0})\}.\end{aligned}$$

For ease of readability in Secs. 4.2–4.7, we strike through the collision operators that vanish when f_{e}^{0} and f_{i}^{0}, $i \in \mathrm{H}$, are isotropic functions. Likewise, the heavy-particle Boltzmann equation (2.30) is found to be

$$\mathscr{D}_{i}^{0}(f_{i}^{0}) + \varepsilon\mathscr{D}_{i}^{1}(f_{i}^{0}, \phi_{i}) = \varepsilon^{-1}\mathfrak{J}_{i}^{-1} + \mathfrak{J}_{i}^{0} + \varepsilon\mathfrak{J}_{i}^{1} + \mathcal{O}(\varepsilon^{2}), \quad i \in \mathrm{H}, \tag{4.6}$$

where the heavy-particle streaming operators read at successive orders

$$\begin{aligned}\mathscr{D}_{i}^{0}(f_{i}^{0}) &= \partial_t f_{i}^{0} + \left(\frac{1}{M_h}\mathbf{C}_i + \mathbf{v}_h\right)\cdot\boldsymbol{\partial}_{\mathbf{x}} f_{i}^{0} + \frac{q_i}{m_i}\left(\frac{1}{M_h}\mathbf{E}' + \delta_{b1}\mathbf{C}_i\wedge\mathbf{B}\right)\cdot\boldsymbol{\partial}_{\mathbf{C}_i} f_{i}^{0}\\ &\qquad - M_h\frac{\mathrm{D}\mathbf{v}_h}{\mathrm{D}t}\cdot\boldsymbol{\partial}_{\mathbf{C}_i} f_{i}^{0} - (\boldsymbol{\partial}_{\mathbf{C}_i} f_{i}^{0}\otimes\mathbf{C}_i){:}\boldsymbol{\partial}_{\mathbf{x}}\mathbf{v}_h,\\ \mathscr{D}_{i}^{1}(f_{i}^{0}, \phi_i) &= \partial_t(f_{i}^{0}\phi_i) + \left(\frac{1}{M_h}\mathbf{C}_i + \mathbf{v}_h\right)\cdot\boldsymbol{\partial}_{\mathbf{x}}(f_{i}^{0}\phi_i)\\ &\qquad + \frac{q_i}{m_i}\delta_{b0}[(\mathbf{C}_i + M_h\mathbf{v}_h)\wedge\mathbf{B}]\cdot\boldsymbol{\partial}_{\mathbf{C}_i} f_{i}^{0}\\ &\qquad + \frac{q_i}{m_i}\left(\frac{1}{M_h}\mathbf{E}' + \delta_{b1}\mathbf{C}_i\wedge\mathbf{B}\right)\cdot\boldsymbol{\partial}_{\mathbf{C}_i}(f_{i}^{0}\phi_i)\\ &\qquad - M_h\frac{\mathrm{D}\mathbf{v}_h}{\mathrm{D}t}\cdot\boldsymbol{\partial}_{\mathbf{C}_i}(f_{i}^{0}\phi_i) - (\boldsymbol{\partial}_{\mathbf{C}_i}(f_{i}^{0}\phi_i)\otimes\mathbf{C}_i):\boldsymbol{\partial}_{\mathbf{x}}\mathbf{v}_h.\end{aligned}$$

The heavy-particle collision operators are given by

$$\begin{aligned}\mathfrak{J}_{i}^{-1} &= \sum_{j\in\mathrm{H}} \cancel{\mathfrak{J}_{ij}(f_{i}^{0}, f_{j}^{0})} + \cancel{\mathfrak{J}_{i\mathrm{e}}^{1}(f_{i}^{0}, f_{\mathrm{e}}^{0})},\\ \mathfrak{J}_{i}^{0} &= \sum_{j\in\mathrm{H}} \mathfrak{J}_{ij}(f_{i}^{0}\phi_i, f_{j}^{0}) + \mathfrak{J}_{ij}(f_{i}^{0}, f_{j}^{0}\phi_j) + \cancel{\mathfrak{J}_{i\mathrm{e}}^{1}(f_{i}^{0}\phi_i, f_{\mathrm{e}}^{0})} + \hat{\mathfrak{J}}_{i}^{0},\\ \hat{\mathfrak{J}}_{i}^{0} &= \mathfrak{J}_{i\mathrm{e}}^{1}(f_{i}^{0}, f_{\mathrm{e}}^{0}\phi_{\mathrm{e}}) + \mathfrak{J}_{i\mathrm{e}}^{2}(f_{i}^{0}, f_{\mathrm{e}}^{0}),\\ \mathfrak{J}_{i}^{1} &= \sum_{j\in\mathrm{H}} \mathfrak{J}_{ij}(f_{i}^{0}\phi_{i}^{2}, f_{j}^{0}) + \mathfrak{J}_{ij}(f_{i}^{0}\phi_i, f_{j}^{0}\phi_j) + \mathfrak{J}_{ij}(f_{i}^{0}, f_{j}^{0}\phi_{j}^{2})\\ &\qquad + \cancel{\mathfrak{J}_{i\mathrm{e}}^{1}(f_{i}^{0}\phi_{i}^{2}, f_{\mathrm{e}}^{0})} + \hat{\mathfrak{J}}_{i}^{1},\\ \hat{\mathfrak{J}}_{i}^{1} &= \mathfrak{J}_{i\mathrm{e}}^{1}(f_{i}^{0}\phi_i, f_{\mathrm{e}}^{0}\phi_{\mathrm{e}}) + \mathfrak{J}_{i\mathrm{e}}^{1}(f_{i}^{0}, f_{\mathrm{e}}^{0}\phi_{\mathrm{e}}^{2}) + \mathfrak{J}_{i\mathrm{e}}^{2}(f_{i}^{0}\phi_i, f_{\mathrm{e}}^{0})\\ &\qquad + \mathfrak{J}_{i\mathrm{e}}^{2}(f_{i}^{0}, f_{\mathrm{e}}^{0}\phi_{\mathrm{e}}) + \cancel{\mathfrak{J}_{i\mathrm{e}}^{3}(f_{i}^{0}, f_{\mathrm{e}}^{0})}.\end{aligned}$$

In the Chapman–Enskog method, the plasma is described at successive orders of the ε parameter as equivalent to as many timescales. The micro- and macroscopic

Table 2. Chapman–Enskog steps.

Order	Time	Heavy particles	Electrons
ε^{-2}	t_{e}^0		Expression for f_{e}^0 Thermalization (T_{e})
ε^{-1}	t_h^0	Expression for f_i^0, $i \in \mathrm{H}$ Thermalization (T_h)	Equation for ϕ_{e} Zeroth-order momentum relation
ε^0	t^0	Equation for ϕ_i, $i \in \mathrm{H}$ Euler equations	Equation for ϕ_{e}^2 Zeroth-order drift-diffusion equations First-order momentum relation
ε	$\frac{t^0}{\varepsilon}$	Navier–Stokes equations	First-order drift-diffusion equations

equations derived at each order are reviewed in Table 2. The second-order electron momentum equation, relevant to the second-order heavy-particle macroscopic equations, is not established in this study.

4.1. *Order ε^{-2}: Electron thermalization*

We solve the electron Boltzmann equation (4.5) at order ε^{-2} corresponding to the kinetic timescale t_{e}^0. The electron population is shown to thermalize in the heavy-particle reference frame to a quasi-equilibrium state described by a Maxwell–Boltzmann distribution function at temperature T_{e}. In contrast, heavy particles do not exhibit any ensemble property at this order.

Proposition 4.1. *Considering a family of functions f_i^0, $i \in \mathrm{H}$, sufficiently regular so that the collision operators $\mathcal{J}_{\mathrm{e}i}^0(f_{\mathrm{e}}^0, f_i^0)$, $i \in \mathrm{H}$, exist, the zeroth-order electron distribution function f_{e}^0, solution to Eq.* (4.5) *at order ε^{-2}, i.e. $\mathscr{D}_{\mathrm{e}}^{-2}(f_{\mathrm{e}}^0) = \mathcal{J}_{\mathrm{e}}^{-2}$, that satisfies the scalar constraints* (4.3), *is a Maxwell–Boltzmann distribution function at the electron temperature*

$$f_{\mathrm{e}}^0 = n_{\mathrm{e}} \left(\frac{1}{2\pi T_{\mathrm{e}}} \right)^{3/2} \exp\left(-\frac{1}{2T_{\mathrm{e}}} \mathbf{C}_{\mathrm{e}} \cdot \mathbf{C}_{\mathrm{e}} \right). \tag{4.7}$$

Proof. Multiplying the equation $\mathscr{D}_{\mathrm{e}}^{-2}(f_{\mathrm{e}}^0) = \mathcal{J}_{\mathrm{e}}^{-2}$ by $\ln[(2\pi)^{3/2} n^0 f_{\mathrm{e}}^0 / Q_{\mathrm{e}}^0]$ and integrating over $\mathrm{d}\mathbf{C}_{\mathrm{e}}$ yields the zeroth-order entropy production rate

$$\Upsilon_{\mathrm{ee}}^0 + \sum_{j \in \mathrm{H}} \Upsilon_{\mathrm{e}j}^0 + \delta_{b1} q_{\mathrm{e}} \int (\mathbf{C}_{\mathrm{e}} \wedge \mathbf{B}) \cdot \boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}} f_{\mathrm{e}}^0 \ln[(2\pi)^{3/2} n^0 f_{\mathrm{e}}^0 / Q_{\mathrm{e}}^0] \mathrm{d}\mathbf{C}_{\mathrm{e}} = 0,$$

with $\Upsilon_{\mathrm{ee}}^0 = -\int \mathcal{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0, f_{\mathrm{e}}^0)(\mathbf{C}_{\mathrm{e}}) \ln[(2\pi)^{3/2} n^0 f_{\mathrm{e}}^0 / Q_{\mathrm{e}}^0]\, \mathrm{d}\mathbf{C}_{\mathrm{e}}$. Using the equality $\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}} f_{\mathrm{e}}^0 \ln[(2\pi)^{3/2} n^0 f_{\mathrm{e}}^0 / Q_{\mathrm{e}}^0] = \boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}} \{ f_{\mathrm{e}}^0 \ln[(2\pi)^{3/2} n^0 f_{\mathrm{e}}^0 / Q_{\mathrm{e}}^0] - f_{\mathrm{e}}^0 \}$ and integrating by parts, the entropy production rate is found to be $\Upsilon_{\mathrm{ee}}^0 + \sum_{j \in \mathrm{H}} \Upsilon_{\mathrm{e}j}^0 = 0$. Moreover, a well-established derivation yields[14]

$$\Upsilon_{\mathrm{ee}}^0 = \frac{1}{4} \int \Omega(f_{\mathrm{e}}^0 f_{\mathrm{e}1}^0, f_{\mathrm{e}}^{0\prime} f_{\mathrm{e}1}^{0\prime}) |\mathbf{C}_{\mathrm{e}} - \mathbf{C}_{\mathrm{e}1}| \sigma_{\mathrm{ee}1} \mathrm{d}\boldsymbol{\omega} \mathrm{d}\mathbf{C}_{\mathrm{e}} \mathrm{d}\mathbf{C}_{\mathrm{e}1} \geq 0.$$

Using Corollary 3.4, we first obtain that $\Upsilon^0_{\mathrm{e}i} \geq 0$, $i \in \mathrm{H}$, so that both terms $\Upsilon^0_{\mathrm{ee}} = 0$ and $\Upsilon^0_{\mathrm{e}i} = 0$, $i \in \mathrm{H}$. Then, Corollary 3.4 implies that f^0_{e} is isotropic in the heavy-particle reference frame. Seeing that $\Upsilon^0_{\mathrm{ee}} = 0$, $\ln f^0_{\mathrm{e}}$ is thus a collisional invariant, i.e. is in the space $\mathcal{I}_{\mathrm{e}}$. By using the macroscopic constraints, expression (4.7) is readily obtained. □

The choice of the reference frame in which electrons thermalize turns out to be crucial for the rest of the development. In the $\mathbf{u} = \mathbf{v}_h$ frame, the quasi-equilibrium electron velocity distribution function is isotropic and the electrons follow the bulk movement associated with the heavy particles, leading to a physically plausible scenario. As already mentioned, the mean heavy-particle velocity $\mathbf{v}_h$ does not depend on the small ε parameter while still being close to the actual hydrodynamic velocity $\mathbf{v}$ of the entire mixture; this property is essential in order to conduct a rigorous multiscale analysis in the framework of the present Chapman–Enskog expansion. The relevance of such a choice of reference frame will be thoroughly investigated in Sec. 4.8.

4.2. *Order ε^{-1}: Heavy-particle thermalization*

We solve the heavy-particle Boltzmann equation (4.6) at order ε^{-1} corresponding to the kinetic timescale t^0_h. The heavy-particle population is shown to thermalize in the heavy-particle reference frame to a quasi-equilibrium state described by a Maxwell–Boltzmann distribution function at temperature T_h.

Proposition 4.2. *Considering f^0_{e} given by Eq.* (4.7), *the zeroth-order family of heavy-particle distribution functions f^0_h solution to Eq.* (4.6) *at order ε^{-1}, i.e. $\mathcal{J}^{-1}_i = 0$, $i \in \mathrm{H}$, that satisfies the scalar constraints* (4.4), *is a family of Maxwell–Boltzmann distribution functions at the heavy-particle temperature*

$$f^0_i = n_i \left(\frac{m_i}{2\pi T_h}\right)^{3/2} \exp\left(-\frac{m_i}{2T_h}\mathbf{C}_i \cdot \mathbf{C}_i\right), \quad i \in \mathrm{H}. \tag{4.8}$$

Proof. As the zeroth-order electron distribution function f^0_{e} is isotropic in the heavy-particle reference frame, Corollary 3.1 yields that the heavy-particle Boltzmann equation (4.6) reads at order ε^{-1}

$$\sum_{j \in \mathrm{H}} \mathcal{J}_{ij}(f^0_i, f^0_j) = 0, \quad i \in \mathrm{H}.$$

After some classical algebra,[14] we obtain expression (4.8) for the zeroth-order heavy-particle distribution functions. □

Thus, Propositions 4.1 and 4.2 describe electron and heavy-particle quasi-equilibrium states at different temperatures.

4.3. *Order ε^{-1}: Electron momentum relation*

We conduct the solution and derive a momentum relation based on the electron Boltzmann equation (4.5) at order ε^{-1} corresponding to the kinetic timescale t_h^0. We then emphasize an original property of the Chapman–Enskog expansion at this order associated with both the absence of a momentum constraint in Eq. (3.21) and our multiscale analysis.

With the previously obtained Maxwell–Boltzmann electron distribution function, we first define the electron linearized collision operator.

Definition 4.1. The electron linearized collision operator $\mathcal{F}_\mathrm{e}$ reads

$$\mathcal{F}_\mathrm{e}(\phi_\mathrm{e}) = -\frac{1}{f_\mathrm{e}^0}\left[\mathcal{J}_\mathrm{ee}(f_\mathrm{e}^0\phi_\mathrm{e}, f_\mathrm{e}^0) + \mathcal{J}_\mathrm{ee}(f_\mathrm{e}^0, f_\mathrm{e}^0\phi_\mathrm{e}) + \sum_{j\in\mathrm{H}}\mathcal{J}_{\mathrm{e}j}^0(f_\mathrm{e}^0\phi_\mathrm{e}, f_j^0)\right],$$

where f_e^0 is given by Eq. (4.7) and f_i^0 by Eq. (4.8).

The kernel of this operator is given in the following property.

Property 4.1. The kernel of the linearized collision operator $\mathcal{F}_\mathrm{e}$ is the space of scalar electron collisional invariants $\mathcal{I}_\mathrm{e}$.

Proof. The linearized collision operator $\mathcal{F}_\mathrm{e}$ is rewritten in the form

$$\begin{aligned}\mathcal{F}_\mathrm{e}(\phi_\mathrm{e}) = &-\int f_\mathrm{e1}^0\,(\phi_\mathrm{e}' + \phi_\mathrm{e1}' - \phi_\mathrm{e} - \phi_\mathrm{e1})|\mathbf{C}_\mathrm{e} - \mathbf{C}_\mathrm{e1}|\sigma_\mathrm{ee1}\mathrm{d}\boldsymbol{\omega}\mathrm{d}\mathbf{C}_\mathrm{e1}\\ &-\sum_{j\in\mathrm{H}} n_j\int\sigma_{\mathrm{e}j}\left(|\mathbf{C}_\mathrm{e}|^2, \boldsymbol{\omega}\cdot\frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\right)|\mathbf{C}_\mathrm{e}|(\phi_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega}) - \phi_\mathrm{e}(\mathbf{C}_\mathrm{e}))\mathrm{d}\boldsymbol{\omega}.\end{aligned}$$

We then obtain that the space $\mathcal{I}_\mathrm{e}$ is in the kernel of $\mathcal{F}_\mathrm{e}$. Conversely, if $\mathcal{F}_\mathrm{e}(\phi_\mathrm{e}) = 0$, multiplying $\mathcal{F}_\mathrm{e}(\phi_\mathrm{e})$ by $f_\mathrm{e}^0\phi_\mathrm{e}$ and integrating over $\mathrm{d}\mathbf{C}_\mathrm{e}$ yields

$$\begin{aligned}&\frac{1}{4}\int f_\mathrm{e}^0 f_\mathrm{e1}^0\,(\phi_\mathrm{e}' + \phi_\mathrm{e1}' - \phi_\mathrm{e} - \phi_\mathrm{e1})^2|\mathbf{C}_\mathrm{e} - \mathbf{C}_\mathrm{e1}|\sigma_\mathrm{ee1}\mathrm{d}\boldsymbol{\omega}\mathrm{d}\mathbf{C}_\mathrm{e}\mathrm{d}\mathbf{C}_\mathrm{e1}\\ &\quad+\frac{1}{2}\sum_{j\in\mathrm{H}} n_j\int\sigma_{\mathrm{e}j}\left(|\mathbf{C}_\mathrm{e}|^2, \boldsymbol{\omega}\cdot\frac{\mathbf{C}_\mathrm{e}}{|\mathbf{C}_\mathrm{e}|}\right)|\mathbf{C}_\mathrm{e}|f_\mathrm{e}^0(\phi_\mathrm{e}(|\mathbf{C}_\mathrm{e}|\boldsymbol{\omega}) - \phi_\mathrm{e}(\mathbf{C}_\mathrm{e}))^2\mathrm{d}\boldsymbol{\omega}\mathrm{d}\mathbf{C}_\mathrm{e} = 0,\end{aligned}$$

so that ϕ_e is in the space $\mathcal{I}_\mathrm{e}$. □

Based on Corollaries 3.4 and 3.5, the electron Boltzmann equation (4.5) is found to be at order ε^{-1}

$$f_\mathrm{e}^0\mathcal{F}_\mathrm{e}(\phi_\mathrm{e}) + \delta_{b1}q_\mathrm{e}\boldsymbol{\partial}_{\mathbf{C}_\mathrm{e}}(f_\mathrm{e}^0\phi_\mathrm{e})\cdot\mathbf{C}_\mathrm{e}\wedge\mathbf{B} = -\hat{\mathscr{D}}_\mathrm{e}^{-1}(f_\mathrm{e}^0), \tag{4.9}$$

with the constraints

$$\langle\!\langle f_\mathrm{e}^0\phi_\mathrm{e}, \hat{\psi}_\mathrm{e}^l\rangle\!\rangle_\mathrm{e} = 0, \quad l\in\{1,2\}. \tag{4.10}$$

The terms $\boldsymbol{\partial}_{\mathbf{C}_\mathrm{e}}(f_\mathrm{e}^0\phi_\mathrm{e})\cdot\mathbf{C}_\mathrm{e}\wedge\mathbf{B}$ and $\hat{\mathscr{D}}_\mathrm{e}^{-1}(f_\mathrm{e}^0)$ are orthogonal to the kernel of $\mathcal{F}_\mathrm{e}$ for the scalar product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_\mathrm{e}$. Consequently, no macroscopic conservation equations of mass and energy can be derived at this order.

Actually, for *any* value of $\mathbf{w}$, defining the shifted Maxwell–Boltzmann distribution

$$f_{\mathrm{e}}^{\mathbf{w}0} = n_{\mathrm{e}} \left(\frac{1}{2\pi T_{\mathrm{e}}} \right)^{3/2} \exp\left(-\frac{1}{2T_{\mathrm{e}}} (\mathbf{C}_{\mathrm{e}} - \varepsilon M_h \mathbf{w})^2 \right), \tag{4.11}$$

we can expand it as a function of ε

$$f_{\mathrm{e}}^{\mathbf{w}0} = f_{\mathrm{e}}^{0} \left(1 + \varepsilon \frac{M_h}{T_{\mathrm{e}}} \mathbf{C}_{\mathrm{e}} \cdot \mathbf{w} + \varepsilon^2 \frac{M_h^2}{2T_{\mathrm{e}}} \left[-\mathbf{w} \cdot \mathbf{w} + \frac{(\mathbf{C}_{\mathrm{e}} \cdot \mathbf{w})^2}{T_{\mathrm{e}}} \right] \right) + \mathcal{O}(\varepsilon^3), \tag{4.12}$$

which still yields, at leading order, the same distribution as defined in Eq. (4.7). We then realize that the Chapman–Enskog expansion can be rewritten in a different way at this order

$$f_{\mathrm{e}}^{0}(1 + \varepsilon \phi_{\mathrm{e}} + \varepsilon^2 \phi_{\mathrm{e}}^2) = f_{\mathrm{e}}^{\mathbf{w}0}(1 + \varepsilon \phi_{\mathrm{e}}^{\mathbf{w}} + \varepsilon^2 \phi_{\mathrm{e}}^{\mathbf{w}0}) + \mathcal{O}(\varepsilon^3), \tag{4.13}$$

with

$$\begin{aligned} \phi_{\mathrm{e}} &= \phi_{\mathrm{e}}^{\mathbf{w}} + \frac{M_h}{T_{\mathrm{e}}} \mathbf{C}_{\mathrm{e}} \cdot \mathbf{w}, \\ \phi_{\mathrm{e}}^2 &= \phi_{\mathrm{e}}^{\mathbf{w}2} + \frac{M_h}{T_{\mathrm{e}}} (\mathbf{C}_{\mathrm{e}} \cdot \mathbf{w}) \phi_{\mathrm{e}}^{\mathbf{w}} + \frac{M_h^2}{2T_{\mathrm{e}}} \left[-\mathbf{w} \cdot \mathbf{w} + \frac{(\mathbf{C}_{\mathrm{e}} \cdot \mathbf{w})^2}{T_{\mathrm{e}}} \right]. \end{aligned} \tag{4.14}$$

It is interesting to notice that, whatever the choice of $\mathbf{w}$, the part of the hydrodynamic velocity of the full mixture

$$(\rho_h + \varepsilon^2 \rho_{\mathrm{e}})\mathbf{v} = \rho_h \mathbf{v}_h + \varepsilon^2 \rho_{\mathrm{e}} \mathbf{v}_{\mathrm{e}},$$

associated with the electrons $\rho_{\mathrm{e}} \mathbf{v}_{\mathrm{e}}$ will be split into two parts at the same order of the multiscale expansion

$$\mathbf{v}_{\mathrm{e}} = \mathbf{v}_h + \frac{1}{M_h} \mathbf{V}_{\mathrm{e}} + \mathcal{O}(\varepsilon) = \mathbf{v}_h + \mathbf{w} + \frac{1}{M_h} \mathbf{V}_{\mathrm{e}}^{\mathbf{w}} + \mathcal{O}(\varepsilon),$$

with $\rho_{\mathrm{e}} \mathbf{V}_{\mathrm{e}}^{\mathbf{w}} = \int \mathbf{C}_{\mathrm{e}} f_{\mathrm{e}}^{\mathbf{w}0} \phi_{\mathrm{e}}^{\mathbf{w}} \mathrm{d}\mathbf{C}_{\mathrm{e}}$. Thus, as opposed to the standard expansion, since no momentum constraint is to be found for the electrons, the definition of the mixture hydrodynamic velocity does not allow to uniquely define electron diffusion velocities. In any case, the hydrodynamic velocity of the mixture is $\mathbf{v}_h$ at order ε^{-1}. It is then necessary to properly delineate the possible choices for the $\mathbf{w}$ velocity, which should not be confused with a change of reference frame, since it only influences the electron Chapman–Enskog expansion.

Lemma 4.1. *In the chosen reference frame, any velocity* $\mathbf{w}$ *leads to a new definition of* $\phi_{\mathrm{e}}^{\mathbf{w}}$ *for which property* 4.1 *is preserved and thus leads to an equivalent solvability condition for* $\phi_{\mathrm{e}}^{\mathbf{w}}$ *as for* ϕ_{e}. *Moreover, the solution for* $\phi_{\mathrm{e}}^{\mathbf{w}}$ *is completely equivalent to the solution for* ϕ_{e}.

Proof. It is sufficient to note that the difference $\delta\phi_{\mathrm{e}}^{\mathbf{w}} = \phi_{\mathrm{e}}^{\mathbf{w}} - \phi_{\mathrm{e}} = -M_h \mathbf{C}_{\mathrm{e}} \cdot \mathbf{w} / T_{\mathrm{e}}$ is orthogonal to the collisional invariants $\langle\!\langle f_{\mathrm{e}}^{0} \delta\phi_{\mathrm{e}}^{\mathbf{w}}, \hat{\psi}_{\mathrm{e}}^{l} \rangle\!\rangle_{\mathrm{e}} = 0$, $l \in \{1, 2\}$. □

For our choice of moving frame $\mathbf{u} = \mathbf{v}_h$, the electron thermalization naturally occurs in the "appropriate" reference frame in close connection to the physics of the problem, and there is no need to use the above-mentioned property in order to conduct the solution at order ε^{-1}. Therefore, we take $\mathbf{w} = 0$ in the following. We will also have to check the validity of such a strategy at higher orders; we will come back to this point in Sec. 4.5.

As mentioned earlier, the partial collision operators $\mathcal{J}^0_{\mathrm{e}i}$, $i \in \mathrm{H}$, are not orthogonal to the space spanned by the vector $\mathbf{C}_\mathrm{e}$. However, an electron momentum relation is obtained by projecting Eq. (4.9) onto this space. First, the electron pressure, diffusion velocity, mean velocity, conduction current density in the mean heavy-particle velocity frame, and conduction current density in the inertial reference frame are defined as

$$p_\mathrm{e} = n_\mathrm{e} T_\mathrm{e}, \tag{4.15}$$

$$\mathbf{V}_\mathrm{e} = \frac{1}{n_\mathrm{e}} \int \mathbf{C}_\mathrm{e} f^0_\mathrm{e} \phi_\mathrm{e} \mathrm{d}\mathbf{C}_\mathrm{e}, \quad \mathbf{v}_\mathrm{e} = \mathbf{v}_h + \frac{1}{M_h} \mathbf{V}_\mathrm{e}, \tag{4.16}$$

$$\mathbf{J}_\mathrm{e} = n_\mathrm{e} q_\mathrm{e} \mathbf{V}_\mathrm{e}, \quad \mathbf{j}_\mathrm{e} = n_\mathrm{e} q_\mathrm{e} \mathbf{v}_\mathrm{e}. \tag{4.17}$$

Then, we have the following proposition.

Proposition 4.3. *Considering f^0_e given by Eq. (4.7) and f^0_i, $i \in \mathrm{H}$, by Eq. (4.8), the zeroth-order momentum transferred from electrons to heavy particles reads*

$$\sum_{j \in \mathrm{H}} \langle\!\langle \mathcal{J}^0_{\mathrm{e}j}(f^0_\mathrm{e}\phi_\mathrm{e}, f^0_j), \mathbf{C}_\mathrm{e} \rangle\!\rangle_\mathrm{e} = \frac{1}{M_h} \boldsymbol{\partial}_\mathbf{x} p_\mathrm{e} - \frac{n_\mathrm{e} q_\mathrm{e}}{M_h} \mathbf{E} - \delta_{b1} \mathbf{j}_\mathrm{e} \wedge \mathbf{B}. \tag{4.18}$$

Proof. Equation (4.9) is projected onto the space spanned by the vector $\mathbf{C}_\mathrm{e}$

$$-\langle\!\langle f^0_\mathrm{e} \mathcal{F}_\mathrm{e}(\phi_\mathrm{e}), \mathbf{C}_\mathrm{e} \rangle\!\rangle_\mathrm{e} = \langle\!\langle \hat{\mathscr{D}}^{-1}_\mathrm{e}, \mathbf{C}_\mathrm{e} \rangle\!\rangle_\mathrm{e} + \delta_{b1} q_\mathrm{e} \langle\!\langle \boldsymbol{\partial}_{\mathbf{C}_\mathrm{e}} (f^0_\mathrm{e}\phi_\mathrm{e}) \cdot (\mathbf{C}_\mathrm{e} \wedge \mathbf{B}), \mathbf{C}_\mathrm{e} \rangle\!\rangle_\mathrm{e}.$$

Then, Eq. (4.18) is readily established by simplifying the left-hand side by means of Eq. (2.36), $\langle\!\langle \mathbf{C}_\mathrm{e}, \mathcal{J}_\mathrm{ee} \rangle\!\rangle_\mathrm{e} = 0$, at order ε and by integrating by parts the right-hand side. □

The zeroth-order momentum transferred from electrons to heavy particles is thus expressed in terms of the electron pressure and electric force. In addition, the following lemma allows for the momentum transferred from heavy particles to electrons to be calculated at order zero.

Lemma 4.2. *Considering f^0_e given by Eq. (4.7) and f^0_i, $i \in \mathrm{H}$, by Eq. (4.8), the net zeroth-order momentum exchanged between electrons and heavy particles vanishes, i.e.*

$$\langle\!\langle \mathcal{J}^1_{h\mathrm{e}}(f^0_h, f^0_\mathrm{e}\phi_\mathrm{e}), \hat{\psi}^{n_h+\nu}_h \rangle\!\rangle_h + \sum_{j \in \mathrm{H}} \langle\!\langle \mathcal{J}^0_{\mathrm{e}j}(f^0_\mathrm{e}\phi_\mathrm{e}, f^0_j), C_{\mathrm{e}\nu} \rangle\!\rangle_\mathrm{e} = 0, \tag{4.19}$$

for $\nu \in \{1, 2, 3\}$.

Proof. Equation (4.19) is derived from Eq. (3.19) at order ε^2 based on Corollaries 3.1–3.5. □

Moreover, the zeroth-order momentum transferred from heavy particles to electrons can be directly calculated after introducing the average force of an electron acting on a heavy particle i given by

$$\mathbf{F}_{i\mathrm{e}} = \int Q_{i\mathrm{e}}^{(1)}(|\boldsymbol{\gamma}_\mathrm{e}|^2)|\boldsymbol{\gamma}_\mathrm{e}|\boldsymbol{\gamma}_\mathrm{e} f_\mathrm{e}^0(\boldsymbol{\gamma}_\mathrm{e})\phi_\mathrm{e}(\boldsymbol{\gamma}_\mathrm{e})\mathrm{d}\boldsymbol{\gamma}_\mathrm{e}, \quad i \in \mathrm{H}. \tag{4.20}$$

Lemma 4.3. *Considering* f_e^0 *given by Eq.* (4.7) *and* f_i^0, $i \in \mathrm{H}$, *by Eq.* (4.8), *the zeroth-order momentum transferred from heavy particles to electrons reads*

$$\langle\!\langle \mathcal{J}_{h\mathrm{e}}^1(f_h^0, f_\mathrm{e}^0\phi_\mathrm{e}), \hat{\psi}_h^{n_h+\nu} \rangle\!\rangle_h = \sum_{j\in\mathrm{H}} n_j F_{j\mathrm{e}\nu}, \tag{4.21}$$

for $\nu \in \{1, 2, 3\}$.

Proof. Equation (4.21) is derived by means of Lemma 4.2, Theorem 3.2, and definitions (3.10) and (4.20). □

We will see that the average forces $\mathbf{F}_{i\mathrm{e}}$, $i \in \mathrm{H}$, contribute to the heavy-particle diffusion driving forces and, in particular, yield anisotropic diffusion velocities for the heavy particles in the $b = 1$ case.

4.4. *Order ε^0: Heavy-particle Euler equations*

We derive Euler equations based on the heavy-particle Boltzmann equation (4.6) at order ε^0 corresponding to the macroscopic timescale t^0. First, a linearized collision operator is introduced for heavy particles.

Definition 4.2. The linearized collision operator $\mathcal{F}_h = (\mathcal{F}_i)_{i\in\mathrm{H}}$ reads

$$\mathcal{F}_i(\phi_h) = -\frac{1}{f_i^0}\sum_{j\in\mathrm{H}}[\mathcal{J}_{ij}(f_i^0\phi_i, f_j^0) + \mathcal{J}_{ij}(f_i^0, f_j^0\phi_j)], \quad i \in \mathrm{H},$$

where f_i^0, $i \in \mathrm{H}$, is given by Eq. (4.8), for a family $\phi_h = (\phi_i)_{i\in\mathrm{H}}$.

The first nonvanishing term of the partial collision operator $\mathcal{J}_{h\mathrm{e}}$ is not included in the linearized collision operator since it is not orthogonal to $\mathcal{I}_h$ for the scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_h$. The kernel of $\mathcal{F}_h$ is given in the following property, the proof of which is omitted since it is a well-established result.[26]

Property 4.2. The kernel of the linearized collision operator $\mathcal{F}_h$ is the space of scalar collisional invariants $\mathcal{I}_h$.

Furthermore, we define the heavy-particle pressure, $p_h = n_h T_h$, the global pressure, $p = p_\mathrm{e} + p_h$, the heavy-particle charge, $n_h q_h = \sum_{j\in\mathrm{H}} n_j q_j$, the global charge,

$nq = n_\mathrm{e}q_\mathrm{e} + n_h q_h$, and the total current density $\mathbf{I}_0 = n_h q_h \mathbf{v}_h + n_\mathrm{e} q_\mathrm{e} \mathbf{v}_\mathrm{e}$. The energy transferred from heavy particles to electrons reads at order zero

$$\Delta E_h^0 = \langle\!\langle \mathcal{J}_{h\mathrm{e}}^2(f_h^0, f_\mathrm{e}^0), \hat{\psi}_h^{n_h+4} \rangle\!\rangle_h. \tag{4.22}$$

This quantity is of the order of the thermal energy divided by the macroscopic timescale, $n^0 \mathrm{k_B} T^0 / t^0$. The accurate Landau–Teller expression is calculated by means of Corollary 3.2

$$\Delta E_h^0 = \frac{\frac{3}{2} n_\mathrm{e}(T_\mathrm{e} - T_h)}{\tau}, \quad \frac{1}{\tau} = \sum_{j \in \mathrm{H}} \frac{2 n_j}{3 n_\mathrm{e} m_j} \nu_{j\mathrm{e}}, \tag{4.23}$$

where τ is the average collision time at which this energy transfer occurs. Then, the heavy-particle Euler equations are derived in the following proposition.

Proposition 4.4. *If ϕ_h is a solution to Eq.* (4.6) *at order ε^0, i.e.*

$$f_i^0 \mathcal{F}_i(\phi_h) = -\mathscr{D}_i^0(f_i^0) + \hat{\mathcal{J}}_i^0, \quad i \in \mathrm{H}, \tag{4.24}$$

where f_e^0 is given by Eq. (4.7), *f_i^0, $i \in \mathrm{H}$, by Eq.* (4.8), *and ϕ_e by Eqs.* (4.9)–(4.10), *and if $f_h^0 \phi_h = (f_i^0 \phi_i)_{i \in \mathrm{H}}$ satisfies the constraints*

$$\langle\!\langle f_h^0 \phi_h, \hat{\psi}_h^l \rangle\!\rangle_h = 0, \quad l \in \{1, \ldots, n^\mathrm{H} + 4\}, \tag{4.25}$$

then, the zeroth-order conservation equations of heavy-particle mass, momentum and energy read

$$\partial_t \rho_i + \boldsymbol{\partial}_\mathbf{x} \cdot (\rho_i \mathbf{v}_h) = 0, \quad i \in \mathrm{H}, \tag{4.26}$$

$$\partial_t(\rho_h \mathbf{v}_h) + \boldsymbol{\partial}_\mathbf{x} \cdot \left(\rho_h \mathbf{v}_h \otimes \mathbf{v}_h + \frac{1}{M_h^2} p \mathbb{I} \right) = \frac{1}{M_h^2} nq\mathbf{E} + \delta_{b1} \mathbf{I}_0 \wedge \mathbf{B}, \tag{4.27}$$

$$\partial_t(\rho_h e_h) + \boldsymbol{\partial}_\mathbf{x} \cdot (\rho_h e_h \mathbf{v}_h) = -p_h \boldsymbol{\partial}_\mathbf{x} \cdot \mathbf{v}_h + \Delta E_h^0. \tag{4.28}$$

Proof. Fredholm's alternative[31] represents the solvability condition of Eq. (4.24)

$$\langle\!\langle \mathscr{D}_h^0, \hat{\psi}_h^l \rangle\!\rangle_h = \langle\!\langle \hat{\mathcal{J}}_h^0, \hat{\psi}_h^l \rangle\!\rangle_h,$$

$l \in \{1, \ldots, n^\mathrm{H} + 4\}$. Integrating by parts the left-hand side and simplifying the right-hand side based on Theorem 3.1 and Corollary 3.2, one obtains Eqs. (4.26), (4.28), and the following momentum conservation equation

$$\begin{aligned} &-M_h \rho_h \frac{\mathrm{D}\mathbf{v}_h}{\mathrm{D}t} - \frac{1}{M_h} \boldsymbol{\partial}_\mathbf{x} p_h + \frac{1}{M_h} n_h q_h \mathbf{E}' \\ &\qquad + \langle\!\langle \mathcal{J}_{h\mathrm{e}}^1\big(f_h^0, f_\mathrm{e}^0 \phi_\mathrm{e}\big), (\hat{\psi}_h^{n_h+\nu})_{\nu \in \{1,2,3\}} \rangle\!\rangle_h = 0. \end{aligned} \tag{4.29}$$

Simplifying this equation by means of the heavy-particle mass conservation equation $\partial_t \rho_h + \boldsymbol{\partial}_\mathbf{x} \cdot (\rho_h \mathbf{v}_h) = 0$ and Lemma 4.2, yields Eq. (4.27). □

4.5. *Order ε^0: Zeroth-order electron drift-diffusion equations*

We derive zeroth-order electron drift-diffusion equations and a momentum relation based on the electron Boltzmann equation (4.5) at order ε^0 corresponding to the macroscopic timescale t^0. We also prove, at this order of the solution, that any nonzero shift introduced at the previous order leads to a series of difficulties at the present order. It thus demonstrates that the initial choice of reference frame leads to a quite natural solution at successive orders.

Based on the Maxwell–Boltzmann electron distribution function previously obtained in Eq. (4.7) we introduce the electron heat flux

$$\mathbf{q}_{\mathrm{e}} = \int \frac{1}{2}\mathbf{C}_{\mathrm{e}} \cdot \mathbf{C}_{\mathrm{e}}\mathbf{C}_{\mathrm{e}} f_{\mathrm{e}}^0 \phi_{\mathrm{e}} \mathrm{d}\mathbf{C}_{\mathrm{e}}. \tag{4.30}$$

The energy transferred from electrons to heavy particles reads at order zero

$$\Delta E_{\mathrm{e}}^0 = \sum_{j \in \mathrm{H}} \langle\!\langle \mathcal{J}_{\mathrm{e}j}^2(f_{\mathrm{e}}^0, f_j^0), \hat{\psi}_{\mathrm{e}}^2 \rangle\!\rangle_{\mathrm{e}}. \tag{4.31}$$

This expression is calculated by means of Eq. (3.20) at order ε^2

$$\Delta E_{\mathrm{e}}^0 + \Delta E_h^0 = 0, \tag{4.32}$$

where ΔE_h^0 is given by Eq. (4.23). Then, the zeroth-order electron drift-diffusion equations are derived in the following proposition.

Proposition 4.5. *If ϕ_{e}^2 is a solution to Eq.* (4.5) *at order ε^0, i.e.*

$$\begin{aligned} f_{\mathrm{e}}^0 \mathcal{F}_{\mathrm{e}}(\phi_{\mathrm{e}}^2) &+ \delta_{b1} q_{\mathrm{e}} \boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}(f_{\mathrm{e}}^0 \phi_{\mathrm{e}}^2) \cdot \mathbf{C}_{\mathrm{e}} \wedge \mathbf{B} \\ &= -\hat{\mathscr{D}}_{\mathrm{e}}^0(f_{\mathrm{e}}^0, \phi_{\mathrm{e}}) + \mathcal{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0 \phi_{\mathrm{e}}, f_{\mathrm{e}}^0 \phi_{\mathrm{e}}) + \hat{\mathcal{J}}_{\mathrm{e}}^0, \end{aligned} \tag{4.33}$$

where f_{e}^0 is given by Eq. (4.7), *f_i^0, $i \in \mathrm{H}$, by Eq.* (4.8), *ϕ_{e} by Eqs.* (4.9)–(4.10), *and ϕ_i, $i \in \mathrm{H}$ by Eqs.* (4.24)–(4.25), *and if $f_{\mathrm{e}}^0 \phi_{\mathrm{e}}^2$ satisfies the constraints*

$$\langle\!\langle f_{\mathrm{e}}^0 \phi_{\mathrm{e}}^2, \hat{\psi}_{\mathrm{e}}^l \rangle\!\rangle_{\mathrm{e}} = 0, \quad l \in \{1, 2\}, \tag{4.34}$$

then, the zeroth-order conservation equations of electron mass and energy read

$$\partial_t \rho_{\mathrm{e}} + \boldsymbol{\partial}_{\mathbf{x}} \cdot \left(\rho_{\mathrm{e}} \mathbf{v}_h + \frac{1}{M_h} \rho_{\mathrm{e}} \mathbf{V}_{\mathrm{e}} \right) = 0, \tag{4.35}$$

$$\partial_t(\rho_{\mathrm{e}} e_{\mathrm{e}}) + \boldsymbol{\partial}_{\mathbf{x}} \cdot (\rho_{\mathrm{e}} e_{\mathrm{e}} \mathbf{v}_h) = -p_{\mathrm{e}} \boldsymbol{\partial}_{\mathbf{x}} \cdot \mathbf{v}_h - \frac{1}{M_h} \boldsymbol{\partial}_{\mathbf{x}} \cdot \mathbf{q}_{\mathrm{e}} + \frac{1}{M_h} \mathbf{J}_{\mathrm{e}} \cdot \mathbf{E}' + \Delta E_{\mathrm{e}}^0. \tag{4.36}$$

Proof. Fredholm's alternative[31] represents the solvability condition of Eq. (4.33)

$$\langle\!\langle \hat{\mathscr{D}}_{\mathrm{e}}^0, \hat{\psi}_{\mathrm{e}}^l \rangle\!\rangle_{\mathrm{e}} = \langle\!\langle \hat{\mathcal{J}}_{\mathrm{e}}^0, \hat{\psi}_{\mathrm{e}}^l \rangle\!\rangle_{\mathrm{e}}, \quad l \in \{1, 2\}.$$

Integrating by parts the left-hand side and simplifying the right-hand side based on Theorem 3.2 and Corollary 3.5 yields Eqs. (4.35) and (4.36). □

Lemma 4.4. *In the chosen reference frame, any velocity* $\mathbf{w}$ *leads to a new definition of* $\phi_{\mathrm{e}}^{\mathbf{w}2}$ *in Eq.* (4.14), *for which property* 4.1 *is preserved, and thus leads to an equivalent solvability condition for* $\phi_{\mathrm{e}}^{\mathbf{w}2}$ *and* ϕ_{e}^{2}. *However, the solution for* $\phi_{\mathrm{e}}^{\mathbf{w}2}$ *is not equivalent to the solution for* ϕ_{e}^{2}: *in particular, the expansion corresponding to* $\mathbf{w} \neq 0$ *yields a nonstandard Chapman–Enskog expansion where the second-order perturbation function does not satisfy the scalar constraints* (4.34).

Proof. Using Eq. (4.14), the difference between $\phi_{\mathrm{e}}^{\mathbf{w}2}$ and ϕ_{e}^{2} reads

$$\delta\phi_{\mathrm{e}}^{\mathbf{w}2} = \phi_{\mathrm{e}}^{\mathbf{w}2} - \phi_{\mathrm{e}}^{2} = -\frac{M_h}{T_{\mathrm{e}}}(\mathbf{C}_{\mathrm{e}} \cdot \mathbf{w})\phi_{\mathrm{e}} + \frac{M_h^2}{2T_{\mathrm{e}}}\left[\mathbf{w}\cdot\mathbf{w} + \frac{(\mathbf{C}_{\mathrm{e}}\cdot\mathbf{w})^2}{T_{\mathrm{e}}}\right].$$

The projection of $\delta\phi_{\mathrm{e}}^{\mathbf{w}2}$ onto the collisional invariants is given by

$$\langle\!\langle f_{\mathrm{e}}^0 \delta\phi_{\mathrm{e}}^{\mathbf{w}2}, \hat{\psi}_{\mathrm{e}}^1 \rangle\!\rangle_{\mathrm{e}} = \frac{M_h}{T_{\mathrm{e}}} n_{\mathrm{e}} \mathbf{w} \cdot (M_h \mathbf{w} - \mathbf{V}_{\mathrm{e}}),$$

$$\langle\!\langle f_{\mathrm{e}}^0 \delta\phi_{\mathrm{e}}^{\mathbf{w}2}, \hat{\psi}_{\mathrm{e}}^2 \rangle\!\rangle_{\mathrm{e}} = M_h \mathbf{w} \cdot \left(2M_h n_{\mathrm{e}} \mathbf{w} - \frac{1}{T_{\mathrm{e}}}\mathbf{q}_{\mathrm{e}}\right).$$

The difference $\delta\phi_{\mathrm{e}}^{\mathbf{w}2}$ is then orthogonal to the collisional invariants if and only if $\mathbf{w} = 0$. To conclude, the solution for $\phi_{\mathrm{e}}^{\mathbf{w}2}$ yields a linearized Boltzmann equation where the right-hand side is orthogonal to the collisional invariants — a direct calculation shows that $\mathcal{F}_{\mathrm{e}}(\delta\phi_{\mathrm{e}}^{\mathbf{w}2}) + \delta_{b1} q_{\mathrm{e}} \boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}(\delta\phi_{\mathrm{e}}^{\mathbf{w}2}) \cdot \mathbf{C}_{\mathrm{e}} \wedge \mathbf{B}$ is orthogonal to the collisional invariants — whereas the scalar constraints on the unknown function $\phi_{\mathrm{e}}^{\mathbf{w}2}$ are not zero. □

Consequently, for the reasons invoked so far, we will not try to shift the center of the Maxwell–Boltzmann distribution for electrons and stick with $\mathbf{w} = 0$ at all orders.

We define the electron viscous tensor, second-order electron diffusion velocity, and second-order current density as

$$\boldsymbol{\Pi}_{\mathrm{e}} = \int \mathbf{C}_{\mathrm{e}} \otimes \mathbf{C}_{\mathrm{e}} f_{\mathrm{e}}^0 \phi_{\mathrm{e}} \mathrm{d}\mathbf{C}_{\mathrm{e}}, \tag{4.37}$$

$$\mathbf{V}_{\mathrm{e}}^2 = \frac{1}{n_{\mathrm{e}}} \int \mathbf{C}_{\mathrm{e}} f_{\mathrm{e}}^0 \phi_{\mathrm{e}}^2 \mathrm{d}\mathbf{C}_{\mathrm{e}}, \tag{4.38}$$

$$\mathbf{J}_{\mathrm{e}}^2 = n_{\mathrm{e}} q_{\mathrm{e}} \mathbf{V}_{\mathrm{e}}^2. \tag{4.39}$$

A first-order electron momentum relation is given in the following proposition.

Proposition 4.6. *Considering* f_{e}^0 *given by Eq.* (4.7), f_i^0, $i \in \mathrm{H}$, *by Eq.* (4.8), ϕ_{e} *by Eqs.* (4.9)–(4.10), ϕ_i, $i \in \mathrm{H}$, *by Eqs.* (4.24)–(4.25), *and* ϕ_{e}^2 *by Eqs.* (4.33)–(4.34), *the first-order momentum transferred from electrons to heavy particles reads*

$$\sum_{j\in\mathrm{H}} \langle\!\langle \mathcal{J}_{\mathrm{e}j}^0(f_{\mathrm{e}}^0\phi_{\mathrm{e}}^2, f_j^0), \mathbf{C}_{\mathrm{e}} \rangle\!\rangle_{\mathrm{e}} + \langle\!\langle \hat{\mathcal{J}}_{\mathrm{e}}^0, \mathbf{C}_{\mathrm{e}} \rangle\!\rangle_{\mathrm{e}} = \frac{1}{M_h} \boldsymbol{\partial}_{\mathbf{x}} \cdot \boldsymbol{\Pi}_{\mathrm{e}} - \left(\delta_{b0}\mathbf{j}_{\mathrm{e}} + \delta_{b1}\mathbf{J}_{\mathrm{e}}^2\right) \wedge \mathbf{B}. \tag{4.40}$$

Proof. Equation (4.33) is projected onto the space spanned by the vector $\mathbf{C}_\mathrm{e}$

$$-\langle\!\langle f_\mathrm{e}^0 \mathcal{F}_\mathrm{e}(\phi_\mathrm{e}^2), \mathbf{C}_\mathrm{e}\rangle\!\rangle_\mathrm{e} + \langle\!\langle \hat{\mathcal{J}}_\mathrm{e}^0, \mathbf{C}_\mathrm{e}\rangle\!\rangle_\mathrm{e}$$
$$= \hat{\mathscr{D}}_\mathrm{e}^0(f_\mathrm{e}^0, \phi_\mathrm{e}) + \delta_{b1} q_\mathrm{e} \langle\!\langle \boldsymbol{\partial}_{\mathbf{C}_\mathrm{e}}(f_\mathrm{e}^0 \phi_\mathrm{e}^2) \cdot \mathbf{C}_\mathrm{e} \wedge \mathbf{B}, \mathbf{C}_\mathrm{e}\rangle\!\rangle_\mathrm{e}.$$

Then, Eq. (4.40) is readily established by simplifying the left-hand side by means of Eq. (2.36) at order ε^2, $\langle\!\langle \mathbf{C}_\mathrm{e}, \mathcal{J}_\mathrm{ee}\rangle\!\rangle_\mathrm{e} = 0$, and by integrating by parts the right-hand side. □

The first-order momentum transferred from electrons to heavy particles is thus expressed in terms of the electron viscous tensor and electric force. The following lemma allows for the momentum transferred from heavy particles to electrons to be calculated at order ε.

Lemma 4.5. *Considering* f_e^0 *given by Eq.* (4.7), f_i^0, $i \in \mathrm{H}$, *by Eq.* (4.8), ϕ_e *by Eqs.* (4.9)–(4.10), ϕ_i, $i \in \mathrm{H}$, *by Eqs.* (4.24)–(4.25), *and* ϕ_e^2 *by Eqs.* (4.33)–(4.34), *the net first-order momentum exchanged between electrons and heavy particles vanishes, i.e.*

$$\langle\!\langle \hat{\mathcal{J}}_h^1, \hat{\psi}_h^{n_h+\nu}\rangle\!\rangle_h + \sum_{j\in\mathrm{H}} \langle\!\langle \mathcal{J}_{\mathrm{e}j}^0(f_\mathrm{e}^0\phi_\mathrm{e}^2, f_j^0), C_{\mathrm{e}\nu}\rangle\!\rangle_\mathrm{e} + \langle\!\langle \hat{\mathcal{J}}_\mathrm{e}^0, C_{\mathrm{e}\nu}\rangle\!\rangle_\mathrm{e} = 0, \quad \nu \in \{1,2,3\}. \tag{4.41}$$

Proof. Equation (4.41) is derived from Eq. (3.19) at order ε^2 based on Corollaries 3.1–3.6. □

4.6. *Order ε: Heavy-particle Navier–Stokes equations*

We derive Navier–Stokes equations based on the heavy-particle Boltzmann equation (4.6) at order ε. First, we introduce the diffusion velocity and mean velocity of species $i \in \mathrm{H}$,

$$\mathbf{V}_i = \frac{1}{n_i}\int \mathbf{C}_i f_i^0 \phi_i \mathrm{d}\mathbf{C}_i, \quad \mathbf{v}_i = \mathbf{v}_h + \frac{\varepsilon}{M_h}\mathbf{V}_i, \;\; i \in \mathrm{H}, \tag{4.42}$$

the heavy-particle viscous tensor,

$$\mathbf{\Pi}_h = \sum_{j\in\mathrm{H}} \int m_j \mathbf{C}_j \otimes \mathbf{C}_j f_j^0 \phi_j \mathrm{d}\mathbf{C}_j, \tag{4.43}$$

the second-order electron mean velocity,

$$\mathbf{v}_\mathrm{e}^2 = \mathbf{v}_h + \frac{1}{M_h}\mathbf{V}_\mathrm{e} + \frac{\varepsilon}{M_h}\mathbf{V}_\mathrm{e}^2, \tag{4.44}$$

the heavy-particle heat flux,

$$\mathbf{q}_h = \sum_{j\in\mathrm{H}} \int \frac{1}{2} m_j \mathbf{C}_j \cdot \mathbf{C}_j \mathbf{C}_j f_j^0 \phi_j \mathrm{d}\mathbf{C}_j, \tag{4.45}$$

the heavy-particle conduction current density in the mean heavy-particle velocity frame, the heavy-particle conduction current density in the inertial reference frame, the second-order electron conduction current density in the inertial reference frame, and the total current density,

$$\mathbf{J}_h = \sum_{j\in\mathrm{H}} n_j q_j \mathbf{V}_j, \quad \mathbf{j}_h = \sum_{j\in\mathrm{H}} n_j q_j \mathbf{v}_j, \quad \mathbf{j}_\mathrm{e}^2 = n_\mathrm{e} q_\mathrm{e} \mathbf{v}_\mathrm{e}^2, \quad \mathbf{I} = \mathbf{j}_h + \mathbf{j}_\mathrm{e}^2. \tag{4.46}$$

Furthermore, we define the energy transferred from heavy particles to electrons at order ε as

$$\begin{aligned}\Delta E_h^1 = {} & \langle\!\langle \mathcal{J}_{h\mathrm{e}}^1(f_h^0\phi_h, f_\mathrm{e}^0\phi_\mathrm{e}), \hat{\psi}_h^{n_h+4}\rangle\!\rangle_h + \langle\!\langle \mathcal{J}_{h\mathrm{e}}^2(f_h^0, f_\mathrm{e}^0\phi_\mathrm{e}), \hat{\psi}_h^{n_h+4}\rangle\!\rangle_h \\ & + \langle\!\langle \mathcal{J}_{h\mathrm{e}}^2(f_h^0\phi_h, f_\mathrm{e}^0), \hat{\psi}_h^{n_h+4}\rangle\!\rangle_h. \end{aligned} \tag{4.47}$$

The first term can be calculated by means of Theorem 3.1

$$\langle\!\langle \mathcal{J}_{h\mathrm{e}}^1(f_h^0\phi_h, f_\mathrm{e}^0\phi_\mathrm{e}), \hat{\psi}_h^{n_h+4}\rangle\!\rangle_h = \sum_{j\in\mathrm{H}} n_j \mathbf{V}_j \cdot \mathbf{F}_{j\mathrm{e}}, \tag{4.48}$$

and the two other terms will be shown to vanish in Sec. 5. Then, we establish the following lemma used in the derivation of the heavy-particle Navier–Stokes equations.

Lemma 4.6. *Considering f_e^0 given by Eq.* (4.7), *f_i^0, $i \in \mathrm{H}$, by Eq.* (4.8), *ϕ_e by Eqs.* (4.9)–(4.10), *ϕ_i, $i \in \mathrm{H}$, by Eqs.* (4.24)–(4.25), *and ϕ_e^2 by Eqs.* (4.33)–(4.34), *the mass transferred at order ε from heavy particles to electrons vanishes, i.e.*

$$\langle\!\langle \hat{\mathcal{J}}_h^1, \hat{\psi}_h^l\rangle\!\rangle_h = 0, \quad l \in \{1, \ldots, n_h\}. \tag{4.49}$$

Proof. Equation (4.49) is readily derived from Eq. (3.18) at order ε^3. □

Proposition 4.7. *If ϕ_h^2 is a solution to Eq.* (2.30) *at order ε^1, i.e.*

$$f_i^0 \mathcal{F}_i(\phi_h^2) = -\mathscr{D}_i^1(f_i^0, \phi_i) + \sum_{j\in\mathrm{H}} \mathcal{J}_{ij}(f_i^0\phi_i, f_j^0\phi_j) + \hat{\mathcal{J}}_i^1, \quad i \in \mathrm{H}, \tag{4.50}$$

where f_e^0 is given by Eq. (4.7), *f_i^0, $i \in \mathrm{H}$, by Eq.* (4.8), *ϕ_e by Eqs.* (4.9)–(4.10), *ϕ_i, $i \in \mathrm{H}$, by Eqs.* (4.24)–(4.25), *and ϕ_e^2 by Eqs.* (4.33)–(4.34), *and if $f_h^0\phi_h^2 = (f_i^0\phi_i^2)_{i\in\mathrm{H}}$ satisfies the constraints*

$$\langle\!\langle f_h^0\phi_h^2, \hat{\psi}_h^l\rangle\!\rangle_h = 0, \quad l \in \{1, \ldots, n^\mathrm{H} + 4\}, \tag{4.51}$$

then, the first-order conservation equations of heavy-particle mass, momentum and energy read

$$\partial_t \rho_i + \boldsymbol{\partial}_\mathbf{x} \cdot \left(\rho_i \mathbf{v}_h + \frac{\varepsilon}{M_h} \rho_i \mathbf{V}_i\right) = 0, \quad i \in \mathrm{H}, \tag{4.52}$$

$$\begin{aligned} & \partial_t(\rho_h \mathbf{v}_h) + \boldsymbol{\partial}_\mathbf{x} \cdot \left(\rho_h \mathbf{v}_h \otimes \mathbf{v}_h + \frac{1}{M_h^2} p \mathbb{I}\right) \\ & \quad = -\frac{\varepsilon}{M_h^2} \boldsymbol{\partial}_\mathbf{x} \cdot (\boldsymbol{\Pi}_h + \boldsymbol{\Pi}_\mathrm{e}) + \frac{1}{M_h^2} nq\mathbf{E} + [\delta_{b0}\mathbf{I}_0 + \delta_{b1}\mathbf{I}] \wedge \mathbf{B}, \end{aligned} \tag{4.53}$$

$$\partial_t(\rho_h e_h) + \boldsymbol{\partial}_{\mathbf{x}} \cdot (\rho_h e_h \mathbf{v}_h)$$
$$= -(p_h \mathbb{I} + \varepsilon \boldsymbol{\Pi}_h){:}\boldsymbol{\partial}_{\mathbf{x}} \mathbf{v}_h - \frac{\varepsilon}{M_h} \boldsymbol{\partial}_{\mathbf{x}} \cdot \mathbf{q}_h + \frac{\varepsilon}{M_h} \mathbf{J}_h \cdot \mathbf{E}' + \Delta E_h^0 + \varepsilon \Delta E_h^1. \quad (4.54)$$

Proof. The Chapman–Enskog method allows for the following conservation equations to be derived

$$\langle\!\langle \mathscr{D}_h^0, \hat{\psi}_h^l \rangle\!\rangle_h + \varepsilon \langle\!\langle \mathscr{D}_h^1, \hat{\psi}_h^l \rangle\!\rangle_h = \langle\!\langle \hat{\mathfrak{J}}_h^0, \hat{\psi}_h^l \rangle\!\rangle_h + \varepsilon \langle\!\langle \hat{\mathfrak{J}}_h^1, \hat{\psi}_h^l \rangle\!\rangle_h,$$

$l \in \{1, \ldots, n^{\mathrm{H}} + 4\}$. Integrating by parts the left-hand side and simplifying the right-hand side based on the proof of heavy-particle Euler Eqs. (4.26)–(4.28), Proposition 4.6, Lemmas 4.5 and 4.6, one obtains Eqs. (4.52)–(4.54). □

Remark 4.1. When only one single type of heavy particles is considered, the first-order energy transfer term, heavy-particle diffusion velocities, and conduction current degenerate, $\Delta E_h^1 = 0$, $\mathbf{V}_i = 0$, $i \in \mathrm{H}$, $\mathbf{J}_h = 0$, the total current is simplified as well, $\mathbf{I} = nq\mathbf{v}_h + n_{\mathrm{e}} q_{\mathrm{e}} \mathbf{V}_{\mathrm{e}}/M_h$. Therefore, we retrieve the formalism of Degond and Lucquin. In such a case, the Navier–Stokes system can be coupled to the system of drift-diffusion equations for the electrons obtained at order ε^0 in the previous section. Since no energy transfer occurs at order ε^1, there is no need to solve the electrons at order ε^1 to obtain a conservative model which insures positivity of the entropy production. However, this oversimplified case hides the details of the complex interaction between the electrons and heavy particles which is exhibited by the system of conservation Eqs. (4.52)–(4.54). For a multicomponent mixture of heavy particles, thus, we have to extend one order further the model obtained so far for the electrons, as done in the following section.

4.7. *Order ε: First-order electron drift-diffusion equations*

We derive first-order electron drift-diffusion equations based on the electron Boltzmann equation (4.5) at order ε^1.

We define the second-order electron heat flux

$$\mathbf{q}_{\mathrm{e}}^2 = \int \frac{1}{2} \mathbf{C}_{\mathrm{e}} \cdot \mathbf{C}_{\mathrm{e}} \mathbf{C}_{\mathrm{e}} f_{\mathrm{e}}^0 \phi_{\mathrm{e}}^2 \mathrm{d}\mathbf{C}_{\mathrm{e}}. \quad (4.55)$$

The energy transferred from electrons to heavy particles at order ε is calculated by means of Eq. (3.20) at order ε^3

$$\Delta E_{\mathrm{e}}^1 + \Delta E_h^1 = 0, \quad (4.56)$$

where ΔE_h^1 is given by Eq. (4.47). Moreover, we establish the following lemma used in the derivation of the first-order electron drift-diffusion equations.

Lemma 4.7. *Considering f_{e}^0 given by Eq.* (4.7), f_i^0, $i \in \mathrm{H}$, *by Eq.* (4.8), ϕ_{e} *by Eqs.* (4.9)–(4.10), ϕ_i, $i \in \mathrm{H}$, *by Eqs.* (4.24)–(4.25), *and* ϕ_{e}^2 *by Eqs.* (4.33)–(4.34), *the mass transferred at order* ε *from electrons to heavy particles vanishes, i.e.*

$$\langle\!\langle \hat{\mathcal{J}}_{\mathrm{e}}^1, \hat{\psi}_{\mathrm{e}}^1 \rangle\!\rangle_{\mathrm{e}} = 0. \tag{4.57}$$

Proof. Equation (4.57) is readily derived from Eq. (3.18) at order ε^3. □

Proposition 4.8. *If* ϕ_{e}^3 *is a solution to Eq.* (4.5) *at order* ε^1, *i.e.*

$$\begin{aligned} f_{\mathrm{e}}^0 \mathcal{F}_{\mathrm{e}}(\phi_{\mathrm{e}}^3) + \delta_{b1} q_{\mathrm{e}} \boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}(f_{\mathrm{e}}^0 \phi_{\mathrm{e}}^3) \cdot \mathbf{C}_{\mathrm{e}} \wedge \mathbf{B} = -\hat{\mathscr{D}}_{\mathrm{e}}^1(f_{\mathrm{e}}^0, \phi_{\mathrm{e}}, \phi_{\mathrm{e}}^2) + \mathcal{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0 \phi_{\mathrm{e}}^2, f_{\mathrm{e}}^0 \phi_{\mathrm{e}}) \\ + \mathcal{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0 \phi_{\mathrm{e}}, f_{\mathrm{e}}^0 \phi_{\mathrm{e}}^2) + \hat{\mathcal{J}}_{\mathrm{e}}^1, \end{aligned} \tag{4.58}$$

where f_{e}^0 *is given by Eq.* (4.7), f_i^0, $i \in \mathrm{H}$, *by Eq.* (4.8), ϕ_{e} *by Eqs.* (4.9)–(4.10), ϕ_i, $i \in \mathrm{H}$, *by Eqs.* (4.24)–(4.25), ϕ_{e}^2 *by Eqs.* (4.33)–(4.34), *and* ϕ_i^2, $i \in \mathrm{H}$, *by Eqs.* (4.50)–(4.51), *and if* $f_{\mathrm{e}}^0 \phi_{\mathrm{e}}^3$ *satisfies the constraints*

$$\langle\!\langle f_{\mathrm{e}}^0 \phi_{\mathrm{e}}^3, \hat{\psi}_{\mathrm{e}}^l \rangle\!\rangle_{\mathrm{e}} = 0, \quad l \in \{1, 2\}, \tag{4.59}$$

then, the first-order conservation equations of electron mass and energy read

$$\partial_t \rho_{\mathrm{e}} + \boldsymbol{\partial}_{\mathbf{x}} \cdot \left[\rho_{\mathrm{e}} \left(\mathbf{v}_h + \frac{1}{M_h} (\mathbf{V}_{\mathrm{e}} + \varepsilon \mathbf{V}_{\mathrm{e}}^2) \right) \right] = 0, \tag{4.60}$$

$$\begin{aligned} \partial_t(\rho_{\mathrm{e}} e_{\mathrm{e}}) + \boldsymbol{\partial}_{\mathbf{x}} \cdot (\rho_{\mathrm{e}} e_{\mathrm{e}} \mathbf{v}_h) = -p_{\mathrm{e}} \boldsymbol{\partial}_{\mathbf{x}} \cdot \mathbf{v}_h - \frac{1}{M_h} \boldsymbol{\partial}_{\mathbf{x}} \cdot (\mathbf{q}_{\mathrm{e}} + \varepsilon \mathbf{q}_{\mathrm{e}}^2) \\ + \frac{1}{M_h} (\mathbf{J}_{\mathrm{e}} + \varepsilon \mathbf{J}_{\mathrm{e}}^2) \cdot \mathbf{E}' + \delta_{b0} \varepsilon M_h \mathbf{J}_{\mathrm{e}} \cdot \mathbf{v}_h \wedge \mathbf{B} \\ + \Delta E_{\mathrm{e}}^0 + \varepsilon \Delta E_{\mathrm{e}}^1. \end{aligned} \tag{4.61}$$

Proof. The Chapman–Enskog method allows for the following conservation equations to be derived

$$\langle\!\langle \hat{\mathscr{D}}_{\mathrm{e}}^0, \hat{\psi}_{\mathrm{e}}^l \rangle\!\rangle_{\mathrm{e}} + \varepsilon \langle\!\langle \hat{\mathscr{D}}_{\mathrm{e}}^1, \hat{\psi}_{\mathrm{e}}^l \rangle\!\rangle_{\mathrm{e}} = \langle\!\langle \hat{\mathcal{J}}_{\mathrm{e}}^0, \hat{\psi}_{\mathrm{e}}^l \rangle\!\rangle_{\mathrm{e}} + \varepsilon \langle\!\langle \hat{\mathcal{J}}_{\mathrm{e}}^1, \hat{\psi}_{\mathrm{e}}^l \rangle\!\rangle_{\mathrm{e}}, \quad l \in \{1, 2\}.$$

Integrating by parts the left-hand side and simplifying the right-hand side based on Lemma 4.7, one obtains Eqs. (4.60)–(4.61). □

Before reaching Sec. 5 in which the transport flux expressions are evaluated, we come back to the question of the influence of the choice of reference frame.

4.8. *About the necessity of working in the* $\mathbf{v}_h$ *frame*

As mentioned earlier, the mean heavy-particle velocity frame is not commonly adopted in the literature to conduct the Chapman–Enskog expansion. We have already emphasized that the choice of the hydrodynamic velocity frame is not appropriate insofar as the global hydrodynamic velocity $\mathbf{v}$ depends on the ε parameter.

Besides, the choice of the inertial reference frame gives a vanishing mean velocity of the electrons; Degond and Lucquin[21] and Lucquin[37, 38] reach such a conclusion. However, since the expansion of the collision operators in terms of ε depends on the choice of reference frame (see Remark 3.1) and since the choice of the inertial reference frame prevents some terms from vanishing (such as $\mathcal{J}^{\mathbf{u}1}_{ei}(f^{\mathbf{u}0}_{\mathrm{e}}, f^{\mathbf{u}0}_i)$, $i \in \mathrm{H}$), we will first show that these authors compensate the presence of nonzero terms in the integro-differential equations by the help of the $\mathbf{w}$ velocity introduced in Sec. 4.3. This is acceptable for the solution for ϕ_{e}, as proved in the following.

Let us review the Chapman–Enskog expansion in a general frame. Considering a frame moving with the velocity $\mathbf{u}$, the peculiar velocities are given by

$$\mathbf{C}^{\mathbf{u}}_{\mathrm{e}} = \mathbf{c}_{\mathrm{e}} - \varepsilon M_h \mathbf{u}, \quad \mathbf{C}^{\mathbf{u}}_i = \mathbf{c}_i - M_h \mathbf{u}, \ \ i \in \mathrm{H}. \tag{4.62}$$

The space of scalar electron collisional invariants $\mathcal{I}^{\mathbf{u}}_{\mathrm{e}}$ is spanned by the following elements

$$\begin{cases} \hat{\psi}^{\mathbf{u},1}_{\mathrm{e}} = 1, \\ \hat{\psi}^{\mathbf{u},2}_{\mathrm{e}} = \dfrac{1}{2}\mathbf{C}^{\mathbf{u}}_{\mathrm{e}} \cdot \mathbf{C}^{\mathbf{u}}_{\mathrm{e}}, \end{cases}$$

the space of scalar heavy-particle collisional invariants $\mathcal{I}^{\mathbf{u}}_h$ by

$$\begin{cases} \hat{\psi}^{\mathbf{u},j}_h = (m_i \delta_{ij})_{i\in\mathrm{H}}, & j \in \mathrm{H}, \\ \hat{\psi}^{\mathbf{u},n^{\mathrm{H}}+\nu}_h = (m_i C^{\mathbf{u}}_{i\nu})_{i\in\mathrm{H}}, & \nu \in \{1,2,3\}, \\ \hat{\psi}^{\mathbf{u},n^{\mathrm{H}}+4}_h = \left(\dfrac{1}{2} m_i \mathbf{C}^{\mathbf{u}}_i \cdot \mathbf{C}^{\mathbf{u}}_i\right)_{i\in\mathrm{H}}, \end{cases}$$

and the macroscopic properties are expressed as partial scalar products of the distribution functions and the collisional invariants

$$\begin{cases} \langle\!\langle f^{\mathbf{u}}_{\mathrm{e}}, \hat{\psi}^{\mathbf{u},1}_{\mathrm{e}} \rangle\!\rangle^{\mathbf{u}}_{\mathrm{e}} = \rho_{\mathrm{e}} \\ \langle\!\langle f^{\mathbf{u}}_{\mathrm{e}}, \hat{\psi}^{\mathbf{u},2}_{\mathrm{e}} \rangle\!\rangle^{\mathbf{u}}_{\mathrm{e}} = \dfrac{3}{2} n_{\mathrm{e}} T_{\mathrm{e}} + M_h n_{\mathrm{e}} \mathbf{V}_{\mathrm{e}} \cdot (\mathbf{u} - \mathbf{v}_h)\varepsilon^2 + \dfrac{1}{2} M_h^2 n_{\mathrm{e}} |\mathbf{u} - \mathbf{v}_h|^2 \varepsilon^2 + \mathcal{O}(\varepsilon^3), \end{cases}$$

and

$$\begin{cases} \langle\!\langle f^{\mathbf{u}}_h, \hat{\psi}^{\mathbf{u},i}_h \rangle\!\rangle^{\mathbf{u}}_h = \rho_i, & i \in \mathrm{H}, \\ \langle\!\langle f^{\mathbf{u}}_h, \hat{\psi}^{\mathbf{u},n^{\mathrm{H}}+\nu}_h \rangle\!\rangle^{\mathbf{u}}_h = \rho_h M_h (v_{h\nu} - u_\nu), & \nu \in \{1,2,3\}, \\ \langle\!\langle f^{\mathbf{u}}_h, \hat{\psi}^{\mathbf{u},n^{\mathrm{H}}+4}_h \rangle\!\rangle^{\mathbf{u}}_h = \dfrac{3}{2} n_h T_h + \dfrac{1}{2} M_h^2 \rho_h |\mathbf{u} - \mathbf{v}_h|^2, \end{cases}$$

where the temperatures are defined in the heavy-particle reference frame, as usual. Similarly to the low Mach number approximation for neutral gases, we decouple the electron thermal energy from the mixture kinetic energy in the limit $\varepsilon \to 0$.

Then, we rewrite the Chapman–Enskog expansion of Sec. 4 in the $\mathbf{u}$ frame. First, let us reformulate two propositions from the beginning of this section.

Proposition 4.9. (Order ε^{-2}: electron thermalization) *The zeroth-order electron distribution function* $f_{\mathrm{e}}^{\mathbf{u}0}$, *solution to Eq.* (4.5) *at order* ε^{-2}, *i.e.* $\mathscr{D}_{\mathrm{e}}^{\mathbf{u},-2}(f_{\mathrm{e}}^{\mathbf{u}0}) = \mathfrak{J}_{\mathrm{e}}^{\mathbf{u},-2}$, *that satisfies the scalar constraints* $\langle\!\langle f_{\mathrm{e}}^{\mathbf{u}0}, \hat{\psi}_{\mathrm{e}}^{\mathbf{u},l}\rangle\!\rangle_{\mathrm{e}}^{\mathbf{u}} = \lim_{\varepsilon\to 0}\langle\!\langle f_{\mathrm{e}}^{\mathbf{u}}, \hat{\psi}_{\mathrm{e}}^{\mathbf{u},l}\rangle\!\rangle_{\mathrm{e}}^{\mathbf{u}}$, $l \in \{1,2\}$, *is a Maxwell–Boltzmann distribution function at the electron temperature*

$$f_{\mathrm{e}}^{\mathbf{u}0} = n_{\mathrm{e}} \left(\frac{1}{2\pi T_{\mathrm{e}}}\right)^{3/2} \exp\left(-\frac{1}{2T_{\mathrm{e}}}\mathbf{C}_{\mathrm{e}}^{\mathbf{u}}\cdot\mathbf{C}_{\mathrm{e}}^{\mathbf{u}}\right). \tag{4.63}$$

Proof. The proof is identical to the one of Proposition 4.1. □

Proposition 4.10. (Order ε^{-1}: heavy-particle thermalization) *Considering* $f_{\mathrm{e}}^{\mathbf{u}0}$ *given by Eq.* (4.63), *the zeroth-order family of heavy-particle distribution functions* $f_h^{\mathbf{u}0}$ *solution to Eq.* (4.6) *at order* ε^{-1}, *i.e.* $\mathfrak{J}_i^{\mathbf{u},-1} = 0$, $i \in \mathrm{H}$, *that satisfies the scalar constraints* $\langle\!\langle f_h^{\mathbf{u}0}, \hat{\psi}_h^{\mathbf{u},l}\rangle\!\rangle_{\mathrm{e}}^{\mathbf{u}} = \langle\!\langle f_h^{\mathbf{u}}, \hat{\psi}_h^{\mathbf{u},l}\rangle\!\rangle_{\mathrm{e}}^{\mathbf{u}}$, $l \in \{1,\ldots, n^{\mathrm{H}}+4\}$, *is a family of Maxwell–Boltzmann distribution functions at the heavy-particle temperature*

$$f_i^{\mathbf{u}0} = n_i \left(\frac{m_i}{2\pi T_h}\right)^{3/2} \exp\left(-\frac{m_i}{2T_h}|\mathbf{C}_i^{\mathbf{u}} - M_h(\mathbf{v}_h - \mathbf{u})|^2\right), \quad i \in \mathrm{H}. \tag{4.64}$$

Proof. The proof is identical to the one of Proposition 4.2. □

At this step, two properties appear: the electron thermalization takes place in any velocity frame, whereas the zeroth-order heavy-particle distribution functions do not depend on the selected frame. Indeed, we clearly have $f_i^{\mathbf{u}0} = f_i^0$, $i \in \mathrm{H}$, for all velocity $\mathbf{u}$.

Considering then the Boltzmann equation at order ε^{-1}, the first-order electron perturbation function $\phi_{\mathrm{e}}^{\mathbf{u}}$ satisfies the linearized Boltzmann equation

$$\begin{aligned} &\mathcal{F}_{\mathrm{e}}(\phi_{\mathrm{e}}^{\mathbf{u}}) + \delta_{b1} q_{\mathrm{e}} \boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}^{\mathbf{u}}}(\phi_{\mathrm{e}}^{\mathbf{u}}) \cdot \mathbf{C}_{\mathrm{e}} \wedge \mathbf{B} \\ &\quad = -\frac{1}{f_{\mathrm{e}}^{\mathbf{u}0}} \hat{\mathscr{D}}_{\mathrm{e}}^{\mathbf{u},-1}(f_{\mathrm{e}}^{\mathbf{u}0}) + \sum_{i\in\mathrm{H}} n_i \frac{M_h}{T_{\mathrm{e}}} Q_{\mathrm{e}i}^{(1)}(|\mathbf{C}_{\mathrm{e}}^{\mathbf{u}}|^2)|\mathbf{C}_{\mathrm{e}}^{\mathbf{u}}|(\mathbf{v}_h - \mathbf{u})\cdot\mathbf{C}_{\mathrm{e}}^{\mathbf{u}}, \end{aligned} \tag{4.65}$$

with the constraints

$$\langle\!\langle f_{\mathrm{e}}^{\mathbf{u}0}\phi_{\mathrm{e}}^{\mathbf{u}}, \hat{\psi}_{\mathrm{e}}^{\mathbf{u},l}\rangle\!\rangle_{\mathrm{e}}^{\mathbf{u}} = 0, \quad l \in \{1,2\}. \tag{4.66}$$

The right-hand side of Eq. (4.65) is orthogonal to the collisional invariants, that is the solvability condition. Moreover, in order to avoid treating the newly introduced term in the integro-differential equation, one can use the absence of momentum constraints on the electron distribution function and introduce a velocity shift $\mathbf{w} = \mathbf{v}_h - \mathbf{u}$ and notice that[37]

$$\mathcal{F}_{\mathrm{e}}(\mathbf{C}_{\mathrm{e}}^{\mathbf{u}}) = -\sum_{i\in\mathrm{H}} n_i Q_{\mathrm{e}i}^{(1)}(|\mathbf{C}_{\mathrm{e}}^{\mathbf{u}}|^2)|\mathbf{C}_{\mathrm{e}}^{\mathbf{u}}|\mathbf{C}_{\mathrm{e}}^{\mathbf{u}}, \tag{4.67}$$

we thus obtain that the conduction of the Chapman–Enskog expansion in the $\mathbf{u}$ frame is equivalent to that in the $\mathbf{v}_h$ frame with

$$\phi_{\mathrm{e}}^{\mathbf{u}} = \phi_{\mathrm{e}} + \frac{M_h}{T_{\mathrm{e}}}(\mathbf{v}_h - \mathbf{u}) \cdot \mathbf{C}_{\mathrm{e}}^{\mathbf{u}}.$$

As already mentioned in Sec. 4.3, the electron velocity $\mathbf{v}_{\mathrm{e}}$ can be split into two parts at the same order of the multiscale expansion $\mathbf{v}_{\mathrm{e}} = \mathbf{u} + \mathbf{V}_{\mathrm{e}}^{\mathbf{u}}/M_h + \mathcal{O}(\varepsilon)$, with $\mathbf{V}_{\mathrm{e}}^{\mathbf{u}} = \mathbf{V}_{\mathrm{e}} + M_h(\mathbf{v}_h - \mathbf{u})$. We have thus provided a nice interpretation of the algebra proposed in Lucquin[37] where the use of $\mathbf{w} = \mathbf{v}_h$ allows then to eliminate the term $\sum_{j\in \mathrm{H}} \mathcal{J}_{\mathrm{e}j}^1(f_{\mathrm{e}}^{\mathbf{u}0}, f_j^{\mathbf{u}0})$ in the integro-differential equation for $\phi_{\mathrm{e}}^{\mathbf{u}}$ obtained when working in the inertial reference frame $\mathbf{u} = 0$.

It amounts to "coming back" to the heavy-particle reference frame. Let us emphasize at this point, that the set of equations obtained for the heavy-particle Euler Eqs. (4.26)–(4.28) coupled to the zeroth-order electron drift-diffusion Eqs. (4.35)–(4.36) is identical to the set obtained in Lucquin.[37] At this order of the expansion, while still equivalent to our study and yielding the same macroscopic equations, the inertial reference frame leads to an artificial complexity. This is a justification of the choices made in Sec. 2.5 in terms of the reference frame and associated simplified algebra. At order ε, which yields heavy-particle Navier–Stokes equations coupled to first-order electron drift-diffusion equations, we realize that such a compensation used through the velocity shift $\mathbf{w}$ has an undesirable influence on the structure of the expansion at the next order (see Lemma 4.4) and hence makes the solution for $\phi_{\mathrm{e}}^{\mathbf{w}2}$ difficult. Concerning the heavy-particle Boltzmann equation at order ε^0, the first-order perturbation functions $\phi_i^{\mathbf{u}}$, $i \in \mathrm{H}$, also satisfy Eq. (4.24), and that implies that $\phi_i^{\mathbf{u}} = \phi_i$, $i \in \mathrm{H}$.

5. Transport Coefficients

In this section, we investigate the electron and heavy-particle perturbation functions in order to obtain expressions for the transport fluxes. We deal with strongly magnetized plasmas ($b = 1$) having anisotropic transport coefficients, both the cases of weakly magnetized plasmas ($b = 0$) and unmagnetized plasmas ($b < 0$) are investigated at the end of this section.

5.1. *Extra notations for anisotropy*

We introduce some extra notations in order to conveniently express the solution to the Boltzmann equation in the presence of a strong magnetic field. First, we define a unit vector for the magnetic field $\boldsymbol{\mathcal{B}} = \mathbf{B}/|\mathbf{B}|$ and also three direction matrices

$$\mathbf{M}^{\|} = \boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{B}}, \quad \mathbf{M}^{\perp} = \mathbb{I} - \boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{B}}, \quad \mathbf{M}^{\odot} = \begin{pmatrix} 0 & -\mathcal{B}_3 & \mathcal{B}_2 \\ \mathcal{B}_3 & 0 & -\mathcal{B}_1 \\ -\mathcal{B}_2 & \mathcal{B}_1 & 0 \end{pmatrix},$$

so that we have for any vector $\mathbf{x}$ in three dimensions

$$\mathbf{x}^{\parallel} = \mathbf{M}^{\parallel}\mathbf{x} = \mathbf{x}\cdot\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}}, \quad \mathbf{x}^{\perp} = \mathbf{M}^{\perp}\mathbf{x} = \mathbf{x} - \mathbf{x}\cdot\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}}, \quad \mathbf{x}^{\odot} = \mathbf{M}^{\odot}\mathbf{x} = \boldsymbol{\mathcal{B}}\wedge\mathbf{x}.$$

In the $(\mathbf{x},\boldsymbol{\mathcal{B}})$ plane, the vector $\mathbf{x}^{\parallel}$ is the component of $\mathbf{x}$ parallel to the magnetic field and $\mathbf{x}^{\perp}$ its component perpendicular to the magnetic field. Thus, we have $\mathbf{x} = \mathbf{x}^{\parallel} + \mathbf{x}^{\perp}$. The vector $\mathbf{x}^{\odot}$ lies in the direction transverse to the $(\mathbf{x},\boldsymbol{\mathcal{B}})$ plane. The three vectors $\mathbf{x}^{\parallel}$, $\mathbf{x}^{\perp}$ and $\mathbf{x}^{\odot}$ are then mutually orthogonal. We will show that the transport coefficients are anisotropic, as expressed by means of the matrix notation $\bar{\bar{\mu}} = \mu^{\parallel}\mathbf{M}^{\parallel} + \mu^{\perp}\mathbf{M}^{\perp} + \mu^{\odot}\mathbf{M}^{\odot}$. In the $b = 0$ and $b < 0$ cases, the transport coefficients are identical in the parallel and perpendicular directions, $\mu^{\parallel} = \mu^{\perp}$, and vanish in the transverse direction, $\mu^{\odot} = 0$.

Finally, the direction matrices satisfy the following two properties. The matrices $\mathbf{M}^{\parallel}$, $\mathbf{M}^{\perp}$ and $\mathbf{M}^{\odot}$ are linearly independent, that is

$$\bar{\bar{\mu}} = 0 \Rightarrow \mu^{\parallel} = \mu^{\perp} = \mu^{\odot} = 0.$$

Moreover, the space spanned by the matrices $\mathbf{M}^{\parallel}$, $\mathbf{M}^{\perp}$ and $\mathbf{M}^{\odot}$ is stable under multiplication, since we have the following relations

$$\begin{array}{lll} \mathbf{M}^{\parallel}\mathbf{M}^{\parallel} = \mathbf{M}^{\parallel}, & \mathbf{M}^{\parallel}\mathbf{M}^{\perp} = \mathbf{0}, & \mathbf{M}^{\parallel}\mathbf{M}^{\odot} = \mathbf{0}, \\ \mathbf{M}^{\perp}\mathbf{M}^{\parallel} = \mathbf{0}, & \mathbf{M}^{\perp}\mathbf{M}^{\perp} = \mathbf{M}^{\perp}, & \mathbf{M}^{\perp}\mathbf{M}^{\odot} = \mathbf{M}^{\odot}, \\ \mathbf{M}^{\odot}\mathbf{M}^{\parallel} = \mathbf{0}, & \mathbf{M}^{\odot}\mathbf{M}^{\perp} = \mathbf{M}^{\odot}, & \mathbf{M}^{\odot}\mathbf{M}^{\odot} = -\mathbf{M}^{\perp}. \end{array}$$

5.2. *First-order electron perturbation function*

The first-order perturbation function ϕ_{e} is a solution to Eq. (4.9)

$$\mathcal{F}_{\mathrm{e}}(\phi_{\mathrm{e}}) + q_{\mathrm{e}}\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}(\phi_{\mathrm{e}})\cdot\mathbf{C}_{\mathrm{e}}\wedge\mathbf{B} = \Psi_{\mathrm{e}}, \tag{5.1}$$

and satisfies the constraints (4.10), where Ψ_{e} is given by the expression $\Psi_{\mathrm{e}} = -\hat{\mathscr{D}}_{\mathrm{e}}^{-1}(f_{\mathrm{e}}^0)/f_{\mathrm{e}}^0$ and f_{e}^0 by Eq. (4.7). After some algebra based on the expression for f_{e}^0, the quantity Ψ_{e} is transformed into

$$\Psi_{\mathrm{e}} = -p_{\mathrm{e}}\boldsymbol{\Psi}_{\mathrm{e}}^{D_{\mathrm{e}}}\cdot\mathbf{d}_{\mathrm{e}} - \boldsymbol{\Psi}_{\mathrm{e}}^{\lambda_{\mathrm{e}}'}\cdot\boldsymbol{\partial}_{\mathbf{x}}\left(\frac{1}{T_{\mathrm{e}}}\right), \tag{5.2}$$

where the electron diffusion driving force $\mathbf{d}_{\mathrm{e}}$ is defined by the relation

$$\mathbf{d}_{\mathrm{e}} = \frac{1}{p_{\mathrm{e}}}\boldsymbol{\partial}_{\mathbf{x}}p_{\mathrm{e}} - \frac{n_{\mathrm{e}}q_{\mathrm{e}}}{p_{\mathrm{e}}}\mathbf{E}', \tag{5.3}$$

and with

$$\boldsymbol{\Psi}_{\mathrm{e}}^{D_{\mathrm{e}}} = \frac{1}{M_h p_{\mathrm{e}}}\mathbf{C}_{\mathrm{e}}, \quad \boldsymbol{\Psi}_{\mathrm{e}}^{\lambda_{\mathrm{e}}'} = \frac{1}{M_h}\left(\frac{5}{2}T_{\mathrm{e}} - \frac{1}{2}\mathbf{C}_{\mathrm{e}}\cdot\mathbf{C}_{\mathrm{e}}\right)\mathbf{C}_{\mathrm{e}}. \tag{5.4}$$

The right-hand side of Eq. (5.1) does not depend on the heavy-particle driving forces. Therefore, the first-order electron perturbation function is decoupled from the heavy particles.

The existence and uniqueness of a solution to Eq. (5.2) is given in the following proposition.

Proposition 5.1. *The scalar function* $\phi_{\rm e}$ *given by*

$$\phi_{\rm e} = -p_{\rm e}\Re[\mathbf{M}^{\|}\boldsymbol{\varphi}_{\rm e}^{D_{\rm e}(1)} + (\mathbf{M}^{\perp} + \mathtt{i}\mathbf{M}^{\odot})\boldsymbol{\varphi}_{\rm e}^{D_{\rm e}(2)}]\cdot \mathbf{d}_{\rm e}$$
$$- \Re[\mathbf{M}^{\|}\boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(1)} + (\mathbf{M}^{\perp} + \mathtt{i}\mathbf{M}^{\odot})\boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}]\cdot \boldsymbol{\partial}_{\mathbf{x}}\left(\frac{1}{T_{\rm e}}\right), \tag{5.5}$$

is the solution to Eq. (5.1) *under the constraints* (4.10), *where the vectorial functions* $\boldsymbol{\varphi}_{\rm e}^{D_{\rm e}(1)}$, $\boldsymbol{\varphi}_{\rm e}^{D_{\rm e}(2)}$, $\boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(1)}$, *and* $\boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}$ *are the solutions to the equation*

$$\mathcal{F}_{\rm e}(\boldsymbol{\varphi}_{\rm e}^{\mu(1)}) = \boldsymbol{\Psi}_{\rm e}^{\mu}, \tag{5.6}$$

$$(\mathcal{F}_{\rm e} + \mathtt{i}|\mathbf{B}|\mathcal{F}_{\rm e}^{q_{\rm e}})(\boldsymbol{\varphi}_{\rm e}^{\mu(2)}) = \boldsymbol{\Psi}_{\rm e}^{\mu}, \tag{5.7}$$

where $\mathcal{F}_{\rm e}^{q_{\rm e}}(\mathbf{u}) = q_{\rm e}\mathbf{u}$, *under the constraints*

$$\langle\!\langle f_{\rm e}^0\boldsymbol{\varphi}_{\rm e}^{\mu(1)}, \hat{\psi}_{\rm e}^l\rangle\!\rangle_{\rm e} = 0, \quad l \in \{1,2\}, \tag{5.8}$$

$$\langle\!\langle f_{\rm e}^0\boldsymbol{\varphi}_{\rm e}^{\mu(2)}, \hat{\psi}_{\rm e}^l\rangle\!\rangle_{\rm e} = 0, \quad l \in \{1,2\}, \tag{5.9}$$

with $\mu \in \{D_{\rm e}, \lambda'_{\rm e}\}$.

Proof. By linearity and isotropy of the linearized Boltzmann operator $\mathcal{F}_{\rm e}$, the development (5.2) of $\Psi_{\rm e}$ can be followed through for $\phi_{\rm e}$ as well to give

$$\phi_{\rm e} = -p_{\rm e}\boldsymbol{\phi}_{\rm e}^{D_{\rm e}}\cdot\mathbf{d}_{\rm e} - \boldsymbol{\phi}_{\rm e}^{\lambda'_{\rm e}}\cdot\boldsymbol{\partial}_{\mathbf{x}}\left(\frac{1}{T_{\rm e}}\right).$$

The functions $\boldsymbol{\phi}_{\rm e}^{\mu}$, $\mu \in \{D_{\rm e}, \lambda'_{\rm e}\}$, are now vectorial and satisfy the equations

$$\mathcal{F}_{\rm e}(\boldsymbol{\phi}_{\rm e}^{\mu}) + q_{\rm e}\mathbf{C}_{\rm e}\wedge\mathbf{B}\cdot\boldsymbol{\partial}_{\mathbf{C}_{\rm e}}\boldsymbol{\phi}_{\rm e}^{\mu} = \boldsymbol{\Psi}_{\rm e}^{\mu}, \tag{5.10}$$

and the scalar constraints

$$\langle\!\langle f_{\rm e}^0\boldsymbol{\phi}_{\rm e}^{\mu}, \hat{\psi}_{\rm e}^l\rangle\!\rangle_{\rm e} = 0, \quad l \in \{1,2\}. \tag{5.11}$$

We seek a solution $\boldsymbol{\phi}_{\rm e}^{\mu}$ in the form

$$\boldsymbol{\phi}_{\rm e}^{\mu} = \phi_{\rm e}^{\mu(1)}\mathbf{C}_{\rm e} + \phi_{\rm e}^{\mu(2)}\mathbf{C}_{\rm e}\wedge\mathbf{B} + \phi_{\rm e}^{\mu(3)}\mathbf{C}_{\rm e}\cdot\mathbf{B}\mathbf{B},$$

where $\phi_{\rm e}^{\mu(1)}$, $\phi_{\rm e}^{\mu(2)}$ and $\phi_{\rm e}^{\mu(3)}$ are scalar functions of $\mathbf{C}_{\rm e}\cdot\mathbf{C}_{\rm e}$, $(\mathbf{C}_{\rm e}\cdot\mathbf{B})^2$ and $\mathbf{B}\cdot\mathbf{B}$, since $\boldsymbol{\phi}_{\rm e}^{\mu}$ must be invariant under a change of coordinates. Substituting this expansion in (5.10), and using isotropy, Eq. (5.10) splits into three separate coupled equations

$$\mathcal{F}_{\rm e}(\phi_{\rm e}^{\mu(1)}\mathbf{C}_{\rm e}) - q_{\rm e}\mathbf{B}\cdot\mathbf{B}\phi_{\rm e}^{\mu(2)}\mathbf{C}_{\rm e} = \boldsymbol{\Psi}_{\rm e}^{\mu}, \tag{5.12}$$

$$\mathcal{F}_{\rm e}(\phi_{\rm e}^{\mu(2)}\mathbf{C}_{\rm e}\wedge\mathbf{B}) + q_{\rm e}\phi_{\rm e}^{\mu(1)}\mathbf{C}_{\rm e}\wedge\mathbf{B} = 0, \tag{5.13}$$

$$\mathcal{F}_{\rm e}(\phi_{\rm e}^{\mu(3)}\mathbf{C}_{\rm e}\cdot\mathbf{B}\mathbf{B}) + q_{\rm e}\mathbf{C}_{\rm e}\cdot\mathbf{B}\phi_{\rm e}^{\mu(2)}\mathbf{B} = 0. \tag{5.14}$$

Further simplification is now obtained if, instead of three real quantities $\phi_{\rm e}^{\mu(1)}$, $\phi_{\rm e}^{\mu(2)}$ and $\phi_{\rm e}^{\mu(3)}$, we introduce one real and one complex unknown defined by

$$\varphi_{\rm e}^{\mu(1)} = \phi_{\rm e}^{\mu(1)} + \mathbf{B}\cdot\mathbf{B}\phi_{\rm e}^{\mu(3)}, \quad \varphi_{\rm e}^{\mu(2)} = \phi_{\rm e}^{\mu(1)} + \mathtt{i}|\mathbf{B}|\phi_{\rm e}^{\mu(2)}.$$

Upon introducing $\boldsymbol{\varphi}_{\rm e}^{\mu(1)} = \varphi_{\rm e}^{\mu(1)}\mathbf{C}_{\rm e}$ and $\boldsymbol{\varphi}_{\rm e}^{\mu(2)} = \varphi_{\rm e}^{\mu(2)}\mathbf{C}_{\rm e}$, Eqs. (5.12)–(5.14) can be conveniently rewritten in terms of these new functions as Eqs. (5.6) and (5.7). Furthermore, the constraints (5.11) are easily rewritten in the form given in Eqs. (5.8) and (5.9). Moreover, expression (5.5) for ϕ_{e} is immediately obtained using the recombination formula

$$\boldsymbol{\phi}_{\mathrm{e}}^{\mu} = \mathbf{M}^{\|}\varphi_{\mathrm{e}}^{\mu(1)} + \mathbf{M}^{\perp}\Re(\boldsymbol{\varphi}_{\mathrm{e}}^{\mu(2)}) - \mathbf{M}^{\odot}\Im(\boldsymbol{\varphi}_{\mathrm{e}}^{\mu(2)}).$$

□

The structure of the integral equation (5.6) under the constraints (5.8) is standard and the structure of Eq. (5.7) under the constraints (5.9) is similar in a complex framework. More specifically, the operator $\mathcal{F}_{\mathrm{e}} + \mathtt{i}|\mathbf{B}|\mathcal{F}_{\mathrm{e}}^{q_{\mathrm{e}}}$ and the associated bilinear form $\mathbf{a}(\mathbf{u},\mathbf{v}) = \langle\!\langle f_{\mathrm{e}}^{0}\mathbf{u}, (\mathcal{F}_{\mathrm{e}} + \mathtt{i}|\mathbf{B}|\mathcal{F}_{\mathrm{e}}^{q_{\mathrm{e}}})\mathbf{v}\rangle\!\rangle_{\mathrm{e}}$, defined on the proper Hilbert space of complex isotropic squared integrable functions associated with the scalar product $[\cdot,\cdot]$, are such that $|\mathbf{a}(\mathbf{u},\mathbf{u})| \geq [\mathbf{u},\mathbf{u}]$, which yields existence and uniqueness thanks to the constraints. Moreover, from the isotropy of the operator $\mathcal{F}_{\mathrm{e}}$, the expressions $\varphi_{\mathrm{e}}^{\mu(1)}$ and $\varphi_{\mathrm{e}}^{\mu(2)}$ cannot be functions of $(\mathbf{C}_{\mathrm{e}}\cdot\mathbf{B})^2$ as shown in Ref. 26.

We further introduce the electron bracket operators $[\![\cdot,\cdot]\!]_{\mathrm{e}}$ and $((\cdot,\cdot))_{\mathrm{e}}$ associated with the two operators $\mathcal{F}_{\mathrm{e}}$ and $\mathcal{F}_{\mathrm{e}}^{q_{\mathrm{e}}}$. For any ξ_{e} and ζ_{e}, we define

$$[\![\xi_{\mathrm{e}},\zeta_{\mathrm{e}}]\!]_{\mathrm{e}} = \langle\!\langle f_{\mathrm{e}}^{0}\xi_{\mathrm{e}}, \mathcal{F}_{\mathrm{e}}(\zeta_{\mathrm{e}})\rangle\!\rangle_{\mathrm{e}}, \quad ((\xi_{\mathrm{e}},\zeta_{\mathrm{e}}))_{\mathrm{e}} = |\mathbf{B}|\,\langle\!\langle f_{\mathrm{e}}^{0}\xi_{\mathrm{e}}, \mathcal{F}_{\mathrm{e}}^{q_{\mathrm{e}}}(\zeta_{\mathrm{e}})\rangle\!\rangle_{\mathrm{e}}.$$

These bracket operators expand into

$$\begin{aligned}[\![\xi_{\mathrm{e}},\zeta_{\mathrm{e}}]\!]_{\mathrm{e}} = \frac{1}{2}\sum_{j\in\mathrm{H}} n_j \int & \sigma_{\mathrm{e}j}(|\mathbf{C}_{\mathrm{e}}|^2, \boldsymbol{\omega}\cdot\mathbf{e})|\mathbf{C}_{\mathrm{e}}|^3 f_{\mathrm{e}}^{0}(|\mathbf{C}_{\mathrm{e}}|\mathbf{e})\\
&\times [\xi_{\mathrm{e}}(|\mathbf{C}_{\mathrm{e}}|\mathbf{e}) - \xi_{\mathrm{e}}(|\mathbf{C}_{\mathrm{e}}|\boldsymbol{\omega})] \odot \overline{[\zeta_{\mathrm{e}}(|\mathbf{C}_{\mathrm{e}}|\mathbf{e}) - \zeta_{\mathrm{e}}(|\mathbf{C}_{\mathrm{e}}|\boldsymbol{\omega})]}\mathrm{d}\boldsymbol{\omega}\mathrm{d}\mathbf{e}\mathrm{d}|\mathbf{C}_{\mathrm{e}}|\\
&+ \frac{1}{4}\int \sigma_{\mathrm{ee1}}|\mathbf{C}_{\mathrm{e}} - \mathbf{C}_{\mathrm{e1}}| f_{\mathrm{e}}^{0} f_{\mathrm{e1}}^{0}\\
&\times (\xi_{\mathrm{e}} + \xi_{\mathrm{e1}} - \xi_{\mathrm{e}}' - \xi_{\mathrm{e1}}') \odot \overline{(\zeta_{\mathrm{e}} + \zeta_{\mathrm{e1}} - \zeta_{\mathrm{e}}' - \zeta_{\mathrm{e1}}')}\mathrm{d}\boldsymbol{\omega}\mathrm{d}\mathbf{C}_{\mathrm{e}}\mathrm{d}\mathbf{C}_{\mathrm{e1}},\end{aligned}$$

and

$$((\xi_{\mathrm{e}},\zeta_{\mathrm{e}}))_{\mathrm{e}} = |\mathbf{B}|q_{\mathrm{e}}\int f_{\mathrm{e}}^{0}\xi_{\mathrm{e}}\odot\bar{\zeta}_{\mathrm{e}}\mathrm{d}\mathbf{C}_{\mathrm{e}}.$$

The bracket operator $[\![\cdot,\cdot]\!]_{\mathrm{e}}$ is hermitian $[\![\xi_{\mathrm{e}},\zeta_{\mathrm{e}}]\!]_{\mathrm{e}} = \overline{[\![\zeta_{\mathrm{e}},\xi_{\mathrm{e}}]\!]_{\mathrm{e}}}$, positive semi-definite $[\![\xi_{\mathrm{e}},\xi_{\mathrm{e}}]\!]_{\mathrm{e}} \geq 0$, and its kernel is spanned by the collisional invariants, i.e. $[\![\xi_{\mathrm{e}},\xi_{\mathrm{e}}]\!]_{\mathrm{e}} = 0$ implies that ξ_{e} is a (tensorial) collisional invariant, or in other words, all its tensorial components are in the space $\mathcal{I}_{\mathrm{e}}$. The bracket operator $((\cdot,\cdot))_{\mathrm{e}}$ is hermitian $((\xi_{\mathrm{e}},\zeta_{\mathrm{e}}))_{\mathrm{e}} = \overline{((\zeta_{\mathrm{e}},\xi_{\mathrm{e}}))_{\mathrm{e}}}$ and negative definite $((\xi_{\mathrm{e}},\xi_{\mathrm{e}}))_{\mathrm{e}} < 0$ if $\xi_{\mathrm{e}} \neq 0$.

Remark 5.1. In the limit case in which $\mathbf{B}$ tends to zero, expression (5.5) for the first-order electron perturbation function reduces to an isotropic form. We prove indeed that, for $\mu \in \{D_{\mathrm{e}}, \lambda_{\mathrm{e}}'\}$, $\boldsymbol{\varphi}_{\mathrm{e}}^{\mu(1)}$ does not depend on the magnetic field and that $\boldsymbol{\varphi}_{\mathrm{e}}^{\mu(2)}$ converges to $\boldsymbol{\varphi}_{\mathrm{e}}^{\mu(1)}$ for a vanishing magnetic field.

The expression for the electron diffusion velocity is given in the following proposition.

Proposition 5.2. *The electron diffusion velocity* $\mathbf{V}_\mathrm{e}$ *reads*

$$\mathbf{V}_\mathrm{e} = -\bar{\bar{\bar{D}}}_\mathrm{e}\mathbf{d}_\mathrm{e} - \bar{\bar{\bar{\theta}}}_\mathrm{e}\boldsymbol{\partial}_\mathbf{x}\ln T_\mathrm{e}, \tag{5.15}$$

where the diffusion coefficients and thermal diffusion coefficients are given by

$$\begin{aligned} D_\mathrm{e}^{\parallel} &= \frac{1}{3}p_\mathrm{e}T_\mathrm{e}M_h[\![\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(1)}, \boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(1)}]\!]_\mathrm{e}, & \theta_\mathrm{e}^{\parallel} &= -\frac{1}{3}M_h[\![\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(1)}, \boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(1)}]\!]_\mathrm{e}, \\ D_\mathrm{e}^{\perp} &= \frac{1}{3}p_\mathrm{e}T_\mathrm{e}M_h[\![\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}, \boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}]\!]_\mathrm{e}, & \theta_\mathrm{e}^{\perp} &= -\frac{1}{3}M_h[\![\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}, \boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(2)}]\!]_\mathrm{e}, \\ D_\mathrm{e}^{\odot} &= -\frac{1}{3}p_\mathrm{e}T_\mathrm{e}M_h((\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}, \boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}))_\mathrm{e}, & \theta_\mathrm{e}^{\odot} &= \frac{1}{3}M_h((\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}, \boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(2)}))_\mathrm{e}. \end{aligned} \tag{5.16}$$

Note that these expressions are real, in particular for $\theta_\mathrm{e}^{\perp}$ *and* $\theta_\mathrm{e}^{\odot}$*, although functions* $\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}$ *and* $\boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(2)}$ *are complex.*

Proof. Using definition (4.16) of the diffusion velocity $\mathbf{V}_\mathrm{e}$ and expression (5.4) for $\boldsymbol{\Psi}_\mathrm{e}^{D_\mathrm{e}}$ yields

$$\mathbf{V}_\mathrm{e} = T_\mathrm{e}M_h\langle\!\langle \boldsymbol{\Psi}_\mathrm{e}^{D_\mathrm{e}}, f_\mathrm{e}^0\phi_\mathrm{e}\rangle\!\rangle_\mathrm{e}.$$

Further substituting expansion (5.5) into this equation, and using isotropy, we obtain expression (5.15) for the diffusion velocity $\mathbf{V}_\mathrm{e}$, where the transport coefficients are defined by $D_\mathrm{e}^{\parallel} = \frac{1}{3}p_\mathrm{e}T_\mathrm{e}M_h\langle\!\langle f_\mathrm{e}^0\varphi_\mathrm{e}^{D_\mathrm{e}(1)}, \boldsymbol{\Psi}_\mathrm{e}^{D_\mathrm{e}}\rangle\!\rangle_\mathrm{e}$, $\theta_\mathrm{e}^{\parallel} = -\frac{1}{3}M_h\langle\!\langle f_\mathrm{e}^0\varphi_\mathrm{e}^{\lambda'_\mathrm{e}(1)}, \boldsymbol{\Psi}_\mathrm{e}^{D_\mathrm{e}}\rangle\!\rangle_\mathrm{e}$, $D_\mathrm{e}^{\perp} + \mathtt{i}D_\mathrm{e}^{\odot} = \frac{1}{3}p_\mathrm{e}T_\mathrm{e}M_h\langle\!\langle f_\mathrm{e}^0\varphi_\mathrm{e}^{D_\mathrm{e}(2)}, \boldsymbol{\Psi}_\mathrm{e}^{D_\mathrm{e}}\rangle\!\rangle_\mathrm{e}$, $\theta_\mathrm{e}^{\perp} + \mathtt{i}\theta_\mathrm{e}^{\odot} = -\frac{1}{3}M_h\langle\!\langle f_\mathrm{e}^0\varphi_\mathrm{e}^{\lambda'_\mathrm{e}(2)}, \boldsymbol{\Psi}_\mathrm{e}^{D_\mathrm{e}}\rangle\!\rangle_\mathrm{e}$. Equations (5.6) and (5.7) for $\mu = D_\mathrm{e}$ classically yields[26,28]

$$D_\mathrm{e}^{\parallel} = \frac{1}{3}p_\mathrm{e}T_\mathrm{e}M_h[\![\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(1)}, \boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(1)}]\!]_\mathrm{e},$$

$$D_\mathrm{e}^{\perp} + \mathtt{i}D_\mathrm{e}^{\odot} = \frac{1}{3}p_\mathrm{e}T_\mathrm{e}M_h([\![\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}, \boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}]\!]_\mathrm{e} - \mathtt{i}((\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}, \boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}))_\mathrm{e}),$$

$$\theta_\mathrm{e}^{\parallel} = -\frac{1}{3}M_h[\![\boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(1)}, \boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(1)}]\!]_\mathrm{e},$$

$$\theta_\mathrm{e}^{\perp} + \mathtt{i}\theta_\mathrm{e}^{\odot} = -\frac{1}{3}M_h([\![\boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(2)}, \boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}]\!]_\mathrm{e} - \mathtt{i}((\boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(2)}, \boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}))_\mathrm{e}).$$

As the bracket operators $[\![\cdot,\cdot]\!]_\mathrm{e}$ and $((\cdot,\cdot))_\mathrm{e}$ are hermitian, we immediately obtain the expressions for $D_\mathrm{e}^{\parallel}$, $D_\mathrm{e}^{\perp}$, $D_\mathrm{e}^{\odot}$ and $\theta_\mathrm{e}^{\parallel}$. Concerning $\theta_\mathrm{e}^{\perp}$ and $\theta_\mathrm{e}^{\odot}$, we use the imaginary part of Eq. (5.7) for $\mu \in \{D_\mathrm{e}, \lambda'_\mathrm{e}\}$ to establish the relation

$$\Im[(\mathcal{F}_\mathrm{e} + \mathtt{i}|\mathbf{B}|\mathcal{F}_\mathrm{e}^{q_\mathrm{e}})(\boldsymbol{\varphi}_\mathrm{e}^{\mu(2)})] = 0, \quad \mu \in \{D_\mathrm{e}, \lambda'_\mathrm{e}\}.$$

Taking the scalar product of this equation with $\boldsymbol{\varphi}_\mathrm{e}^{\mu(2)}$, $\mu \in \{D_\mathrm{e}, \lambda'_\mathrm{e}\}$, yields the following four relations

$$\begin{aligned} &[\![\Re\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}, \Im\boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(2)}]\!]_\mathrm{e} + ((\Re\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}, \Re\boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(2)}))_\mathrm{e} = 0, \\ &[\![\Im\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}, \Im\boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(2)}]\!]_\mathrm{e} + ((\Im\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}, \Re\boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(2)}))_\mathrm{e} = 0, \\ &[\![\Im\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}, \Im\boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(2)}]\!]_\mathrm{e} + ((\Re\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}, \Im\boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(2)}))_\mathrm{e} = 0, \\ &[\![\Im\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}, \Re\boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(2)}]\!]_\mathrm{e} + ((\Re\boldsymbol{\varphi}_\mathrm{e}^{D_\mathrm{e}(2)}, \Re\boldsymbol{\varphi}_\mathrm{e}^{\lambda'_\mathrm{e}(2)}))_\mathrm{e} = 0. \end{aligned}$$

Then, a direct calculation implies that

$$\Re[\![\boldsymbol{\varphi}_{\rm e}^{D_{\rm e}(2)}, \boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}]\!]_{\rm e} = -\frac{3}{M_h}\theta_{\rm e}^{\perp}, \quad \Re((\boldsymbol{\varphi}_{\rm e}^{D_{\rm e}(2)}, \boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}))_{\rm e} = \frac{3}{M_h}\theta_{\rm e}^{\odot},$$
$$\Im[\![\boldsymbol{\varphi}_{\rm e}^{D_{\rm e}(2)}, \boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}]\!]_{\rm e} = 0, \qquad \Im((\boldsymbol{\varphi}_{\rm e}^{D_{\rm e}(2)}, \boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}))_{\rm e} = 0,$$

so that $\theta_{\rm e}^{\perp} = -\frac{1}{3}M_h[\![\boldsymbol{\varphi}_{\rm e}^{D_{\rm e}(2)}, \boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}]\!]_{\rm e}$ and $\theta_{\rm e}^{\odot} = \frac{1}{3}M_h((\boldsymbol{\varphi}_{\rm e}^{D_{\rm e}(2)}, \boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}))_{\rm e}$. □

An alternative form of the diffusion velocity is given by

$$\mathbf{V}_{\rm e} = -\bar{\bar{D}}_{\rm e}(\mathbf{d}_{\rm e} + \bar{\bar{\chi}}_{\rm e}\boldsymbol{\partial}_{\mathbf{x}}\ln T_{\rm e}), \tag{5.17}$$

where the real-valued thermal diffusion ratios $\chi_{\rm e}^{\parallel}$, $\chi_{\rm e}^{\perp}$, $\chi_{\rm e}^{\odot}$ are defined by the relation

$$\bar{\bar{\theta}}_{\rm e} = \bar{\bar{D}}_{\rm e}\bar{\bar{\chi}}_{\rm e}. \tag{5.18}$$

Indeed the compact matrix relation (5.18) well defines the matrix thermal diffusion ratio $\bar{\bar{\chi}}_{\rm e}$ because (5.18) is equivalent to both the split equations

$$\theta_{\rm e}^{\parallel} = D_{\rm e}^{\parallel}\chi_{\rm e}^{\parallel}, \quad \theta_{\rm e}^{\perp} + \mathtt{i}\theta_{\rm e}^{\odot} = (D_{\rm e}^{\perp} + \mathtt{i}D_{\rm e}^{\odot})(\chi_{\rm e}^{\perp} + \mathtt{i}\chi_{\rm e}^{\odot}).$$

The viscous tensor is calculated in the following proposition.

Proposition 5.3. *The electron viscous tensor vanishes, i.e.*

$$\boldsymbol{\Pi}_{\rm e} = 0. \tag{5.19}$$

Proof. Using definition (4.37) of the stress tensor and expression (5.5) for $\phi_{\rm e}$, one readily obtains that $\boldsymbol{\Pi}_{\rm e} = 0$. □

The electron heat flux is given in the following proposition.

Proposition 5.4. *The electron heat flux* $\mathbf{q}_{\rm e}$ *reads*

$$\mathbf{q}_{\rm e} = -\bar{\bar{\lambda}}'_{\rm e}\boldsymbol{\partial}_{\mathbf{x}}T_{\rm e} - p_{\rm e}\bar{\bar{\theta}}_{\rm e}\mathbf{d}_{\rm e} + \rho_{\rm e}h_{\rm e}\mathbf{V}_{\rm e} \tag{5.20}$$

where the partial thermal conductivities are given by

$$\begin{aligned} \lambda_{\rm e}'^{\parallel} &= \frac{1}{3T_{\rm e}^2}M_h[\![\boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(1)}, \boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(1)}]\!]_{\rm e}, \\ \lambda_{\rm e}'^{\perp} &= \frac{1}{3T_{\rm e}^2}M_h[\![\boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}, \boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}]\!]_{\rm e}, \\ \lambda_{\rm e}'^{\odot} &= -\frac{1}{3T_{\rm e}^2}M_h((\boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}, \boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}))_{\rm e}. \end{aligned} \tag{5.21}$$

Proof. Using definition (4.30) of the heat flux $\mathbf{q}_{\rm e}$ and expression (5.4) of $\boldsymbol{\Psi}_{\rm e}^{\lambda'_{\rm e}}$ yields

$$\mathbf{q}_{\rm e} = \rho_{\rm e}h_{\rm e}\mathbf{V}_{\rm e} - M_h\langle\!\langle\boldsymbol{\Psi}_{\rm e}^{\lambda'_{\rm e}}, f_{\rm e}^0\phi_{\rm e}\rangle\!\rangle_{\rm e}.$$

Further substituting expansion (5.5) into this equation, and using isotropy, we obtain expression (5.20) for the heat flux $\mathbf{q}_{\rm e}$ where the transport coefficients $\theta_{\rm e}^{\parallel}$, $\theta_{\rm e}^{\perp}$, $\theta_{\rm e}^{\odot}$ are given in Eq. (5.16) and the partial thermal conductivities

$\lambda_e^{\prime\|}$, $\lambda_e^{\prime\perp}$, $\lambda_e^{\prime\odot}$ are defined by $\lambda_e^{\prime\|} = \frac{1}{3T_e^2} M_h \langle\langle f_e^0 \varphi_e^{\lambda_e'(1)}, \boldsymbol{\Psi}_e^{\lambda_e'} \rangle\rangle_e$, $\lambda_e^{\prime\perp} + \mathtt{i}\lambda_e^{\prime\odot} = \frac{1}{3T_e^2} M_h \langle\langle f_e^0 \varphi_e^{\lambda_e'(2)}, \boldsymbol{\Psi}_e^{\lambda_e'} \rangle\rangle_e$. Equations (5.6) and (5.7) for $\mu = \lambda_e$ classically yields[26,28]

$$\lambda_e^{\prime\|} = \frac{1}{3T_e^2} M_h \, [\![\boldsymbol{\varphi}_e^{\lambda_e'(1)}, \boldsymbol{\varphi}_e^{\lambda_e'(1)}]\!]_e,$$

$$\lambda_e^{\prime\perp} + \mathtt{i}\lambda_e^{\prime\odot} = \frac{1}{3T_e^2} M_h ([\![\boldsymbol{\varphi}_e^{\lambda_e'(2)}, \boldsymbol{\varphi}_e^{\lambda_e'(2)}]\!]_e - \mathtt{i} ((\boldsymbol{\varphi}_e^{\lambda_e'(2)}, \boldsymbol{\varphi}_e^{\lambda_e'(2)}))_e).$$

As the bracket operators $[\![\cdot,\cdot]\!]_e$ and $((\cdot,\cdot))_e$ are hermitian, we immediately obtain the expressions for $\lambda_e^{\prime\|}$, $\lambda_e^{\prime\perp}$ and $\lambda_e^{\prime\odot}$. □

Using the thermal diffusion ratios defined in Eq. (5.18), the electron heat flux is rewritten

$$\mathbf{q}_e = -\bar{\bar{\lambda}}_e \boldsymbol{\partial}_{\mathbf{x}} T_e + p_e \bar{\bar{\chi}}_e \mathbf{V}_e + \rho_e h_e \mathbf{V}_e, \tag{5.22}$$

where the thermal conductivities $\lambda_e^{\|}$, $\lambda_e^{\perp}$, $\lambda_e^{\odot}$ are real quantities given by

$$\bar{\bar{\lambda}}_e = \bar{\bar{\lambda}}_e' - n_e \bar{\bar{\chi}}_e \bar{\bar{\theta}}_e.$$

Finally, the first-order electron mass-energy flux vector

$$\mathsf{F}_e = [\mathbf{q}_e - \rho_e h_e \mathbf{V}_e, \mathbf{V}_e]^{\mathrm{T}}$$

is proportional to the electron diffusion force vector $\mathsf{X}_e = (\boldsymbol{\partial}_{\mathbf{x}} \ln T_e,\ p_e \mathbf{d}_e)^{\mathrm{T}}$, as expressed by the relation

$$\mathsf{F}_e = -\bar{\bar{\mathbf{A}}}_e \mathsf{X}_e, \tag{5.23}$$

where the electron mass-energy transport matrix is given by

$$\bar{\bar{\mathbf{A}}}_e = \begin{pmatrix} T_e \bar{\bar{\lambda}}_e' & \bar{\bar{\theta}}_e \\ & \\ \bar{\bar{\theta}}_e & \dfrac{1}{p_e} \bar{\bar{D}}_e \end{pmatrix}. \tag{5.24}$$

The positivity properties associated with the heat flux and diffusion velocities are given in the following.

Property 5.1. Considering any two-dimensional real vectors $\mathbf{x}^{\|}$, $\mathbf{x}^{\perp}$ and $\mathbf{x}^{\odot}$, the following two inequalities are satisfied:

$$\langle \mathbf{A}_e^{\|} \mathbf{x}^{\|}, \mathbf{x}^{\|} \rangle \geq 0, \tag{5.25}$$

$$\langle \mathbf{A}_e^{\perp} \mathbf{x}^{\perp} - \mathbf{A}_e^{\odot} \mathbf{x}^{\odot}, \mathbf{x}^{\perp} \rangle + \langle \mathbf{A}_e^{\perp} \mathbf{x}^{\odot} + \mathbf{A}_e^{\odot} \mathbf{x}^{\perp}, \mathbf{x}^{\odot} \rangle \geq 0. \tag{5.26}$$

Proof. Introducing $\mathbf{x}^{\|} = (x_1^{\|}, x_2^{\|})$, $\mathbf{x}^{\perp} = (x_1^{\perp}, x_2^{\perp})$, and $\mathbf{x}^{\odot} = (x_1^{\odot}, x_2^{\odot})$, expressions (5.16) and (5.21) for transport coefficients yield

$$\langle \mathbf{A}_e^{\|} \mathbf{x}^{\|}, \mathbf{x}^{\|} \rangle = \frac{T_e M_h}{3} [\![\mathbf{y}^{(1)}, \mathbf{y}^{(1)}]\!]_e,$$

$$\langle \mathbf{A}_e^{\perp} \mathbf{x}^{\perp} - \mathbf{A}_e^{\odot} \mathbf{x}^{\odot}, \mathbf{x}^{\perp} \rangle + \langle \mathbf{A}_e^{\perp} \mathbf{x}^{\odot} + \mathbf{A}_e^{\odot} \mathbf{x}^{\perp}, \mathbf{x}^{\odot} \rangle = \frac{T_e M_h}{3} [\![\mathbf{y}^{(2)}, \mathbf{y}^{(2)}]\!]_e,$$

with

$$\mathbf{y}^{(1)} = x_2^{\|}\boldsymbol{\varphi}_{\mathrm{e}}^{D_{\mathrm{e}}(1)} - \frac{1}{T_{\mathrm{e}}} x_1^{\|}\boldsymbol{\varphi}_{\mathrm{e}}^{\lambda'_{\mathrm{e}}(1)},$$

$$\mathbf{y}^{(2)} = (x_2^{\perp} + \mathtt{i}x_2^{\odot})\boldsymbol{\varphi}_{\mathrm{e}}^{D_{\mathrm{e}}(2)} - \frac{1}{T_{\mathrm{e}}}(x_1^{\perp} + \mathtt{i}x_1^{\odot})\boldsymbol{\varphi}_{\mathrm{e}}^{\lambda'_{\mathrm{e}}(2)}.$$

Inequalities (5.25) and (5.26) are then obtained, thanks to the positivity of the bracket operator $[\![\cdot,\cdot]\!]_{\mathrm{e}}$. □

Remark 5.2. In the limit case in which $\mathbf{B}$ tends to zero, the behavior of the transport coefficients can be investigated. We formally prove that the matrix $\mathbf{A}_{\mathrm{e}}^{\|}$ does not depend on the magnetic field, that $\mathbf{A}_{\mathrm{e}}^{\perp}$ converges to $\mathbf{A}_{\mathrm{e}}^{\|}$, and that $\mathbf{A}_{\mathrm{e}}^{\odot}$ vanishes.

5.3. *First-order heavy-particle perturbation function*

The first-order perturbation function $\phi_h = (\phi_i)_{i\in\mathrm{H}}$ is a solution to Eq. (4.24), i.e.

$$\mathcal{F}_i(\phi_h) = \Psi_i + \frac{1}{f_i^0}\hat{\mathcal{J}}_i^0, \quad i \in \mathrm{H},$$

and satisfies the constraints (4.25), where $\Psi_i = -\mathscr{D}_i^0(f_i^0)/f_i^0$, $i \in \mathrm{H}$. After some lengthy calculation based on the expression (4.8) for f_h^0, the Euler equations (4.26), (4.28) and (4.29), Theorem 3.1 and Corollary 3.2, one obtains

$$\mathcal{F}_i(\phi_h) = -\boldsymbol{\Psi}_i^{\eta_h} : \boldsymbol{\partial}_{\mathbf{x}}\mathbf{v}_h - p_h \sum_{j\in\mathrm{H}} \boldsymbol{\Psi}_i^{D_j} \cdot \hat{\mathbf{d}}_j - \boldsymbol{\Psi}_i^{\lambda'_h} \cdot \boldsymbol{\partial}_{\mathbf{x}}\left(\frac{1}{T_h}\right) - \Psi_i^{\Theta}(T_{\mathrm{e}} - T_h), \tag{5.27}$$

where

$$\begin{cases} \boldsymbol{\Psi}_i^{\eta_h} = \dfrac{m_i}{T_h}\left(\mathbf{C}_i \otimes \mathbf{C}_i - \dfrac{1}{3}\mathbf{C}_i\cdot\mathbf{C}_i\mathbb{I}\right) & i \in \mathrm{H}, \\ \boldsymbol{\Psi}_i^{D_j} = \dfrac{1}{M_h p_i}\left(\delta_{ij} - \dfrac{\rho_i}{\rho_h}\right)\mathbf{C}_i & i,j \in \mathrm{H}, \\ \boldsymbol{\Psi}_i^{\lambda'_h} = \dfrac{1}{M_h}\left(\dfrac{5}{2}T_h - \dfrac{1}{2}m_i\mathbf{C}_i\cdot\mathbf{C}_i\right)\mathbf{C}_i & i \in \mathrm{H}, \\ \Psi_i^{\Theta} = \dfrac{2}{T_h^2}\left(\dfrac{\nu_{i\mathrm{e}}}{3m_i} - \displaystyle\sum_{j\in\mathrm{H}}\dfrac{n_j\nu_{j\mathrm{e}}}{n_h m_j}\right)\left(\dfrac{3}{2}T_h - \dfrac{1}{2}m_i\mathbf{C}_i\cdot\mathbf{C}_i\right) & i \in \mathrm{H}. \end{cases} \tag{5.28}$$

Quantity $p_i = n_i T_h$ stands for the partial pressure of species $i \in \mathrm{H}$. A linearly independent family of diffusion driving forces is also introduced

$$\hat{\mathbf{d}}_i = \frac{1}{p_h}\boldsymbol{\partial}_{\mathbf{x}} p_i - \frac{n_i q_i}{p_h}\mathbf{E}' - \frac{n_i M_h}{p_h}\mathbf{F}_{i\mathrm{e}}, \quad i \in \mathrm{H}. \tag{5.29}$$

The average electron forces acting on the heavy particles belong to the category of diffusion driving forces and allow for a coupling between the heavy particles and electrons. Expression for ϕ_{e} given in Eq. (5.5) and definition (4.20) implies that $\mathbf{F}_{i\mathrm{e}}$, $i \in \mathrm{H}$, is proportional to the electron diffusion driving force and the electron temperature gradient. Thus, the heavy-particle transport fluxes to be derived are also expected to be proportional to the electron forces.

The existence and uniqueness of a solution to Eq. (5.27) is then established in the following proposition.

Proposition 5.5. *The family of scalar functions* $\phi_h = (\phi_i)_{i \in \mathrm{H}}$, *given by*

$$\phi_i = -\boldsymbol{\phi}_i^{\eta_h} : \boldsymbol{\partial}_{\mathbf{x}} \mathbf{v}_h - p_h \sum_{j \in \mathrm{H}} \boldsymbol{\phi}_i^{D_j} \cdot \hat{\mathbf{d}}_j - \boldsymbol{\phi}_i^{\lambda'_h} \cdot \boldsymbol{\partial}_{\mathbf{x}} \left(\frac{1}{T_h} \right) - \phi_i^{\Theta} (T_{\mathrm{e}} - T_h), \quad i \in \mathrm{H}, \tag{5.30}$$

is the solution to Eq. (5.27) *under the constraints* (4.25), *where the family of tensorial functions* $\boldsymbol{\phi}_h^{\eta_h} = (\boldsymbol{\phi}_i^{\eta_h})_{i \in \mathrm{H}}$, *the families of vectorial functions* $\boldsymbol{\phi}_h^{D_j} = (\boldsymbol{\phi}_i^{D_j})_{i \in \mathrm{H}}$, $j \in \mathrm{H}$, *and* $\boldsymbol{\phi}_h^{\lambda'_h} = (\boldsymbol{\phi}_i^{\lambda'_h})_{i \in \mathrm{H}}$, *and the family of scalar functions* $\phi_h^{\Theta} = (\phi_i^{\Theta})_{i \in \mathrm{H}}$ *are the solutions to the equations*

$$\mathcal{F}_i(\boldsymbol{\phi}_h^{\mu}) = \boldsymbol{\Psi}_i^{\mu}, \quad i \in \mathrm{H}, \tag{5.31}$$

under the scalar constraints

$$\langle\!\langle f_h^0 \boldsymbol{\phi}_h^{\mu}, \hat{\psi}_h^l \rangle\!\rangle_h = 0, \quad l \in \{1, \ldots, n^{\mathrm{H}} + 4\}, \tag{5.32}$$

with $\mu \in \{\eta_h, (D_j)_{j \in \mathrm{H}}, \lambda'_h, \Theta\}$.

Proof. By linearity and isotropy of the linearized Boltzmann operator $\mathcal{F}_i$, the development of Ψ_i can be followed through for ϕ_i as well. Therefore, ϕ_i is given by Eq. (5.30) where the function families $\boldsymbol{\phi}_h^{\mu}$, for $\mu \in \{\eta_h, (D_j)_{j \in \mathrm{H}}, \lambda'_h, \Theta\}$ satisfy Eq. (5.31) under the scalar constraints (5.32). We seek a solution in the form

$$\begin{aligned} \boldsymbol{\phi}_i^{\mu} &= \phi_i^{\mu(1)} \mathbf{C}_i, && i \in \mathrm{H}, \ \mu \in \{(D_j)_{j \in \mathrm{H}}, \lambda'_h\}, \\ \boldsymbol{\phi}_i^{\eta_h} &= \phi_i^{\eta(1)} \left(\mathbf{C}_i \otimes \mathbf{C}_i - \frac{1}{3} \mathbf{C}_i \cdot \mathbf{C}_i \mathbb{I} \right), && i \in \mathrm{H}. \end{aligned}$$

Quantities $\phi_i^{\mu(1)}$, $\mu \in \{\eta_h, (D_j)_{j \in \mathrm{H}}, \lambda'_h\}$, and ϕ_i^{Θ} are scalar functions of $\mathbf{C}_i \cdot \mathbf{C}_i$, for $i \in \mathrm{H}$, since $\boldsymbol{\phi}_h^{\mu}$, $\mu \in \{\eta_h, (D_j)_{j \in \mathrm{H}}, \lambda'_h, \Theta\}$ must be invariant under a change of coordinates. Uniqueness of the solution is readily proved based on the linearity property of the operator $\mathcal{F}_h$, its kernel given in property 4.2, and the constraints (4.25) satisfied by ϕ_h. □

We further introduce the heavy-particle bracket operator $[\![\cdot, \cdot]\!]_h$ associated with the operator $\mathcal{F}_h$. For any ξ_h, ζ_h, we define

$$[\![\xi_h, \zeta_h]\!]_h = \langle\!\langle f_h^0 \xi_h, \mathcal{F}_h(\zeta_h) \rangle\!\rangle_h.$$

This bracket operator expands into

$$
\begin{aligned}
[\![\xi_h, \zeta_h]\!]_h = \frac{1}{4} \sum_{i,j \in \mathrm{H}} \int & f_i^0 f_j^0 (\xi_i + \xi_j - \xi_i' - \xi_j') \\
& \odot \, (\overline{\zeta_i + \zeta_j - \zeta_i' - \zeta_j'}) |\mathbf{C}_i - \mathbf{C}_j| \sigma_{ij} \mathrm{d}\boldsymbol{\omega} \mathrm{d}\mathbf{C}_i \mathrm{d}\mathbf{C}_j.
\end{aligned}
$$

This bracket operator $[\![\cdot,\cdot]\!]_h$ is hermitian, $[\![\xi_h, \zeta_h]\!]_h = \overline{[\![\zeta_h, \xi_h]\!]_h}$, positive semi-definite, $[\![\xi_h, \xi_h]\!]_h \geq 0$, and its kernel is spanned by the collisional invariants, i.e. $[\![\xi_h, \xi_h]\!]_h = 0$ implies that ξ_h is a (tensorial) collisional invariant, or in other words, that all its tensorial components are in the space $\mathcal{I}_h$. The expression for the heavy-particle diffusion velocities is given in the following proposition.

Proposition 5.6. *The diffusion velocity of species* $i \in \mathrm{H}$ *reads*

$$
\mathbf{V}_i = -\sum_{j \in \mathrm{H}} D_{ij} \hat{\mathbf{d}}_j - \theta_i^h \boldsymbol{\partial}_{\mathbf{x}} \ln T_h, \tag{5.33}
$$

where the diffusion coefficients and thermal diffusion coefficients are given by

$$
\begin{aligned}
D_{ij} &= \frac{1}{3} p_h T_h M_h [\![\boldsymbol{\phi}_h^{D_i}, \boldsymbol{\phi}_h^{D_j}]\!]_h, \quad i, j \in \mathrm{H}, \\
\theta_i^h &= -\frac{1}{3} M_h [\![\boldsymbol{\phi}_h^{D_i}, \boldsymbol{\phi}_h^{\lambda_h'}]\!]_h, \qquad i \in \mathrm{H}.
\end{aligned} \tag{5.34}
$$

Proof. Using definition (4.42) of the diffusion velocity and expression (5.28) for $\boldsymbol{\Psi}_h^{D_i}$, $i \in \mathrm{H}$, yields

$$
\mathbf{V}_i = T_h M_h \langle\!\langle \boldsymbol{\Psi}_h^{D_i}, f_h^0 \phi_h \rangle\!\rangle_h, \quad i \in \mathrm{H}.
$$

Further substituting expansion (5.30) into this equation, we obtain expression (5.33) for the diffusion velocities. □

From the properties of the bracket operator, we infer that the diffusion matrix D is symmetric. Moreover, an alternative form for the diffusion velocities is given by

$$
\mathbf{V}_i = -\sum_{j \in \mathrm{H}} D_{ij} \left(\hat{\mathbf{d}}_j + \chi_j^h \boldsymbol{\partial}_{\mathbf{x}} \ln T_h \right), \quad i \in \mathrm{H}, \tag{5.35}
$$

where the thermal diffusion ratios are defined from the relations

$$
\begin{cases}
\displaystyle\sum_{j \in \mathrm{H}} D_{ij} \chi_j^h = \theta_i^h, \quad i \in \mathrm{H}, \\
\displaystyle\sum_{j \in \mathrm{H}} \chi_j^h = 0.
\end{cases} \tag{5.36}
$$

Then, we introduce the tensor

$$
\mathsf{S} = [\boldsymbol{\partial}_{\mathbf{x}} \mathbf{v}_h + (\boldsymbol{\partial}_{\mathbf{x}} \mathbf{v}_h)^{\mathrm{T}}] - \frac{2}{3} \boldsymbol{\partial}_{\mathbf{x}} \cdot \mathbf{v}_h \mathbb{I},
$$

in order to express the viscous tensor in the following proposition.

Proposition 5.7. *The heavy-particle viscous tensor reads*

$$\boldsymbol{\Pi}_h = -\eta_h \mathsf{S}, \tag{5.37}$$

where the shear viscosity is given by

$$\eta_h = \frac{T_h}{10} [\![\boldsymbol{\phi}_h^{\eta_h}, \boldsymbol{\phi}_h^{\eta_h}]\!]_h. \tag{5.38}$$

Proof. Using definition (4.43) of the viscous tensor and expression (5.28) for $\boldsymbol{\Psi}_h^{\eta_h}$ yields

$$\boldsymbol{\Pi}_h = T_h \langle\!\langle \boldsymbol{\Psi}_h^{\eta_h}, f_h^0 \phi_h \rangle\!\rangle_h.$$

Further substituting expansion (5.30) into this equation, we obtain expression (5.37) for the viscous tensor. □

The expression for the heavy-particle heat flux is given in the following proposition.

Proposition 5.8. *The heavy-particle heat flux reads*

$$\mathbf{q}_h = -\lambda_h' \boldsymbol{\partial}_{\mathbf{x}} T_h - p_h \sum_{j \in \mathrm{H}} \theta_j^h \hat{\mathbf{d}}_j + \sum_{j \in \mathrm{H}} \rho_j h_j \mathbf{V}_j, \tag{5.39}$$

where the partial thermal conductivity is given by

$$\lambda_h' = \frac{1}{3T_h^2} M_h [\![\boldsymbol{\phi}_h^{\lambda_h'}, \boldsymbol{\phi}_h^{\lambda_h'}]\!]_h. \tag{5.40}$$

Proof. Using definition (4.45) of the heavy-particle heat flux and expression (5.28) for $\boldsymbol{\Psi}_i^{\lambda_h'}$ yields

$$\mathbf{q}_h = -M_h \langle\!\langle \boldsymbol{\Psi}_h^{\lambda_h'}, f_h^0 \phi_h \rangle\!\rangle_h + \frac{5}{2} T_h \sum_{j \in \mathrm{H}} n_j \mathbf{V}_j.$$

Further substituting expansion (5.30) into this equation, we obtain expression (5.39) for the heat flux. □

Remark 5.3. The heavy-particle diffusion velocities and heat flux are thus proportional to the electron driving force and electron temperature gradient through the $\mathbf{F}_{i\mathrm{e}}$ contribution to $\hat{\mathbf{d}}_i$, $i \in \mathrm{H}$. Kolesnikov[35] has already introduced electron heavy-particle diffusion coefficients and thermal diffusion coefficients and ratios to couple the heavy-particle diffusion velocities and heat flux to the electron forces. Therefore, we propose to refer to this phenomenon as the Kolesnikov effect for the heavy particles.

Using the thermal diffusion ratios defined in Eq. (5.36), the heavy-particle heat flux can be rewritten

$$\mathbf{q}_h = -\lambda_h \boldsymbol{\partial}_{\mathbf{x}} T_h + p_h \sum_{j \in \mathrm{H}} \chi_j^h \mathbf{V}_j + \sum_{j \in \mathrm{H}} \rho_j h_j \mathbf{V}_j, \tag{5.41}$$

where the thermal conductivity is given by

$$\lambda_h = \lambda'_h - n_h \sum_{j\in \mathrm{H}} \chi_j^h \theta_j^h. \tag{5.42}$$

Finally, the first-order heavy-particle mass-energy flux vector

$$\mathsf{F}_h = \left(\mathbf{q}_h - \sum_{j\in\mathrm{H}} \rho_j h_j \mathbf{V}_j,\ (\mathbf{V}_i)_{i\in\mathrm{H}} \right)^{\mathrm{T}},$$

is proportional to the heavy-particle diffusion force vector

$$\mathsf{X}_h = (\boldsymbol{\partial}_{\mathbf{x}} \ln T_h,\ p_h(\hat{\mathbf{d}}_i)_{i\in\mathrm{H}})^{\mathrm{T}},$$

as expressed by the relation

$$\mathsf{F}_h = -\mathbf{A}_h \mathsf{X}_h, \tag{5.43}$$

where the heavy-particle mass-energy transport matrix is given by

$$\mathbf{A}_h = \begin{pmatrix} T_h \lambda'_h & [(\theta_i^h)_{i\in\mathrm{H}}]^{\mathrm{T}} \\ (\theta_i^h)_{i\in\mathrm{H}} & \dfrac{1}{p_h}(D_{ij})_{i,j\in\mathrm{H}} \end{pmatrix}.$$

The positivity properties associated with the heat flux and diffusion velocities are given in the following.

Property 5.2. The heavy-particle mass-energy transport matrix $\mathbf{A}_h$ is symmetric, positive semi-definite, and its kernel is one dimensional and spanned by the vector $[0, (\rho_i)_{i\in\mathrm{H}}]^{\mathrm{T}}$.

Proof. We consider the vector $\mathbf{x} = [x_{T_h}, (x_i)_{i\in\mathrm{H}}]^{\mathrm{T}}$ and the family $\mathbf{y}_h = (\mathbf{y}_i)_{i\in\mathrm{H}}$, where

$$\mathbf{y}_i = \sum_{j\in\mathrm{H}} x_j \boldsymbol{\phi}_i^{D_j} - \frac{1}{T_h} x_{T_h} \boldsymbol{\phi}_i^{\lambda'_h}.$$

Seeing the scalar constraints (5.32), this family is orthogonal to the collisional invariants. Expressions (5.34) and (5.40) for transport coefficients yield

$$\langle \mathbf{A}_h \mathbf{x}, \mathbf{x} \rangle = \frac{1}{3} T_h M_h [\![\mathbf{y}_h, \mathbf{y}_h]\!]_h.$$

Given the properties of the heavy-particle bracket operator $[\![\cdot,\cdot]\!]_h$, we have $\langle \mathbf{A}_h \mathbf{x}, \mathbf{x} \rangle \geq 0$, and $\langle \mathbf{A}_h \mathbf{x}, \mathbf{x} \rangle = 0$ implies that $\mathbf{y}_h$ is a collisional invariant, hence $\mathbf{y}_h = 0$. Moreover, the linear rank of the family $(\boldsymbol{\phi}_i^{\lambda'_h}, \boldsymbol{\phi}_i^{D_1}, \dots, \boldsymbol{\phi}_i^{D_{n^{\mathrm{H}}}})$ is exactly n^{H} because it is the rank of the corresponding right-hand side $(\boldsymbol{\Psi}_i^{\lambda'_h}, \boldsymbol{\Psi}_i^{D_1}, \dots, \boldsymbol{\Psi}_i^{D_{n^{\mathrm{H}}}})$ of Eq. (5.31). We then conclude that $\mathbf{y}_h = 0$ if and only if $\mathbf{x}$ lies in the space spanned by the vector $[0, (\rho_i)_{i\in\mathrm{H}}]^{\mathrm{T}}$. □

5.4. *Second-order electron perturbation function*

The second-order perturbation function ϕ_{e}^2 is a solution to Eq. (4.33), i.e.

$$\mathcal{F}_{\mathrm{e}}(\phi_{\mathrm{e}}^2) + q_{\mathrm{e}}\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}(\phi_{\mathrm{e}}^2)\cdot\mathbf{C}_{\mathrm{e}}\wedge\mathbf{B} = \Psi_{\mathrm{e}}^2, \tag{5.44}$$

and satisfies the constraints (4.34), where

$$\Psi_{\mathrm{e}}^2 = \frac{1}{f_{\mathrm{e}}^0}(-\hat{\mathscr{D}}_{\mathrm{e}}^0(f_{\mathrm{e}}^0,\phi_{\mathrm{e}}) + \mathcal{J}_{\mathrm{ee}}(f_{\mathrm{e}}^0\phi_{\mathrm{e}}, f_{\mathrm{e}}^0\phi_{\mathrm{e}}) + \hat{\mathcal{J}}_{\mathrm{e}}^0).$$

Introducing second-order heavy-particle diffusion driving forces $\mathbf{d}_i^2 = -\mathbf{V}_i$, $i\in\mathrm{H}$, one obtains after some lengthy calculation

$$\Psi_{\mathrm{e}}^2 = -\boldsymbol{\Psi}_{\mathrm{e}}^{\eta_{\mathrm{e}}}{:}\boldsymbol{\partial}_{\mathbf{x}}\mathbf{v}_h - p_{\mathrm{e}}\sum_{j\in\mathrm{H}}\boldsymbol{\Psi}_{\mathrm{e}}^{D_j}\cdot\mathbf{d}_j^2 - \widetilde{\Psi}_{\mathrm{e}}^2,$$

where $\widetilde{\Psi}_{\mathrm{e}}^2$ is a scalar function of $\mathbf{C}_{\mathrm{e}}\cdot\mathbf{C}_{\mathrm{e}}$, and

$$\begin{cases} \boldsymbol{\Psi}_{\mathrm{e}}^{\eta_{\mathrm{e}}} = \dfrac{1}{T_{\mathrm{e}}}\left(\mathbf{C}_{\mathrm{e}}\otimes\mathbf{C}_{\mathrm{e}} - \dfrac{1}{3}\mathbf{C}_{\mathrm{e}}\cdot\mathbf{C}_{\mathrm{e}}\mathbb{I}\right), \\ \boldsymbol{\Psi}_{\mathrm{e}}^{D_i} = \dfrac{n_i}{p_{\mathrm{e}}T_{\mathrm{e}}}Q_{\mathrm{e}i}^{(1)}(|\mathbf{C}_{\mathrm{e}}|^2)|\mathbf{C}_{\mathrm{e}}|\mathbf{C}_{\mathrm{e}}, \quad i\in\mathrm{H}. \end{cases} \tag{5.45}$$

The coupling of the electrons with the heavy particles occurs in the integral equation for the second-order perturbation function through the $\mathbf{d}_i^2$ forces, $i\in\mathrm{H}$. Thus, the second-order electron transport fluxes to be derived are also expected to be proportional to the heavy-particle forces.

The complete solution to Eq. (5.44) is not necessary since we only need to express the second-order transport fluxes $\mathbf{V}_{\mathrm{e}}^2$ and $\mathbf{q}_{\mathrm{e}}^2$ in terms of bracket operators. Consequently, we only have to examine the contribution of the two vectorial terms $\boldsymbol{\Psi}_{\mathrm{e}}^{D_{\mathrm{e}}}$ and $\boldsymbol{\Psi}_{\mathrm{e}}^{D_i}$, $i\in\mathrm{H}$.

Proposition 5.9. *The scalar function ϕ_{e}^2 given by*

$$\phi_{\mathrm{e}}^2 = -\boldsymbol{\phi}_{\mathrm{e}}^{\eta_{\mathrm{e}}}:\boldsymbol{\partial}_{\mathbf{x}}\mathbf{v}_h - p_{\mathrm{e}}\sum_{j\in\mathrm{H}}\Re[\mathbf{M}^{\|}\boldsymbol{\varphi}_{\mathrm{e}}^{D_j(1)} + (\mathbf{M}^{\perp}+\mathrm{i}\mathbf{M}^{\odot})\boldsymbol{\varphi}_{\mathrm{e}}^{D_j(2)}]\cdot\mathbf{d}_j^2 - \widetilde{\phi}_{\mathrm{e}}^2, \tag{5.46}$$

is the solution to Eq. (5.44) *under the constraints* (4.34). *The vectorial functions* $\boldsymbol{\varphi}_{\mathrm{e}}^{D_i(1)}$, $\boldsymbol{\varphi}_{\mathrm{e}}^{D_i(2)}$, $i\in\mathrm{H}$, *are the solutions to the equations*

$$\mathcal{F}_{\mathrm{e}}(\boldsymbol{\varphi}_{\mathrm{e}}^{D_i(1)}) = \boldsymbol{\Psi}_{\mathrm{e}}^{D_i}, \tag{5.47}$$

$$(\mathcal{F}_{\mathrm{e}} + \mathrm{i}|\mathbf{B}|\mathcal{F}_{\mathrm{e}}^{q_{\mathrm{e}}})(\boldsymbol{\varphi}_{\mathrm{e}}^{D_i(2)}) = \boldsymbol{\Psi}_{\mathrm{e}}^{D_i}, \tag{5.48}$$

under the constraints

$$\langle\!\langle f_{\mathrm{e}}^0\boldsymbol{\varphi}_{\mathrm{e}}^{D_i(1)}, \hat{\psi}_{\mathrm{e}}^l\rangle\!\rangle_{\mathrm{e}} = 0, \quad l\in\{1,2\}, \tag{5.49}$$

$$\langle\!\langle f_{\mathrm{e}}^0\boldsymbol{\varphi}_{\mathrm{e}}^{D_i(2)}, \hat{\psi}_{\mathrm{e}}^l\rangle\!\rangle_{\mathrm{e}} = 0, \quad l\in\{1,2\}. \tag{5.50}$$

The tensorial function $\boldsymbol{\phi}_{\mathrm{e}}^{\eta_{\mathrm{e}}}$ satisfies

$$\mathcal{F}_{\mathrm{e}}(\boldsymbol{\phi}_{\mathrm{e}}^{\eta_{\mathrm{e}}}) + q_{\mathrm{e}}\boldsymbol{\partial}_{\mathbf{C}_{\mathrm{e}}}(\boldsymbol{\phi}_{\mathrm{e}}^{\eta_{\mathrm{e}}})\cdot\mathbf{C}_{\mathrm{e}}\wedge\mathbf{B} = \boldsymbol{\Psi}_{\mathrm{e}}^{\eta_{\mathrm{e}}},$$

under the constraints

$$\langle\!\langle f_{\rm e}^0 \boldsymbol{\phi}_{\rm e}^{\eta_{\rm e}}, \hat{\psi}_{\rm e}^l \rangle\!\rangle_{\rm e} = 0, \quad l \in \{1,2\},$$

and the function $\widetilde{\phi}_{\rm e}^2$ *is a scalar function of* $\mathbf{C}_{\rm e} \cdot \mathbf{C}_{\rm e}$ *and* $(\mathbf{C}_{\rm e} \cdot \mathbf{B})^2$.

Proof. The proof for this proposition is similar to the one for Proposition 5.1 since Eqs. (5.1) and (5.44) for $\phi_{\rm e}$ and $\phi_{\rm e}^2$ only differ with their right-hand side. □

The expressions for the second-order electron diffusion velocity and heat flux and of the average electron force are given in the following proposition.

Proposition 5.10. *The second-order electron diffusion velocity* $\mathbf{V}_{\rm e}^2$ *is given by*

$$\mathbf{V}_{\rm e}^2 = -\sum_{j \in {\rm H}} \bar{\bar{\alpha}}_{{\rm e}j} \mathbf{d}_j^2 \tag{5.51}$$

the second-order electron heat flux $\mathbf{q}_{\rm e}^2$

$$\mathbf{q}_{\rm e}^2 = -p_{\rm e} \sum_{j \in {\rm H}} \bar{\bar{\chi}}_j^{\rm e} \mathbf{d}_j^2 + \rho_{\rm e} h_{\rm e} \mathbf{V}_{\rm e}^2, \tag{5.52}$$

and the average electron force $\mathbf{F}_{i{\rm e}}$ *acting on* i *heavy particles*

$$\mathbf{F}_{i{\rm e}} = -\frac{p_{\rm e}}{n_i M_h} \bar{\bar{\alpha}}_{{\rm e}i} \mathbf{d}_{\rm e} - \frac{p_{\rm e}}{n_i M_h} \bar{\bar{\chi}}_i^{\rm e} \boldsymbol{\partial}_{\mathbf{x}} \ln T_{\rm e}, \quad i \in {\rm H}. \tag{5.53}$$

The element of the matrices of $\bar{\bar{\alpha}}_{{\rm e}i}$ *coefficients and second-order electron thermal diffusion ratios read*

$$\begin{aligned}
\alpha_{{\rm e}i}^{\parallel} &= \frac{1}{3} p_{\rm e} T_{\rm e} M_h [\![\boldsymbol{\varphi}_{\rm e}^{D_{\rm e}(1)}, \boldsymbol{\varphi}_{\rm e}^{D_i(1)}]\!]_{\rm e}, & \chi_i^{{\rm e}\parallel} &= -\frac{1}{3} M_h [\![\boldsymbol{\varphi}_{\rm e}^{D_i(1)}, \boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(1)}]\!]_{\rm e}, \quad i \in {\rm H}, \\
\alpha_{{\rm e}i}^{\perp} &= \frac{1}{3} p_{\rm e} T_{\rm e} M_h [\![\boldsymbol{\varphi}_{\rm e}^{D_{\rm e}(2)}, \boldsymbol{\varphi}_{\rm e}^{D_i(2)}]\!]_{\rm e}, & \chi_i^{{\rm e}\perp} &= -\frac{1}{3} M_h [\![\boldsymbol{\varphi}_{\rm e}^{D_i(2)}, \boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}]\!]_{\rm e}, \quad i \in {\rm H}, \\
\alpha_{{\rm e}i}^{\odot} &= -\frac{1}{3} p_{\rm e} T_{\rm e} M_h ((\boldsymbol{\varphi}_{\rm e}^{D_{\rm e}(2)}, \boldsymbol{\varphi}_{\rm e}^{D_i(2)}))_{\rm e}, & \chi_i^{{\rm e}\odot} &= \frac{1}{3} M_h ((\boldsymbol{\varphi}_{\rm e}^{D_i(2)}, \boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}))_{\rm e}, \quad i \in {\rm H}.
\end{aligned} \tag{5.54}$$

Note that these expressions are real, in particular for $\alpha_{{\rm e}i}^{\perp}$, $\alpha_{{\rm e}i}^{\odot}$, $\chi_i^{{\rm e}\perp}$ *and* $\chi_i^{{\rm e}\odot}$, $i \in {\rm H}$, *although functions* $\boldsymbol{\varphi}_{\rm e}^{\lambda'_{\rm e}(2)}$, $\boldsymbol{\varphi}_{\rm e}^{D_{\rm e}(2)}$, *and* $\boldsymbol{\varphi}_{\rm e}^{D_i(2)}$ *are complex.*

Proof. Using definition (4.38) (respectively (4.55) and (4.20)) of the second-order diffusion velocity $\mathbf{V}_{\rm e}^2$ (respectively the second-order electron heat flux $\mathbf{q}_{\rm e}^2$ and average electron force $\mathbf{F}_{i{\rm e}}$, $i \in {\rm H}$), the same proof as that of Proposition 5.2 yields the conclusion. □

Remark 5.4. The term $\boldsymbol{\phi}_{\rm e}^{\eta_{\rm e}} : \boldsymbol{\partial}_{\mathbf{x}} \mathbf{v}_h$ of Eq. (5.46) contributes to a second-order electron momentum relation not investigated here.

Remark 5.5. The second-order electron diffusion velocity and heat flux are thus proportional to the heavy-particle diffusion velocities, that is the Kolesnikov effect for the electrons. To the author's knowledge, it is the first time that such second-order transport fluxes are rigorously derived from a multiscale analysis. However, since the electron collision operator is of the order of $1/\varepsilon^2$ in the electron Boltzmann equation (2.29), it is important to mention that they should not be confused with Burnett transport fluxes[26] based on a second-order perturbation function and a collision operator of the order of $1/\varepsilon$.

We rewrite the mass and energy transport fluxes in terms of the diffusion forces by replacing expression (5.53) for $\mathbf{F}_{ie}$ in Eq. (5.29). The heavy-particle diffusion velocities given in Eq. (5.33) read

$$\mathbf{V}_i = -\bar{\bar{\theta}}_i^{\mathrm{e}} \boldsymbol{\partial}_{\mathbf{x}} \ln T_{\mathrm{e}} - \theta_i^h \boldsymbol{\partial}_{\mathbf{x}} \ln T_h - \bar{\bar{D}}_{i\mathrm{e}} \mathbf{d}'_{\mathrm{e}} - \sum_{j \in \mathrm{H}} D_{ij} \mathbf{d}'_j, \quad i \in \mathrm{H}, \tag{5.55}$$

with the modified driving forces

$$\mathbf{d}'_{\mathrm{e}} = \frac{p_{\mathrm{e}}}{p_h} \mathbf{d}_{\mathrm{e}}, \quad \mathbf{d}'_i = \frac{1}{p_h} \boldsymbol{\partial}_{\mathbf{x}} p_i - \frac{n_i q_i}{p_h} \mathbf{E}', \quad i \in \mathrm{H}.$$

The matrices of heavy-particle electron diffusion coefficients and electron thermal diffusion coefficients, defined as

$$\bar{\bar{D}}_{i\mathrm{e}} = \sum_{j \in \mathrm{H}} D_{ij} \bar{\bar{\alpha}}_{\mathrm{e}j}, \quad \bar{\bar{\theta}}_i^{\mathrm{e}} = \frac{p_{\mathrm{e}}}{p_h} \sum_{j \in \mathrm{H}} D_{ij} \bar{\bar{\chi}}_j^{\mathrm{e}}, \quad i \in \mathrm{H},$$

exhibit the following properties

$$\sum_{j \in \mathrm{H}} \rho_j \bar{\bar{D}}_{j\mathrm{e}} = 0, \quad \sum_{j \in \mathrm{H}} \rho_j \bar{\bar{\theta}}_j^{\mathrm{e}} = 0.$$

The alternative formulation (5.35) is found to be

$$\mathbf{V}_i = -\bar{\bar{D}}_{i\mathrm{e}} \mathbf{d}'_{\mathrm{e}} - \sum_{j \in \mathrm{H}} D_{ij} \left(\mathbf{d}'_j + \frac{p_{\mathrm{e}}}{p_h} \bar{\bar{\chi}}_j^{\mathrm{e}} \boldsymbol{\partial}_{\mathbf{x}} \ln T_{\mathrm{e}} + \chi_j^h \boldsymbol{\partial}_{\mathbf{x}} \ln T_h \right), \quad i \in \mathrm{H}. \tag{5.56}$$

The heavy-particle heat flux given in Eq. (5.39) reads

$$\mathbf{q}_h = -\bar{\bar{\lambda}}_{\mathrm{e}}^{h\prime} \boldsymbol{\partial}_{\mathbf{x}} T_{\mathrm{e}} - \lambda'_h \boldsymbol{\partial}_{\mathbf{x}} T_h - p_h \bar{\bar{\theta}}_{\mathrm{e}}^h \mathbf{d}'_{\mathrm{e}} - p_h \sum_{j \in \mathrm{H}} \theta_j^h \mathbf{d}'_j + \sum_{j \in \mathrm{H}} \rho_j h_j \mathbf{V}_j, \tag{5.57}$$

with the matrices of partial thermal conductivity and thermal diffusion coefficient

$$\bar{\bar{\lambda}}_{\mathrm{e}}^{h\prime} = n_{\mathrm{e}} \sum_{j \in \mathrm{H}} \theta_j^h \bar{\bar{\chi}}_j^{\mathrm{e}}, \quad \bar{\bar{\theta}}_{\mathrm{e}}^h = \sum_{j \in \mathrm{H}} \theta_j^h \bar{\bar{\alpha}}_{\mathrm{e}j}.$$

Then, substituting the previous expression (5.55) for $\mathbf{V}_i$, $i \in \mathrm{H}$, into the expression (5.51) for the second-order electron diffusion velocity $\mathbf{V}_{\mathrm{e}}^2$, one has

$$\mathbf{V}_{\mathrm{e}}^2 = -\bar{\bar{\theta}}_{\mathrm{e}}^{\mathrm{e}} \boldsymbol{\partial}_{\mathbf{x}} \ln T_{\mathrm{e}} - \bar{\bar{\theta}}_{\mathrm{e}}^h \boldsymbol{\partial}_{\mathbf{x}} \ln T_h - \bar{\bar{D}}_{\mathrm{ee}} \mathbf{d}'_{\mathrm{e}} - \sum_{j \in \mathrm{H}} \bar{\bar{D}}_{\mathrm{e}j} \mathbf{d}'_j, \tag{5.58}$$

with the following second-order matrices of electron diffusion coefficients, electron heavy-particle diffusion coefficients, and electron thermal diffusion coefficient

$$\bar{\bar{D}}_{\mathrm{ee}} = \sum_{j \in \mathrm{H}} \bar{\bar{\alpha}}_{\mathrm{e}j} \bar{\bar{D}}_{j\mathrm{e}}, \quad \bar{\bar{D}}_{\mathrm{e}i} = \bar{\bar{D}}_{i\mathrm{e}}, \quad i \in \mathrm{H}, \quad \bar{\bar{\theta}}_{\mathrm{e}}^{\mathrm{e}} = \sum_{j \in \mathrm{H}} \bar{\bar{\alpha}}_{\mathrm{e}j} \bar{\bar{\theta}}_j^{\mathrm{e}}.$$

The alternative formulation reads

$$\mathbf{V}_{\mathrm{e}}^{2} = -\bar{\bar{D}}_{\mathrm{ee}}\mathbf{d}'_{\mathrm{e}} - \sum_{j\in\mathrm{H}} \bar{\bar{D}}_{\mathrm{e}j}\left(\mathbf{d}'_j + \frac{p_{\mathrm{e}}}{p_h}\bar{\bar{\chi}}_j^{\mathrm{e}}\boldsymbol{\partial}_{\mathbf{x}}\ln T_{\mathrm{e}} + \chi_j^h\boldsymbol{\partial}_{\mathbf{x}}\ln T_h\right). \tag{5.59}$$

Regarding the electron heat flux given in Eq. (5.52), one obtains

$$\mathbf{q}_{\mathrm{e}}^{2} = -\bar{\bar{\lambda}}_{\mathrm{e}}^{2\prime}\boldsymbol{\partial}_{\mathbf{x}}T_{\mathrm{e}} - \frac{T_{\mathrm{e}}}{T_h}\bar{\bar{\lambda}}_{\mathrm{e}}^{h\prime}\boldsymbol{\partial}_{\mathbf{x}}T_h - p_h\,\bar{\bar{\theta}}_{\mathrm{e}}^{\mathrm{e}}\mathbf{d}'_{\mathrm{e}} - p_h\sum_{j\in\mathrm{H}}\bar{\bar{\theta}}_j^{\mathrm{e}}\mathbf{d}'_j + \rho_{\mathrm{e}}h_{\mathrm{e}}\mathbf{V}_{\mathrm{e}}^{2}, \tag{5.60}$$

with the second-order matrix of electron partial thermal conductivity

$$\bar{\bar{\lambda}}_{\mathrm{e}}^{2\prime} = n_{\mathrm{e}}\sum_{j\in\mathrm{H}}\bar{\bar{\chi}}_j^{\mathrm{e}}\,\bar{\bar{\theta}}_j^{\mathrm{e}}.$$

5.5. *Weakly magnetized and unmagnetized plasmas*

We recall that the intensity of the magnetic field is expressed by means of the b parameter used to define the scaling of the nondimensional electron Hall parameter $q^0B^0t_{\mathrm{e}}^0/m_{\mathrm{e}}^0 = \varepsilon^{1-b}$. We deal with weakly magnetized plasmas ($b = 0$) and unmagnetized plasmas ($b < 0$) by reviewing the whole section in this simplified frame.

The first-order electron perturbation function ϕ_{e} is a solution to Eq. (4.9), i.e.

$$\mathcal{F}_{\mathrm{e}}(\phi_{\mathrm{e}}) = \Psi_{\mathrm{e}}, \tag{5.61}$$

and satisfies the constraints (4.10). The transport coefficients are shown to be isotropic since the operator $\mathcal{F}_{\mathrm{e}}^{q_{\mathrm{e}}}$ does not appear in Eq. (5.61). Defining the electron driving force $\mathbf{d}_{\mathrm{e}} = (\boldsymbol{\partial}_{\mathbf{x}}p_{\mathrm{e}} - n_{\mathrm{e}}q_{\mathrm{e}}\mathbf{E})/p_{\mathrm{e}}$, the electron perturbation function is given in the following proposition.

Proposition 5.11. *The scalar function ϕ_{e} given by*

$$\phi_{\mathrm{e}} = -p_{\mathrm{e}}\boldsymbol{\phi}_{\mathrm{e}}^{D_{\mathrm{e}}}\cdot\mathbf{d}_{\mathrm{e}} - \boldsymbol{\phi}_{\mathrm{e}}^{\lambda'_{\mathrm{e}}}\cdot\boldsymbol{\partial}_{\mathbf{x}}\left(\frac{1}{T_{\mathrm{e}}}\right), \tag{5.62}$$

is the solution to Eq. (5.61) *under the constraints* (4.10), *where the vectorial functions $\boldsymbol{\phi}_{\mathrm{e}}^{D_{\mathrm{e}}}$ and $\boldsymbol{\phi}_{\mathrm{e}}^{\lambda'_{\mathrm{e}}}$ are the solutions to the equations*

$$\mathcal{F}_{\mathrm{e}}(\boldsymbol{\phi}_{\mathrm{e}}^{\mu}) = \boldsymbol{\Psi}_{\mathrm{e}}^{\mu}, \tag{5.63}$$

under the scalar constraints

$$\langle\!\langle f_{\mathrm{e}}^0\boldsymbol{\phi}_{\mathrm{e}}^{\mu}, \hat{\psi}_{\mathrm{e}}^{l}\rangle\!\rangle_{\mathrm{e}} = 0, \quad l\in\{1,2\}, \tag{5.64}$$

with $\mu\in\{D_{\mathrm{e}},\lambda'_{\mathrm{e}}\}$.

Proof. By linearity and isotropy of the linearized collision Boltzmann operator $\mathcal{F}_{\mathrm{e}}$, the development of Ψ_{e} can be followed through for ϕ_{e} as well. Therefore, the functions $\boldsymbol{\phi}_{\mathrm{e}}^{\mu}$ for $\mu\in\{D_{\mathrm{e}},\lambda'_{\mathrm{e}}\}$ satisfy Eq. (5.63) under the scalar constraints (5.64). We seek a solution in the form $\boldsymbol{\phi}_{\mathrm{e}}^{\mu} = \phi_{\mathrm{e}}^{\mu}\mathbf{C}_{\mathrm{e}}$. Quantities ϕ_{e}^{μ}, $\mu\in\{D_{\mathrm{e}},\lambda'_{\mathrm{e}}\}$, are scalar functions of $\mathbf{C}_e \cdot \mathbf{C}_e$ since $\boldsymbol{\phi}_e^\mu$, $\mu \in \{D_e, \lambda'_e\}$, must be invariant under a change of coordinates. Uniqueness of the solution is readily proved based on the linearity property of the operator $\mathcal{F}_e$, its kernel given in property 4.1, and the constraints (4.10) satisfied by ϕ_e. □

The expressions for the electron transport fluxes are given in the following proposition.

Proposition 5.12. *The electron diffusion velocity reads*

$$\mathbf{V}_e = -D_e \mathbf{d}_e - \theta_e \boldsymbol{\partial}_{\mathbf{x}} \ln T_e, \tag{5.65}$$

the electron heat flux,

$$\mathbf{q}_e = -\lambda'_e \boldsymbol{\partial}_{\mathbf{x}} T_e - p_e \theta_e \mathbf{d}_e + \rho_e h_e \mathbf{V}_e, \tag{5.66}$$

where the diffusion coefficients, thermal diffusion coefficients, and partial thermal conductivity are given by

$$\begin{aligned} D_e &= \frac{1}{3} p_e T_e M_h [\![\boldsymbol{\phi}_e^{D_e}, \boldsymbol{\phi}_e^{D_e}]\!]_e, \quad \theta_e = -\frac{1}{3} M_h [\![\boldsymbol{\phi}_e^{D_e}, \boldsymbol{\phi}_e^{\lambda'_e}]\!]_e, \\ \lambda'_e &= \frac{1}{3T_e^2} M_h [\![\boldsymbol{\phi}_e^{\lambda'_e}, \boldsymbol{\phi}_e^{\lambda'_e}]\!]_e, \end{aligned} \tag{5.67}$$

and the electron viscous tensor vanishes, i.e.

$$\boldsymbol{\Pi}_e = 0. \tag{5.68}$$

Proof. The proof of this proposition is based on definition (4.16) (respectively (4.30) and (4.37)) of the first-order diffusion velocity $\mathbf{V}_e$ (respectively the first-order electron heat flux $\mathbf{q}_e$ and viscous tensor $\boldsymbol{\Pi}_e$) and on expression (5.62) for ϕ_e. □

Alternative forms of the electron diffusion velocity and heat flux are also introduced

$$\begin{aligned} \mathbf{V}_e &= -D_e(\mathbf{d}_e + \chi_e \boldsymbol{\partial}_{\mathbf{x}} \ln T_e), \\ \mathbf{q}_e &= -\lambda_e \boldsymbol{\partial}_{\mathbf{x}} T_e + p_e \chi_e \mathbf{V}_e + \rho_e h_e \mathbf{V}_e, \end{aligned}$$

where the thermal diffusion ratio χ_e is defined by the relation $\theta_e = D_e \chi_e$, and the thermal conductivity by $\lambda_e = \lambda'_e - n_e \chi_e \theta_e$.

Concerning the heavy particles, the entire Sec. 5.3 remains valid since Eq. (4.24) is identical for all cases ($b = 1$, $b = 0$ and $b < 0$).

The second-order electron perturbation function ϕ_e^2 is a solution to Eq. (4.33)

$$\mathcal{F}_e(\phi_e^2) = \Psi_e^2, \tag{5.69}$$

and satisfies the constraints (4.34). Introducing a second-order electron diffusion driving force $\mathbf{d}_\mathrm{e}^2 = -n_\mathrm{e} q_\mathrm{e} M_h^2 \mathbf{v}_h \wedge \mathbf{B}/p_\mathrm{e}$, one obtains after some lengthy calculation

$$\Psi_\mathrm{e}^2 = -\boldsymbol{\Psi}_\mathrm{e}^{\eta_\mathrm{e}} : \boldsymbol{\partial}_\mathbf{x} \mathbf{v}_h - \delta_{b0} p_\mathrm{e} \boldsymbol{\Psi}_\mathrm{e}^{D_\mathrm{e}} \cdot \mathbf{d}_\mathrm{e}^2 - p_\mathrm{e} \sum_{j \in \mathrm{H}} \boldsymbol{\Psi}_\mathrm{e}^{D_j} \cdot \mathbf{d}_j^2 - \widetilde{\Psi}_\mathrm{e}^2.$$

Proposition 5.13. *The scalar function ϕ_e^2 given by*

$$\phi_\mathrm{e}^2 = -\boldsymbol{\phi}_\mathrm{e}^{\eta_\mathrm{e}} : \boldsymbol{\partial}_\mathbf{x} \mathbf{v}_h - \delta_{b0} p_\mathrm{e} \boldsymbol{\phi}_\mathrm{e}^{D_\mathrm{e}} \cdot \mathbf{d}_\mathrm{e}^2 - p_\mathrm{e} \sum_{j \in \mathrm{H}} \boldsymbol{\phi}_\mathrm{e}^{D_j} \cdot \mathbf{d}_j^2 - \widetilde{\phi}_\mathrm{e}^2, \tag{5.70}$$

is the solution to Eq. (5.69) *under the constraints* (4.34)*. The vectorial functions $\boldsymbol{\phi}_\mathrm{e}^{D_\mathrm{e}}$, $\boldsymbol{\phi}_\mathrm{e}^{D_i}$, $i \in \mathrm{H}$, and the tensorial function $\boldsymbol{\phi}_\mathrm{e}^{\eta_\mathrm{e}}$ are the solutions to the equations $\mathcal{F}_\mathrm{e}(\boldsymbol{\phi}_\mathrm{e}^\mu) = \Psi_\mathrm{e}^\mu$, under the constraints $\langle\!\langle f_\mathrm{e}^0 \boldsymbol{\phi}_\mathrm{e}^\mu, \hat{\psi}_\mathrm{e}^l \rangle\!\rangle_\mathrm{e} = 0$, $l \in \{1, 2\}$, with $\mu \in \{D_\mathrm{e}, (D_i)_{i \in \mathrm{H}}, \eta_\mathrm{e}\}$. The function $\widetilde{\phi}_\mathrm{e}^2$ is a scalar function of $\mathbf{C}_\mathrm{e} \cdot \mathbf{C}_\mathrm{e}$.*

Proof. The proof of this proposition is similar to the one of Proposition 5.11 since Eqs. (5.61) and (5.69) only differ with their right-hand side. □

The expressions for the second-order electron diffusion velocity and heat flux and of the average electron force are given in the following proposition.

Proposition 5.14. *The second-order electron diffusion velocity is given by*

$$\mathbf{V}_\mathrm{e}^2 = -\delta_{b0} D_\mathrm{e} \mathbf{d}_\mathrm{e}^2 - \sum_{j \in \mathrm{H}} \alpha_{\mathrm{e}j} \mathbf{d}_j^2, \tag{5.71}$$

the second-order electron heat flux,

$$\mathbf{q}_\mathrm{e}^2 = -\delta_{b0} p_\mathrm{e} \theta_\mathrm{e} \mathbf{d}_\mathrm{e}^2 - p_\mathrm{e} \sum_{j \in \mathrm{H}} \chi_j^\mathrm{e} \mathbf{d}_j^2 + \rho_\mathrm{e} h_\mathrm{e} \mathbf{V}_\mathrm{e}^2, \tag{5.72}$$

and the average electron force acting on i heavy particles

$$\mathbf{F}_{i\mathrm{e}} = -\frac{p_\mathrm{e}}{n_i M_h} \alpha_{\mathrm{e}i} \mathbf{d}_\mathrm{e} - \frac{p_\mathrm{e}}{n_i M_h} \chi_i^\mathrm{e} \boldsymbol{\partial}_\mathbf{x} \ln T_\mathrm{e} \quad i \in \mathrm{H}. \tag{5.73}$$

The $\alpha_{\mathrm{e}i}$ coefficients and second-order electron thermal diffusion ratios read

$$\alpha_{\mathrm{e}i} = \frac{1}{3} p_\mathrm{e} T_\mathrm{e} M_h [\![\boldsymbol{\phi}_\mathrm{e}^{D_\mathrm{e}}, \boldsymbol{\phi}_\mathrm{e}^{D_i}]\!]_\mathrm{e} \quad \chi_i^\mathrm{e} = -\frac{1}{3} M_h [\![\boldsymbol{\phi}_\mathrm{e}^{D_i}, \boldsymbol{\phi}_\mathrm{e}^{\lambda'_\mathrm{e}}]\!]_\mathrm{e}, \quad i \in \mathrm{H}. \tag{5.74}$$

Proof. The proof of this proposition is based on the definition (4.38) (respectively (4.55) and (4.20)) of the second-order diffusion velocity $\mathbf{V}_\mathrm{e}^2$ (respectively the second-order electron heat flux $\mathbf{q}_\mathrm{e}^2$ and average electron force $\mathbf{F}_{i\mathrm{e}}$, $i \in \mathrm{H}$) and on the expression (5.70) for ϕ_e^2. □

Table 3. Magnetic field influence.

b	Conservation equations	Transport properties
< 0	—	—
0	Bulk magnetic force Electron magnetic force	Electron bulk magnetic driving force
1	Bulk magnetic force Electron magnetic force Heavy-particle magnetic force	Electron bulk magnetic driving force Heavy-particle bulk magnetic driving forces Anisotropic electron transport properties

Three categories of plasmas are reviewed in Table 3. A value of $b < 0$ corresponds to unmagnetized plasmas, $b = 0$, weakly magnetized plasmas, and $b = 1$, strongly magnetized plasmas.

6. Conservation Equations

We review the heavy-particle Navier–Stokes equations (4.52)–(4.54) and electron drift-diffusion equations (4.60) and (4.61). We also derive a total energy equation and an entropy equation.

6.1. *Mass*

The species mass conservation equations read

$$\partial_t \rho_{\mathrm{e}} + \boldsymbol{\partial}_{\mathbf{x}} \cdot \left[\rho_{\mathrm{e}} \left(\mathbf{v}_h + \frac{1}{M_h} (\mathbf{V}_{\mathrm{e}} + \varepsilon \mathbf{V}_{\mathrm{e}}^2) \right) \right] = 0, \tag{6.1}$$

$$\partial_t \rho_i + \boldsymbol{\partial}_{\mathbf{x}} \cdot \left[\rho_i \left(\mathbf{v}_h + \frac{\varepsilon}{M_h} \mathbf{V}_i \right) \right] = 0, \quad i \in \mathrm{H}. \tag{6.2}$$

Summing Eq. (6.2) over $i \in \mathrm{H}$ and using the constraint $\sum_{j \in \mathrm{H}} \rho_j \mathbf{V}_j = 0$ given in Eq. (4.25), a heavy-particle mass conservation equation is obtained

$$\partial_t \rho_h + \boldsymbol{\partial}_{\mathbf{x}} \cdot (\rho_h \mathbf{v}_h) = 0. \tag{6.3}$$

The heavy-particle mass is conserved in the mean heavy-particle velocity frame. Then, adding the electron drift Eq. (6.1) to Eq. (6.3) and using Eq. (2.27), i.e.

$$\rho \mathbf{v} = \rho_h \mathbf{v}_h + \varepsilon^2 \rho_{\mathrm{e}} \left(\mathbf{v}_h + \frac{1}{M_h} (\mathbf{V}_{\mathrm{e}} + \varepsilon \mathbf{V}_{\mathrm{e}}^2) \right),$$

a conservation equation of global mass $\rho = \rho_h + \varepsilon^2 \rho_{\mathrm{e}}$ is also established

$$\partial_t \rho + \boldsymbol{\partial}_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0. \tag{6.4}$$

The global mass is conserved in the hydrodynamic velocity frame, although the transport fluxes are calculated in the mean heavy-particle velocity frame. It is the only place where the difference between the global hydrodynamic velocity and the mean heavy-particle velocity, of the order of ε^2, plays an essential role. This fact is another evidence of the coherence of our formalism compared to other approaches found in the literature.

6.2. *Momentum*

The momentum conservation is expressed by

$$\partial_t(\rho_h \mathbf{v}_h) + \boldsymbol{\partial}_{\mathbf{x}} \cdot \left(\rho_h \mathbf{v}_h \otimes \mathbf{v}_h + \frac{1}{M_h^2} p \mathbb{I}\right)$$
$$= -\frac{\varepsilon}{M_h^2} \boldsymbol{\partial}_{\mathbf{x}} \cdot \boldsymbol{\Pi}_h + \frac{1}{M_h^2} nq\mathbf{E} + [\delta_{b0}\mathbf{I}_0 + \delta_{b1}\mathbf{I}] \wedge \mathbf{B}. \tag{6.5}$$

It is important to recall that the electrons participate to the momentum balance through the pressure gradient and the Lorentz force.

6.3. *Energy*

A flow kinetic energy equation is obtained by projecting Eq. (6.5) onto the mean heavy-particle velocity

$$\partial_t \left(\frac{1}{2}\rho_h |\mathbf{v}_h|^2\right) + \boldsymbol{\partial}_{\mathbf{x}} \cdot \left[\mathbf{v}_h \left(\frac{1}{2}\rho_h |\mathbf{v}_h|^2 + \frac{1}{M_h^2} p\right)\right]$$
$$= \frac{1}{M_h^2} p \boldsymbol{\partial}_{\mathbf{x}} \cdot \mathbf{v}_h - \frac{\varepsilon}{M_h^2} \mathbf{v}_h \cdot \boldsymbol{\partial}_{\mathbf{x}} \cdot \boldsymbol{\Pi}_h + \frac{1}{M_h^2} nq\mathbf{E} \cdot \mathbf{v}_h + \mathbf{v}_h \cdot (\delta_{b0}\mathbf{I}_0 + \delta_{b1}\mathbf{I}) \wedge \mathbf{B}. \tag{6.6}$$

The electron energy equation reads

$$\partial_t(\rho_e e_e) + \boldsymbol{\partial}_{\mathbf{x}} \cdot (\rho_e e_e \mathbf{v}_h) = -p_e \boldsymbol{\partial}_{\mathbf{x}} \cdot \mathbf{v}_h - \frac{1}{M_h} \boldsymbol{\partial}_{\mathbf{x}} \cdot \left(\mathbf{q}_e + \varepsilon \mathbf{q}_e^2\right) + \frac{1}{M_h} \left(\mathbf{J}_e + \varepsilon \mathbf{J}_e^2\right) \cdot \mathbf{E}'$$
$$+ \delta_{b0} \varepsilon M_h \mathbf{J}_e \cdot \mathbf{v}_h \wedge \mathbf{B} + \Delta E_e^0 + \varepsilon \Delta E_e^1, \tag{6.7}$$

and the heavy-particle energy equation

$$\partial_t(\rho_h e_h) + \boldsymbol{\partial}_{\mathbf{x}} \cdot (\rho_h e_h \mathbf{v}_h) = -(p_h \mathbb{I} + \varepsilon \boldsymbol{\Pi}_h) : \boldsymbol{\partial}_{\mathbf{x}} \mathbf{v}_h - \frac{\varepsilon}{M_h} \boldsymbol{\partial}_{\mathbf{x}} \cdot \mathbf{q}_h + \frac{\varepsilon}{M_h} \mathbf{J}_h \cdot \mathbf{E}'$$
$$+ \Delta E_h^0 + \varepsilon \Delta E_h^1. \tag{6.8}$$

So that a global energy equation is derived by summing Eqs. (6.7) and (6.8)

$$\partial_t(\rho e) + \boldsymbol{\partial}_{\mathbf{x}} \cdot (\rho e \mathbf{v}_h) = -(p\mathbb{I} + \varepsilon \boldsymbol{\Pi}_h) : \boldsymbol{\partial}_{\mathbf{x}} \mathbf{v}_h - \frac{1}{M_h} \boldsymbol{\partial}_{\mathbf{x}} \cdot \boldsymbol{\mathcal{Q}}$$
$$+ \frac{1}{M_h} (\mathbf{J}_e + \varepsilon \mathbf{J}_e^2 + \varepsilon \mathbf{J}_h) \cdot \mathbf{E}' + \delta_{b0} \varepsilon M_h \mathbf{J}_e \cdot \mathbf{v}_h \wedge \mathbf{B}, \tag{6.9}$$

where quantity $\boldsymbol{\mathcal{Q}} = \mathbf{q}_e + \varepsilon \mathbf{q}_e^2 + \varepsilon \mathbf{q}_h$ is the total heat flux and $\rho e = \rho_e e_e + \rho_h e_h$, the mixture energy in the heavy-particle reference frame. Finally, a total energy equation is derived by adding Eq. (6.6)

$$\partial_t(\mathcal{E}) + \boldsymbol{\partial}_{\mathbf{x}} \cdot (\mathcal{H} \mathbf{v}_h) = -\varepsilon \boldsymbol{\partial}_{\mathbf{x}} \cdot (\boldsymbol{\Pi}_h \cdot \mathbf{v}_h) - \frac{1}{M_h} \boldsymbol{\partial}_{\mathbf{x}} \cdot \boldsymbol{\mathcal{Q}} + \mathbf{I} \cdot \mathbf{E}, \tag{6.10}$$

where quantity $\mathcal{E} = \rho e + M_h^2 \rho_h \frac{1}{2} |\mathbf{v}_h|^2$ stands for the total energy and $\mathcal{H} = \mathcal{E} + p$, the total enthalpy. The term $\mathbf{I} \cdot \mathbf{E}$ of Eq. (6.10) represents the power developed by the electromagnetic field. It has the form prescribed by Poynting's theorem. Hence, the first law of thermodynamics is satisfied.

6.4. *Entropy*

In addition to the thermal energy, we introduce other relevant thermodynamic functions. First, the species Gibbs free energy is defined by the relations

$$\rho_{\mathrm{e}} g_{\mathrm{e}} = n_{\mathrm{e}} T_{\mathrm{e}} \ln\left(\frac{n_{\mathrm{e}} n^0}{T_{\mathrm{e}}^{3/2} Q_{\mathrm{e}}^0}\right), \quad \rho_i g_i = n_i T_h \ln\left[\frac{n_i n^0}{(m_i T_h)^{3/2} Q_h^0}\right], \quad i \in \mathrm{H}, \tag{6.11}$$

where the translational partition functions read

$$Q_{\mathrm{e}}^0 = \left(\frac{2\pi m_{\mathrm{e}}^0 \mathrm{k_B} T^0}{\mathrm{h_P^2}}\right)^{3/2}, \quad Q_h^0 = \left(\frac{2\pi m_h^0 \mathrm{k_B} T^0}{\mathrm{h_P^2}}\right)^{3/2}. \tag{6.12}$$

Then, the species enthalpy is given by

$$\rho_{\mathrm{e}} h_{\mathrm{e}} = \frac{5}{2} n_{\mathrm{e}} T_{\mathrm{e}}, \quad \rho_i h_i = \frac{5}{2} n_i T_h, \quad i \in \mathrm{H}. \tag{6.13}$$

Finally, the species entropy is introduced as

$$s_{\mathrm{e}} = \frac{h_{\mathrm{e}} - g_{\mathrm{e}}}{T_{\mathrm{e}}}, \quad s_i = \frac{h_i - g_i}{T_h}, \quad i \in \mathrm{H}. \tag{6.14}$$

Therefore, the mixture entropy reads $\rho s = \sum_{j \in \mathrm{S}} \rho_j s_j$. The thermodynamic functions exhibit a wider range of validity than in classical thermodynamics, introduced for stationary homogeneous equilibrium states.[27] Indeed, they are interpreted in the framework of kinetic theory by establishing a relation between the thermodynamic entropy and the kinetic entropy. This quantity is based upon the distribution functions

$$\mathcal{S}^{\mathrm{kin}} = \sum_{j \in \mathrm{H}} \int f_j \left\{1 - \ln\left[\frac{(2\pi)^{3/2} n^0}{m_j^3 Q_h^0} f_j\right]\right\} \mathrm{d}\mathbf{C}_j + \int f_{\mathrm{e}} \left\{1 - \ln\left[\frac{(2\pi)^{3/2} n^0}{Q_{\mathrm{e}}^0} f_{\mathrm{e}}\right]\right\} \mathrm{d}\mathbf{C}_{\mathrm{e}}. \tag{6.15}$$

Proposition 6.1. *The kinetic entropy and the thermodynamic entropy are asymptotically equal at order* ε^2, *i.e.*

$$\mathcal{S}^{\mathrm{kin}} = \rho s + \mathcal{O}(\varepsilon^2), \tag{6.16}$$

provided that the distribution functions follow the Enskog expansion given in Eqs. (4.1) *and* (4.2).

Proof. Using definition (6.15) and expansions (4.1) and (4.2), the kinetic entropy is found to be

$$
\begin{aligned}
&\sum_{j\in\mathrm{H}}\int f_j^0\left\{1-\ln\left[\frac{(2\pi)^{3/2}n^0}{m_j^3 Q_h^0}f_j^0\right]\right\}\mathrm{d}\mathbf{C}_j+\int f_\mathrm{e}^0\left\{1-\ln\left[\frac{(2\pi)^{3/2}n^0}{Q_\mathrm{e}^0}f_\mathrm{e}^0\right]\right\}\mathrm{d}\mathbf{C}_\mathrm{e}\\
&\quad+\varepsilon\sum_{j\in\mathrm{H}}\int f_j^0\phi_j\ln\left[\frac{(2\pi)^{3/2}n^0}{m_j^3 Q_h^0}f_j^0\right]\mathrm{d}\mathbf{C}_j+\varepsilon\int f_\mathrm{e}^0\phi_\mathrm{e}\ln\left[\frac{(2\pi)^{3/2}n^0}{Q_\mathrm{e}^0}f_\mathrm{e}^0\right]\mathrm{d}\mathbf{C}_\mathrm{e}\\
&\quad+\mathcal{O}(\varepsilon^2).
\end{aligned}
$$

The first-order term vanishes by the constraints (4.10) and (4.25). Then, using expressions (4.7) and (4.8) and definition (6.14), Eq. (6.16) is readily obtained. □

Consequently, a first-order conservation equation of thermodynamic entropy can be used instead of a conservation equation of kinetic entropy to ensure that the second law of thermodynamics is satisfied. First, we introduce the heavy-particle entropy $\rho_h s_h = \sum_{j\in\mathrm{H}}\rho_j s_j$ and derive the entropy equations.

Proposition 6.2. *The electron and heavy-particle entropy equations associated with the macroscopic conservation equations* (6.1)–(6.8) *read*

$$\partial_t(\rho_\mathrm{e}s_\mathrm{e})+\boldsymbol{\partial}_\mathbf{x}\cdot(\rho_\mathrm{e}s_\mathrm{e}\mathbf{v}_h)+\boldsymbol{\partial}_\mathbf{x}\cdot(\boldsymbol{\mathcal{J}}_\mathrm{e}^0+\varepsilon\boldsymbol{\mathcal{J}}_\mathrm{e}^1)=\Upsilon_\mathrm{e}^0+\varepsilon\Upsilon_\mathrm{e}^1, \tag{6.17}$$

$$\partial_t(\rho_h s_h)+\boldsymbol{\partial}_\mathbf{x}\cdot(\rho_h s_h\mathbf{v}_h)+\varepsilon\boldsymbol{\partial}_\mathbf{x}\cdot\boldsymbol{\mathcal{J}}_h^1=\Upsilon_h^0+\varepsilon\Upsilon_h^1, \tag{6.18}$$

where the electron and heavy-particle entropy fluxes are given by

$$\boldsymbol{\mathcal{J}}_\mathrm{e}^0=\frac{1}{M_hT_\mathrm{e}}(\mathbf{q}_\mathrm{e}-\rho_\mathrm{e}g_\mathrm{e}\mathbf{V}_\mathrm{e}),\quad \boldsymbol{\mathcal{J}}_\mathrm{e}^1=\frac{1}{M_hT_\mathrm{e}}(\mathbf{q}_\mathrm{e}^2-\rho_\mathrm{e}g_\mathrm{e}\mathbf{V}_\mathrm{e}^2), \tag{6.19}$$

$$\boldsymbol{\mathcal{J}}_h^1=\frac{1}{M_hT_h}\left(\mathbf{q}_h-\sum_{j\in\mathrm{H}}\rho_j g_j\mathbf{V}_j\right), \tag{6.20}$$

and the electron and heavy-particle entropy production rates by

$$\Upsilon_\mathrm{e}^0=\frac{1}{T_\mathrm{e}}\Delta E_\mathrm{e}^0-\frac{p_\mathrm{e}}{M_hT_\mathrm{e}}\mathbf{d}_\mathrm{e}\cdot\mathbf{V}_\mathrm{e}-\frac{1}{M_hT_\mathrm{e}}\boldsymbol{\partial}_\mathbf{x}\ln T_\mathrm{e}\cdot(\mathbf{q}_\mathrm{e}-\rho_\mathrm{e}h_\mathrm{e}\mathbf{V}_\mathrm{e}), \tag{6.21}$$

$$
\begin{aligned}
\Upsilon_\mathrm{e}^1=&\frac{1}{T_\mathrm{e}}\Delta E_\mathrm{e}^1-\frac{p_\mathrm{e}}{M_hT_\mathrm{e}}(\mathbf{d}_\mathrm{e}\cdot\mathbf{V}_\mathrm{e}^2+\delta_{b0}\mathbf{d}_\mathrm{e}^2\cdot\mathbf{V}_\mathrm{e})\\
&-\frac{1}{M_hT_\mathrm{e}}\boldsymbol{\partial}_\mathbf{x}\ln T_\mathrm{e}\cdot(\mathbf{q}_\mathrm{e}^2-\rho_\mathrm{e}h_\mathrm{e}\mathbf{V}_\mathrm{e}^2),
\end{aligned} \tag{6.22}
$$

$$\Upsilon_h^0=\frac{1}{T_h}\Delta E_h^0, \tag{6.23}$$

$$\Upsilon_h^1 = \frac{1}{T_h}\Delta E_h^1 - \frac{1}{T_h}\boldsymbol{\Pi}_h : \boldsymbol{\partial}_{\mathbf{x}}\mathbf{v}_h - \frac{p_h}{M_h T_h}\sum_{j\in \mathrm{H}} \frac{1}{p_h}\big(\boldsymbol{\partial}_{\mathbf{x}} p_j - n_j q_j \mathbf{E}'\big)\cdot \mathbf{V}_j$$

$$- \frac{1}{M_h T_h}\boldsymbol{\partial}_{\mathbf{x}} \ln T_h \cdot \left(\mathbf{q}_h - \sum_{j\in\mathrm{H}} \rho_j h_j \mathbf{V}_j\right). \tag{6.24}$$

Proof. Based on the relations

$$\rho_{\mathrm{e}} \mathrm{d}\left(\frac{g_{\mathrm{e}}}{T_{\mathrm{e}}}\right) = \mathrm{d}n_{\mathrm{e}} - \frac{3n_{\mathrm{e}}}{2T_{\mathrm{e}}}\mathrm{d}T_{\mathrm{e}}, \quad \rho_i \mathrm{d}\left(\frac{g_i}{T_h}\right) = \mathrm{d}n_i - \frac{3n_i}{2T_h}\mathrm{d}T_h, \quad i \in \mathrm{H},$$

and definition (6.14), one obtains

$$\begin{aligned}\partial_t(\rho_{\mathrm{e}} s_{\mathrm{e}}) + \boldsymbol{\partial}_{\mathbf{x}}\cdot(\rho_{\mathrm{e}} s_{\mathrm{e}} \mathbf{v}_h) &= \frac{1}{T_{\mathrm{e}}}[\partial_t(\rho_{\mathrm{e}} e_{\mathrm{e}}) + \boldsymbol{\partial}_{\mathbf{x}}\cdot(\rho_{\mathrm{e}} e_{\mathrm{e}} \mathbf{v}_h)] + n_{\mathrm{e}}\boldsymbol{\partial}_{\mathbf{x}}\cdot\mathbf{v}_h \\ &\quad - [\partial_t \rho_{\mathrm{e}} + \boldsymbol{\partial}_{\mathbf{x}}\cdot(\rho_{\mathrm{e}}\mathbf{v}_h)]\frac{g_{\mathrm{e}}}{T_{\mathrm{e}}}, \\ \partial_t(\rho_h s_h) + \boldsymbol{\partial}_{\mathbf{x}}\cdot(\rho_h s_h \mathbf{v}_h) &= \frac{1}{T_h}[\partial_t(\rho_h e_h) + \boldsymbol{\partial}_{\mathbf{x}}\cdot(\rho_h e_h \mathbf{v}_h)] + n_h\boldsymbol{\partial}_{\mathbf{x}}\cdot\mathbf{v}_h \\ &\quad - \sum_{j\in\mathrm{H}}[\partial_t \rho_j + \boldsymbol{\partial}_{\mathbf{x}}\cdot(\rho_j\mathbf{v}_h)]\frac{g_j}{T_h}.\end{aligned}$$

Then, using Eqs. (6.1), (6.2), (6.7), (6.8), and the relations

$$\mathrm{d}\left(\frac{g_{\mathrm{e}}}{T_{\mathrm{e}}}\right) = -\frac{h_{\mathrm{e}}}{T_{\mathrm{e}}^2}\mathrm{d}T_{\mathrm{e}} + \frac{1}{p_{\mathrm{e}}}\mathrm{d}p_{\mathrm{e}}, \quad \mathrm{d}\left(\frac{g_i}{T_h}\right) = -\frac{h_i}{T_h^2}\mathrm{d}T_h + \frac{1}{m_i p_i}\mathrm{d}p_i, \quad i \in \mathrm{H},$$

and expressions for the second-order electron diffusion velocity $\mathbf{V}_{\mathrm{e}}^2$, heat flux $\mathbf{q}_{\mathrm{e}}^2$, first-order energy transfer terms ΔE_{e}^1, ΔE_h^1, and average electron forces $\mathbf{F}_{i\mathrm{e}}$, $i \in \mathrm{H}$, given in Sec. 5, we readily obtain Eqs. (6.17) and (6.18), with the entropy fluxes given in Eqs. (6.19) and (6.20) and the entropy production rates given in Eqs. (6.21)–(6.24). □

Remark 6.1. In Eq. (6.22), the entropy produced by the first-order energy transfer term ΔE_{e}^1 is partially compensated by the transport fluxes associated with the Kolesnikov effect (second-order electron transport fluxes) in the magnetized cases ($b = 1$ and $b = 0$) and is exactly compensated in the unmagnetized case ($b = 0$). In Eq. (6.24), the entropy produced by the first-order energy transfer term ΔE_h^1 is exactly compensated by the transport fluxes associated with the Kolesnikov effect (average electron forces acting on the heavy particles).

Adding Eqs. (6.17) and (6.18), a global entropy equation is found

$$\partial_t(\rho s) + \boldsymbol{\partial}_{\mathbf{x}}\cdot(\rho s \mathbf{v}_h) + \boldsymbol{\partial}_{\mathbf{x}}\cdot\boldsymbol{\mathcal{J}} = \Upsilon, \tag{6.25}$$

where the global entropy flux is given by

$$\boldsymbol{\mathcal{J}} = \boldsymbol{\mathcal{J}}_{\mathrm{e}}^{0} + \varepsilon \boldsymbol{\mathcal{J}}_{\mathrm{e}}^{1} + \varepsilon \boldsymbol{\mathcal{J}}_{h}^{1}, \tag{6.26}$$

and the global entropy production rate by

$$\Upsilon = \Upsilon_{\mathrm{e}}^{0} + \varepsilon \Upsilon_{\mathrm{e}}^{1} + \Upsilon_{h}^{0} + \varepsilon \Upsilon_{h}^{1}. \tag{6.27}$$

Proposition 6.3. *Defining* $\mathbf{x}_h = (\boldsymbol{\partial}_{\mathbf{x}} \ln T_h, p_h(\tilde{\mathbf{d}}_i)_{i\in\mathrm{H}})^{\mathrm{T}}$, $\mathbf{x}_{\mathrm{e}}^{\|} = ((\boldsymbol{\partial}_{\mathbf{x}} \ln T_{\mathrm{e}})^{\|}, p_{\mathrm{e}}\mathbf{d}_{\mathrm{e}}^{\|})^{\mathrm{T}}$, *and* $\mathbf{x}_{\mathrm{e}}^{\perp} = ((\boldsymbol{\partial}_{\mathbf{x}} \ln T_{\mathrm{e}})^{\perp}, p_{\mathrm{e}}\mathbf{d}_{\mathrm{e}}^{\perp})^{\mathrm{T}}$, *where*

$$\tilde{\mathbf{d}}_i = \hat{\mathbf{d}}_i - \frac{n_{\mathrm{e}}}{n_h}(\alpha_{\mathrm{e}i}^{\odot}\mathbf{d}_{\mathrm{e}}^{\odot} + \chi_i^{\mathrm{e}\odot}(\boldsymbol{\partial}_{\mathbf{x}} \ln T_{\mathrm{e}})^{\odot}), \quad i \in \mathrm{H},$$

the global entropy production rate Υ *defined in Eq.* (6.27) *can be rewritten for strongly magnetized plasmas* ($b = 1$) *in the following form*

$$\Upsilon = \frac{(T_{\mathrm{e}} - T_h)^2}{T_{\mathrm{e}}T_h} \sum_{j\in\mathrm{H}} \frac{n_j}{m_j}\nu_{j\mathrm{e}} + \varepsilon\eta_h \mathbf{S}:\mathbf{S} + \varepsilon\frac{1}{M_hT_h}\langle \mathbf{A}_h\mathbf{x}_h, \mathbf{x}_h\rangle$$
$$+ \frac{1}{M_hT_{\mathrm{e}}}\langle \mathbf{A}_{\mathrm{e}}^{\|}\mathbf{x}_{\mathrm{e}}^{\|}, \mathbf{x}_{\mathrm{e}}^{\|}\rangle + \frac{1}{M_hT_{\mathrm{e}}}\langle \mathbf{A}_{\mathrm{e}h}^{\perp}\mathbf{x}_{\mathrm{e}}^{\perp}, \mathbf{x}_{\mathrm{e}}^{\perp}\rangle, \tag{6.28}$$

where the matrix $\mathbf{A}_{\mathrm{e}h}^{\perp}$

$$\mathbf{A}_{\mathrm{e}h}^{\perp} = \begin{pmatrix} T_{\mathrm{e}}\lambda_{\mathrm{e}}'^{\perp} & \theta_{\mathrm{e}}^{\perp} \\ \theta_{\mathrm{e}}^{\perp} & \dfrac{1}{p_{\mathrm{e}}}D_{\mathrm{e}}^{\perp} \end{pmatrix} - \varepsilon p_{\mathrm{e}}\frac{n_{\mathrm{e}}}{n_h}\sum_{i,j\in\mathrm{H}} D_{ij}\begin{pmatrix} \chi_i^{\mathrm{e}\odot}\chi_j^{\mathrm{e}\odot} & \dfrac{1}{p_{\mathrm{e}}}\alpha_{\mathrm{e}i}^{\odot}\chi_j^{\mathrm{e}\odot} \\ \dfrac{1}{p_{\mathrm{e}}}\alpha_{\mathrm{e}j}^{\odot}\chi_i^{\mathrm{e}\odot} & \dfrac{1}{p_{\mathrm{e}}^2}\alpha_{\mathrm{e}i}^{\odot}\alpha_{\mathrm{e}j}^{\odot} \end{pmatrix},$$

is a perturbation of the mass-energy transport matrix $\mathbf{A}_{\mathrm{e}}^{\perp}$ *defined in Eq.* (5.24). *In particular, the global entropy production rate is non-negative provided that* ε *is small enough and that the collision frequencies* $\nu_{i\mathrm{e}}$, $i \in \mathrm{H}$, *are non-negative.*

Proof. Expression (6.28) is obtained after some lengthy calculation based on the expressions for the diffusion velocities $\mathbf{V}_{\mathrm{e}}$, $\mathbf{V}_{\mathrm{e}}^{2}$, $\mathbf{V}_i$, $i \in \mathrm{H}$, heat fluxes $\mathbf{q}_{\mathrm{e}}$, $\mathbf{q}_{\mathrm{e}}^{2}$, $\mathbf{q}_h$, viscous stress tensor $\mathbf{\Pi}_h$, energy transfer terms $\Delta E_{\mathrm{e}}^{0}$, $\Delta E_{\mathrm{e}}^{1}$, ΔE_{h}^{0}, ΔE_{h}^{1}, and average forces $\mathbf{F}_{i\mathrm{e}}$, $i \in \mathrm{H}$, given in Sec. 5.

The positivity of the collision frequencies $\nu_{i\mathrm{e}}$, $i \in \mathrm{H}$, (respectively the viscosity η_h) immediately yields the positivity of the first term of Eq. (6.28) $(T_{\mathrm{e}} - T_h)^2/(T_{\mathrm{e}}T_h)\sum_{j\in\mathrm{H}} n_j\nu_{j\mathrm{e}}/m_j$ (respectively the second term $\eta_h\mathbf{S}:\mathbf{S}$). Moreover, Propositions 5.1 and 5.2 ensure that both terms $\langle \mathbf{A}_h\mathbf{x}_h, \mathbf{x}_h\rangle$ and $\langle \mathbf{A}_{\mathrm{e}}^{\|}\mathbf{x}_{\mathrm{e}}^{\|}, \mathbf{x}_{\mathrm{e}}^{\|}\rangle$ are non-negative. Finally, the last term is expanded as

$$\frac{1}{M_hT_{\mathrm{e}}}\langle \mathbf{A}_{\mathrm{e}h}^{\perp}\mathbf{x}_{\mathrm{e}}^{\perp}, \mathbf{x}_{\mathrm{e}}^{\perp}\rangle = \frac{1}{3}[\![\mathbf{y}, \mathbf{y}]\!]_{\mathrm{e}} - \varepsilon\frac{1}{3}[\![\mathbf{z}, \mathbf{z}]\!]_{h}, \tag{6.29}$$

where the vectors $\mathbf{y}$ and $\mathbf{z}_i$, $i \in \mathrm{H}$, are defined as

$$\mathbf{y} = p_{\mathrm{e}}\mathbf{d}_{\mathrm{e}}^{\perp} \otimes \boldsymbol{\varphi}_{\mathrm{e}}^{D_{\mathrm{e}}(2)} - \frac{1}{T_{\mathrm{e}}}(\boldsymbol{\partial}_{\mathbf{x}} \ln T_{\mathrm{e}})^{\perp} \otimes \boldsymbol{\varphi}_{\mathrm{e}}^{\lambda'_{\mathrm{e}}(2)},$$

$$\mathbf{z}_i = \frac{1}{3} p_{\mathrm{e}} T_h M_h \sum_{j \in \mathrm{H}} ((\mathbf{y}, \boldsymbol{\varphi}_{\mathrm{e}}^{D_j(2)}))_{\mathrm{e}} \otimes \boldsymbol{\phi}_i^{D_j}.$$

We conclude after noticing that the standard term $[\![\mathbf{y}, \mathbf{y}]\!]_{\mathrm{e}}$ is non-negative and vanishes if and only if $\mathbf{y} = 0$ thanks to the scalar constraints (5.8) and (5.9). □

For weakly magnetized plasmas ($b = 0$) and unmagnetized plasmas ($b < 0$), we define $\mathbf{x}_h = (\boldsymbol{\partial}_{\mathbf{x}} \ln T_h, p_h(\hat{\mathbf{d}}_i)_{i \in \mathrm{H}})^{\mathrm{T}}$ and $\mathbf{x}_{\mathrm{e}} = [\boldsymbol{\partial}_{\mathbf{x}} \ln T_{\mathrm{e}}, p_{\mathrm{e}}(\mathbf{d}_{\mathrm{e}} + \varepsilon \delta_{b0} \mathbf{d}_{\mathrm{e}}^2)]^{\mathrm{T}}$. Hence, the global entropy production rate reads

$$\Upsilon = \frac{(T_{\mathrm{e}} - T_h)^2}{T_{\mathrm{e}} T_h} \sum_{j \in \mathrm{H}} \frac{n_j}{m_j} \nu_{j\mathrm{e}} + \varepsilon \eta_h \mathsf{S} : \mathsf{S} + \varepsilon \frac{1}{M_h T_h} \langle \mathbf{A}_h \mathbf{x}_h, \mathbf{x}_h \rangle$$
$$+ \frac{1}{M_h T_{\mathrm{e}}} \langle \mathbf{A}_{\mathrm{e}} \mathbf{x}_{\mathrm{e}}, \mathbf{x}_{\mathrm{e}} \rangle - \varepsilon^2 \delta_{b0} \frac{n_{\mathrm{e}}}{M_h} D_{\mathrm{e}} \mathbf{d}_{\mathrm{e}}^2 \cdot \mathbf{d}_{\mathrm{e}}^2. \tag{6.30}$$

This quantity is non-negative provided that ε is small enough in the $b = 0$ case and that the collision frequencies $\nu_{i\mathrm{e}}$, $i \in \mathrm{H}$, are non-negative in the two $b = 0$ and $b < 0$ cases.

The non-negativity of the global entropy production rate implies that the second law of thermodynamics is satisfied. This statement could be equivalently formulated by means of a H-Theorem. In addition, the electron and heavy-particle temperatures must be equal when an equilibrium state is reached. Provided that the collision frequencies $\nu_{i\mathrm{e}}$, $i \in \mathrm{H}$, are positive, the quasi-equilibrium states described by the Maxwell–Boltzmann distribution functions given in Eqs. (4.7) and (4.8) create some non-negative entropy expressed by the term $(T_{\mathrm{e}} - T_h)^2/(T_{\mathrm{e}} T_h) \sum_{j \in \mathrm{H}} n_j \nu_{j\mathrm{e}}/m_j$. This term vanishes when the electron and heavy-particle temperatures are identical.

6.5. *Onsager's reciprocal relations*

In this section, we deduce from kinetic theory the Onsager reciprocal relations for strongly magnetized plasmas. The expressions for the transport fluxes, denoted by the vector F, are proportional to the diffusion forces, denoted by the vector X, i.e.

$$\mathsf{F}_\alpha = -\sum_{\beta} \mathsf{L}_{\alpha\beta}\, \mathsf{X}_\beta.$$

Onsager's reciprocal relations are symmetry constraints which must hold between the transport coefficients[22, 55]

$$\mathsf{L}_{\alpha\beta}(-\mathbf{B}) = [\mathsf{L}_{\beta\alpha}(\mathbf{B})]^{\mathrm{T}}. \tag{6.31}$$

They result from microscopic reversibility, the magnetic field appearing with a minus sign to achieve motion reversal for charged particles. We identify the diffusion forces from the quadratic form of the entropy production rate given in Eq. (6.28) and use the transport coefficient expressions established in Sec. 5. Alternatively, the derivation could be based on the symetrization of the system of macroscopic equations expressed in (extensive) conservative variables, the resulting system of equations being expressed in (intensive) entropic variables obtained by Legendre transform of the conservative variables.

At order ε^0, the first-order electron mass-energy flux vector is proportional to the electron diffusion force vector as shown in Eq. (5.23). The generalized Onsager reciprocal relations for the first-order electron transport coefficients are given by

$$\bar{\bar{\lambda}}'_{\mathrm{e}}(-\mathbf{B}) = [\,\bar{\bar{\lambda}}'_{\mathrm{e}}(\mathbf{B})]^{\mathrm{T}}, \quad \bar{\bar{\theta}}_{\mathrm{e}}(-\mathbf{B}) = [\bar{\bar{\theta}}_{\mathrm{e}}(\mathbf{B})]^{\mathrm{T}}, \quad \bar{\bar{D}}_{\mathrm{e}}(-\mathbf{B}) = [\bar{\bar{D}}_{\mathrm{e}}(\mathbf{B})]^{\mathrm{T}}.$$

At order ε^1, the momentum flux is decoupled from the mass and energy fluxes.[22] The heavy-particle viscous tensor obeys standard Onsager reciprocal relations. The global mass-energy flux vector

$$\mathsf{F} = \left(\mathbf{q}_{\mathrm{e}}^2 - \rho_{\mathrm{e}} h_{\mathrm{e}} \mathbf{V}_{\mathrm{e}}^2,\ \mathbf{q}_h - \sum_{j\in\mathrm{H}} \rho_j h_j \mathbf{V}_j,\ \mathbf{V}_{\mathrm{e}}^2,\ (\mathbf{V}_i)_{i\in\mathrm{H}}\right)^{\mathrm{T}},$$

is proportional to the global diffusion force vector

$$\mathsf{X} = (\boldsymbol{\partial}_{\mathbf{x}} \ln T_{\mathrm{e}},\ \boldsymbol{\partial}_{\mathbf{x}} \ln T_h,\ p_h \mathbf{d}'_{\mathrm{e}},\ p_h(\mathbf{d}'_i)_{i\in\mathrm{H}})^{\mathrm{T}},$$

as expressed by the relation $\mathsf{F} = -\bar{\bar{\mathsf{A}}}\,\mathsf{X}$, where the mass-energy transport matrix has the following block structure

$$\bar{\bar{\mathsf{A}}} = \begin{pmatrix} (\bar{\bar{\lambda}}_{\mathrm{e}}^{2\prime} T_{\mathrm{e}}) & (\bar{\bar{\lambda}}_{\mathrm{e}}^{h\prime} T_{\mathrm{e}}) & (\bar{\bar{\theta}}_{\mathrm{e}}^{\mathrm{e}}) & [(\bar{\bar{\theta}}_i^{\mathrm{e}})_{i\in\mathrm{H}}]^{\tilde{T}} \\ (\bar{\bar{\lambda}}_{\mathrm{e}}^{h\prime} T_{\mathrm{e}}) & (\lambda'_h T_h \mathbb{I}) & (\bar{\bar{\theta}}_{\mathrm{e}}^{h}) & [(\theta_i^h \mathbb{I})_{i\in\mathrm{H}}]^{\tilde{T}} \\ (\bar{\bar{\theta}}_{\mathrm{e}}^{\mathrm{e}}) & (\bar{\bar{\theta}}_{\mathrm{e}}^{h}) & \left(\dfrac{1}{p_h}\bar{\bar{D}}_{\mathrm{ee}}\right) & \left[\left(\dfrac{1}{p_h}\bar{\bar{D}}_{\mathrm{e}i}\right)_{i\in\mathrm{H}}\right]^{\tilde{T}} \\ (\bar{\bar{\theta}}_i^{\mathrm{e}})_{i\in\mathrm{H}} & (\theta_i^h \mathbb{I})_{i\in\mathrm{H}} & \left(\dfrac{1}{p_h}\bar{\bar{D}}_{i\mathrm{e}}\right)_{i\in\mathrm{H}} & \left(\dfrac{1}{p_h} D_{ij}\mathbb{I}\right)_{i,j\in\mathrm{H}} \end{pmatrix}. \tag{6.32}$$

The notation $\tilde{T}$ has been introduced for the transpose operation restricted to the species components, excluding the space components. Concerning the mass-energy transport, the generalized Onsager reciprocal relations for the second-order electron transport coefficients and first-order heavy-particle transport coefficients are given by

$$\bar{\bar{\lambda}}_{\mathrm{e}}^{2\prime}(-\mathbf{B}) = [\,\bar{\bar{\lambda}}_{\mathrm{e}}^{2\prime}(\mathbf{B})]^{\mathrm{T}}, \quad \bar{\bar{\lambda}}_{\mathrm{e}}^{h\prime}(-\mathbf{B}) = [\,\bar{\bar{\lambda}}_{\mathrm{e}}^{h\prime}(\mathbf{B})]^{\mathrm{T}},$$
$$\bar{\bar{\theta}}_{\mathrm{e}}^{\mathrm{e}}(-\mathbf{B}) = [\,\bar{\bar{\theta}}_{\mathrm{e}}^{\mathrm{e}}(\mathbf{B})]^{\mathrm{T}}, \quad \bar{\bar{\theta}}_{\mathrm{e}}^{h}(-\mathbf{B}) = [\,\bar{\bar{\theta}}_{\mathrm{e}}^{h}(\mathbf{B})]^{\mathrm{T}}, \quad \bar{\bar{\theta}}_i^{\mathrm{e}}(-\mathbf{B}) = [\,\bar{\bar{\theta}}_i^{\mathrm{e}}(\mathbf{B})]^{\mathrm{T}}, \quad i\in\mathrm{H},$$
$$\bar{\bar{D}}_{\mathrm{ee}}(-\mathbf{B}) = [\bar{\bar{D}}_{\mathrm{ee}}(\mathbf{B})]^{\mathrm{T}}, \quad \bar{\bar{D}}_{i\mathrm{e}}(-\mathbf{B}) = [\bar{\bar{D}}_{i\mathrm{e}}(\mathbf{B})]^{\mathrm{T}}, \quad i\in\mathrm{H}.$$

6.6. *Mathematical structure*

A purely convective system extracted from the mass, momentum, electron and heavy-particle energy, and entropy equations (6.1), (6.2), (6.5), (6.7) and (6.8) is written in a quasi-linear form

$$\partial_t W + \mathbf{A} \cdot \boldsymbol{\partial}_{\mathbf{x}} W = \Omega'_W, \tag{6.33}$$

by means of the variables

$$W = [\rho_{\mathrm{e}}, (\rho_i)_{i\in\mathrm{H}}, \mathbf{v}_h, p_{\mathrm{e}}, p_h]^{\mathrm{T}},$$

the source terms

$$\Omega'_W = \left[0,\ 0,\ \frac{nq}{M_h^2 \rho_h}\mathbf{E} + \frac{1}{\rho_h}(\delta_{b0} + \delta_{b1})\mathbf{I}' \wedge \mathbf{B},\ \frac{2}{3}\Delta E_{\mathrm{e}}^0,\ \frac{2}{3}\Delta E_h^0\right]^{\mathrm{T}},$$

with the current $\mathbf{I}' = nq\mathbf{v}_h$ and the Jacobian matrices

$$A_\nu = \begin{pmatrix} v_{h\nu} & 0 & \rho_{\mathrm{e}}\mathbf{e}_\nu^T & 0 & 0 \\ 0 & v_{h\nu}(\delta_{ij})_{i,j\in\mathrm{H}} & (\rho_i)_{i\in\mathrm{H}}\mathbf{e}_\nu^T & 0 & 0 \\ 0 & 0 & v_{h\nu}\mathbb{I} & \dfrac{1}{M_h^2\rho_h}\mathbf{e}_\nu & \dfrac{1}{M_h^2\rho_h}\mathbf{e}_\nu \\ 0 & 0 & \gamma p_{\mathrm{e}}\mathbf{e}_\nu^T & v_{h\nu} & 0 \\ 0 & 0 & \gamma p_h \mathbf{e}_\nu^T & 0 & v_{h\nu} \end{pmatrix}, \quad \nu \in \{1,2,3\}, \tag{6.34}$$

where the specific heat ratio reads $\gamma = 5/3$ and symbol $\mathbf{e}_\nu$ stands for the unit vector in the ν direction. For any direction defined by the unit vector $\mathbf{n}$, the matrix $\mathbf{n} \cdot \mathbf{A}$ is shown to be diagonalizable with real eigenvalues and a complete set of eigenvectors. There are two nonlinear acoustic fields with the eigenvalues $\mathbf{v}_h \cdot \mathbf{n} \pm c$, where the sound speed is given by $c^2 = \gamma p/(\rho_h M_h^2)$, and linearly degenerate fields with the eigenvalue $\mathbf{v}_h \cdot \mathbf{n}$ of multiplicity $n^{\mathrm{S}} + 3$. Thus, the macroscopic system of conservation equations derived from kinetic theory in the proposed mixed hyperbolic-parabolic scaling has a hyperbolic structure from a fluid point of view, as far as the convective part of the system is concerned. Such a property is far from being obvious since the obtained sound speed involves the electron pressure and the rigorous derivation of the momentum equation of the heavy particles involves the many analytic steps shown in the paper.

7. Conclusions

In the present study, we have derived from kinetic theory a unified fluid model for multicomponent plasmas by accounting for electromagnetic field influence, neglecting particle internal energy and reactive collisions. Given the strong disparity of mass between the electrons and heavy particles, such as molecules, atoms, and ions, we have conducted a dimensional analysis of the Boltzmann equation following Petit and Darrozes[47] and introduced a scaling based on the ε parameter, or square root of the ratio of the electron mass to a characteristic heavy-particle mass. The multiscale analysis occurs at three levels: in the kinetic equations, the collisional invariants, and the collision operators. The Boltzmann equation has been expressed in the heavy-particle reference frame allowing for the first- and second-order electron perturbation function equations to be solved, as opposed to the inertial reference frame chosen by Degond and Lucquin.[20,21] Then, the solvability of the electron and heavy-particle perturbation functions has been based on the identification of the kernel of the linearized collision operators and the space of scalar collisional invariants for both types of species. The system has been examined at successive orders of approximation by means of a generalized Chapman–Enskog method. The micro- and macroscopic equations derived at each order are reviewed in Table 2. Depending on the type of species, the quasi-equilibrium solutions are Maxwell–Boltzmann velocity distribution functions at either the electron temperature or the heavy-particle temperature, thereby, allowing for thermal non-equilibrium to occur. At order ε^1, the set of macroscopic conservation equations of mass, momentum, and energy comprises multicomponent Navier–Stokes equations for the heavy particles, which results from a hyperbolic scaling, and first-order drift-diffusion equations for the electrons, which results from a parabolic scaling. The expressions for the transport fluxes have also been derived: first- and second-order diffusion velocity and heat flux for the electrons, and first-order diffusion velocities, heat flux, and viscous tensor for the heavy particles. The transport coefficients have been written in terms of bracket operators; both electron and heavy-particle transport coefficients exhibit anisotropy, provided that the magnetic field is strong enough. We have also proposed a complete description of the Kolesnikov effect, i.e. the crossed contributions to the mass and energy transport fluxes coupling the electrons and heavy particles. This effect, appearing in multicomponent plasmas, is essential to obtain a positive entropy production. It also contains, as a degenerate case, the single heavy-species plasmas considered by Degond and Lucquin for which the Kolesnikov effect is not present. The properties of electron and heavy-particle mass-energy transport matrices have been established by using the mathematical structure of the bracket operators. In particular, the properties of symmetry and positivity imply that the second law of thermodynamics is satisfied, as shown by deriving an entropy equation. Moreover, Onsager's reciprocal relations hold between the transport coefficients. The first law of thermodynamics was also verified by deriving a total energy equation. Finally, the purely convective system of equations was found to be hyperbolic, thus leading to a well defined structure.

The proposed formalism remains valid for collision operators of Fokker–Planck–Landau type. These operators can be used to model the charged particle interaction, instead of Boltzmann operators associated with a Coulomb potential screened at the Debye distance. In addition, the explicit expression for the diffusion coefficients, thermal diffusion coefficients, viscosity, and partial thermal conductivities can be obtained by means of a variational procedure to solve the integral equations

(Galerkin spectral method[14]). The expressions for the thermal conductivity, thermal diffusion ratios, and Stefan–Maxwell equations for the diffusion velocities can be derived by means of a Goldstein expansion of the perturbation function, as proposed by Kolesnikov and Tirskiy.[36] Finally, the mathematical structure of the transport matrices obtained by the variational procedure can readily be used to build efficient transport algorithms, as already shown by Ern and Giovangigli[25] for neutral gases, or Magin and Degrez[41] for unmagnetized plasmas.

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