Gelfand-Fuchs cohomology in algebraic geometry and factorization algebras

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Abstract

Let X be a smooth affine variety over a field $\mathbf{k}$ of characteristic 0 and $T(X)$ be the Lie algebra of regular vector fields on X. We compute the Lie algebra cohomology of $T(X)$ with coefficients in $\mathbf{k}$. The answer is given in topological terms relative to any embedding $\mathbf{k} \subset \mathbb{C}$ and is analogous to the classical Gelfand-Fuks computation for smooth vector fields on a C^∞-manifold. Unlike the C^∞-case, our setup is purely algebraic: no topology on $T(X)$ is present. The proof is based on the techniques of factorization algebras, both in algebro-geometric and topological contexts.

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Introduction

A. Description of the result. Let $\mathbf{k}$ be a field of characteristic 0 and X be a smooth affine algebraic variety over $\mathbf{k}$. Denote by $T(X) = \operatorname{Der} \mathbf{k}[X]$ the Lie algebra of regular vector fields on X. In this paper we determine $H^\bullet_{\mathrm{Lie}}(T(X))$, the Lie algebra cohomology of $T(X)$ with coefficients in $\mathbf{k}$.

Clearly, extending the field of definition of X from $\mathbf{k}$ to $\mathbf{k}' \supset \mathbf{k}$ results in extending the scalars in $H^\bullet_{\mathrm{Lie}}(T(X))$ from $\mathbf{k}$ to $\mathbf{k}'$. Since any X can be defined over a field $\mathbf{k}$ finitely generated over $\mathbb{Q}$ and any such field embeds into the complex field $\mathbb{C}$, the problem of finding $H^\bullet_{\text{Lie}}(T(X))$ reduces to the case $\mathbf{k} = \mathbb{C}$, when we can speak about X_{an}, the space of complex points of X with the transcendental topology. In this case our main result, Theorem 6.3.2, implies that $H^\bullet_{\text{Lie}}(T(X))$ is finite-dimensional in each degree and is an invariant of $\dim(X)$, of the rational homotopy type of X_{an} and of its rational Chern classes. More precisely, it is identified with $H^\bullet_{\text{top}}(\text{Sect}(\underline{Y}_X/X_{\text{an}}))$, the topological cohomology of the space of continuous sections of a natural fibration $\underline{Y}_X \to X_{\text{an}}$ over X_{an}. This allows one to easily compute $H^\bullet_{\text{Lie}}(T(X))$ in many examples, using elementary rational homotopy type theory, cf. §6.4.

The interest and importance of this problem stems from its relation to the algebro-geometric study of higher-dimensional analogs of vertex algebras, in particular, of Kac-Moody [FHK] [GW] and Virasoro algebras. While the full study eventually involves non-affine varieties (see n° D. below), the affine case already presents considerable difficulties which we address in this paper. Thus, we learned that Theorem 6.3.2 was conjectured by B. L. Feigin back in the 1980's but there has been no proof even in the case of curves, despite some work for holomorphic vector fields and continuous cohomology [Ka] [Wag1], [Wag2].

B. Relation to Gelfand-Fuchs theory. Theorem 6.3.2 is an algebro-geometric analog of the famous results by Gelfand-Fuchs [GF] [Fu], Haefliger [Hae2] and Bott-Segal [BS] on the cohomology of $\text{Vect}(M)$, the Lie algebra of smooth vector fields on a C^∞-manifold M. We recall the main features of that theory.

(1) First, one considers $W_n = \text{Der}\,\mathbb{R}[[z_1, \dots, z_n]]$, the Lie algebra of formal vector fields on $\mathbb{R}^n$, with its adic topology. Its cohomology is identified with the topological cohomology of a certain CW-complex Y_n with action of $GL_n(\mathbb{R})$, see [Fu] §2.2.

(2) Given an n-dimensional C^∞-manifold M, the tangent bundle of M gives an associated fibration $\underline{Y}_M \to M$, and $H^\bullet_{\text{Lie}}(\text{Vect}(M))$ is identified with the topological cohomology of $\text{Sect}(\underline{Y}_M/M)$, the space of continuous sections [Hae2] [BS], [Fu] §2.4.

We notice that Y_n can be realized as a complex algebraic variety acted upon by $GL_n(\mathbb{C}) \supset GL_n(\mathbb{R})$ and so any complex manifold X carries the associated fibraton $\underline{Y}_X$ with fiber Y_n (even though the real dimension of

X is $2n$). It is this fibration that is used in Theorem 6.3.2. While in the C^∞-theory $\mathrm{Vect}(M)$ is considered with its natural Fréchet topology and $H^\bullet_{\mathrm{Lie}}$ is understood accordingly (continuous cochains), in our approach $T(X)$ is considered purely algebraically.

C. Method of proof: factorization homology. Our proof of Theorem 6.3.2 is based on the theory of factorization algebras and factorization homology, both in topological [Lu-HA] [CoG1] [Gi] and algebro-geometric [BD] [G] [FG] [GL] contexts. In particular, we work systematically with the algebro-geometric version of the Ran space (§3.1).

This theory provides, first of all, a simple treatment of the C^∞-case. That is, the correspondence

$$U \mapsto \mathcal{A}(U) = \mathrm{CE}^\bullet(\mathrm{Vect}(U))$$

(Chevalley-Eilenberg complex of continuous cochains) is a locally constant factorization algebra $\mathcal{A}$ on M. As $\mathcal{A}$ is natural in M, it is, by Lurie's theorem [Lu-HA] [Gi] §6.3, determined by an $\mathbf{e}_n$-algebra A_n with a homotopy action of $GL_n(\mathbb{R})$, so that $H^\bullet_{\mathrm{Lie}}(\mathrm{Vect}(M)$ is identified with $\int_M(A_n)$, the factorization homology of M with coefficients in A_n. The Gelfand-Fuchs computation of $H^\bullet_{\mathrm{Lie}}(W_n)$ identifies A_n with $C^\bullet(Y_n)$, the cochain algebra of Y_n, and the identification with the cohomology of the space of sections follows from *non-abelian Poincaré duality*, see [Lu-HA] §5.5.6, [GL] [Gi] and Proposition 1.2.8 and Theorem 1.3.17 below.

Passing to the algebraic case, we find that $H^\bullet_{\mathrm{Lie}}(T(X))$ can also be interpreted as the factorization homology on the algebro-geometric Ran space (cf. [BD] §4.8 for $n = 1$ and [FG] Cor. 6.4.4 in general) but the corresponding factorization algebra $\check{\mathcal{C}}^\bullet$ is far from being locally constant. Already for $n = 1$ it corresponds to the vertex algebra Vir_0 (the vacuum module over the Virasoro algebra with central charge 0) which gives a holomorphic but not at all locally constant factorization algebra.

The crucial ingredient in our approach is the *covariant Verdier duality* of Gaitsgory and Lurie [GL] which is a correspondence ψ between (ordinary, or $*$-) sheaves and !-sheaves on the Ran space. For a sheaf $\mathcal{F}$ its covariant Verdier dual $\psi(\mathcal{F})$ is the collection $(i_p^! \mathcal{F})_{p\geqslant 1}$ where i_p is the embedding of the pth diagonal skeleton of the Ran space. In our case $\psi(\check{\mathcal{C}}^\bullet)$ is the algebro-geometric analog of the *diagonal filtration* of Gelfand-Fuchs [GF]. It turns out that $\psi(\check{\mathcal{C}}^\bullet)$ is a locally constant factorization algebra even though $\check{\mathcal{C}}^\bullet$ itself is not. This appearance of locally constant objects from holomorphic ones is perhaps the most surprising phenomenon that we came across in this work.

By using non-abelian Poincaré duality we show that the factorization homology of $\psi(\check{\mathcal{C}}^\bullet)$ is identified with $H^\bullet_{\rm top}(\mathrm{Sect}(\underline{Y}_X/X))$, and our main result follows from comparing the factorization homology of $\psi(\check{\mathcal{C}}^\bullet)$ and $\check{\mathcal{C}}^\bullet$ for an affine X ("completeness of the diagonal filtration").

We find it remarkable that the classical Gelfand-Fuchs theory has anticipated, in many ways, the modern theory of factorization algebras. Thus, the Ran space appears and is used explicitly (under the name "configuration space") in the 1977 paper of Haefliger [Hae2] while the diagram of diagonal embeddings of Cartesian powers M^I is fundamental in the analysis of [GF].

D. Non-affine varieties and future directions. When X is an arbitrary (not necessarily affine) smooth variety, we can understand $T(X)$ as a dg-Lie algebra $R\Gamma(X, T_X)$ and its Lie algebra cohomology is also of great interest. If X is projective, $H^\bullet_{\rm Lie}(T(X))$ plays a fundamental role on Derived Deformation theory (DDT), see [F] [HS2] [Lu-DAGX] [CaG]. The corresponding Chevalley-Eilenberg complex is identified

$$\mathrm{CE}^\bullet(T(X)) \simeq \big(\widehat{\mathcal{O}}^\bullet_{\mathcal{M},[X]}, d\big)$$

with the commutative dg-algebra of functions on the formal germ of the derived moduli space $\mathcal{M}$ of complex structures on X so $H^0_{\rm Lie}(T(X))$ is the space of formal functions on the usual moduli space. In particular, $H^\bullet_{\rm Lie}(T(X))$ is no longer a topological invariant of X and $c_i(T_X)$.

In the case of arbitrary X we still have an interpretation of $H^\bullet_{\rm Lie}(T(X))$ as the factorization homology of $\check{\mathcal{C}}^\bullet$ and $\psi(\check{\mathcal{C}}^\bullet)$ is still locally constant. So our analysis (Theorem 6.3.1) gives a canonical map

$$\tau_X : H^\bullet_{\rm top}(\mathrm{Sect}(\underline{Y}_X)/X) \longrightarrow H^\bullet_{\rm Lie}(T(X)).$$

While τ_X may no longer be an isomorphism, it provides an interesting supply of "topological" classes in $H^\bullet_{\rm Lie}(T(X))$. For example, when X is projective of dimension n, "integration over the fundamental class of X" produces, out of τ_X, a map

$$H^{2n+1}(Y_n) \longrightarrow H^1_{\rm Lie}(T(X)),$$

i.e., a supply of *characters* (1-dimensional representations) of $T(X)$. Cohomology of $T(X)$ with coefficients in such representations should describe, by extending the standard DDT, formal sections of natural determinantal bundles on $\mathcal{M}$, cf. [F]. We recall that

$$H^{2n+1}(Y_n) \simeq \mathbb{C}[x_1, \dots, x_n]^{S_n}_{\deg = n+1}, \quad \deg(x_i) = 1$$

is identified with the space of symmetric polynomials in n variables of degree $n+1$, which have the meaning of polynomials in the Chern classes.

In a similar vein, for $X = \mathbb{A}^n - \{0\}$ (the "n-dimensional punctured disk"), the space $H^{2n+1}(Y_n)$ maps to $H^2_{\mathrm{Lie}}(T(X))$, i. e., we get a supply of *central extensions* of $T(X)$, generalizing the classical Virasoro extension for $n = 1$.

E. The structure of the paper. In Chapter 1 we reformulate, using the point of view of factorization algebras, the classical Gelfand-Fuchs theory. We firsr recall, in §1.1, the theory of factorization algebras and factorization homology on a C^∞-manifold M, in the form given in [CoG1], i.e., as dealing with pre-cosheaves on M itself, rather than on the Ran space or on an appropriate category of disks. In our case the factorization algebras carry additional structures of commutative dg-algebras (cdga's), so the theory simplifies and reduces to cosheaves of cdga's with no further structure. This simplification is due to [Gi] (Prop. 48), and in §1.2 we review its applications. Unfortunately, we are not aware of a similar simplification in the algebro-geometric setting.

In §1.3 we review the concept of factorization homology of G-structured manifolds in the setting of G-equivariant cdga's, where a self-contained treatment is possible. In particular, we review non-abelian Poincaré duality which will be our main tool in relating global objects to the cohomology of the spaces of sections. It allows us to give a concise proof of the Haefliger-Bott-Segal theorem in §1.4. The identifications in §1.4 are formulated in such a way that they can be re-used later, in §6.2, with the full $GL_n(\mathbb{C})$-equivariance taken into account.

Chapter 2 is dedicated to the formalism of $\mathcal{D}$-modules which we need in a form more flexible than it is usually done. More precisely, our factorization algebras, in their $\mathcal{D}$-module incarnation, are not holonomic, but we need functorialities that are traditionally available only for holonomic modules. So in §2.3 we introduce two "non-standard" functorialities on the category of pro-objects. Thus, for a map $f : Z \to W$ of varieties we introduce the functor $f^{[[*]]}$ (formal inverse image), which for f a closed embedding and an induced $\mathcal{D}$-module $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{D}$ on W corresponds to restriction of sections of $\mathcal{F}$ to the formal neighborhood of Z. We also introduce the functor $f_{[[!]]}$ (formal direct image with proper support) which for an induced $\mathcal{D}$-module on Z corresponds to the functor $f_!$ on pro-coherent sheaves introduced by Deligne [De]. With such definitions we have, for instance, algebraic Serre duality on non-proper algebraic varieties.

In Chapter 3 we review the algebro-geometric Ran space (§3.1) and define, in §3.2, two main types of $\mathcal{D}$-modules on it, corresponding to the concepts $*$-sheaves and !-sheaves. Since we understand the $*$-inverse image in the formal series sense (for not necessarily holonomic modules), this understanding propagates into the definition of $\mathcal{D}$-module analogs of $*$-sheaves, which we call $[[\mathcal{D}]]$-modules. We also make a distinction between lax and strict modules of both types. In practice, lax modules are more easy to construct and are of more finitistic nature. They can be strictified which usually produces much larger objects but with the same factorization homology. In §3.3 we adapt to our situation the concept of covariant Verdier duality from [GL].

Chapter 4 is devoted to the theory of factorization algebras in our algebro-geometric (and higher-dimensional) context. Here the main technical issue is to show that covariant Verdier duality preserves factorizable objects. This is not obvious in the standard setting when the Ran space is represented by the diagram of the X^I for all nonempty finite sets I and their surjections. In fact, this necessitates an alternative approach to factorization algebras themselves: defining them as collections of data not on the X^I (as it is usually done and as recalled in §4.2) but on varieties labelled by all surjective maps $I \to J$ of finite sets. This is done in Sections 4.3 and 4.4 (we need, moreover, *two forms* of such a definition, each one good for a particular class of properties). This allows us to prove, in §4.6, that covariant Verdier duality indeed preserves factorization algebras (Theorem 4.6.1).

After these preparations, in Chapter 5 we study the factorization algebras $\mathcal{C}_\bullet$ and $\check{\mathcal{C}}^\bullet$ that lead, for any smooth variety X, to the (homological and cohomological) Chevalley-Eilenberg complexes of $R\Gamma(X, L)$ where L is any local Lie algebra on X. They are introduced in §5.1 and can be considered as natural "sheafifications" of these complexes. In §5.2 we specialize to the case of affine X and prove Theorem 5.2.1 and Corollary 5.2.4 which imply that $\phi(\check{\mathcal{C}}^\bullet)$ and $\check{\mathcal{C}}^\bullet$ have the same factorization homology.

Finally, in Chapter 6 we compare the algebro-geometric theory with the topological one. We start by outlining a general procedure of comparison in §6.1. We make a particular emphasis the holonomic regular case, when we can pass between de Rham cohomology on the Zariski topology (which will eventually be related to the purely algebraic object $H^\bullet_{\mathrm{Lie}}(T(X))$) and the topological cohomology on the complex topology (which will be eventually related to the cohomology of $\mathrm{Sect}(\underline{Y}_X/X_{\mathrm{an}})$). In §6.2 we specialize to the case of the tangent bundle where, as we show, the factorization algebra $\psi(\check{\mathcal{C}}^\bullet)$ is indeed holonomic regular. This allows us to identify the corresponding locally constant factorization algebra and in §6.3 we prove our main result, Theorem 6.3.2. The final §6.4 contains some explicit computations of $H^\bullet_{\mathrm{Lie}}(T(X))$ following from Theorem 6.3.2.

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0 Notations and conventions

A. Basic notations. $\mathbf{k}$: a field of characteristic 0, specialized to $\mathbb{R}$ or $\mathbb{C}$ as needed.

dgVect: the category of cochain complexes (dg-vector spaces) $V = \{V^i, d_i : V^i \to V^{i+1}\}$ over $\mathbf{k}$, with its standard symmetric monoidal structure $\otimes_{\mathbf{k}}$ (tensor product over $\mathbf{k}$).

We use the abbreviation *cdga* for "commutative dg-algebra".

CDGA: the category of cdga's over $\mathbf{k}$, i.e., of commutative algebra objects in dgVect. It is also symmetric monoidal with respect to $\otimes_{\mathbf{k}}$.

Top: the category of topological spaces homotopy equivalent to a CW-complex.

$\Delta^\circ\mathcal{C}$ resp. $\Delta\mathcal{C}$: the category of simplicial, resp. cosimplicial objects in a category $\mathcal{C}$. In particular, we use the category $\Delta^\circ \mathcal{S}et$ of simplicial sets.

$\mathrm{Sing}_\bullet(T)$: the singular simplicial set of a topological space T.

B. Categorial language. The categories dgVect, CDGA, Top, $\Delta^\circ \mathcal{S}et$ are symmetric monoidal *model categories*, see [Ho] [Lu-HTT] for background on model structures. We denote by W the classes of weak equivalences in these categories (thus W consists of quasi-isomorphisms for dgVect and CDGA).

We will mostly use the weaker structure: that of a *homotopical category* [DHKS] which is a category $\mathcal{C}$ with just one class W of morphisms, called weak equivalences and satisfying suitable axioms. A homotopical category $(\mathcal{C}, W)$ gives rise to a simplicially enriched category $L_W(\mathcal{C})$ (Dwyer-Kan localization). Taking π_0 of the simplicial Hom-sets in $L_W(\mathcal{C})$ gives the usual localization $\mathcal{C}[W^{-1}]$. We refer to $L_W(\mathcal{C})$ as the *homotopy category* of $(\mathcal{C}, W)$ (often, this term is reserved for $\mathcal{C}[W^{-1}]$). In particular, $(\mathcal{C}, W)$ has standard notions of homotopy limits and colimits which we denote $\underleftarrow{\mathrm{holim}}$ and $\underrightarrow{\mathrm{holim}}$. By an *equivalence of homotopical categories* $(\mathcal{C}, W) \to (\mathcal{C}', W')$ we mean a functor $F : \mathcal{C} \to \mathcal{C}'$ such that:

(1) $F(W) \subset W'$.

(2) The induced functor of simplicially enriched categories $L_W(\mathcal{C}) \to L_{W'}(\mathcal{C}')$ is a *quasi-equivalence*, that is:

- (2a) It gives an equivalence of the usual categories $\mathcal{C}[W^{-1}] \to \mathcal{C}'[(W')^{-1}]$.
- (2b) It induces weak equivalences on the simplicial Hom-sets.

We will freely use the language of ∞-categories [Lu-HTT]. In particular, any simplicially enriched category gives rise, in a standard way, to an ∞-category with the same objects, and we will simply consider it as an ∞-category. This applies to the Dwyer-Kan localizations $L_W(\mathcal{C})$ above. For instances, various derived categories will be "considered as ∞-categories" when needed.

C. Thom-Sullivan cochains. We recall the *Thom-Sullivan functor*

$$\mathrm{Th}^\bullet : \Delta\mathrm{dgVect} \longrightarrow \mathrm{dgVect}.$$

Explicitly, for a cosimplicial dg-vector space $V^\bullet$, the dg-vector space $\mathrm{Th}^\bullet(V^\bullet)$ is defined as the end, in the sense of [Mac], of the simplicial-cosimplicial dg-vector space $\Omega^\bullet_{\mathrm{pol}}(\Delta^\bullet) \otimes V^\bullet$ where $\Omega^\bullet_{\mathrm{pol}}(\Delta^\bullet)$ consists of polynomial differential forms on the standard simplices, see [HS1] [FHT].

Note that $\mathrm{Th}^\bullet(V^\bullet)$ is quasi-isomorphic to the naive total complex

$$\mathrm{Tot}(V^\bullet) = \Big(\bigoplus V^n[-n],\, d_V + \sum (-1)^i \delta_i \Big), \tag{0.1}$$

where δ_i are the coface maps of $V^\bullet$. The quasi-isomorphism is given by the Whitney forms on the Δ^n, see [Get] §3.

The functor $\mathrm{Th}^\bullet$ is compatible with symmetric monoidal structures and so sends cosimplicial cdga's to cdga's. It can, therefore, be used to represent the homotopy limit of any diagram of cdga's as an explicit cdga. In particular, we have a cdga structure on the cohomology of any sheaf of cdga's, a structure of a sheaf of cdga's in any direct images of a sheaf of cdga's and so on. We note some particular cases.

Let $S_\bullet$ be a simplicial set. We write $\mathrm{Th}^\bullet(S_\bullet) = \mathrm{Th}^\bullet(\mathbf{k}^{S_\bullet})$, where $\mathbf{k}^{S_\bullet}$ is the cosimplicial cdga $(\mathbf{k}^{S_p})_{p\geqslant 0}$ (simplicial cochains). The cdga $\mathrm{Th}^\bullet(S_\bullet)$ is called the *Thom-Sullivan cochain algebra* of $S_\bullet$. It consists of compatible systems of polynomial differential forms on all the geometric simplices of $S_\bullet$.

Let T be a topological space. We write $\mathrm{Th}^\bullet(T) = \mathrm{Th}^\bullet(\mathrm{Sing}_\bullet(T))$. This is a cdga model for the cochain algebra of T with coefficients in $\mathbf{k}$.

1 C^∞ Gelfand-Fuchs cohomology and factorization algebras

1.1 Factorization algebras on C^∞ manifolds

A. Factorization algebras. We follow the approach of [CoG1] [Gi].

Definition 1.1.1. Let M be a C^∞-manifold and $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category. A *pre-factorization algebra* on M with values in $\mathcal{C}$ is a rule $\mathcal{A}$ associating:

(1) To each open subset $U \subset M$ an object $\mathcal{A}(U) \in \mathcal{C}$, with $\mathcal{A}(\emptyset) = \mathbf{1}$.

(2) To each finite family of open sets $U_0, U_1, \dots, U_r$, $r \geqslant 0$, such that $U_1, \dots, U_r$ are disjoint and contained in U_0, a permutation invariant morphism in $\mathcal{C}$

$$\mu^{U_0}_{U_1,\dots,U_r} : \mathcal{A}(U_1) \otimes \cdots \otimes \mathcal{A}(U_r) \longrightarrow \mathcal{A}(U_0),$$

these morphisms satisfying the obvious associativity conditions.

Taking $r = 1$ in (2), we see that a pre-factorization algebra defines an $\mathcal{C}$-valued pre-cosheaf on M, i.e., a covariant functor from the poset of opens in M to $\mathcal{C}$.

Let now $(\mathcal{C}, \otimes, \mathbf{1}, W)$ be a symmetric monoidal homotopical category. We assume that $\mathcal{C}$ has small coproducts which we denote by $\coprod$. Let $U \subset M$ be an open subset and $\mathfrak{U} = (U_i)_{i\in I}$ be an open cover of U. We write $U_{ij} = U_i \cap U_j$ etc. If $\mathcal{F}$ is an $\mathcal{C}$-valued pre-cosheaf on M, we have then the standard simplicial object in $\mathcal{C}$ (co-descent diagram)

$$\mathcal{N}_\bullet(\mathfrak{U}, \mathcal{F}) \;=\; \left\{ \cdots \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; \coprod_{i,j,k\in I} \mathcal{F}(U_{ijk}) \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; \coprod_{i,j\in I} \mathcal{F}(U_{ij}) \;\substack{\longrightarrow\\[-1ex]\longrightarrow}\; \coprod_{i\in I} \mathcal{F}(U_i) \right\}$$

and a morphism

$$\gamma_{\mathcal{U}} : \underrightarrow{\operatorname{holim}}\, \mathcal{N}_\bullet(\mathfrak{U}, \mathcal{F}) \;\longrightarrow\; \mathcal{F}(U). \tag{1.1.2}$$

Definition 1.1.3. Let $U \subset M$ be an open subset and $\mathfrak{U} = (U_i)_{i\in I}$ be an open cover of U. We say that $\mathfrak{U}$ is a *Weiss cover*, if for any finite subset $S \subset U$ there is i such that $S \subset U_i$.

As pointed out in [CoG1], Weiss covers are typically very large (consist of infinitely many opens).

Definition 1.1.4. Let $(\mathcal{C}, \otimes, \mathbf{1}, W)$ be a symmetric monoidal homotopical category with coproducts. A pre-factorization algebra $\mathcal{A}$ on M with values in $\mathcal{C}$ is called a *factorization algebra*, if the following conditions hold:

(3) For any disjoint open $U_1, \dots, U_r \subset M$, the morphism $\mu^{U_1\cup\cdots\cup U_r}_{U_1,\dots,U_r}$ in (2) is a weak equivalence.

(4) For any open $U \subset M$ and any Weiss cover $\mathfrak{U} = (U_i)_{i\in I}$ of U, the morphism $\gamma_{\mathfrak{U}}$ is a weak equivalence.

For a factorization algebra $\mathcal{A}$ the object of global cosections will be also denoted by

$$\int_M \mathcal{A} = \mathcal{A}(M)$$

and called the *factorization homology* of M with coefficients in $\mathcal{A}$.

In particular, we will use factorization algebras with values in the categories dgVect, CDGA, Top, $\Delta^\circ\mathcal{S}et$, see §0.

Definition 1.1.5. (a) By a *disk* in M we mean an open subset homeomorphic to $\mathbb{R}^n$ for $n = \dim(M)$.

(b) A (pre-)factorization algebra $\mathcal{A}$ with values in $\mathcal{C}$ is called *locally constant*, if for any embeddings $U_1 \subset U_0$ of disks in M the morphism $\mu_{U_1}^{U_0}$ is a weak equivalence.

1.2 Factorization algebras of cdga's

A. Sheaves and cosheaves. We start with a more familiar sheaf-theoretic analog of the fornalism of §1.1.

Let T be a topological space. Denote by $\mathrm{Op}(T)$ the poset of open sets in T considered as a category.

Definition 1.2.1. (a) Let $(\mathcal{C}, W)$ be a homotopical category. A $\mathcal{C}$-valued pre-cosheaf $\mathcal{A} : \mathrm{Op}(T) \to \mathcal{C}$ on T is called a *homotopy cosheaf* , if, for each $U \in \mathrm{Op}(T)$ and each cover $\mathfrak{U}$ of U, the canonical morphism $\gamma_{\mathfrak{U}}$ defined as in (1.1.2), is a weak equivalence.

(b) By a $\mathcal{C}$-valued *homotopy sheaf* on T we mean a homotopy cosheaf with values in $\mathcal{C}^\circ$.

In the sequel we will drop the word "homotopy" when discussing homotopy sheaves and cosheaves. We denote by $\mathrm{Sh}_T(\mathcal{C})$ and $\mathrm{Cosh}_T(\mathcal{C})$ the categories of $\mathcal{C}$-valued sheaves and cosheaves.

Definition 1.2.2. Let M be a C^∞ manifold. A $\mathcal{C}$-valued cosheaf $\mathcal{A}$ on $\mathcal{M}$ is called *locally constant* if for any two disks $U_1 \subset U_0 \subset M$ the co-restriction map $\mathcal{A}(U_1) \to \mathcal{A}(U_0)$ is a weak equivalence. $\mathcal{C}$-valued locally constant sheaves on M are defined similarly.

We denote by $\mathrm{Sh}_M^{\mathrm{lc}}(\mathcal{C})$ and $\mathrm{Cosh}_M^{\mathrm{lc}}(\mathcal{C})$ the homotopical categories of locally constant $\mathcal{C}$-valued sheaves and cosheaves.

Proposition 1.2.3. *The homotopical categories* $\mathrm{Sh}^{\mathrm{lc}}_M(\mathcal{C})$ *and* $\mathrm{Cosh}^{\mathrm{lc}}_M(\mathcal{C})$ *are equivalent.*

Proof: Let $D(M) \subset \mathrm{Op}(M)$ be the poset of disks in M. As disks form a basis of topology in M, any sheaf $\mathcal{B}$ on M is determined by its values on $D(M)$. More precisely, $\mathcal{B}$ is the (homotopy) right Kan extension [DHKS] of $\mathcal{B}|_{D(M)}$. This implies that $\mathrm{Sh}^{\mathrm{lc}}_M(\mathcal{C})$ is identified with the category formed by contravariant functors, $D(M) \to \mathcal{C}$ sending each morphism to a weak equivalence, i.e., with the category of simplicially enriched functors $L(D(M)^\circ) \longrightarrow L_W(\mathcal{C})$. Here and below L without subscript stands for the Dwyer-Kan localization with respect to all morphisms.

Dually, any cosheaf $\mathcal{A}$ on M is the homotopy left Kan extension of $\mathcal{A}|_{D(M)}$. This means that $\mathrm{Cosh}^{\mathrm{lc}}_M(\mathcal{C})$ is identified with the category formed by covariant functors $D(M) \to \mathcal{C}$ sending each morphism to a weak equivalence, i.e., with the category of simplicially enriched functors $L(D(M)) \longrightarrow L_W(\mathcal{C})$. Now notice that $L(D(M))$ and $L(D(M)^\circ)$ are canonically identified. □

Definition 1.2.4. For a locally constant sheaf $\mathcal{B}$ we will denote by $\mathcal{B}^{-1}$ and call the *inverse* of $\mathcal{B}$ the locally constant cosheaf corresponding to $\mathcal{B}$.

B. Cosheaves of cdga's as factorization algebras. We use the abbreviation *cdga* for "commutative dg-algebra". The category CDGA of cdga's over $\mathbf{k}$ is a symmetric monoidal homotopical category, with monoidal operation $\otimes_{\mathbf{k}}$ (tensor product of cdga's) and weak equivalences being quasi-isomorphisms of cdga's.

Proposition 1.2.5. *(a) Let* $\mathcal{A}$ *be a factorization algebra on* M *with values in* CDGA*. Then* $\mathcal{A}$ *is a cosheaf on* M *with values in* CDGA*.*

(b) The construction in (a) (i.e., forgetting all the $\mu^{U_0}_{U_1,\dots,U_r}$ *for* $r > 1$*) establishes an equivalence between the category of* CDGA*-valued factorization algebras on* M *and the category of* $\mathrm{Cosh}_M(\mathrm{CDGA})$*. Under this equivalences, locally constant factorization algebras correspond to locally constant cosheaves.*

Proof: This is proved in [Gi], Prop. 48. It is based on the fact that $\otimes_{\mathbf{k}}$, the monoidal operation in CDGA, is at the same time the categorical coproduct. □

Remark 1.2.6. Note that a cosheaf of cdga's is typically not a cosheaf of dg-vector spaces, as the coproduct in dgVect is $\oplus$, not $\otimes_{\mathbf{k}}$. On the other hand, a sheaf of cdga's is indeed a sheaf a sheaf of dg-vector spaces, as $\prod$, the product in dgVect, is also the product in CDGA.

C. Non-abelian Poincaré duality I. Let $p : Z \to M$ be a Serre fibration with base a C^∞ manifold M and fiber Y. We then have the following pre-sheaf and pre-cosheaf of cdga's on M:

$$(1.2.7) \qquad \begin{aligned} Rp_*(\underline{\mathbf{k}}_Z) &: U \mapsto \mathrm{Th}^\bullet(p^{-1}(U), \\ Rp_*^{\otimes}(\underline{\mathbf{k}}_Z) &: U \mapsto \mathrm{Th}^\bullet(\mathrm{Sect}(p^{-1}(U)/U)), \end{aligned}$$

where $\mathrm{Sect}(p^{-1}(U)/U)$ is the space of continuous sections of the fibration $p^{-1}(U) \to U$. Thus $Rp_*(\underline{\mathbf{k}}_Z)$ is in fact a locally constant sheaf of cdga's, namely the direct image of the constant sheaf $\underline{\mathbf{k}}_Z$, made into a sheaf of cdga by using Thom-Sullivan cochains.

Proposition 1.2.8. *Suppose Y is n-connected, where $n = \dim(M)$. Then $Rp_*^{\otimes}(\underline{\mathbf{k}}_Z)$ is a locally constant cosheaf of cdga's. Further, $Rp_*^{\otimes}(\underline{\mathbf{k}}_Z)$ is inverse to $Rp_*(\underline{\mathbf{k}}_Z)$ (see Proposition 1.2.3).*

Proof: The first statement is a consequence of [BS] Cor. 5.4. The second statement is clear since for a disk $U \subset X$ the space $\mathrm{Sect}(p^{-1}(U)/U)$ is homotopy equivalent to $p^{-1}(x)$ for any $x \in U$. □

1.3 Equivariant cdga's and factorization homology

We first compare several notions of a Lie group acting on a cdga.

A. Classifying space approach. Let $(\mathcal{C}, W)$ be a homotopical category.

Definition 1.3.1. Let $T_\bullet$ be a simplicial topological space so that for each morphism $s : [p] \to [q]$ in Δ we have a morphism of topological spaces $s^* : T_q \to T_p$. A $\mathcal{C}$-valued sheaf on $T_\bullet$ is a datum $\mathcal{F}$ consisting of sheaves $\mathcal{F}_p$ on T_p for each p and of weak equivalences of sheaves $\alpha_s : (s^*)^{-1}(\mathcal{F}_q) \to \mathcal{F}_p$ given for each $s : [p] \to [q]$ and compatible with the compositions. We denote by $\mathrm{Sh}_{T_\bullet}(\mathcal{C})$ the category of $\mathcal{C}$-valued sheaves on $T_\bullet$.

Let G be a topological group and $N_\bullet G \in \Delta^\circ\mathrm{Top}$ its simplicial nerve. Thus $BG = |N_\bullet G|$ is the classifying space of G. We denote, following [BL] Appendix B,

$$D_G(\mathrm{pt}) = \mathrm{Sh}_{N_\bullet G}(\mathrm{dgVect}), \quad \mathrm{CDGA}_G = \mathrm{Sh}_{N_\bullet G}(\mathrm{CDGA}). \tag{1.3.2}$$

Proposition 1.3.3. *For any object $\mathcal{F}$ of $D_G(\mathrm{pt})$ (resp. of CDGA_G) we have the following:*

(a) Each $\mathcal{F}_p$ is a locally constant (in fact, constant) sheaf of dg-vector spaces resp. cdga's on N_pG.

(b) $\mathcal{F}$ gives a locally constant sheaf $|\mathcal{F}|$ of dg-vector spaces (resp. cdga's) on BG. □

Proof: (a) Consider any morphism $s : [0] \to [q]$, so $s^* : N_qG \to N_0G = \mathrm{pt}$. Then α_s identifies $\mathcal{F}_q$ with the constant sheaf $(s^*)^{-1}(\mathcal{F}_0)$. Part (b) follows from (a). □

Corollary 1.3.4. *A morphism $G' \to G$ of topological groups which is a homotopy equivalence, induces equivalences of homotopical categories $D_G(\mathrm{pt}) \to D_{G'}(pt)$ and $\mathrm{CDGA}_G \to \mathrm{CDGA}_{G'}$.*

Proof: Follows from homotopy invariance of locally constant sheaves. □

Given $\mathcal{F} \in D_G(\mathrm{pt})$, the component $V^\bullet = \mathcal{F}_0 \in \mathrm{Sh}_{N_0G}$ is just a dg-vector space,

Definition 1.3.5. Let $V^\bullet$ be a dg-vector space (resp. a cdga). A *BL-action* of G on $V^\bullet$ is an object $\mathcal{F}$ in $D_G(\mathrm{pt})$ (resp. in CDGA_G) together with identification of dg-vector spaces (resp. of cdga's) $\mathcal{F}_0 \simeq V^\bullet$.

For a cdga A we denote by dgMod_A, resp. CDGA_A the category of dg-modules over A resp. commutative differential graded A-algebras.

Let $H^\bullet(BG)$ be the cohomology ring of BG with coefficients in $\mathbf{k}$, i.e., the cohomology algebra of the cdga $\mathrm{Th}^\bullet(BG)$. Any sheaf of dg-vector spaces $\mathcal{F}$ on BG gives rise to the dg-space of cochains $C^\bullet(BG, \mathcal{F})$. We note that $C^\bullet(BG, \mathcal{F})$ is a kind of homotopy limit (dg-vector space associated to a cosimplicial dg-vector space) and so we can define it using the Thom-Sullivan construction. Thus defined, $C^\bullet(BG, \mathcal{F})$ is a dg-module over $\mathrm{Th}^\bullet(BG) = C^\bullet(BG, \underline{\mathbf{k}}_{BG})$. Further, if $\mathcal{F}$ is a sheaf of cdga's, then $C^\bullet(BG, \mathcal{F})$ is a cdga over $\mathrm{Th}^\bullet(BG)$.

Proposition 1.3.6. *Let G be a connected compact Lie group. Then:*

(a) We have a quasi-isomorphism $H^\bullet(BG) \to \mathrm{Th}^\bullet(BG)$.

(b) The functor $\mathcal{F} \mapsto C^\bullet(BG, |\mathcal{F}|)$ defines symmetric monoidal equivalences of homotopical categories

$$D_G^+(\mathrm{pt}) \xrightarrow{\sim} \mathrm{dgMod}^+_{H^\bullet(BG)}, \quad \mathrm{CDGA}_G^{\geqslant 0} \xrightarrow{\sim} \mathrm{CDGA}^{\geqslant 0}_{H^\bullet(BG)},$$

where the subscripts "+" signify the subcategories formed by sheaves and dg-modules bounded below as complexes, and "$\geqslant 0$" signifies the subcategories formed by dg-algebras graded by $\mathbb{Z}_{\geqslant 0}$.

Proof: Part (a) is classical (invariant forms on BG). The first equivalence in (b) is [BL] Th. 12.7.2. The second equivalence follows from the first by passing to commutative algebra objects.

□

B. Cartan-Weil approach.

Definition 1.3.7. (cf. [GS] Def. 2.3.1 and [BS] Def. 3.1.) Let $\mathbb{G}$ be an affine algebraic group over $\mathbf{k}$ with Lie algebra $\mathfrak{g}$ and $V^\bullet \in \mathrm{dgVect}$ be a cochain complex. A $\mathbb{G}^*$-*action* on $V^\bullet$ is a datum consisting of:

(1) A regular action of $\mathbb{G}$ on $V^\bullet$. In particular, for each $\xi \in \mathfrak{g}$ we have the infinitesimal automorphism $L_\xi \in \mathrm{End}^0_{\mathbf{k}}(V^\bullet)$.

(2) An $\mathbf{k}$-linear map $i : \mathfrak{g} \to \mathrm{End}^{-1}_{\mathbf{k}}(V^\bullet)$ such that;

  (2a) i is $\mathbb{G}$-equivariant with respect to the adjoint action of $\mathbb{G}$ on $\mathfrak{g}$ and the $\mathbb{G}$-action on $\mathrm{End}^{-1}_{\mathbf{k}}(V^\bullet)$ coming from (1).

  (2b) We have $[d, i(\xi)] = L_\xi$ for each $\xi \in \mathfrak{g}$.

  (2c) For any $\xi_1, \xi_2 \in \mathfrak{g}$ we have $[i(\xi_1), i(\xi_2)] = 0$.

We denote by $\mathbb{G}^*$-dgVect the category of cochain complexes with $\mathbb{G}^*$-actions. This category has a symmetric monoidal structure $\otimes_{\mathbf{k}}$ with the operators $i(\xi)$ defined on the tensor products by the Leibniz rule. Commutative algebra objects in $\mathbb{G}^*$-dgVect will be called $\mathbb{G}^*$-*cdga's.* Such an algebra is a cdga with $\mathbb{G}$ acting regularly by automorphisms, so $\mathfrak{g}$ acts by derivations of degree 0, and with $i(\xi)$ being derivations of degree (-1). We denote by $\mathbb{G}^*$-CDGA the category of $\mathbb{G}^*$-cdga's.

Let $\Omega^\bullet_{\mathrm{reg}}(\mathbb{G})$ be the cdga of regular differential forms on $\mathbb{G}$. The group structure on $\mathbb{G}$ makes $\Omega^\bullet_{\mathrm{reg}}(\mathbb{G})$ into a commutative Hopf dg-algebra over $\mathbf{k}$. We note the following.

Proposition 1.3.8. *(a) A $\mathbb{G}^*$-action on a cochain complex $V^\bullet$ is the same as a structure of a comodule over the dg-coalgebra $\Omega^\bullet_{\mathrm{reg}}(\mathbb{G})$. This identification is compatible with tensor products.*

(b) A structure of a $\mathbb{G}^*$-cdga on a cdga A is the same as a coaction $A \to A \otimes \Omega^\bullet_{\mathrm{reg}}(\mathbb{G})$ from (a) which is a morphism of cdga's.

Proof: We prove (a), since (b) follows by passing to commutative algebra objects.

A regular action of $\mathbb{G}$ is by definition, a coaction of the coalgebra $\mathcal{O}(\mathbb{G}) = \Omega^0_{\mathrm{reg}}(\mathbb{G})$ of regular functions. More explicitly, the coaction map $c_0 : V^\bullet \to \Omega^0_{\mathrm{reg}}(\mathbb{G}) \otimes V^\bullet$ is just the action map $\rho : \mathbb{G} \to \mathrm{End}_{\mathbf{k}}(V^\bullet)$ considered as an element of $\Omega^0_{\mathrm{reg}}(\mathbb{G}) \otimes \mathrm{End}_{\mathbf{k}}(V^\bullet)$. The fact that c_0 is a coaction, i.e., that ρ is multiplicative, is equivalent to the fact that ρ is equivariant with respect to $\mathbb{G}$ acting on itself by right translations and on $\mathrm{End}_{\mathbf{k}}(V^\bullet)$ by the action induced by ρ.

Let us view $\mathfrak{g}$ as consisting of left invariant vector fields on $\mathbb{G}$. The adjoint action of $\mathbb{G}$ on $\mathfrak{g}$ is the action on such vector fields by right translations. Let $\omega = g^{-1}dg \in \Omega^1_{\mathrm{reg}}(\mathbb{G}) \otimes \mathfrak{g}$ be the canonical $\mathfrak{g}$-valued left invariant 1-form on $\mathbb{G}$. Composing ω with i we get an element $c_1 \in \Omega^1_{\mathrm{reg}}(\mathbb{G}) \otimes \mathrm{End}^{-1}_{\mathbf{k}}(V^\bullet)$ which is equivariant with respect to $\mathbb{G}$ acting on $\Omega^1_{\mathrm{reg}}(\mathbb{G})$ by right translations and on $\mathrm{End}^{-1}_{\mathbf{k}}(V^\bullet)$ by the action induced by ρ. Further, for each $p \geqslant 1$ we define, using (2c):

$$c_p \;=\; \Lambda^p(c_1) \;\in\; \Omega^p_{\mathrm{reg}}(\mathbb{G}) \otimes \mathrm{End}^{-p}_{\mathbf{k}}(V^\bullet).$$

We claim that the

$$c = \sum c_p \;\in\; \bigoplus_p \Omega^p_{\mathrm{reg}}(\mathbb{G}) \otimes \mathrm{End}^{-p}_{\mathbf{k}}(V^\bullet) \;=\; \mathrm{Hom}^0_{\mathbf{k}}(V^\bullet, \Omega^\bullet_{\mathrm{reg}}(\mathbb{G}) \otimes V^\bullet)$$

is a coaction of $\Omega^\bullet_{\mathrm{reg}}(\mathbb{G})$ on $V^\bullet$. Indeed, considering c_1 as a morphism of dg-vector space $V^\bullet \to \Omega^1_{\mathrm{reg}}(\mathbb{G}) \otimes V^\bullet$, we translate its equivariance property above into saying that c_1 is compatible with c_0. Further compatibilities follow since the c_p are defined as exterior powers of c_1. □

Given a $\mathbb{G}^*$-action on $V^\bullet$, we form, in a standard way, the *cobar-construction* of the $\Omega^\bullet_{\rm reg}(\mathbb{G})$-comodule structure. This is the cosimplicial object

$$\mathrm{Cob}(V^\bullet) = \left\{ V^\bullet \underset{1\otimes -}{\overset{c}{\rightrightarrows}} \Omega^\bullet_{\rm reg}(\mathbb{G}) \otimes V^\bullet \rightrightarrows\!\!\!\!\to \Omega^\bullet_{\rm reg}(\mathbb{G}^2) \otimes V^\bullet \rightrightarrows\!\!\!\!\rightrightarrows \cdots \right\}.$$

For example, the first two coface maps are given by the coaction c and by multiplication with $1 \in \Omega^\bullet_{\rm reg}(\mathbb{G})$ respectively.

Recall that $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$. Taking $\mathbf{k}$-points of $\mathbb{G}$, we get a Lie group $G = \mathbb{G}(\mathbf{k})$. Let $\underline{\Omega}^\bullet_{G^p}$ be the sheaf of smooth forms on G^p. Thus $\Omega^\bullet_{\rm reg}(\mathbb{G}^p)$ maps into the global sections of this sheaf. This allows us to define a dgVect-valued sheaf $\mathfrak{S}(V^\bullet)$ on the simplicial space $N_\bullet G$ by "localizing $\mathrm{Cob}(V^\bullet)$". More precisely, we define $\mathfrak{S}(V^\bullet)_p = \underline{\Omega}^\bullet_{G^p} \otimes V^\bullet$ and the compatibility maps α_s, $s \in \mathrm{Mor}(\Delta)$ are induced by the corresponding maps of $\mathrm{Cob}(V^\bullet)$. This gives a symmetric monoidal functor

$$\mathfrak{S} : \mathbb{G}^*\text{-dgVect} \longrightarrow D_G(\mathrm{pt}) = \mathrm{Sh}_{N_\bullet(G)}(\mathrm{dgVect}), \quad V^\bullet \mapsto \mathfrak{S}(V^\bullet) \qquad (1.3.9)$$

which we call the *sheafification functor*.

We now recall a version of the standard comparison between the "Weil model" and the "classifying space model" for G-equivariant cohomology, cf. [GS]. An element $v \in V^\bullet$ is called *basic*, if v is $\mathbb{G}$-invariant and annihilated by the operators $i(\xi), \xi \in \mathfrak{g}$. Basic elements form a subcomplex $V^\bullet_{\rm basic}$. Denote $I(\mathfrak{g}) = S^\bullet(\mathfrak{g}^*)^{\mathfrak{g}}$. Let also

$$\mathrm{We}(\mathfrak{g}) \;=\; \big(S^\bullet(\mathfrak{g}^*) \otimes \Lambda^\bullet(\mathfrak{g}^*), d\big) \qquad (1.3.10)$$

be the *Weil algebra of* $\mathfrak{g}$. Here the first $\mathfrak{g}^*$ has degree 2, while the second $\mathfrak{g}^*$ has degree 1. With this grading, $\mathrm{We}(\mathfrak{g})$ is a cdga with $\mathbb{G}^*$-action, quasi-isomorphic to $\mathbf{k}$. Moreover, $I(\mathfrak{g})$ with trivial differential, is a dg-subalgebra in $\mathrm{We}(\mathfrak{g})$.

Proposition 1.3.11. *Let $\mathbb{G}$ be reductive and assume, in the case $\mathbf{k} = \mathbb{R}$, that $G \hookrightarrow \mathbb{G}(\mathbb{C})$ is a homotopy equivalence. Then:*

(a) We have an isomorphism $I(\mathfrak{g}) \simeq H^\bullet(BG, \mathbf{k})$.

For a dg-vector-space (resp, cdga) with a $\mathbb{G}^*$-action, we have a natural quasi-isomorphism of dg-modules (resp. cdga's) over $I(\mathfrak{g}) = H^\bullet(BG, \mathbf{k})$

$$C^\bullet(BG, |\mathfrak{S}(V^\bullet)|) \;\simeq\; \big(\mathrm{We}(\mathfrak{g}) \otimes_{\mathbf{k}} V^\bullet\big)_{\rm basic}.$$

Proof: (a) is well known. To prove (b), note, first, that we have a natural identification

$$C^\bullet(BG, |\mathfrak{S}(V^\bullet)|) \simeq \mathrm{Th}^\bullet(\mathrm{Cob}(V^\bullet)),$$

compatible with the symmetric monoidal structures. Indeed, this follows from the fact that for any m, the restriction map $r_m : \Omega^\bullet_{\mathrm{reg}}(\mathbb{G}^m) \to \Omega^\bullet(G^m)$ quasi-isomorphism. The target of r_m calculates $H^\bullet(G^m, \mathbf{k})$ by the de Rham theorem, and the source calculates $H^\bullet(\mathbb{G}(\mathbb{C})^m, \mathbf{k})$, by Grothendieck's algebro-geometric version.

Note next that for any dg-vector space $E^\bullet$ with $\mathbb{G}^*$-action

$$E^\bullet_{\mathrm{basic}} \;=\; \mathrm{Ker}\big\{ \; E^\bullet \underset{1\otimes -}{\overset{c}{\rightrightarrows}} \Omega^\bullet_{\mathrm{reg}}(\mathbb{G}) \otimes E^\bullet \; \big\}$$

is the kernel of the first two cofaces in $\mathrm{Cob}(E^\bullet)$.

We apply this to $E^\bullet = \mathrm{We}(\mathfrak{g}) \otimes V^\bullet$ which is quasi-isomorphic to $V^\bullet$. For a dg-vector space $F^\bullet$ let $F^\sharp$ denote the graded vector space obtained from $F^\bullet$ by forgetting the differential. So to prove part (b), it is enough to show the following acyclicity statement: the embedding of the constant cosimplicial graded vector space associated to $E^\sharp_{\mathrm{basic}}$, into the cosimplicial graded vector space $\mathrm{Cob}(E^\sharp)$ is a weak equivalence. This is equivalent to saying that the complex

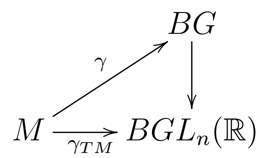
$$E^\sharp \longrightarrow \Omega^\bullet_{\mathrm{reg}}(\mathbb{G}) \otimes E^\sharp \longrightarrow \Omega^\bullet_{\mathrm{reg}}(\mathbb{G}^2) \otimes E^\sharp \longrightarrow \cdots$$

with differential $\sum(-1)^i\delta_i$, is exact everywhere except the leftmost term. But this complex calculates $\mathrm{Cotor}^\bullet_{\Omega^\sharp(\mathbb{G})}(\mathbf{k}, E^\sharp)$, i.e., the derived functors of the functor $\beta : E \mapsto E_{\mathrm{basic}}$ on the category of graded $\Omega^\sharp(\mathbb{G})$-comodules. So it is enough to show that for $E = (\mathrm{We}(\mathfrak{g}) \otimes V^\bullet)^\sharp$, the higher derived functors $R^k\beta(E)$ vanish for $k > 0$.

Now, $\beta(E)$ is obtained by first, taking invariants with respect to the abelian Lie superalgebra $i(\mathfrak{g})$ and then taking $\mathbb{G}$-invariants. Since $\mathbb{G}$ is reductive, taking $\mathbb{G}$-invariants is an exact functor. So vanishing of $R^{>0}\beta(E)$ will be assured if $\Lambda^\bullet(\mathfrak{g})$, the enveloping algebra of $i(\mathfrak{g})$, acts on E freely. This is the case for $E = (\mathrm{We}(\mathfrak{g}) \otimes V^\bullet)^\sharp$. □

C. Equivariant cdga's and factorization homology. We now fix $n \geqslant 1$ and let $G \subset GL(n, \mathbb{R})$ be a closed subgroup.

Definition 1.3.12. Let M be a C^∞-manifold of dimension n. By a *G-structure* on M we will mean a reduction of structure group of the tangent bundle T_M from $GL_n(\mathbb{R})$ to G in the homotopy sense, i.e., a homotopy class of maps γ making the diagram



homotopy commutative. Here γ_{TM} is the map classifying the tangent bundle TM.

Proposition 1.3.13. *Let A be a cdga with a BL-action of G. . Then, for any n-dimensional manifold M with G-structure, one can associate to A a locally constant cosheaf of cdga's $\underline{A}_M$ on M compatible with unramified coverings, so that $\underline{A}_{\mathbb{R}^n}$ is the constant cosheaf corresponding to A.*

Proof: We note first that by Proposition 1.2.3 it is enough to associate to A a locally constant *sheaf* of cdga's $[A]_M$ on M, so that $\underline{A}_M$ will be defined as the inverse cosheaf $([A]_M)^{-1}$. But a BL-action on A is, by definition, a sheaf $\mathcal{B}$ on $N_\bullet(G)$ which gives a locally constant sheaf $|\mathcal{B}|$ on BG, and we define $[A]_M = \gamma^{-1}|\mathcal{B}|$. □

We will refer to $\underline{A}_M$ as the *cosheaf of cdga's associated to A* on a G-structured manifold M.

Remark 1.3.14. Proposition 1.3.13 is a particular case of a result due to Lurie [Lu-HA] which relates G-equivariant E_n-algebras with locally constant factorization algebras on G-structured manifolds, see also [AF3], Prop. 3.14 and [Gi] §6.3. Our case corresponds, in virtue of Proposition 1.2.5, to factorization algebras with values in CDGA, for which the E_n-structure reduces to a commutative one. In this case, the (co)sheaf language leads to a simple direct construction.

We will refer to $\underline{A}_M$ as the *cosheaf of cdga's associated to A* on a G-structured manifold M.

Definition 1.3.15. Let A be a cdga with a BL-action of G, and M be a G-structured manifold. The *factorization homology* of M with coefficients in A is defined as the space of global cosections of the cosheaf $\underline{A}_M$ and denoted by

$$\int_M (A) \;=\; \underline{A}_M(M).$$

D. Non-abelian Poincare's duality II. Let $G \subset GL_n(\mathbb{R})$ be a closed subgroup, M be a n-dimensional manifold with G-structure. Let Y be a CW-complex with G-action. This G-action gives rise to the *associated fibration*

$$\underline{Y}_M = P_M \times_G Y \xrightarrow{p} Y \tag{1.3.16}$$

with fiber Y. Here P_M is the principal $GL_n(\mathbb{R})$-bundle of frames in the tangent bundle TM. Continuous sections of $\underline{Y}_M$ form a sheaf $\mathrm{Sect}_{\underline{Y}_M}$ on M with values in topological spaces. Taking Thom-Sullivan cochains, we get a pre-factorization algebra

$$\mathrm{Th}^\bullet \mathrm{Sect}_{\underline{Y}_M} : U \mapsto \mathrm{Th}^\bullet(\mathrm{Sect}(\underline{Y}_U/U))$$

on M with values in CDGA. Note that $\mathrm{Th}^\bullet\mathrm{Sect}_{\underline{Y}_M}$ is locally constant.

At the same time, the G-action on Y gives a fibration over $N_\bullet G$ with fiber Y and so a BL-action of G on the cdga $\mathrm{Th}^\bullet(Y)$.

Theorem 1.3.17 (Non-abelian Poincaré duality). *Suppose Y is n-connected. Then, the pre-factorization algebra $\mathrm{Th}^\bullet\mathrm{Sect}_{\underline{Y}_M}$ on M is a factorization algebra, i.e., a cosheaf of cdga's. Further, it is identified with $\underline{\mathrm{Th}^\bullet(Y)}_M$, the cosheaf associated to the G-equivariant cdga $\mathrm{Th}^\bullet(Y)$. Therefore*

$$\mathrm{Th}^\bullet(\mathrm{Sect}(Y_M/M)) \simeq \int_M (\mathrm{Th}^\bullet(Y)).$$

Proof: Direct consequence of Proposition 1.2.8. Indeed, the locally constant sheaf $[\mathrm{Th}^\bullet(Y)]_M = (\underline{\mathrm{Th}^\bullet}_M)^{-1}$ is, by construction, $Rp_*(\underline{\mathbf{k}}_{Y_M})$. □

Remark 1.3.18. As with Proposition 1.3.13, Theorem 1.3.17 is an adaptation of a result of Lurie about G-equivariant E_n-algebras to the much simpler case of E_n-algebras being commutative. It is often formulated in a "dual" version involving the non-commutative G-equivariant E_n-algebra $C_\bullet(\Omega^n(Y,y))$, the singular chain complex of the n-fold loop space of Y at a point y (assumed G-invariant). In this case y gives rise to a distinguished section $\underline{y}$ of $\underline{Y}_M$ and we have

$$\int_M \big(C_\bullet(\Omega^n(Y,y))\big) \simeq C_\bullet(\mathrm{Sect}_c(\underline{Y}_M/M)),$$

where Sect_c stands for sections with compact support (those which coincide with $\underline{y}$ outside of a compact subset of M). The relation between this formulation and Theorem 1.3.17 comes from a Koszul duality quasi-isomorphism over the (Koszul self-dual, up to a shift) operad $\mathbf{e}_n$ (singular chains on E_n):

$$\mathrm{Th}^\bullet(Y)^!_{\mathbf{e}_n} \;\simeq\; C_\bullet(\Omega^n(Y,y_0)),$$

see [AF2] [AF1].

1.4 Classical Gelfand-Fuchs theory: $\mathrm{CE}^\bullet(W_n)$ and $C^\bullet(Y_n)$ as $GL_n(\mathbb{C})$-cdga's

A. The Gelfand-Fuks skeleton. Let $\mathbf{k}$ be either $\mathbb{R}$ or $\mathbb{C}$. We denote by $W_n(\mathbf{k}) = \mathrm{Der}\,\mathbf{k}[[z_1,\dots,z_n]]$ be the $\mathbf{k}$-Lie algebra of formal vector fields on $\mathbf{k}^n$, equipped with its natural adic topology. By $\mathrm{CE}^\bullet(W_n(\mathbf{k}))$ we denote the $\mathbf{k}$-linear Chevalley-Eilenberg cochain complex of $W_n(\mathbf{k})$ formed by cochains $\Lambda^p(W_n(\mathbf{k}))\to\mathbf{k}$ which are continuous with respect to the adic topology. By $H^\bullet_{\mathrm{Lie}}(W_n(\mathbf{k}))$ we denote the cohomology of $\mathrm{CE}^\bullet(W_n(\mathbf{k}))$, i.e., the continuous Lie algebra cohomology of $W_n(\mathbf{k})$. We recall the classical calculation of $H^\bullet_{\mathrm{Lie}}(W_n(\mathbf{k}))$, see [GF] [Fu].

Consider the infinite Grassmannian

$$G(n,\mathbb{C}^\infty) \;=\; \varinjlim_{N\geqslant n} G(n,\mathbb{C}^N) \;\simeq\; BGL_n(\mathbb{C})$$

as a CW-complex (union of projective algebraic variety over $\mathbb{C}$. For any $n\geqslant N$ let

$$E(n,\mathbb{C}^N) \;=\; \big\{(e_1,\dots,e_n)\in(\mathbb{C}^n)^N\,\big|e_1,\dots,e_n \text{ are linearly independent}\big\}$$

be the Stiefel variety formed by partial (n-element) frames in $\mathbb{C}^N$. There is a natural projection (principal $GL_n(\mathbb{C})$-bundle)

$$\rho: E(n,\mathbb{C}^N)\longrightarrow G(n,\mathbb{C}^N),\quad (e_1,\dots,e_n)\;\mapsto\;\mathbb{C}e_1+\cdots+\mathbb{C}e_n,$$

which associates to each frame the n-dimensional subspace spanned by it. Then the union

$$E(n,\mathbb{C}^\infty) \;=\; \varinjlim_{N\geqslant n} E(n,\mathbb{C}^N) \;\simeq\; EGL_n(\mathbb{C})$$

is the universal bundle over the classifying space of $GL_n(\mathbb{C})$.

The *Gelfand-Fuchs skeleton* Y_n is defined as the fiber product

$$\begin{array}{ccc} Y_n & \longrightarrow & EGL_n(\mathbb{C}) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ \mathrm{sk}_{2n}BGL_n(\mathbb{C}) & \longrightarrow & BGL_n(\mathbb{C}), \end{array}$$

where $\mathrm{sk}_{2n}BGL_n(\mathbb{C}) \subset G(n, \mathbb{C}^{2n})$ is[1] the $2n$-dimensional skeleton with respect to the standard Schubert cell decomposition. Thus Y_n is a quasi-projective algebraic variety over $\mathbb{C}$ with a free $GL_n(\mathbb{C})$-action which makes it a principal $GL_n(\mathbb{C})$-bundle over $\mathrm{sk}_{2n}BGL_n(\mathbb{C})$. More explicitly, Y_n is a closed subvariety in $E(n, \mathbb{C}^{2n})$ which is, in its turn, a Zariski open subset in the affine space of matrices $\mathrm{Mat}(n, 2n)(\mathbb{C})$. The following is the classical result of Gelfand-Fuks ([Fu] Th. 2.2.4).

Theorem 1.4.1. *Recall that* $\mathbf{k}$ *is either* $\mathbb{R}$ *or* $\mathbb{C}$.

(a) We have an isomorphism $H^\bullet_{\mathrm{Lie}}(W_n, \mathbf{k}) \simeq H^\bullet(Y_n, \mathbf{k})$ *with the topological cohomology of* Y_n *with coefficients in* $\mathbf{k}$. *Further, the cup-product on both sides is equal to zero, as well as all the higher Massey operations.*

(b) The space Y_n *is* $2n$*-connected: its first homology space is* $H^{2n+1}(Y_n, \mathbf{k})$.

B. The result for smooth vector fields. Let now M be an n-dimensional C^∞-manifold. The group $GL_n(\mathbb{C})$ acts on Y_n. In particular, the action of $GL_n(\mathbb{R}) \subset GL_n(\mathbb{C})$ gives rise to the *Gelfand-Fuchs fibration*

$$p : \underline{Y}_M \longrightarrow M$$

with fiber Y_n, associated to the tangent bundle of M, as in (1.3.16).

We denote by $\mathrm{Vect}_{\mathbf{k}}(M)$ the Lie algebra of $\mathbf{k}$-valued smooth vector fields on M equipped with the standard Fréchet topology (as for the space of C^∞-sections of any smooth vector bundle). As before, we denote by $\mathrm{CE}^\bullet(\mathrm{Vect}_{\mathbf{k}}(M))$ and $H^\bullet_{\mathrm{Lie}}(\mathrm{Vect}_{\mathbf{k}}(M))$ the $\mathbf{k}$-linear Chevalley-Eilenberg complex of continuous cochains of $\mathrm{Vect}_{\mathbf{k}}(M)$ and its cohomology. The following is also classical ([Fu] Lemma 1 p. 152).

[1] Note that the $2n$-skeleton in $G(n, \mathbb{C}^{2n})$ agrees with the $2n$-skeleton in $G(n, \mathbb{C}^N)$ for any $N \geqslant 2n$.

Theorem 1.4.2. *Let $M = D \subset \mathbb{R}^n$ be the standard unit ball. Then the homomorphism* $\mathrm{Vect}_{\mathbf{k}}(M) \to W_n(\mathbf{k})$ *given by the Taylor series expansion at* 0 *induces an isomorphism* $H^\bullet_{\mathrm{Lie}}(W_n(\mathbf{k})) \to H^\bullet_{\mathrm{Lie}}(\mathrm{Vect}_{\mathbf{k}}(D))$.

The following theorem was conjectured by Gelfand-Fuchs and proved by Haefliger [Hae1] [Hae2] and Bott-Segal [BS].

Theorem 1.4.3. *For any M we have an isomorphism* $H^\bullet_{\mathrm{Lie}}(\mathrm{Vect}_{\mathbf{k}}(M)) \simeq H^\bullet(\mathrm{Sect}(\underline{Y}_M/M), \mathbf{k})$.

C. Proof using factorization homology. From modern point of view, Theorem 1.4.3 can be seen as a textbook application of the techniques of factorization homology. In the remainder of this section we give its proof using these techniques, as a precursor to the study of the algebro-geometric case. We first recall [CoG1]:

Theorem 1.4.4. *Let $\mathcal{L}$ be a C^∞ local Lie algebra on M, i.e., a smooth* $\mathbf{k}$*-vector bundle with a Lie bracket on sections given by a bi-differential operator. For an open $U \subset M$ let $\mathcal{L}(U)$ be the space of smooth sections of $\mathcal{L}$ over U, considered as a Lie algebra with its Fréchet topology, and* $\mathrm{CE}^\bullet(\mathcal{L}(U))$ *be its Chevalley-Eilenberg complex of continuous sections. Then*

$$\mathrm{CE}^\bullet(\mathcal{L}) : U \mapsto \mathrm{CE}^\bullet(\mathcal{L}(U))$$

is a factorization algebra on M. □

Since $\mathrm{CE}^\bullet(\mathcal{L})$ consists of cdga's, Proposition 1.2.5 implies:

Corollary 1.4.5. *In the situation of Theorem 1.4.4,* $\mathrm{CE}^\bullet(\mathcal{L})$ *is a cosheaf of cdga's on M.* □

We apply this to $\mathcal{L} = T_M^{\mathbf{k}}$ being the tangent bundle of M for $\mathbf{k} = \mathbb{R}$ or its complexification for $\mathbf{k} = \mathbb{C}$. Theorem 1.4.2 implies that the factorization algebra $\mathrm{CE}^\bullet(T_M^{\mathbf{k}})$ is locally constant.

Consider the algebraic group $\mathbb{GL}_n$ over $\mathbf{k}$. It acts on $\mathbf{k}[[z_1, \cdots, z_n]]$ and thus on $W_n(\mathbf{k})$ in a natural way. Moreover, the cdga $\mathrm{CE}^*(W_n(\mathbf{k}))$ has a natural structure of $\mathbb{GL}_n^*$-algebra (see Definition 1.3.7). Indeed, any $\xi \in \mathfrak{gl}_n(\mathbf{k})$ gives a linear vector field on $\mathbf{k}^n$, also denoted ξ, which we can consider as an element of $W_n(\mathbf{k})$. The derivation $i(\xi)$ is given by contraction of the cochains with ξ. Now, the sheafification functor (1.3.9) gives a BL-action of $GL_n(\mathbf{k})$

on $\mathrm{CE}^*(W_n(\mathbf{k}))$. Therefore, we have the cosheaf of cdga's $\underline{\mathrm{CE}^*(W_n(\mathbf{k}))}_M$ on M associated with the BL-action of $GL_n(\mathbb{R}) \subset GL_n(\mathbf{k})$ on the cdga $\mathrm{CE}^*(W_n(\mathbf{k}))$.

Proposition 1.4.6. *The cosheaves of cdga's* $\mathrm{CE}^*(T_M^{\mathbf{k}})$ *and* $\underline{\mathrm{CE}^*(W_n(\mathbf{k}))}_M$ *are weakly equivalent.*

Proof: By definition, the cosheaf $\underline{\mathrm{CE}^*(W_n(\mathbf{k}))}_M$ is the inverse of the sheaf $[\mathrm{CE}^*(W_n(\mathbf{k}))]_M$. So it suffices to construct, for any disk $U \subset M$, a quasi-isomorphism $q_U : [\mathrm{CE}^\bullet(W_n(\mathbf{k})]_M(U) \to \mathrm{CE}^\bullet(\mathrm{Vect}(U))$ so that for any inclusion of disks $U_1 \subset U_0$ we have a commutative diagram

(1.4.7)
$$\begin{array}{ccc} \mathrm{CE}^\bullet(\mathrm{Vect}(U_1)) & \xrightarrow{\mu_{U_1}^{U_0}} & \mathrm{CE}^\bullet(\mathrm{Vect}(U_0)) \\ {\scriptstyle q_{U_1}}\uparrow & & \uparrow{\scriptstyle q_{U_0}} \\ [\mathrm{CE}^\bullet(W_n(\mathbf{k})]_M(U_1) & \xleftarrow{\mathrm{res}} & [\mathrm{CE}^\bullet(W_n(\mathbf{k})]_M(U_0). \end{array}$$

In fact, it suffices to construct, for each U, not a single quasi-isomorphism q_U but a family of such parametrized by a contractible space T_U, so that the commutativity of the diagram will hold for some parameters in T_{U_1} and T_{U_0}. This is what we will do.

For each $x \in M$ let $W_x(\mathbf{k})$ be the Lie algebra of formal vector fields on M at x (tensored with $\mathbb{C}$, if $\mathbf{k} = \mathbb{C}$). Let further $W_{T_xM}(\mathbf{k})$ be the Lie algebra of formal vector fields on the vector space T_xM at 0 (also tensored with $\mathbb{C}$, if $\mathbf{k} = \mathbb{C}$). Thus $W_x(\mathbf{k})$ and $W_{T_xM}(\mathbf{k})$ are isomorphic to $W_n(\mathbf{k})$ but not canonically. We note the following.

Proposition 1.4.8. *(a) The stalk of the sheaf* $[\mathrm{CE}^\bullet(W_n(\mathbf{k}))]_M$ *at* $x \in M$ *is identified with* $W_{T_xM}(\mathbf{k})$.

(b) For any disk $U \subset M$ *and any* $x \in U$ *the pullback map*

$$r_x : \mathrm{CE}^\bullet(W_x(\mathbf{k})) \longrightarrow \mathrm{CE}^\bullet(\mathrm{Vect}(U))$$

is a quasi-isomorphism.

Proof: Part (a) follows by construction of $[\mathrm{CE}^\bullet(W_n(\mathbf{k})]_M$ via the tangent bundle. Part (b) follows from Proposition 1.4.2. □

Notice that the space of formal identifications ϕ between $(T_xM, 0)$ and (M, x) identical on the tangent space, is contractible. Any such identification defines an isomorphism ϕ_* of $\mathrm{CE}^\bullet(W_{T_xM}(\mathbf{k}))$ with $\mathrm{CE}^\bullet(W_x(\mathbf{k}))$. Now, take a disk $U \subset M$. Any choice of $x \in U$ and ϕ as above defines a chain of quasi-isomorphisms

$$
\begin{aligned}
(1.4.9) \qquad [\mathrm{CE}^\bullet(W_n(\mathbf{k})]_M(U) \to [\mathrm{CE}(W_n(\mathbf{k}))]_{M,x} = \mathrm{CE}^\bullet(W_{T_xM}(\mathbf{k})) \xrightarrow{\phi_*} \\
\xrightarrow{\phi_*} \mathrm{CE}^\bullet(W_x(\mathbf{k})) \xrightarrow{r_x} \mathrm{CE}^\bullet(\mathrm{Vect}(U)).
\end{aligned}
$$

By composing the above arrows we get a family of quasi-isomorphisms q_U parametrized by a contractible space T_U (the total space of the family of the identifications ϕ for all $x \in U$). Now, suppose $U_1 \subset U_0$. Then we can use any $x \in U_1$ and any identification ϕ to construct both q_{U_1} and q_{U_0}. With these choices, the diagram (1.4.7) is trivially commutative. □

Theorem 1.4.3 will now follow from Non-Abelian Poincaré Duality (Theorem 1.3.17), if we prove:

Theorem 1.4.10. *Let* $\mathbf{k} = \mathbb{C}$. *There is an identification*

$$\mathrm{CE}^\bullet(W_n(\mathbb{C})) \simeq \mathrm{Th}^\bullet(Y_n)$$

as cdga's with a BL-action of $GL_n(\mathbb{C})$.

Proof: Denote for short $G = GL_n(\mathbb{C})$. Since G is homotopy equivalent to a connected compact Lie group U_n, by Proposition 1.3.6 it suffices to identify the corresponding cdga's over

$$H^\bullet(BG) = \mathbb{C}[e_1, \dots, e_n], \quad \deg(e_i) = 2i.$$

Now, the cdga corresponding to $\mathrm{Th}^\bullet(Y_n)$ is the cochain algebra of the fibration over BG corresponding to the G-space Y_n. This fibration is homotopy equivalent to $\mathrm{sk}_{2n}(BG)$, so the corresponding algebra is $\mathrm{Th}^\bullet(\mathrm{sk}_{2n}(BG))$ which is, as well known, quasi-isomorphic to

$$H^\bullet(\mathrm{sk}_{2n}(BG), \mathbb{C}) = \mathbb{C}[e_1, \dots, e_n]/(\deg > 2n).$$

We now identify the cdga corresponding to $\mathrm{CE}^\bullet(W_n(\mathbb{C}))$. For this we first recall the standard material on relative Lie algebra cohomology [Fu] Ch.1

Let $\mathfrak{g}$ be a Lie subalgebra of a Lie algebra $\mathfrak{w}$. The *relative Chevalley-Eilenberg complex* of $\mathfrak{w}$ modulo $\mathfrak{g}$ (with trivial coefficients) is defined as

$$\mathrm{CE}^\bullet(\mathfrak{w}, \mathfrak{g}) = \left(\Lambda^\bullet((\mathfrak{w}/\mathfrak{g})^*)\right)^{\mathfrak{g}}.$$

let $I(\mathfrak{g})$, $\mathrm{We}(\mathfrak{g})$ be as in Proposition 1.3.11.

We apply this to $\mathfrak{w} = W_n(\mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, so it is the Lie algebra of the algebraic group $\mathbb{G} = \mathbb{GL}_{n,\mathbb{C}}$ with the grop of $\mathbb{C}$-points $G = GL_n(\mathbb{C})$.

Let $I(\mathfrak{g})$, $\mathrm{We}(\mathfrak{g})$ be as in Proposition 1.3.11, so $I(\mathfrak{g}) = H^\bullet(BG, \mathbb{C})$. Further, as a $\mathbb{GL}_{n,\mathbb{C}}$-representation, the topological dual $W_n(\mathbb{C})^*$ splits as the direct sum of $V_{n,k} = \mathrm{Sym}^{k+1}(\mathbb{C}^n) \otimes (\mathbb{C}^n)^*$ for $k \geqslant -1$. This gives a $\mathbb{GL}_{n,\mathbb{C}}$-equivariant projection $q : W_n(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C}) = V_{n,0}^*$. The projection q induces a $\mathbb{GL}_{n,\mathbb{C}}$-equivariant morphism of dg-algebras ("connection") $\nabla : \mathrm{We}(\mathfrak{gl}_n(\mathbb{C})) \to \mathrm{CE}^\bullet(W_n(\mathbb{C}))$, see [Fu]. In particular, $\mathrm{CE}^\bullet(W_n(\mathbb{C}), \mathfrak{gl}_n(\mathbb{C}))$ becomes an algebra over $I(\mathfrak{gl}_n(\mathbb{C})) = H^\bullet(BG, \mathbb{C})$.

Proposition 1.4.11. *We have a canonical quasi-isomorphism of $I(\mathfrak{g})$-cdga's*

$$(\mathrm{We}(\mathfrak{g}) \otimes \mathrm{CE}^\bullet(\mathfrak{w}))_{\mathrm{basic}} \simeq \mathrm{CE}^\bullet(\mathfrak{w})_{\mathrm{basic}} = \mathrm{CE}^\bullet(\mathfrak{w}, \mathfrak{g}).$$

Proof: The second equality is by definition. The first quasi-isomorphism follows from the existence of ∇ by [GS], Thm. 4.3.1. □

Theorem 1.4.10 now follows from Proposition 1.3.11 and from the quasi-isomorphism of cdga's over $H^\bullet(BG, \mathbb{C})$

$$\mathrm{CE}^\bullet(W_n(\mathbb{C}), \mathfrak{gl}_n(\mathbb{C})) \simeq \mathbb{C}[e_1, \dots, e_n]/(\deg > 2n).$$

which is the original computation of Gelfand-Fuchs, see [Fu, proof of Thm. 2.2.4].

2 $\mathcal{D}$-modules and extended functoriality

2.1 $\mathcal{D}$-modules and differential sheaves

A. Generalities on $\mathcal{D}$-modules. For general background see [Bo] [HTT].

Let Z a smooth algebraic variety over $\mathbf{k}$. Put $n = \dim(Z)$. We denote by $\mathrm{Coh}_Z = \mathrm{Coh}_{\mathcal{O}_Z}$ and $\mathrm{QCoh}_Z = \mathrm{QCoh}_{\mathcal{O}_Z}$ the categories of coherent and quasi-coherent sheaves of $\mathcal{O}_Z$-modules.

We denote by $\mathcal{D}_Z$ the sheaf of rings of differential operators from $\mathcal{O}_Z$ to $\mathcal{O}_Z$. By $\mathrm{Coh}_{\mathcal{D}_Z} \subset \mathrm{QCoh}_{\mathcal{D}_Z}$ we denote the categories of coherent and quasi-coherent sheaves of right $\mathcal{D}_Z$-modules. By ${}_{\mathcal{D}_Z}\mathrm{Coh} \subset {}_{\mathcal{D}_Z}\mathrm{QCoh}$ we denote similar categories for left $\mathcal{D}_Z$-modules. We will be mostly interested in right $\mathcal{D}$-modules.

Let $D(\mathrm{Coh}_{\mathcal{D}_Z})$, resp. $D^b(\mathrm{Coh}_{\mathcal{D}_Z})$ denote the full (unbounded), resp. bounded derived categories of coherent right $\mathcal{D}_Z$-modules. We consider them as dg-categories and then as stable ∞-categories in the standard way. Similarly for $\mathrm{QCoh}_{\mathcal{D}_Z}$ etc. Since $\mathcal{D}_Z$ has finite homological dimension (equal to $2\dim(Z)$), we have the identification

$$D^b\mathrm{Coh}_{\mathcal{D}_Z} \simeq \mathrm{Perf}_{\mathcal{D}_Z},$$

where on the right we have the category of perfect complexes.

By ω_Z we denote the sheaf of volume forms on Z, a right $\mathcal{D}_Z$-module. We have the standard equivalence (*volume twist*)

$$(2.1.1)\qquad {}_{\mathcal{D}_Z}\mathrm{QCoh} \longrightarrow \mathrm{QCoh}_{\mathcal{D}_Z}, \quad \mathcal{N} \mapsto \mathcal{N} \otimes_{\mathcal{O}_Z} \omega_Z.$$

We call the *Verdier duality* the anti-equivalence
(2.1.2)

$$D^b(\mathrm{Coh}_{\mathcal{D}_Z})^{\mathrm{op}} \to D^b(\mathrm{Coh}_{\mathcal{D}_Z}), \quad \mathcal{M} \mapsto \mathcal{M}^\vee = \underline{\mathrm{RHom}}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{D}_Z) \otimes_{\mathcal{O}_Z} \omega_Z[n].$$

For a right $\mathcal{D}_Z$-module $\mathcal{M} \in \mathrm{QCoh}_{\mathcal{D}_Z}$ we have the *de Rham complex*

$$\mathrm{DR}(\mathcal{M}) = \mathcal{M} \otimes^L_{\mathcal{D}_Z} \mathcal{O}_Z.$$

For a coherent sheaf $\mathcal{F} \in \mathrm{Coh}_Z$ we have the induced right $\mathcal{D}_Z$-module $\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{D}_Z$. We have a canonical identification

$$\mathrm{DR}(\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{D}_Z) \simeq \mathcal{F}.$$

For two coherent sheaves $\mathcal{F}, \mathcal{G}$ the sheaf

$$\mathrm{Diff}(\mathcal{F}, \mathcal{G}) = \underline{\mathrm{Hom}}_{\mathcal{D}_Z}(\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{D}_Z, \mathcal{G} \otimes_{\mathcal{O}_Z} \mathcal{D}_Z)$$

consists of differential operators from $\mathcal{F}$ to $\mathcal{G}$ in the standard sense. In particular,

$$\mathcal{D}_Z = \mathrm{Diff}(\mathcal{O}_Z, \mathcal{O}_Z), \quad \mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{D}_Z = \mathrm{Diff}(\mathcal{O}_Z, \mathcal{F})$$

with $\mathcal{D}_Z$ acting on the right by composition.

For a vector bundle E on Z we denote by $E^\vee = \omega_Z \otimes E^*$ the *Serre dual vector bundle*. Thus the Verdier dual of the induced $\mathcal{D}$-module $E \otimes_{\mathcal{O}} \mathcal{D}$ is given by

$$\big(E \otimes_{\mathcal{O}_Z} \mathcal{D}_Z\big)^\vee \;=\; (E^\vee) \otimes_{\mathcal{O}_Z} \mathcal{D}_Z[n].$$

For two vector bundles E and F the Verdier duality gives the identification

$$\mathrm{Diff}(E, F) \longrightarrow \mathrm{Diff}(F^\vee, E^\vee), \quad P \mapsto P^\vee$$

(passing to the adjoint differential operator).

B. Differential sheaves and differential complexes.

Definition 2.1.3. A right $\mathcal{D}_Z$-module $\mathcal{M}$ is called *quasi-induced*, if, Zariski locally on Z, it is isomorphic to an induced $\mathcal{D}$-module.

We see that for a quasi-induced $\mathcal{M}$ its de Rham complex can be identified with a single sheaf. It is convenient to introduce the following concept.

Definition 2.1.4. A *differential sheaf* (resp. *differential bundle*) on Z is a sheaf $\mathcal{F}$ which is glued out of coherent sheaves (resp. vector bundles) on Zariski open charts so that transition functions are invertible differential operators. We consider a representation of $\mathcal{F}$ by such gluing a part of the structure of a differential sheaf.

By definition, we can speak about:

- Differential operators $\mathcal{F} \to \mathcal{G}$ where $\mathcal{F}, \mathcal{G}$ are differential sheaves.
- The Serre dual differential bundle $E^\vee$ associated to a differential bundle E.

We denote by DS_Z the category formed by differential sheaves on Z and differential operators between them.

Proposition 2.1.5. *Taking the de Rham complex (which, in our case, reduces to the functor $- \otimes_{\mathcal{D}_Z} \mathcal{O}_Z$) induces an equivalence of categories*

$$\big\{\textit{Quasi-induced } \mathcal{D}_Z\textit{-modules}\big\} \xrightarrow{\mathrm{DR}} \mathrm{DS}_Z. \quad \square$$

We will denote the quasi-inverse to the above equivalence by

$$\mathcal{F} \mapsto \mathrm{DR}^{-1}(\mathcal{F}) = \mathrm{Diff}(\mathcal{O}_Z, \mathcal{F}) = \text{“}\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{D}_Z\text{”}.$$

By a *differential complex* on Z we mean a complex $\mathcal{F}^\bullet$ formed by differential sheaves with differentials being differential operators. We denote by $\mathrm{Com}(\mathrm{DS}_Z)$ the category of differential complexes on Z. In particular, the de Rham complex of $\mathcal{M} \in \mathrm{Coh}_{\mathcal{D}_Z}$ can be quasi-isomorphically identified with a differential complex by taking a locally induced resolution of $\mathcal{M}$.

C. Analytification and holonomic differential complexes. Let $\mathbf{k} = \mathbb{C}$. We denote Z_{an} the space $Z(\mathbb{C})$ with its standard analytic topology and sheaf $\mathcal{O}_{Z_{\mathrm{an}}}$ of analytic functions.

Any differential sheaf (resp. complex) $\mathcal{F}$ has a well defined analytification $\mathcal{F}_{\mathrm{an}}$ which is a sheaf (resp. complex of sheaves) on Z_{an}. In particular, considering the de Rham complex of $\mathcal{M} \in \mathrm{Coh}_{\mathcal{D}_Z}$ as a differential complex as above, we have its analytification which is simply

$$\mathrm{DR}(\mathcal{M})_{\mathrm{an}} \simeq \mathcal{M} \otimes^L_{\mathcal{D}_Z} \mathcal{O}_{Z_{\mathrm{an}}}.$$

The following are standard features of the Riemann-Hilbert correspondence between holonomic $\mathcal{D}$-modules and constructible complexes, see [HTT] for instance.

Proposition 2.1.6. *Let $\mathcal{M}^\bullet \in D(\mathrm{Coh}_{\mathcal{D}_Z})$ be a complex with holonomic regular cohomology modules. Then $\mathrm{DR}(\mathcal{M}^\bullet)_{\mathrm{an}}$ is a constructible complex on Z_{an} and:*

(a) We have

$$R\Gamma(Z_{\mathrm{Zar}}, \mathrm{DR}(\mathcal{M}^\bullet)) \simeq R\Gamma(Z_{\mathrm{an}}, \mathrm{DR}(\mathcal{M}^\bullet)_{\mathrm{an}}).$$

(b) We also have

$$\mathrm{DR}(M^\vee)_{\mathrm{an}} \simeq \mathbb{D}_Z(\mathrm{DR}(\mathcal{M}^\bullet)_{\mathrm{an}}),$$

where $\mathbb{D}_Z$ is the Verdier duality on the derived category of constructible complexes on Z_{an}, see [KS1].

A differential complex $\mathcal{F}^\bullet$ will be called *holonomic* (resp. *holonomic regular*), if $\mathrm{DR}^{-1}(\mathcal{F}^\bullet)$, the corresponding complex of quasi-induced $\mathcal{D}$-modules, has holonomic (resp. holonomic regular) cohomology.

Corollary 2.1.7. *(a) If $\mathcal{F}^\bullet$ is a holonomic differential complex, then $\mathcal{F}^\bullet_{\mathrm{an}}$ is a constructible complex.*

(b) If $\mathcal{F}^\bullet$ is a holonomic regular differential complex, then

$$R\Gamma(X_{\mathrm{Zar}}, \mathcal{F}^\bullet) \simeq R\Gamma(X_{\mathrm{an}}, \mathcal{F}^\bullet_{\mathrm{an}}).$$

2.2 The standard functorialities

We now review the standard functorialities on quasi-coherent $\mathcal{D}$-modules. Our eventual interest is always in *right* modules.

A. Inverse image $f^!$. Let $f : Z \to W$ be a morphism of smooth algebraic varieties. We then have the *transfer bimodule*

$$\mathcal{D}_{Z\to W} = \mathcal{O}_Z \otimes_{f^{-1}\mathcal{O}_W} f^{-1}\mathcal{D}_W.$$

It can be viewed as consisting of differential operators from $f^{-1}\mathcal{O}_W$ to $\mathcal{O}_Z$. It is a left $\mathcal{D}_Z$-module (quasi-coherent but not, in general, coherent) and a right $f^{-1}\mathcal{D}_W$-module.

The inverse image is most easily defined on left $\mathcal{D}$-modules in which case it is given by f^*, the usual (derived) inverse image for underlying $\mathcal{O}$-modules. That is, we have the functor

$$f^* : {}_{\mathcal{D}_W}\mathrm{QCoh} \longrightarrow {}_{\mathcal{D}_Z}\mathrm{QCoh}, \quad f^*\mathcal{N} = \mathcal{O}_Z \otimes^L_{f^{-1}\mathcal{O}_W} f^{-1}\mathcal{N} = \mathcal{D}_{Z\to W} \otimes^L_{f^{-1}\mathcal{D}_W} f^{-1}\mathcal{N}.$$

The corresponding functor on right $\mathcal{D}$-modules is denoted

$$f^! : \mathrm{QCoh}_{\mathcal{D}_W} \longrightarrow \mathrm{QCoh}_{\mathcal{D}_Z}, \quad f^!\mathcal{M} = \omega_Z \otimes_{\mathcal{O}_Z} \big(f^*(\omega_W^{-1} \otimes_{\mathcal{O}_W} \mathcal{M})\big).$$

B. Compatibility of $f^!$ with DR on Zariski topology.

Proposition 2.2.1. *Let $\mathcal{F} \in \mathrm{Coh}_W$. Then*

$$\mathrm{DR}\big(f^!(\mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{D}_W)\big) = f^!\mathcal{F},$$

where in the right hand side the functor $f^!$ is defined on coherent $\mathcal{O}$-modules on the Zariski topology as in [Har1]. □

Let us illustrate the action of functor $f^!$ on coherent $\mathcal{O}$-modules.

Examples 2.2.2. (a) Suppose $f : Z \to W$ is a smooth morphism of relative dimension m. Then $f^! \mathcal{F} = \omega_{Z/W} \otimes f^* \mathcal{F}[m]$.

(b) Suppose that $i : Z \to W$ is a closed embedding of codimension m, and $I_Z \subset \mathcal{O}_W$ be the ideal of Z. Let $\mathcal{F} = E$ be a vector bundle on W. Then

$$i^! E \;=\; i^{-1} \underline{R\Gamma}_Z(E) \;\simeq\; i^{-1} \underline{H}^m_Z(E)[-m],$$

where

$$\underline{H}^m_Z(E) \;=\; \varinjlim_d F_d\, \underline{H}^m_Z(E), \quad F_d\, \underline{H}^m_Z(E) \;=\; \underline{\mathrm{Ext}}^m_{\mathcal{O}_W}(I_Z^d, E)$$

is the local cohomology sheaf with its natural filtration "by the order of poles". We note that each $F_d\, \underline{H}^m_Z(E)$ is a sheaf of $\mathcal{O}$-modules on the dth infinitesimal neighborhood $Z^{(d-1)} \subset W$ but not on Z itself. However, we point out the following.

Proposition 2.2.3. *In the situation of Example 2.2.2(b):*

(a) We have a canonical identification of the quotients

$$F_d \underline{H}^m_Z(E) / F_{d-1} \underline{H}^m_Z(E) \;\simeq\; \det(N_{Z/W}) \otimes \mathrm{Sym}^d(N_{Z/W}),$$

where $N_{Z/W}$ is the normal bundle of Z in W. In particular, each quotient is canonically a vector bundle on Z.

(b) Each $F_d \underline{H}^m_Z(E)$ has a canonical structure of a differential bundle on Z.

Proof: (a) is well known. To see (b), it is enough to show that $\mathcal{O}_{Z^{(d-1)}} = \mathcal{O}_W / I_Z^d$ has a canonical structure of a differential bundle on Z. Let us show this.

We can, locally, project $Z^{(d-1)}$ back to Z. That is, let z be any point of Z. We can find a Zariski neighborhood U of $z \in Z$ and a morphism of schemes $p : U^{(d-1)} \to U$ such that the composition of p with the embedding $U \to U^{(d-1)}$ is the identity. This makes $\mathcal{O}_{Z^{(d-1)}}|_U$ into an $\mathcal{O}_U$-module.

A different choice of a projection will give, in general, a different $\mathcal{O}_U$-module, i.e., the identity map will not be $\mathcal{O}_U$-linear with respect to these structures. However, it will always be a differential operator of order $\leqslant d-1$. Therefore choosing the projections locally, we make $\mathcal{O}_{Z^{(d-1)}}$ into a differential bundle. □

C. Compatibility of $f^!$ with DR on complex topology. Let $\mathbf{k} = \mathbb{C}$.

Proposition 2.2.4. *Let $\mathcal{M}^\bullet$ be a bounded complex of quasi-coherent right $\mathcal{D}_W$-modules with cohomology modules $\underline{H}^j(\mathcal{M}^\bullet)$ being holonomic regular. Then*

$$\mathrm{DR}(f^!\mathcal{M}^\bullet)_{\mathrm{an}} \;\simeq\; f^!(\mathrm{DR}(\mathcal{M}^\bullet)_{\mathrm{an}}),$$

where in the right hand side we have the usual topological functor $f^!$ on constructible complexes. □

Because of the above compatibilities, we use the same notation $f^!$ for the functor on right $\mathcal{D}$-modules as well as for the corresponding functors on the de Rham complexes.

D. Direct image f_*. The direct image of right $\mathcal{D}$-modules is the functor

$$f_* : D(\mathrm{QCoh}_{\mathcal{D}_Z}) \longrightarrow D(\mathrm{QCoh}_{\mathcal{D}_W}), \quad f_*\mathcal{M} \;=\; Rf_\bullet\big(\mathcal{M} \otimes_{\mathcal{D}_Z} \mathcal{D}_{Z\to W}\big),$$

where $Rf_\bullet$ is the usual topological derived direct image functor on sheaves on the Zariski topology. In the particular case when $f = p : Z \to \mathrm{pt}$ is the projection to the point, $\mathcal{D}_{Z\to\mathrm{pt}} = \mathcal{O}_Z$, and we will use the following notation:

$$R\Gamma_{\mathrm{DR}}(Z, \mathcal{M}) \;=\; p_*\mathcal{M} \;=\; R\Gamma(Z, \mathrm{DR}(\mathcal{M})) \;\in\; D(\mathrm{Vect}_{\mathbf{k}}).$$

Here are the standard properties of f_*.

Proposition 2.2.5. *(a) If f is étale, then f_* is right adjoint to $f^!$.*

(b) If f is proper, then f_* takes $D(\mathrm{Coh}_{\mathcal{D}_Z})$ to $D(\mathrm{Coh}_{\mathcal{D}_W})$ as well as $D^b(\mathrm{Coh}_{\mathcal{D}_Z})$ to $D^b(\mathrm{Coh}_{\mathcal{D}_W})$ etc. In this case f_* is left adjoint to $f^!$.

(c) Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_Z$-modules. then

$$\mathrm{DR}\big(f_*(\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{D}_Z)\big) \;\simeq\; Rf_\bullet(\mathcal{F})$$

where in the right hand side $Rf_\bullet$ is the topological direct image of sheaves on the Zariski topology.

(d) Let $\mathbf{k} = \mathbb{C}$ and $\mathcal{M}^\bullet$ be a bounded complex of quasicoherent right $\mathcal{D}_Z$-modules with holonomic regular cohomology modules. Then

$$\mathrm{DR}(f_*\mathcal{M}^\bullet)_{\mathrm{an}} \;\simeq\; Rf_\bullet(\mathrm{DR}(\mathcal{M}^\bullet)_{\mathrm{an}}),$$

where in the right hand side $Rf_\bullet$ is the topological direct image of sheaves on the complex topology.

Let $j : Z \hookrightarrow W$ be an open embedding, with $i : K \hookrightarrow W$ be the closed embedding of the complement. Let $\mathcal{M} \in \mathrm{QCoh}_{\mathcal{D}_W}$.

Proposition 2.2.6. *We have canonical quasi-isomorphisms*

$$i_* i^! \mathcal{M} \simeq \underline{R\Gamma}_K(\mathcal{M}), \quad j_* j^! \mathcal{M} \simeq Rj_\bullet j^{-1}\mathcal{M}$$

where on the right hand side we have purely sheaf-theoretical operations for sheaves on the Zariski topology. We further have the canonical triangle in $D(\mathrm{QCoh}_{\mathcal{D}_W})$

$$i_* i^! \mathcal{M} \longrightarrow \mathcal{M} \longrightarrow j_* j^! \mathcal{M} \to i_* i^! \mathcal{M}[1].$$

Proof: For the first identification, see [S]. The second one is obvious since j is an open embedding. After this, the triangle in question is just the standard sheaf-theoretic triangle

$$\underline{R\Gamma}_K(\mathcal{M}) \longrightarrow \mathcal{M} \longrightarrow Rj_\bullet j^{-1}\mathcal{M} \longrightarrow \underline{R\Gamma}_K(\mathcal{M})[1].$$

E. $\mathcal{D}$-modules on singular varieties. The above formalism is extended, in a standard way, to right $\mathcal{D}$-modules on possibly singular varieties. Let us briefly recall this procedure, following the treatment of [S] for the case of analytic varieties.

Let $i : Z \to \widetilde{Z}$ is a closed embedding of a (possibly singular) variety Z into a smooth variety $\widetilde{Z}$. We define the categories

$$\mathrm{QCoh}_{Z,\mathcal{D}_{\widetilde{Z}}} \subset \mathrm{QCoh}_{\mathcal{D}_{\widetilde{Z}}}, \quad \mathrm{Coh}_{Z,\mathcal{D}_{\widetilde{Z}}} \subset \mathrm{Coh}_{\mathcal{D}_{\widetilde{Z}}}$$

to be the full subcategories of quasi-coherent and coherent $\mathcal{D}_{\widetilde{Z}}$-modules which are, sheaf-theoretically, supported on Z. If Z is smooth, then, as well known (Kashiwara's lemma), the functors

$$i_* : \mathrm{Coh}_{\mathcal{D}_Z} \longleftrightarrow \mathrm{Coh}_{Z,\mathcal{D}_{\widetilde{Z}}} : i^!$$

are mutually quasi-inverse equivalences, and similarly for QCoh. This implies that the categories $\mathrm{QCoh}_{Z,\mathcal{D}_{\widetilde{Z}}}$ and $\mathrm{Coh}_{Z,\mathcal{D}_{\widetilde{Z}}}$ are canonically (up to equivalence which is unique up to a unique isomorphism) independent on the choice of an embedding i and we denote them

$$\mathrm{QCoh}_{\mathcal{D},Z} := \mathrm{QCoh}_{Z,\mathcal{D}_{\widetilde{Z}}}, \quad \mathrm{Coh}_{\mathcal{D},Z} := \mathrm{Coh}_{Z,\mathcal{D}_{\widetilde{Z}}}, \quad \forall\, Z \xrightarrow{i} \widetilde{Z}.$$

We note that

$$D^b(\mathrm{Coh}_{Z,\mathcal{D}_{\widetilde{Z}}}) \;\simeq\; \mathrm{Perf}_{Z,\mathcal{D}_{\widetilde{Z}}}$$

is identified with the category of perfect complexes of right $\mathcal{D}_{\widetilde{Z}}$-modules which are exact outside of Z. We thus define

$$\mathrm{Perf}_{Z,\mathcal{D}} := D^b(\mathrm{Coh}_{Z,\mathcal{D}_{\widetilde{Z}}}) \simeq \mathrm{Perf}_{Z,\mathcal{D}_{\widetilde{Z}}}\,.$$

In particular, we have $D(\mathrm{QCoh}_{Z,\mathcal{D}}) \simeq \mathrm{Ind}(\mathrm{Perf}_{Z,\mathcal{D}})$.

Given a morphism $f : Z \to W$ of possibly singular varieties, we can extend it to a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & W \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ \widetilde{Z} & \xrightarrow[\widetilde{f}]{} & \widetilde{W} \end{array}$$

with i, i' being closed embeddings into smooth varieties. After this, the functor $f_* : D(\mathrm{QCoh}_{\mathcal{D},Z}) \longrightarrow D(\mathrm{QCoh}_{\mathcal{D},W})$ is defined to be given by the functor

$$\widetilde{f}_* : D(\mathrm{QCoh}_{Z,\mathcal{D}_{\widetilde{Z}}}) \longrightarrow D(\mathrm{QCoh}_{W,\mathcal{D}_{\widetilde{W}}})).$$

Further, the functor $f^! : D(\mathrm{QCoh}_{\mathcal{D},W}) \longrightarrow D(\mathrm{QCoh}_{\mathcal{D},Z})$ is defined to be given by the functor

$$\underline{R\Gamma}_Z \circ \widetilde{f}^! : D(\mathrm{QCoh}_{W,\mathcal{D}_{\widetilde{W}}})) \longrightarrow D(\mathrm{QCoh}_{Z,\mathcal{D}_{\widetilde{Z}}})).$$

These definitions are canonically independent on the choices and we have:

Proposition 2.2.7. *(a) The functor f_* is left adjoint to $f^!$ for proper f and right adjoint to $f^!$ for étale f.*

(b) For any Cartesian square of varieties

$$\begin{array}{ccc} Z_{12} & \xrightarrow{g_2} & Z_1 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle f_1} \\ Z_2 & \xrightarrow[f_2]{} & Z \end{array}$$

we have the base change identification

$$f_2^! \circ (f_1)_* \;\simeq\; (g_1)_* \circ g_2^!. \quad \square$$

Further, the Verdier duality

$$D(\mathrm{QCoh}_{\mathcal{D},Z})^{\mathrm{op}} \longrightarrow D(\mathrm{QCoh}_{\mathcal{D},Z}), \quad \mathcal{M} \mapsto \mathcal{M}^\vee$$

is defined to be given by the functor (Verdier duality on $\widetilde{Z}$)

$$D(\mathrm{QCoh}_{Z,\mathcal{D}_{\widetilde{Z}}})^{\mathrm{op}} \to D(\mathrm{QCoh}_{Z,\mathcal{D}_{\widetilde{Z}}}), \quad \mathcal{M} \mapsto \omega_{\widetilde{Z}} \otimes_{\mathcal{O}_{\widetilde{Z}}} \underline{\mathrm{RHom}}_{\mathcal{D}_{\widetilde{Z}}}(\mathcal{M}, \mathcal{D}_{\widetilde{Z}}),$$

which is canonically independent on the choices.

For $\mathcal{M}^\bullet \in D(\mathrm{QCoh}_{\mathcal{D},Z})$ we have the canonically defined de Rham complex $\mathrm{DR}(\mathcal{M})$ of sheaves on Z. If $\mathcal{M}$ is represented by a complex $\widetilde{\mathcal{M}}^\bullet \in D(\mathrm{QCoh}_{Z,\mathcal{D}_{\widetilde{Z}}})$, then $\mathrm{DR}(\mathcal{M}^\bullet)$ is represented by the de Rham complex $\mathrm{DR}(\widetilde{\mathcal{M}}^\bullet)$ which is canonically independent on the choices.

Further, we have a well defined concept of holonomic (resp. holonomic regular) objects of $\mathrm{Coh}_{\mathcal{D},Z}$. We denote by $\mathrm{Hol}_{\mathcal{D},Z}$ the category of holonomic objects of $\mathrm{Coh}_{\mathcal{D},Z}$. The corresponding derived category will be denoted by

$$D^b_{\mathrm{hol}}\mathrm{QCoh}_{\mathcal{D},Z} \simeq D^b\mathrm{Hol}_{\mathcal{D},Z}. \tag{2.2.8}$$

Here the LHS means the category of complexes with holonomic cohomology, the RHS the category of complexes consisting of holonomic modules and the equivalence between two derived categories thus defined is standard (see [Bo, VI, Prop. 1.14]).

2.3 The nonstandard functorialities

A. Reminder on ind- and pro-$\mathcal{D}$-modules. Let Z be a (possibly singular) variety over $\mathbf{k}$. As in [De], we have identifications

$$\mathrm{QCoh}_Z \simeq \mathrm{Ind}(\mathrm{Coh}_Z), \quad \mathrm{QCoh}_{\mathcal{D},Z} \simeq \mathrm{Ind}(\mathrm{Coh}_{\mathcal{D},Z}).$$

For the derived ∞-categories we have identification (see [GR] for instance):

$$D(\mathrm{QCoh}_Z) \simeq \mathrm{Ind}(\mathrm{Perf}_Z), \quad D(\mathrm{QCoh}_{\mathcal{D},Z}) \simeq \mathrm{Ind}(\mathrm{Perf}_{\mathcal{D},Z})).$$

where Perf_Z is the category of perfect complexes of $\mathcal{O}_Z$-modules.

Therefore the Verdier duality gives an anti-equivalence

$$D(\mathrm{QCoh}_{\mathcal{D},Z}) \xrightarrow{\sim} \mathrm{Pro}(\mathrm{Perf}_{\mathcal{D},Z}), \quad \mathcal{M} = \text{“}\varinjlim\text{”}\, \mathcal{M}_\nu \mapsto \mathcal{M}^\vee = \text{“}\varprojlim\text{”}\, \mathcal{M}_\nu^\vee.$$

B. Formal inverse image $f^{[[*]]}$. We keep the notation of the previous section.

Define the functor of *formal inverse image*

$$f^{[[*]]} : \mathrm{Pro}(\mathrm{Perf}_{\mathcal{D},W}) \longrightarrow \mathrm{Pro}(\mathrm{Perf}_{\mathcal{D},Z})$$

by putting, for $\mathcal{M}^\bullet \in \mathrm{Perf}_{\mathcal{D},W}$

$$f^{[[*]]}\mathcal{M} = \big(f^!(\mathcal{M}^\vee)\big)^\vee$$

and then extend to pro-objects in a standard way. If Z and W are smooth, then

$$f^{[[*]]}\mathcal{M} = \underline{\mathrm{RHom}}_{f^{-1}\mathcal{D}_W}(\mathcal{D}_{Z\to W}, f^{-1}\mathcal{M}), \tag{2.3.1}$$

where we notice that $\mathcal{D}_{Z\to W}$ is a quasi-coherent, i.e., ind-coherent right $\mathcal{D}_Z$-module, so taking $\underline{\mathrm{RHom}}$ from it produces a pro-object.

Proposition 2.3.2. *(a) Suppose f is proper, so that f_* takes $\mathrm{Perf}_{\mathcal{D},Z}$ to $\mathrm{Perf}_{\mathcal{D},W}$ and therefore extends to a functor*

$$f_* : \mathrm{Pro}(\mathrm{Perf}_{\mathcal{D},Z}) \longrightarrow \mathrm{Pro}(\mathrm{Perf}_{\mathcal{D},W})$$

denoted by the same symbol. Then the functor $f^{[[*]]}$ is left adjoint to f_* thus defined.

(b) The functor $f^{[[*]]}$ takes $D^b_{\mathrm{hol}}(\mathrm{Coh}_{\mathcal{D},W})$ to $D^b_{\mathrm{hol}}(\mathrm{Coh}_{\mathcal{D},Z})$ (no pro-objects needed).

We note that defining the $*$-inverse image on holonomic $\mathcal{D}$-modules by conjugating $f^!$ with the Verdier duality is a standard procedure. The corresponding functor is usually denoted by f^*, see [Bo]. We use the notation $f^{[[*]]}$ to emphasize the pro-object structure in the general (non-holonomic) case.

Proof: (a) In the case when Z and W are smooth, the statement follows from (2.3.1) and from the adjunction between Hom and $\otimes$. The general case is dual to Proposition 2.2.5 (b).

Part (b) is standard, see [Bo]. □

C. Compatibility of $i^{[[*]]}$ with DR on Zariski topology. Let $i : Z \to W$ be a closed embedding of smooth varieties of codimension m, with the ideal $I_Z \subset \mathcal{O}_W$. Let E be a vector bundle on W. As in Proposition 2.2.3 we see that each $E/I_Z^d E$ is naturally a differential bundle on Z. We define the *formal restriction* of E to Z to be the pro-differential bundle

$$i^{[[*]]}E \;=\; \text{“}\varprojlim_d\text{”}\; E/I_Z^d E \;=\; \widehat{E}_Z \;\in\; \mathrm{Pro}(\mathrm{DS}_Z)$$

given by the formal completion of E along Z.

Proposition 2.3.3. *We have a quasi-isomorphism of sheaves on the Zariski topology of Z:*

$$\mathrm{DR}\big(i^{[[*]]}(E \otimes_{\mathcal{O}_W} \mathcal{D}_W)\big) \;\simeq\; i^{[[*]]}E.$$

D. Compatibility of $f^{[[*]]}$ with DR on complex topology. Let $\mathbf{k} = \mathbb{C}$.

Proposition 2.3.4. *Let $\mathcal{M}^\bullet \in D^b(\mathrm{Coh}_{\mathcal{D},W})$ be a complex with holonomic regular cohomology. Then*

$$\mathrm{DR}(f^{[[*]]}\mathcal{M})_{\mathrm{an}} \;\simeq\; f^{-1}\big(\mathrm{DR}(\mathcal{M}^\bullet)_{\mathrm{an}}\big),$$

where on the right we have the usual inverse image of constructible complexes on the complex topology.

Proof: Follows from Proposition 2.2.4 by Verdier duality. □

Because of these compatibilities we use the same notation $f^{[[*]]}$ for the formal inverse image functor on $\mathcal{D}$-modules and differential sheaves.

E. The formal compactly supported direct image $f_{[[!]]}$. . We define the functor of *formal compactly supported direct image*

$$f_{[[!]]} : \mathrm{Pro}(\mathrm{Perf}_{\mathcal{D},Z}) \longrightarrow \mathrm{Pro}(\mathrm{Perf}_{\mathcal{D},W})$$

by putting, for $\mathcal{M}^\bullet \in \mathrm{Perf}_{\mathcal{D},Z}$

$$f_{[[!]]}\mathcal{M} \;=\; (f_*\mathcal{M}^\vee)^\vee \;\in\; \mathrm{Pro}(\mathrm{Perf}_{\mathcal{D},W})$$

and then extending to pro-objects in the standard way. In the particular case when $f = p : Z \to \mathrm{pt}$ is the projection to the point, we will use the notation

$$R\Gamma^{[[c]]}_{\mathrm{DR}}(Z, \mathcal{M}) \;:=\; p_{[[!]]}\mathcal{M} \;\in\; \mathrm{Pro}(\mathrm{Perf}_{\mathbf{k}}).$$

Proposition 2.3.5. *(a) The functor $f_{[[!]]}$ is right adjoint to $f^{[[*]]}$ if f is proper, and left adjoint to $f^{[[*]]}$ for f étale.*

(b) For any Cartesian square of varieties as in Proposition 2.2.7(b), we have a canonical identification (base change)

$$f_2^{[[*]]} \circ (f_1)_{[[!]]} \;\simeq\; (g_1)_{[[!]]} \circ g_2^{[[*]]}.$$

(c) Let $j : Z \hookrightarrow W$ be an open embedding, with $i : K \hookrightarrow W$ be the closed embedding of the complement. Then, for any $\mathcal{M} \in \mathrm{Pro}(\mathrm{Perf}_{\mathcal{D},W})$ we have the canonical triangle in $\mathrm{Pro}(D^b(\mathrm{QCoh}_{\mathcal{D}_W}))$

$$j_{[[!]]} j^* \mathcal{M} \longrightarrow \mathcal{M} \longrightarrow i_* i^{[[*]]} \mathcal{M} \longrightarrow j_{[[!]]} j^* \mathcal{M}[1].$$

Here $j^*\mathcal{M}$ can be understood either as $j^{[[*]]}\mathcal{M}$ or as $j^!\mathcal{M}$, the two results being the same.

(d) The functor $f_{[[!]]}$ takes $D^b_{\mathrm{hol}}(\mathrm{Coh}_{\mathcal{D},Z})$ to $D^b_{\mathrm{hol}}(\mathrm{Coh}_{\mathcal{D},W})$ (no pro-objects needed).

As with $f^{[[*]]}$, we note that defining the !-direct image on holonomic $\mathcal{D}$-modules by conjugating f_* with the Verdier duality is a standard procedure. The corresponding functor is usually denoted by $f_!$, see [Bo]. We use the notation $f_{[[!]]}$ to emphasize the pro-object structure in the general (non-holonomic) case.

Proof: Follows from the corresponding properties for $f^!$ and f_* by applying Verdier duality. □

F. Compatibility of $f_{[[!]]}$ with DR on Zariski topology. The effect of $f_{[[!]]}$ on induced $\mathcal{D}$-modules can be described directly, following [De].

First, suppose that $j : Z \to \overline{Z}$ is an open embedding with complement K and ideal $I_K \subset \mathcal{O}_{\overline{Z}}$. Let $\mathcal{F} \in \mathrm{Coh}_Z$ be a coherent $\mathcal{O}_Z$-module. Choose a coherent $\mathcal{O}_Z$-module $\overline{\mathcal{F}} \in \mathrm{Coh}_Z$ extending $\mathcal{F}$, i.e., such that $j^*\overline{\mathcal{F}} = \mathcal{F}$. Following [De] we define

$$\tilde{j}_{[[!]]}\mathcal{F} \;=\; \text{“}\varprojlim_d\text{”}\; I_K^d \overline{\mathcal{F}} \;\in\; \mathrm{Pro}(\mathrm{Coh}_{\overline{Z}}). \tag{2.3.6}$$

Proposition 2.3.7. *(a) The object $\tilde{j}_{[[!]]}\mathcal{F}$ is canonically (uniquely up to a unique isomorphism) independent of the choice of $\overline{\mathcal{F}}$. In this way we get* *a canonically defined functor* $\tilde{\jmath}_{[[!]]}\colon \mathrm{Coh}_Z \to \mathrm{Pro}(\mathrm{Coh}_{\overline{Z}})$ *which extends, in a standard way, to a functor*

$$\tilde{\jmath}_{[[!]]}\colon\ \mathrm{Pro}(\mathrm{Coh}_Z) \to \mathrm{Pro}(\mathrm{Coh}_{\overline{Z}}).$$

(b) Further, suppose $\overline{Z}$ *smooth. Then a differential operator* $P\colon \mathcal{F} \to \mathcal{G}$ *between coherent sheaves on* Z *gives rise to a morphism* $\tilde{\jmath}_{[[!]]}P\colon \tilde{\jmath}_{[[!]]}\mathcal{F} \to \tilde{\jmath}_{[[!]]}\mathcal{G}$ *in* $\mathrm{Pro}(DS_{\overline{Z}})$*, thus giving a functor*

$$\tilde{\jmath}_{[[!]]}\colon\ \mathrm{Pro}(\mathrm{DS}_Z) \to \mathrm{Pro}(\mathrm{DS}_{\overline{Z}}).$$

Proof: For part (a), see [De]. Part (b) follows since a differential operator $P\colon \mathcal{F} \to \mathcal{G}$ of order r maps $I_K^m\mathcal{F}$ to $I_K^{m-r}\mathcal{G}$. $\square$.

Next, let $f\colon Z \to W$ be any morphism of algebraic varieties over $\mathbf{k}$. We can always factor f as the composition $f = qj$

$$Z \overset{j}{\hookrightarrow} \overline{Z} \overset{q}{\longrightarrow} W$$

with j being an open embedding and q proper. We define

$$\tilde{f}_{[[!]]} \ = \ (Rq_\bullet) \circ \tilde{\jmath}_{[[!]]}\colon\ \mathrm{Pro}(\mathrm{Perf}_Z) \longrightarrow \mathrm{Pro}(\mathrm{Perf}_W).$$

Proposition 2.3.8. *The functor* $\tilde{f}_{[[!]]}$ *is well defined (values canonically independent on the choices) and is compatible with composition of morphisms.*

In particular, for $W = \mathrm{pt}$ we have the functor of *algebraic cohomology with compact support*, see [Har2]:

(2.3.9)

$$R\Gamma^{[[c]]}(Z,\mathcal{F}) \ = \ R\Gamma(\overline{Z}, \tilde{\jmath}_{[[!]]}\mathcal{F}) \ = \ \text{“}\varprojlim_d\text{”} \ R\Gamma(\overline{Z}, I_K^d\overline{\mathcal{F}}) \ \in \ \mathrm{Pro}(\mathrm{Perf}_{\mathbf{k}}).$$

It satisfies Serre duality. That is, suppose Z is smooth and E is a vector bundle on Z. Then we have the isomorphism

$$R\Gamma^{[[c]]}(Z, \omega_Z \otimes E^*) \ \simeq \ R\Gamma(Z,E)^* \tag{2.3.10}$$

(duality between objects of $\mathrm{Pro}(\mathrm{Perf}_{\mathbf{k}})$ and $\mathrm{Ind}(\mathrm{Perf}_{\mathbf{k}})$).

As before, the functor $\tilde{f}_{[[!]]}$ on coherent sheaves inherits the action of differential operators, not just $\mathcal{O}$-linear morphisms.

Proposition 2.3.11. *Suppose Z is smooth. For any coherent sheaf $\mathcal{F}$ on Z we have*

$$\mathrm{DR}\big(f_{[[!]]}(\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{D}_Z)\big) \;\simeq\; \tilde{f}_{[[!]]}\mathcal{F}.$$

Proof: In our approach, the functor $f_{[[!]]}$ was constructed formally to satisfy duality, while $\tilde{f}_{[[!]]}$ was constructed explicitly using compactification. So the relation between them follows from the results of [De] which establish the duality for the functor constructed via compactification. □

G. Compatibility of $f_{[[!]]}$ with DR on complex topology.

Proposition 2.3.12. *Let $\mathbf{k} = \mathbb{C}$ and let $\mathcal{M}^\bullet \in \mathrm{Perf}_{\mathcal{D},Z}$ be a complex of right $\mathcal{D}_Z$-modules whose cohomology modules are holonomic regular. Then we have a quasi-isomorphism*

$$\mathrm{DR}(f_{[[!]]}\mathcal{M}^\bullet)_{\mathrm{an}} \;\simeq\; f_!(\mathrm{DR}(\mathcal{M}^\bullet)_{\mathrm{an}}),$$

where $f_!$ is the usual functor of direct image with proper support for sheaves on the complex topology.

Proof: Follows from the similar statement about the functor f_* by applying Verdier duality. □

Corollary 2.3.13. *Suppose Z is smooth. Let $\mathcal{F}^\bullet$ be a differential complex on Z such that the complex $\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{F}^\bullet$, of induced $\mathcal{D}_Z$-modules has regular holonomic cohomology sheaves. Then*

$$(\tilde{f}_{[[!]]}\mathcal{F}^\bullet)_{\mathrm{an}} \;\simeq\; f_!(\mathcal{F}^\bullet_{\mathrm{an}}).$$

2.4 Correspondences and base change

Definition 2.4.1. We denote by $\mathrm{Var}_{\mathbf{k}}^{\mathrm{corr}}$ the following 2-category, called the category of correspondences between $\mathbf{k}$-varieties. Its objects are objects of $\mathrm{Var}_{\mathbf{k}}$. A morphism in $\mathrm{Var}_{\mathbf{k}}^{\mathrm{corr}}$ from X to Y is a correspondence: a third variety Z with two morphisms of varieties $X \leftarrow Z \to Y$. The composition of two correspondences $X \leftarrow Z \to Y$ and $Y \leftarrow Z' \to Y'$ is the correspondence $X \leftarrow Z \times_Y Z' \to Y'$. Finally, a transformation (i.e., a 2-morphism) between two correspondences $X \leftarrow Z \to Y$ and $X \leftarrow Z' \to Y$ is the datum of a *proper* morphism $Z \to Z'$ commuting with the maps to X and Y. The vertical composition is the obvious one.

It follows from [GR] that base change between lower-$*$ and upper-! functors can be encoded as an $(\infty, 2)$-functor

$$\mathcal{D}^{\mathrm{corr}}\colon \mathrm{Var}_{\mathbf{k}}^{\mathrm{corr}} \longrightarrow \mathrm{Cat}_\infty$$

mapping a variety X to $D(\mathrm{QCoh}_{\mathcal{D},X})$ and a correspondence $X \xleftarrow{f} Z \xrightarrow{g} Y$ to the functor $g_* f^!$. It finally maps a (proper) transformation $f\colon Z \to Z'$ between two correspondences $X \xleftarrow{g} Z \xrightarrow{h} Y$ and $X \xleftarrow{u} Z' \xrightarrow{v} Y$ to the natural transformation

$$h_* g^! = v_* f_* f^! u^! \Rightarrow v_* u^!$$

induced by the adjunction counit.

Similarly, we have an $(\infty, 2)$- functor

$$\mathcal{D}^{[[\mathrm{corr}]]}\colon \mathrm{Var}_{\mathbf{k}}^{\mathrm{corr}} \longrightarrow \mathrm{Cat}_\infty$$

that maps a variety X to $\mathrm{Pro}(D^b\mathrm{Coh}_{\mathcal{D},X})$, a correspondence $X \xleftarrow{f} Z \xrightarrow{g} Y$ to the functor $g_{[[!]]} f^{[[*]]}$. It finally maps a (proper) transformation $f\colon Z \to Z'$ between two correspondences $X \xleftarrow{g} Z \xrightarrow{h} Y$ and $X \xleftarrow{u} Z' \xrightarrow{v} Y$ to the natural transformation

$$h_{[[!]]} g^{[[*]]} = v_{[[!]]} f_{[[!]]} f^{[[*]]} u^{[[*]]} \Rightarrow v_{[[!]]} u^{[[*]]}$$

induced by the adjunction counit.

3 $\mathcal{D}$-modules on the Ran space

3.1 The Ran space in algebraic geometry

Throughout this section, we will denote by $\mathrm{Var}_{\mathbf{k}}$ the category of varieties over $\mathbf{k}$. Let $X \in \mathrm{Var}_{\mathbf{k}}$.

A. Ran diagram and Ran space.

Definition 3.1.1. (a) Let $\mathscr{S}$ denote the category of non-empty finite sets with surjective maps between them. We define the *Ran diagram* of X as the contravariant functor $X^{\mathscr{S}}\colon \mathscr{S} \to \mathrm{Var}_{\mathbf{k}}$ defined by:

$$X^{\mathscr{S}}\colon I \mapsto X^I, \quad (g\colon I \twoheadrightarrow J) \mapsto (\delta_g\colon X^J \to X^I),$$

where δ_g is the diagonal embedding associated to g.

(b) By an *(algebro-geometric) space* over $\mathbf{k}$ we will mean a sheaf of sets on the big étale site (affine $\mathbf{k}$-schemes with étale coverings). The category of such will be denoted $\mathrm{AGS}_{\mathbf{k}}$. We have the standard embedding $\mathrm{Var}_{\mathbf{k}} \hookrightarrow \mathrm{AGS}_{\mathbf{k}}$ (representable functors).

(c) The *Ran space* of X is defined as

$$\mathrm{Ran}(X) \;=\; \varinjlim X^{\mathscr{S}} \;=\; \varinjlim_{I\in\mathscr{S}} X^I$$

(colimit in $\mathrm{AGS}_{\mathbf{k}}$).

The category $\mathscr{S}$ is not filtering so $\mathrm{Ran}(X)$ is not an ind-variety in the standard sense.

B. Diagonal skeleta.

Definition 3.1.2. (a) For any $I \in \mathscr{S}$ and $q > 0$, we call the *q-fold diagonal* and denote by

$$X^I_q := \bigcup_{\substack{f:\, I\twoheadrightarrow Q\\ |Q|=q}} \delta_f(X^Q)$$

the closed subvariety of X^I whose closed points are families of at most q different points of X.

(b) We denote by $X^{\mathscr{S}}_q$ the functor $\mathscr{S} \to \mathrm{Var}_{\mathbf{k}}$ given by $I \mapsto X^I_q$. We also denote

$$\mathrm{Ran}_q(X) \;=\; \varinjlim X^{\mathscr{S}}_q \;=\; \varinjlim_{I\in\mathscr{S}} X^I_q$$

the space corresponding to $X^{\mathscr{S}}_q$.

Let S_q be the symmetric group on q symbols . We denote by

$$\mathrm{Sym}^q(X) \;=\; X^q/S_q, \quad \mathrm{Sym}^q_{\neq}(X) = (X^q - X^q_{q-1})/S_q$$

the qth symmetric power of X (as a singular variety) and its open part given by complement of all the diagonals.

Proposition 3.1.3. *(a) We have*

$$X^{\mathscr{S}} = \varinjlim_q X_q^{\mathscr{S}}, \quad \mathrm{Ran}(X) = \varinjlim_q \mathrm{Ran}_q(X)$$

(colimit in the category of functors resp. in $\mathrm{AGS}_{\mathbf{k}}$*).*

(b) We have an identification

$$\mathrm{Ran}_q(X) - \mathrm{Ran}_{q-1}(X) \;=\; \mathrm{Sym}^q_{\neq}(X).$$

In particular, $\mathrm{Ran}_1(X) = X$*.*

C. Varieties $\Delta(I, J)$. We will use the following construction from [GL] (9.4.12).

Definition 3.1.4. Let I, J be two nonempty finite sets. We denote by $\Delta(I, J) \subset X^I \times X^J$ the closed algebraic subvariety whose $\overline{\mathbf{k}}$-points are pairs of tuples $\big((x_i)_{i\in I}, (y_j)_{j\in J}\big)$, $x_i, y_j \in X(\overline{\mathbf{k}})$ such that the corresponding unordered subsets

$$\{x_i\}_{i\in I} \;=\; \bigcup_{i\in I} \{x_i\}, \quad \{y_j\}_{j\in J} \;=\; \bigcup_{j\in J} \{y_j\}$$

coincide. We denote

(3.1.5)
$$X^I \xleftarrow{p_{IJ}} \Delta(I, J) \xrightarrow{q_{IJ}} X^J$$

the natural projections. They are finite morphisms.

The following is obvious,

Proposition 3.1.6. *We have*

$$\Delta(I, J) \;=\; \varinjlim^{\mathrm{Var}}_{\{I \overset{u}{\twoheadrightarrow} Q \overset{v}{\twoheadleftarrow} J\}} \mathrm{Im}\big\{(\delta_u, \delta_v) : X^Q \longrightarrow X^I \times X^J\big\},$$

where the colimit is taken over the category whose objects are pairs of surjections $I \overset{u}{\twoheadrightarrow} Q \overset{v}{\twoheadleftarrow} J$ *and morphisms are surjections* $Q \to Q'$ *commuting with the arrows. The colimit reduces to the union inside* $X^I \times X^J$*.* □

Any surjection $g : I \to J$ induces, for each finite nonempty K, a natural morphism (closed embedding) $\Delta(g, K) : \Delta(J, K) \to \Delta(I, K)$. The following is clear by definition.

Proposition 3.1.7. *For any $g : I \to J$ and K as above we have a commutative diagram with the square being Cartesian*

$$\begin{array}{ccccc} X^I & \xleftarrow{p_{IK}} & \Delta(I,K) & \xrightarrow{q_{IK}} & X^K \\ {\scriptstyle \delta_g}\uparrow & & \uparrow{\scriptstyle \Delta(g,K)} & \nearrow{\scriptstyle q_{JK}} & \\ X^J & \xleftarrow[p_{JK}]{} & \Delta(J,K). & & \end{array}$$

□

An alternative way to arrive at Proposition 3.1.7 is via the next statement.

Proposition 3.1.8. *The natural morphism in* $\mathrm{AGS}_{\mathbf{k}}$

$$(p_{IJ}, q_{IJ}) : \Delta(I,J) \longrightarrow X^I \times_{\mathrm{Ran}(X)} X^J$$

is an isomorphism.

Proof: To say that (p_{IJ}, q_{IJ}) is an isomorphism of sheaves on the big étale site, means that it induces a bijection on S-points for any scheme S which is the spectrum of a strictly Henselian local ring. Let S be given and $(p,q) : S \to X^I \times_{\mathrm{Ran}(X)} X^J$ be a morphism, i.e., $p : S \to X^I$ and $q : S \to X^J$ are morphisms of schemes which become equal after map to $\mathrm{Ran}(X)$. We need to show that (p,q) considered as a morphism $S \to X^I \times X^J$ factors through $\Delta(I,J)$.

By definition of $\mathrm{Ran}(X)$ and our assumptions on S,

$$\mathrm{Hom}(S, \mathrm{Ran}(X)) \;=\; \varinjlim^{\mathcal{S}et}_{I \in \mathscr{S}} \mathrm{Hom}(S, X^I) \;=\; \bigsqcup_{I \in \mathscr{S}} \mathrm{Hom}(S, X^I) \Big/ \equiv$$

where $\equiv$ is the equivalence relation generated by the following relation $\equiv_0$. We say that $p : S \to X^I$ and $q : S \to X^J$ satisfy $p \equiv_0 q$, if there is a diagram of surjections $I \overset{a}{\twoheadleftarrow} L \overset{b}{\twoheadrightarrow} J$ such that $\delta_a p = \delta_b q$ in $\mathrm{Hom}(S, X^L)$.

We treat the case $p \equiv_0 q$, the general case follows easily. Let Q be the coproduct, fitting in the coCartesian square of sets below left. Note that all arrows in that square are surjections.

$$\begin{array}{ccc} L & \xrightarrow{a} & I \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ J & \xrightarrow[d]{} & Q \end{array} \qquad\qquad \begin{array}{ccc} X^L & \xleftarrow{\delta_a} & X^I \\ {\scriptstyle \delta_b}\uparrow & & \uparrow{\scriptstyle \delta_c} \\ X^J & \xleftarrow[\delta_d]{} & X^Q \end{array}$$

This square induces a Cartesian square of varieties (above right). So p, q comes from a morphism $S \to X^Q$ and our statement follows from Proposition 3.1.6. $\square$

3.2 $[[\mathcal{D}]]$-modules and $\mathcal{D}^!$-modules

A. Reminder on lax limits. Let $\mathscr{A}$ be a small category and $\mathcal{P}\colon \mathscr{A} \to \mathrm{Cat}_\infty$ an (∞-)functor. In particular, for each object $a \in \mathscr{A}$ we have an ∞-category $\mathcal{P}_a$ and for any morphism $g\colon a \to a'$ in $\mathscr{A}$ we have an ∞-functor $p_g\colon \mathcal{P}_a \to \mathcal{P}_{a'}$. In this setting we have the ∞-category known as the *Cartesian Grothendieck construction* (or *relative nerve*) $\mathscr{G}(\mathcal{P})$. see [Lu-HTT] Def. 3.2.5.2. Thus, in particular,

(GR0) Objects (0-simplices) of $\mathscr{G}(\mathcal{P})$ are pairs (a, x) where $a \in \mathrm{Ob}(\mathscr{A})$ and x is an object (0-simplex) in $\mathcal{P}_a$.

(GR1) Morphisms (1-simplices) in $\mathscr{G}(\mathcal{P})$ from (a, x) to (a', x') are pairs (g, α) where $g\colon a \to a'$ is a morphism in $\mathscr{A}$ and $\alpha\colon x' \to p_g(x)$ is a morphism in $\mathcal{P}_{a'}$,

and so on, see *loc. cit.* for details. Note that there is a "dual" version, called the coCartesian Grothendieck construction $\overline{\mathscr{G}}(\mathcal{P})$ with the same objects but with (higher) morphisms defined in a partially dualized way, for example,

($\overline{\mathrm{GR1}}$) Morphisms (1-simplices) in $\mathscr{G}(\mathcal{P})$ from (a, x) to (a', x') are pairs (g, α) where $g\colon a \to a'$ is a morphism in $\mathscr{A}$ and $\alpha\colon p_g(x) \to x'$ is a morphism in $\mathcal{P}_{a'}$,

and so on. In other words,

$$\overline{\mathscr{G}}(\mathcal{P}) \;=\; \mathscr{G}(\mathcal{P}^{\mathrm{op}})^{\mathrm{op}}, \quad \mathcal{P}^{\mathrm{op}} = (\mathcal{P}_a^{\mathrm{op}},\, p_g^{\mathrm{op}}\colon \mathcal{P}_a^{\mathrm{op}} \to \mathcal{P}_{a'}^{\mathrm{op}}).$$

We have the natural projections

$$q\colon\; \mathscr{G}(\mathcal{P}) \longrightarrow \mathscr{A}^{\mathrm{op}}, \quad \bar{q}\colon\; \overline{\mathscr{G}}(\mathcal{P}) \longrightarrow \mathscr{A}.$$

Definition 3.2.1. The *lax limit* and *op-lax limit* of $\mathcal{P}$ are the ∞-categories

$$\underleftarrow{\mathrm{lax}}(\mathcal{P}) \;=\; \mathrm{Sect}(\mathscr{G}(\mathcal{P})/\mathscr{A}), \quad \underleftarrow{\mathrm{lax}}^\circ(\mathcal{P}) \;=\; \mathrm{Sect}(\overline{\mathscr{G}}(\mathcal{P})/\mathscr{A})$$

formed by sections of q and $\bar{q}$, i.e., by ∞-functors (morphisms of simplicial sets) $s\colon \mathcal{A} \to \mathscr{G}(\mathcal{P})$ (resp. $\bar{s}\colon \mathcal{A} \to \overline{\mathscr{G}}(\mathcal{P})$) such that $qs = \mathrm{Id}$ (resp. $\bar{q}\bar{s} = \mathrm{Id}$).

Examples 3.2.2. (a) Thus, an object of $\underleftarrow{\mathrm{lax}}^{\circ}(\mathcal{P})$ is a following set of data:

(0) For each $a \in \mathrm{Ob}(\mathscr{A})$, an object $x_a \in \mathcal{P}_a$.

(1) For each morphism $g : a_0 \to a_1$ in $\mathcal{A}$, a morphism (not necessarily an isomorphism) $\gamma_g \colon p_g(x_{a_0}) \to x_{a_1}$.

(2) For each composable pair $a_0 \xrightarrow{g_0} a_1 \xrightarrow{g_1} a_2$ in $\mathscr{A}$, a homotopy (necessarily invertible, as we work in an $(\infty, 1)$-category) $\gamma_{g_1} \circ p_{g_1}(\gamma_{g_0}) \Rightarrow \gamma_{g_1 g_0}$.

(p) Similar homotopies for composable chains in $\mathscr{A}$ of length p for any p.

(b) Similarly, an object of $\underleftarrow{\mathrm{lax}}(\mathcal{P})$ is a set of data with part (0) identical to the above, part (1) replaced by morphisms $\beta_g \colon x_{a_1} \to p_g(x_{a_0})$ and so on.

Definition 3.2.3. A morphism $(a_0, x_{a_0}) \to (a_1, x_{a_1})$ in $\mathscr{G}(\mathcal{P})$ is called Cartesian if the corresponding morphism $x_{a_0} \to p_g(x_{a_1})$ is an equivalence. Dually, a morphism $(a_0, x_{a_0}) \to (a_1, x_{a_1})$ in $\overline{\mathscr{G}}(\mathcal{P})$ is called coCartesian if the corresponding morphism $p_g(x_{a_0}) \to x_{a_1}$ is an equivalence.

In particular, the full subcategory of $\underleftarrow{\mathrm{lax}}\,\mathcal{P}$ spanned by sections s such that for any map g in $\mathscr{A}$, $s(g)$ is Cartesian is equivalent to the limit $\underleftarrow{\mathrm{holim}}\,\mathcal{P}$ (see [Lu-HTT, Cor. 3.3.3.2]). Dually, the full subcategory of $\underleftarrow{\mathrm{lax}}^{\circ}\,\mathcal{P}$ spanned by sections mapping every arrow to a coCartesian one is equivalent to $\underleftarrow{\mathrm{holim}}\,\mathcal{P}$.

Let now $\mathcal{Q} \colon \mathscr{B} \to \mathscr{A}$ be another functor. There are pullback diagrams

$$\begin{array}{ccc} \mathscr{G}(\mathcal{P} \circ \mathcal{Q}) & \longrightarrow & \mathscr{G}(\mathcal{P}) \\ \downarrow & & \downarrow \\ \mathscr{B}^{\mathrm{op}} & \xrightarrow[\mathcal{Q}]{} & \mathscr{A}^{\mathrm{op}} \end{array} \qquad \begin{array}{ccc} \overline{\mathscr{G}}(\mathcal{P} \circ \mathcal{Q}) & \longrightarrow & \overline{\mathscr{G}}(\mathcal{P}) \\ \downarrow & & \downarrow \\ \mathscr{B} & \xrightarrow[\mathcal{Q}]{} & \mathscr{A}. \end{array}$$

In particular, pulling back sections defines projections

$$\underleftarrow{\mathrm{lax}}\,\mathcal{P} \longrightarrow \underleftarrow{\mathrm{lax}}\,\mathcal{P} \circ \mathcal{Q} \qquad \text{and} \qquad \underleftarrow{\mathrm{lax}}^{\circ}\,\mathcal{P} \longrightarrow \underleftarrow{\mathrm{lax}}^{\circ}\,\mathcal{P} \circ \mathcal{Q},$$

both compatible with the projection $\underleftarrow{\mathrm{holim}}\,\mathcal{P} \to \underleftarrow{\mathrm{holim}}\,\mathcal{P} \circ \mathcal{Q}$.

B. Lax and strict $[[\mathcal{D}]]$- and $\mathcal{D}^!$-modules on diagrams. By a *diagram* of varieties of finite type over $\mathbf{k}$ we mean a datum of a small category $\mathscr{A}$ and a functor $\mathcal{Y}\colon \mathscr{A} \to \mathrm{Var}_{\mathbf{k}}$. That is, for each object $a \in \mathscr{A}$ we have a (possibly singular) variety Y_a and for each morphism $g\colon a \to b$ in $\mathscr{A}$ we have a morphism of varieties $\xi_g : Y_a \to Y_b$.

Given a diagram $\mathcal{Y}$, we have two functors $\mathscr{A}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$:

$$\mathcal{D}_{\mathcal{Y}}^{[[*]]}\colon a \mapsto \mathrm{Pro}(D^b\mathrm{Coh}_{\mathcal{D},Y_a}), \quad (g\colon a \to a') \mapsto \xi_g^{[[*]]}$$
$$\mathcal{D}_{\mathcal{Y}}^!\colon a \mapsto D(\mathrm{QCoh}_{\mathcal{D},Y_a}), \quad (g\colon a \to a') \mapsto \xi_g^!.$$

Definition 3.2.4. We define the ∞-categories

$$\mathrm{Mod}_{[[\mathcal{D}]]}(\mathcal{Y}) \;=\; \underleftarrow{\mathrm{lax}}^{\circ}(\mathcal{D}_{\mathcal{Y}}^{[[*]]}), \qquad \mathrm{Mod}^!_{\mathcal{D}}(\mathcal{Y}) \;=\; \underleftarrow{\mathrm{lax}}(\mathcal{D}^!_{\mathcal{Y}})$$

whose objects will be called *lax* $[[\mathcal{D}]]$-*modules* and *lax* $\mathcal{D}^!$-*modules* on $\mathcal{Y}$.

Remarks 3.2.5. (a) Thus, a lax $[[\mathcal{D}]]$-module F on $\mathcal{Y}$ can be viewed as a family (F_a) where F_a is a pro-coherent complex of left $\mathcal{D}$-modules over Y_a, together with *transition (compatibility) maps*

$$\gamma_g\colon \xi_g^{[[*]]}F_{a'} \longrightarrow F_a,$$

given for any $g\colon a \to a'$ in $\mathscr{A}$ and further compatible under compositions of the g's. Because of the adjunction between $\xi_g^{[[*]]}$ and $(\xi_g)_*$, we can write transition maps of a lax $[[\mathcal{D}]]$-module in the *dual form*, as morphisms

$$\gamma_g^\dagger\colon F_{a'} \longrightarrow (\xi_g)_*F_a.$$

Since $(\xi_g)_*$ preserves coherent $\mathcal{D}$-modules, this allows us to deal with some lax $[[\mathcal{D}]]$-modules without using pro-objects.

(b) Similarly, we will view a lax $\mathcal{D}^!$-module on $\mathcal{Y}$ as a family $(E^{(a)})$ where $E^{(a)}$ is a quasicoherent (i.e., ind-coherent) complex of right $\mathcal{D}$-modules on Y_a, together with transition maps

$$\beta_g\colon E^{(a)} \longrightarrow \xi_g^!E^{(a')}$$

given for any $g\colon a \to a'$ in $\mathscr{A}$ and further compatible under composition of the g's. As before, we can define the structure maps of a lax $\mathcal{D}^!-$module in *the dual form*, as morphisms

$$\beta_g^\dagger\colon (\xi_g)_*E^{(a')} \longrightarrow E^{(a)},$$

using the adjunction between $(\xi_g)_*$ and $\xi_g^!$.

Definition 3.2.6. (a) A lax $[[\mathcal{D}]]$-module F is called *strict*, if all transition maps γ_g are equivalences. We denote by $\underline{\underline{\mathrm{Mod}}}_{[[\mathcal{D}]]}(\mathcal{Y})$ the full ∞-category of strict $[[\mathcal{D}]]$-modules on $\mathcal{Y}$. It embeds fully faithfully into $\mathrm{Mod}_{[[\mathcal{D}]]}(\mathcal{Y})$.

(b) A lax $\mathcal{D}^!$-module E on $\mathcal{Y}$ is called *strict*, if the transition maps β_g are equivalences. Let $\underline{\underline{\mathrm{Mod}}}^!_{\mathcal{D}}(\mathcal{Y})$ be the ∞-category of $\mathcal{D}^!$-modules on $\mathcal{Y}$. It embeds fully faithfully into $\mathrm{Mod}^!_{\mathcal{D}}(\mathcal{Y})$.

In other words

$$\underline{\underline{\mathrm{Mod}}}_{[[\mathcal{D}]]}(\mathcal{Y}) \;=\; \varprojlim\,(\mathcal{D}_{\mathcal{Y}}^{[[*]]}), \qquad \underline{\underline{\mathrm{Mod}}}^!_{\mathcal{D}}(\mathcal{Y}) \;=\; \varprojlim(\mathcal{D}^!_{\mathcal{Y}})$$

are the strict (∞-categorical) limits of the same functors as above, cf. [FG] §2.1. Objects of the strict limit correspond to Cartesian sections of the Grothendieck construction inside all sections.

C. $[[\mathcal{D}]]$- and $\mathcal{D}^!$-modules on the Ran diagram. We now specialize to the case when $\mathcal{A} = \mathscr{S}^{\mathrm{op}}$ and $\mathcal{Y} = X^{\mathscr{S}}$ is the Ran diagram. Thus, a lax $[[\mathcal{D}]]$-module F on $X^{\mathscr{S}}$ consists of pro-coherent complexes of $\mathcal{D}$-modules F_I on X^I, morphisms $\gamma_g\colon \delta_g^{[[*]]}F_J \to F_I$ plus coherent higher compatibilities for the γ_g. A lax $\mathcal{D}^!$-module E on $X^{\mathscr{S}}$ consists of quasi-coherent complexes of $\mathcal{D}$-modules $E^{(I)}$ on X^I, morphisms $\beta_g\colon E^{(J)} \to \delta_g^! E^{(I)}$ plus coherent higher compatibilities for the β_g.

Recall also, the Ran space $\mathrm{Ran}(X) = \varinjlim X^{\mathscr{S}}$. It is clear that strict modules can be defined invariantly in terms of the space $\mathrm{Ran}(X)$, while the concept of a lax module is tied to the specific diagram $X^{\mathscr{S}}$ representing $\mathrm{Ran}(X)$. Nevertheless, most of our constructions can and will be performed directly on $X^{\mathscr{S}}$.

D. Strictification of lax $\mathcal{D}^!$-modules. We have the full embedding

$$\underline{\underline{\mathrm{Mod}}}^!_{\mathcal{D}}(X^{\mathscr{S}}) \hookrightarrow \mathrm{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}}).$$

The left adjoint functor to this embedding will be called the *strictification* and will be denoted $E \mapsto \underline{\underline{E}}$. Its existence can be guaranteed on general grounds, see [GL] (5.2). Here we give an explicit formula for it.

Definition 3.2.7. Let $E = \big(E^{(I)}, \beta_g \colon E^{(J)} \to \delta_g^! E^{(I)}\big)$ be a lax $\mathcal{D}^!$-module E on $X^{\mathscr{S}}$. Its *strictification* is the strict $\mathcal{D}^!$-module $\underline{\underline{E}}$ on $X^{\mathscr{S}}$ defined by

$$\underline{\underline{E}}^{(I)} \;=\; \underset{K\in\mathscr{S}}{\underrightarrow{\mathrm{holim}}}(p_{IK})_*\, q_{IK}^! E^{(K)},$$

where p_{IK} and q_{IK} are the canonical projections of the variety $\Delta(I,K)$, see (3.1.5). The structure map

$$\underline{\underline{\beta}}_g : \underline{\underline{E}}^{(J)} \xrightarrow{\simeq} \delta_g^! \underline{\underline{E}}^{(I)}, \quad g : I \twoheadrightarrow J$$

comes from identification of the target with

$$\begin{aligned} \underset{K}{\underrightarrow{\mathrm{holim}}}\, \delta_g^!(p_{IK})_* q_{IK}^!\, E^{(K)} \;&\simeq\; \underset{K}{\underrightarrow{\mathrm{holim}}}\, (p_{JK})_*\, \Delta(g,K)^!\, q_{IK}^!\, E^{(K)} \;\simeq \\ &\simeq\; \underset{K}{\underrightarrow{\mathrm{holim}}}\, (p_{JK})_*\, q_{JK}^!\, E^{(K)} \;=\; \underline{\underline{E}}^{(I)}, \end{aligned}$$

where we used the base change theorem for the Cartesian square in Proposition 3.1.7 as well as the commutativity of the triangle there.

E. Strictification of lax $[[\mathcal{D}]]$-modules. The theory here is parallel to the $\mathcal{D}^!$-module case.

Definition 3.2.8. Let $F = \big(F_I, \gamma_g : \delta_g^{[[*]]} F_I \to F_J\big)$ be a lax $[[\mathcal{D}]]$-module on $X^{\mathscr{S}}$. Its *strictification* is the strict $[[\mathcal{D}]]$-module $\underline{\underline{F}}$ on $X^{\mathscr{S}}$ defined by

$$\underline{\underline{F}}_I \;=\; \underset{K\in\mathscr{S}}{\underleftarrow{\mathrm{holim}}}(p_{IK})_{[[!]]}\, q_{IK}^{[[*]]}\, F_K,$$

with the structure map

$$\underline{\underline{\gamma}}_g : \delta_g^{[[*]]} \underline{\underline{F}}_I \xrightarrow{\simeq} \underline{\underline{F}}_J, \quad g : I \twoheadrightarrow J$$

induced by the base change in the square of Proposition 3.1.7

The functor $F \mapsto \underline{\underline{F}}$ is right adjoint to the embedding $\underline{\underline{\mathrm{Mod}}}_{[[\mathcal{D}]]}(X^{\mathscr{S}}) \hookrightarrow \mathrm{Mod}_{[[\mathcal{D}]]}(X^{\mathscr{S}})$.

F. Factorization homology.

Definition 3.2.9. (a) We call the *factorization homology* (resp. *compactly supported factorization homology*) of a (lax) $\mathcal{D}^!$-module E on $X^{\mathscr{S}}$ the complex of $\mathbf{k}$-vector spaces

$$\int_X E \ := \ \underset{I\in\mathscr{S}}{\underrightarrow{\operatorname{holim}}} \ R\Gamma_{\mathrm{DR}}(X^I, E^{(I)}) \ \in \ \operatorname{Ind}(\operatorname{Perf}_{\mathbf{k}}) = C(\mathbf{k}),$$
$$\int_X^{[[c]]} E \ := \ \underset{I\in\mathscr{S}}{\underrightarrow{\operatorname{holim}}} \ R\Gamma_{\mathrm{DR}}^{[[c]]}(X^I, E^{(I)}) \ \in \ \operatorname{Ind}(\operatorname{Pro}(\operatorname{Perf}_{\mathbf{k}})).$$

Thus $\int_X E$ is just a (possibly infinite-dimensional) complex of $\mathbf{k}$-vector spaces, as $\operatorname{Ind}(\operatorname{Perf}_{\mathbf{k}}) = C(\mathbf{k})$ is the category of all chain complexes over $\mathbf{k}$. On the other hand, $\int_X^{[[c]]} E$ is an ind-pro-finite-dimensional complex.

Definition 3.2.10. We call the *compactly supported factorization cohomology* (resp. *factorization homology*) of a (lax) $[[\mathcal{D}]]$-module F on $X^{\mathscr{S}}$ the complexes

$$\oint_X^{[[c]]} F \ := \ \underset{I\in\mathscr{S}}{\underleftarrow{\operatorname{holim}}} \ R\Gamma_{\mathrm{DR}}^{[[c]]}(X^I, F_I) \ \in \ \operatorname{Pro}(\operatorname{Perf}_{\mathbf{k}}),$$
$$\oint_X F \ := \ \underset{I\in\mathscr{S}}{\underleftarrow{\operatorname{holim}}} \ R\Gamma_{\mathrm{DR}}(X^I, F_I) \ \in \ \operatorname{Pro}(\operatorname{Ind}(\operatorname{Perf}_{\mathbf{k}})).$$

Those constructions define exact functors between stable ∞-categories:

$$\int_X : \operatorname{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}}) \longrightarrow C(\mathbf{k}),$$
$$\int_X^{[[c]]} : \operatorname{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}}) \longrightarrow \operatorname{Ind}(\operatorname{Pro}(\operatorname{Perf}_{\mathbf{k}})),$$
$$\oint_X^{[[c]]} : \operatorname{Mod}_{[[\mathcal{D}]]}(X^{\mathscr{S}}) \longrightarrow \operatorname{Pro}(\operatorname{Perf}_{\mathbf{k}}),$$
$$\oint_X : \operatorname{Mod}_{[[\mathcal{D}]]}(X^{\mathscr{S}}) \longrightarrow \operatorname{Pro}(\operatorname{Ind}(\operatorname{Perf}_{\mathbf{k}})).$$

Proposition 3.2.11. *All four types of factorization homology are unchanged* *under strictification, i.e., we have*

$$\int_X E \simeq \int_X \underline{\underline{E}}, \qquad \int_X^{[[c]]} E \simeq \int_X^{[[c]]} \underline{\underline{E}}, \qquad E \in \mathrm{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}}),$$
$$\oint_X^{[[c]]} F \simeq \oint_X^{[[c]]} \underline{\underline{F}}, \qquad \oint_X F \simeq \oint_X \underline{\underline{F}}, \qquad F \in \mathrm{Mod}_{[[\mathcal{D}]]}(X^{\mathscr{S}}).$$

Proposition 3.2.12. *Verdier duality (on each X^I) induces equivalences*

$$(-)^\vee : \mathrm{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}})^{\mathrm{op}} \simeq \mathrm{Mod}_{[[\mathcal{D}]]}(X^{\mathscr{S}}) \quad \textit{and} \quad \underline{\underline{\mathrm{Mod}}}^!_{\mathcal{D}}(X^{\mathscr{S}})^{\mathrm{op}} \simeq \underline{\underline{\mathrm{Mod}}}_{[[\mathcal{D}]]}(X^{\mathscr{S}}),$$

compatible with the inclusion functors. Moreover, there are natural equivalences

$$\oint_X^{[[c]]} (-)^\vee \simeq \left(\int_X - \right)^*, \qquad \int_X^{[[c]]} (-)^\vee \simeq \left(\oint_X - \right)^*.$$

3.3 Covariant Verdier duality and the diagonal filtration

The content of this section is inspired from [GL].

A. For (lax) $\mathcal{D}^!$-modules. We denote by $\mathrm{Coh}^!_{\mathcal{D}}(X^{\mathscr{S}})$ the full subcategory of $\mathrm{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}})$ spanned by lax $\mathcal{D}^!$-modules $E = (E^{(I)})$ such that each $E^{(I)}$ is coherent as a $\mathcal{D}$-module over X^I.

Definition 3.3.1. We call the *covariant Verdier duality* the functor

$$\phi : \mathrm{Coh}^!_{\mathcal{D}}(X^{\mathscr{S}}) \longrightarrow \underline{\underline{\mathrm{Mod}}}_{[[\mathcal{D}]]}(X^{\mathscr{S}}), \quad (\phi(E))_I = \underset{K \in \mathscr{S}}{\underrightarrow{\mathrm{holim}}}\, (p_{IK})_* \, q_{IK}^{[[*]]} \, E^{(K)},$$

where p_{IK} nad q_{IK} are the projections of the subvariety $\Delta(I,K) \subset X^I \times X^K$.

Let us explain the definition in more details.

<u>*The diagram to take the colimit.*</u> Let $h : K' \to K$ be a surjection. We then have a diagram similar to that of Proposition 3.1.7:

$$\begin{array}{ccccc} & & \Delta(I,K') & \xrightarrow{q_{IK'}} & X^{K'} \\ & \swarrow^{p_{IK'}} & \uparrow{\scriptstyle \Delta(I,h)} & & \uparrow{\scriptstyle \delta_h} \\ X^I & \xleftarrow[p_{IK}]{} & \Delta(I,K) & \xrightarrow[q_{IK}]{} & X^K. \end{array}$$

Consider the structure morphism $\beta_h \colon E^{(K)} \to \delta_h^! E^{(K')}$. It corresponds to an adjoint morphism $(\delta_h)_* E^{(K)} \to E^{(K)}$ in $D(\mathrm{Coh}_{\mathcal{D}_{X^{K'}}})$. Applying the functor $(p_{IK'})_* \, q_{IK'}^{[[*]]}$ we find $(p_{IK'})_* \, q_{IK'}^{[[*]]} \, (\delta_h)_* E^{(K)} \to (p_{IK'})_* \, q_{IK'}^{[[*]]} E^{(K)}$. Using base change for the above pullback square, we find

$$(p_{IK'})_* \, q_{IK'}^{[[*]]} \, (\delta_h)_* E^{(K)} \simeq (p_{IK'})_* \, \Delta(I,h)_* \, q_{IK}^{[[*]]} E^{(K)} \simeq (p_{IK})_* \, q_{IK}^{[[*]]} E^{(K)}.$$

We get the natural morphism

$$(p_{IK})_* \, q_{IK}^{[[*]]} E^{(K)} \to (p_{IK'})_* \, q_{IK'}^{[[*]]} E^{(K)}$$

that is used to form the diagram over which we take the homotopy colimit.

<u>*Structure maps for $\phi(E)$ and its strictness.*</u> Let $g : I \to J$ be a surjection. The structure map

$$\begin{aligned}\gamma_g : \delta_g^{[[*]]} \phi(E)_I \; &= \; \underset{\overrightarrow{K}}{\mathrm{holim}} \, \delta_g^{[[*]]} (p_{IK})_* \, q_{IK}^{[[*]]} \, E^{(K)} \longrightarrow \\ &\longrightarrow \underset{\overrightarrow{K}}{\mathrm{holim}} (p_{JK})_* \, q_{JK}^{[[*]]} E^{(K)} \; = \; \phi(E)_J\end{aligned}$$

is obtained by contemplating the diagram in Proposition 3.1.7 which gives, for each K, the identification

$$\delta_g^{[[*]]} (p_{IK})_* \, q_{IK}^{[[*]]} \, E^{(K)} \overset{\mathrm{BC}}{\simeq} (p_{JK})_* \, \Delta(g,K)^{[[*]]} q_{IK}^{[[*]]} \, E^{(K)} \simeq (p_{JK})_* \, q_{JI}^{[[*]]} E^{(K)}.$$

Here BC is the base change identification, where we note that p_{IK} ansd p_{JK} are finite morphisms so the [[!]]-direct image for them coincides with the usual $*$-direct image.

The above identifications also show that each γ_g is an equivalence, and so $\phi(E)$ is a strict [[$\mathcal{D}$]]-module.

Remarks 3.3.2. (a) We restrict to coherent !-sheaves so that $q_{IK}^{[[*]]} E^{(K)}$ is defined in pro-coherent $\mathcal{D}$-modules.

(b) Note the similarity with Definition 3.2.7 of $\underline{\underline{E}}$, the strictification of E.

B. Topological approximation and diagonal filtration. Recall the following general fact. Let $\mathcal{I}, \mathcal{K}$ be two small categories and $(A_{IK})_{I \in \mathcal{I}, K \in \mathcal{K}}$ be a bi-diagram in an ∞-category $\mathscr{C}$, i.e. an ∞-functor $\mathcal{I}\times\mathcal{K}\to\mathscr{C}$. Then there is a canonical morphism

$$\text{can} : \underset{K\in\mathcal{K}}{\underrightarrow{\text{holim}}}\ \underset{I\in\mathcal{I}}{\underleftarrow{\text{holim}}}\ A_{IK} \longrightarrow \underset{I\in\mathcal{I}}{\underleftarrow{\text{holim}}}\ \underset{K\in\mathcal{K}}{\underrightarrow{\text{holim}}}\ A_{IK}. \tag{3.3.3}$$

We apply this to the case when $\mathcal{I}=\mathcal{K}=\mathscr{S}$ and $\mathscr{C}=\text{Pro}(C(k))$.

Let $E\in\text{Coh}^!_{\mathcal{D}}(X^{\mathscr{S}})$ be a coherent lax $\mathcal{D}^!$-module. We then have natural maps

$$\tau : \int_X E \longrightarrow \oint_X \phi(E), \qquad \tau_c : \int_X^{[[c]]} E \longrightarrow \oint_X^{[[c]]} \phi(E), \tag{3.3.4}$$

which we call the *topological approximation* maps and which are defined as follows. Using the standard map

$$\sigma : R\Gamma_{\text{DR}}(X^K, E^{(K)}) \longrightarrow \underset{I}{\underleftarrow{\text{holim}}}\ R\Gamma_{\text{DR}}(\Delta(I,K), q_{IK}^{[[*]]}E^{(K)}), \tag{3.3.5}$$

we first map

$$\int_X E \;=\; \underset{K}{\underrightarrow{\text{holim}}}\, R\Gamma_{\text{DR}}(X^K, E^{(K)}) \to \underset{K}{\underrightarrow{\text{holim}}}\ \underset{I}{\underleftarrow{\text{holim}}}\ R\Gamma_{\text{DR}}(\Delta(I,K), q_{IK}^{[[*]]}E^{(K)})$$

and then map the target by the canonical map (3.3.3) to

$$\begin{aligned} &\underset{I}{\underleftarrow{\text{holim}}}\ \underset{K}{\underrightarrow{\text{holim}}}\ R\Gamma_{\text{DR}}(\Delta(I,K), q_{IK}^{[[*]]}E^{(K)}) \\ &\qquad \simeq\ \underset{I}{\underleftarrow{\text{holim}}}\ \underset{K}{\underrightarrow{\text{holim}}}\ R\Gamma_{\text{DR}}(X^I, (p_{IK})_*\, q_{IK}^{[[*]]}E^{(K)}) \\ &\qquad \simeq\ \underset{I}{\underleftarrow{\text{holim}}}\ R\Gamma_{\text{DR}}(X^I,\, \phi(E)_I) \;=\; \oint_X \phi(E). \end{aligned}$$

This gives τ. The map τ_c is defined similarly by noticing that q_{IK} being finite, we have a map σ_c analogous to σ and featuring compactly supported de Rham cohomology.

For any positive integer d, we denote by i_d the pointwise closed embedding $X_d^{\mathscr{S}}\to X^{\mathscr{S}}$, see Definition 3.1.2.

Definition 3.3.6. We define

$$\int_X^{\leqslant d} E \ := \ \underleftarrow{\operatorname{holim}}_I R\Gamma_{\mathrm{DR}}(X^I_d, i_d^{[[*]]} E^{(I)}) \ \in \ \operatorname{Pro}(C(\mathbf{k})),$$
$$\int_X^{[[c]], \leqslant d} E \ := \ \underleftarrow{\operatorname{holim}}_I R\Gamma_{\mathrm{DR}}^{[[c]]}(X^I_d, i_d^{[[*]]} E^{(I)}) \ \in \ \operatorname{Pro}(\operatorname{Perf}_{\mathbf{k}}),$$

and call them the *factorization homology* (resp. *compactly supported factorization homology*) *of arity at most* d of E.

Remark that $\int_X^{\leqslant d} E$ (and similarly for its compactly supported analog) can be seen as the factorization homology of a lax pro-$\mathcal{D}^!$-module $(i_d)_* i_d^{[[*]]} E$.

As d varies, they fit into sequences

$$\int_X E \longrightarrow \quad \cdots \to \int_X^{\leqslant d} E \to \cdots \to \int_X^{\leqslant 1} E,$$
$$\int_X^{[[c]]} E \longrightarrow \quad \cdots \to \int_X^{[[c]], \leqslant d} E \to \cdots \to \int_X^{[[c]], \leqslant 1} E.$$

Lemma 3.3.7. *For* $E \in \operatorname{Coh}^!_{\mathcal{D}}(X^{\mathscr{S}})$, *there are canonical equivalences*

$$\oint_X \phi(E) \ \simeq \ \underleftarrow{\operatorname{holim}}_d \int_X^{\leqslant d} E, \qquad \oint_X^{[[c]]} \phi(E) \ \simeq \ \underleftarrow{\operatorname{holim}}_d \int_X^{[[c]], \leqslant d} E.$$

Proof: This is a straightforward formal computation on limits. □

C. For (lax) $[[\mathcal{D}]]$-modules. The above constructions admit a dual version, for $[[\mathcal{D}]]$-modules. We sketch it briefly, formulating statements but omitting details.

We denote by $\operatorname{Coh}_{[[\mathcal{D}]]}(X^{\mathscr{S}})$ the category of lax $[[\mathcal{D}]]$-modules

$$F = \big((F_I, (\gamma_g : \delta_g^{[[*]]} F_I \longrightarrow F_J)_{g: I \twoheadrightarrow J}\big)$$

on $X^{\mathscr{S}}$ such that each F_I is coherent on X^I.

Definition 3.3.8. We call the *covariant Verdier duality* for $[[\mathcal{D}]]$-modules the functor

$$\psi : \operatorname{Coh}_{[[\mathcal{D}]]}(X^{\mathscr{S}}) \longrightarrow \underline{\underline{\operatorname{Mod}}}^!_{\mathcal{D}}(X^{\mathscr{S}})$$

defined by

$$\psi(F)^{(I)} = \underleftarrow{\operatorname{holim}}_K (p_{IK})_* \, q_{IK}^! \, F_K.$$

We have canonical *topological approximation* maps

$$\tau : \int_X \psi(F) \longrightarrow \oint_X F, \qquad \tau_c : \int_X^{[[c]]} \psi(F) \longrightarrow \oint_X^{[[c]]} F. \tag{3.3.9}$$

For any lax coherent $[[\mathcal{D}]]$-module F on $X^{\mathscr{S}}$, we denote by $\underline{R\Gamma}_{X_d^{\mathscr{S}}}(F)$ the lax ind-$[[\mathcal{D}]]$-module $(i_d)_* i_d^! F$ on $X^{\mathscr{S}}$.

Definition 3.3.10. We define the *(compactly supported) factorization cohomology of F with d-fold support* as the (compactly supported) factorization cohomology of $\underline{R\Gamma}_{X_d^{\mathscr{S}}}(F)$. We denote these cohomologies by

$$\begin{aligned} R\Gamma_{X_d^{\mathscr{S}},\mathrm{DR}}(X^{\mathscr{S}}, F) &:= \oint_X \underline{R\Gamma}_{X_d^{\mathscr{S}}}(F) \in C(\mathbf{k}), \\ R\Gamma^{[[c]]}_{X_d^{\mathscr{S}},\mathrm{DR}}(X^{\mathscr{S}}, F) &:= \oint_X^{[[c]]} \underline{R\Gamma}_{X_d^{\mathscr{S}}}(F) \in \mathrm{Ind}(\mathrm{Pro}(\mathrm{Perf}_{\mathbf{k}})). \end{aligned}$$

We call the sequences

$$\begin{aligned} &R\Gamma_{X_1^{\mathscr{S}},\mathrm{DR}}(X^{\mathscr{S}}, F) \to \cdots \to R\Gamma_{X_d^{\mathscr{S}},\mathrm{DR}}(X^{\mathscr{S}}, F) \to \cdots, \\ &R\Gamma^{[[c]]}_{X_1^{\mathscr{S}},\mathrm{DR}}(X^{\mathscr{S}}, F) \to \cdots \to R\Gamma^{[[c]]}_{X_d^{\mathscr{S}},\mathrm{DR}}(X^{\mathscr{S}}, F) \to \cdots, \end{aligned}$$

the *diagonal filtration* on the (compactly supported) factorization cohomology of F.

Lemma 3.3.11. *There are canonical equivalences*

$$\begin{aligned} \underset{d}{\underrightarrow{\mathrm{holim}}}\, R\Gamma_{X_d^{\mathscr{S}},\mathrm{DR}}(X^{\mathscr{S}}, F) &\simeq \int_X \psi(F), \\ \underset{d}{\underrightarrow{\mathrm{holim}}}\, R\Gamma^{[[c]]}_{X_d^{\mathscr{S}},\mathrm{DR}}(X^{\mathscr{S}}, F) &\simeq \int_X^{[[c]]} \psi(F). \end{aligned}$$

D. Compatibility with the usual (contravariant) Verdier duality. We start with a lax coherent $\mathcal{D}^!$-module E on $X^{\mathscr{S}}$. As Verdier duality exchanges $i^{[[*]]}$ and $i^!$ and commutes with i_* for a closed immersion i, we get

Proposition 3.3.12. *There are canonical equivalences*

$$R\Gamma^{[[c]]}_{X_d^{\mathscr{S}},\mathrm{DR}}(X^{\mathscr{S}}, E^\vee) \simeq \left(\int_X^{\leqslant d} E\right)^*, \quad R\Gamma_{X_d^{\mathscr{S}},\mathrm{DR}}(X^{\mathscr{S}}, E^\vee) \simeq \left(\int_X^{[[c]],\leqslant d} E\right)^*.$$

Moreover, it is compatible with the transition maps (when d varies) and with the map to

$$\oint_X^{[[c]]} E^\vee \simeq \left(\int_X E\right)^*, \qquad \oint_X E^\vee \simeq \left(\int_X^{[[c]]} E\right)^*.$$

4 Factorization algebras

The goal of this chapter is to introduce factorization algebras in the algebro-geometric setting (due to [BD] in the 1-dimensional case and to [FG] in general). Those are structured (lax) $\mathcal{D}^!$-modules (or, in our non-holonomic formalism, $[[\mathcal{D}]]$-modules). We will then prove that the covariant Verdier duality introduced in the previous chapter preserves this structure. To achieve that, we will need various equivalent definitions of factorization algebras.

4.1 Symmetric monoidal ∞-categories

A. Reminders. Let $\mathscr{S}^*$ be the category of pointed finite sets of cardinal at least 2, with pointed surjections between them. For any non-empty finite set I, we denote by I^* the set $I \amalg \{*\}$ pointed at $*$. For any $i \in I$, we denote by p_i the map $I^* \to \{i\}^*$ given by

$$p_i(j) = \begin{cases} i & \text{if } j = i \\ * & \text{else.} \end{cases}$$

A non-unital symmetric monoidal structure on an ∞-category $\mathcal{C}$ is the datum of a functor $\underline{\mathcal{C}} \colon \mathscr{S}^* \to \mathrm{Cat}_\infty$ such that $\underline{\mathcal{C}}(\{1\}^*) \simeq \mathcal{C}$ and such that for any I, the functor

$$\underline{\mathcal{C}}\left(\prod p_i\right) : \underline{\mathcal{C}}(I^*) \to \prod_{i \in I} \underline{\mathcal{C}}(\{i\}^*)$$

is an equivalence. The tensor product on $\mathcal{C}$ is then the functor

$$\mathcal{C} \times \mathcal{C} \simeq \underline{\mathcal{C}}(\{1,2\}^*) \longrightarrow \underline{\mathcal{C}}(\{1\}^*) \simeq \mathcal{C}$$

induced by $\{1,2\} \to \{1\}$ mapping both elements to 1.

As is done in [Lu-HA], we can encode monoidal categories by their Grothendieck constructions:

$$\mathcal{C}^{\otimes} := \overline{\mathscr{G}}(\underline{\mathcal{C}}) \to \mathscr{S}^*$$
$$\mathcal{C}_{\otimes} := \mathscr{G}(\underline{\mathcal{C}}) \to (\mathscr{S}^*)^{\mathrm{op}}.$$

A monoidal functor between two (non-unital) symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a natural transformation $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$. Equivalently, it corresponds to a functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ over $\mathscr{S}^*$ that preserves coCartesian morphisms, and to a functor $\mathcal{C}_{\otimes} \to \mathcal{D}_{\otimes}$ over $(\mathscr{S}^*)^{\mathrm{op}}$ that preserves Cartesian morphisms.

Definition 4.1.1. A lax monoidal functor $\mathcal{C} \to \mathcal{D}$ is a functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ over $\mathscr{S}^*$ such that any coCartesian morphism in $\mathcal{C}^{\otimes}$ that lies over a projection $p_i \colon \{I\}^* \to \{i\}^*$ is mapped to a coCartesian morphism in $\mathcal{D}^{\otimes}$.

A colax monoidal functor $\mathcal{C} \to \mathcal{D}$ is a functor $\mathcal{C}_{\otimes} \to \mathcal{D}_{\otimes}$ over $(\mathscr{S}^*)^{\mathrm{op}}$ such that any Cartesian morphism in $\mathcal{C}_{\otimes}$ that lies over a projection $p_i \colon \{I\}^* \to \{i\}^*$ is mapped to a Cartesian morphism in $\mathcal{D}_{\otimes}$.

It follows from the above definitions that a monoidal functor is in particular both a lax and colax monoidal functor.

Informally, a functor $F \colon \mathcal{C} \to \mathcal{D}$ is lax (resp. colax) monoidal if we have functorial morphisms

$$F(c_1) \otimes F(c_2) \longrightarrow F(c_1 \otimes c_2)$$
$$(\text{resp. } F(c_1 \otimes c_2) \longrightarrow F(c_1) \otimes F(c_2)\ ).$$

It is monoidal if those morphisms are equivalences.

B. Day convolution product: Fix two (non-unital) symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$. The (right) Day convolution product is a (non-unital) symmetric monoidal structure on the category of functors $\mathcal{C} \to \mathcal{D}$. It is given on two functors F and $G \colon \mathcal{C} \to \mathcal{D}$ by the formula

$$(F \otimes G)(c) = \underset{c_1 \otimes c_2 \to c}{\underleftarrow{\mathrm{holim}}}\, F(c_1) \otimes G(c_2).$$

Note that the existence of the right Day convolution is not automatic and relies on the existence of such limits. For a complete account on the Day convolution in the context of ∞-categories, we refer to [Gla].

Remark 4.1.2. In [Gla], the author develops the left Day convolution which is dual to the right one that we are considering. The left convolution (if it exists) is given by the formula

$$(F \otimes G)(c) = \underset{c \to c_1 \otimes c_2}{\underrightarrow{\operatorname{holim}}} F(c_1) \otimes G(c_2).$$

Definition 4.1.3. Let $\mathcal{C}$ be a (non-unital) symmetric monoidal category. We say that $\mathcal{C}$ is of finite decompositions if the following hold

(a) For any $c, c_1, c_2 \in \mathcal{C}$, any morphism $c_1 \otimes c_2 \to c$ factors essentially uniquely as

$$c_1 \otimes c_2 \xrightarrow{f_1 \otimes f_2} d_1 \otimes d_2 \xrightarrow{\sim} c.$$

(b) For any object $c \in \mathcal{C}$, the number of decompositions $c \simeq d_1 \otimes d_2$ is essentially finite.

Proposition 4.1.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal categories. Assume that $\mathcal{C}$ is of finite decompositions, and that $\mathcal{D}$ admits finite products (resp. coproducts). Then the right (resp. left) Day convolution on the category of functors $\mathcal{C} \to \mathcal{D}$ exists.*

If moreover $\mathcal{D}$ admits finite sums (ie finite products and coproducts coincide), then the left and the right Day convolution products coincide. In this case, we call it the Day convolution.

Proof. We fix $c \in \mathcal{C}$ and denote by $\mathcal{C} \times \mathcal{C}/c$ the category of triples (c_1, c_2, f) where $c_1, c_2 \in \mathcal{C}$ and $f\colon c_1 \otimes c_2 \to c$. We denote by $\otimes^{-1}(c)$ the full subcategory of $\mathcal{C} \times \mathcal{C}/c$ spanned by triples as above for which f is an equivalence.

Because $\mathcal{C}$ is of finite decompositions, the inclusion functor $\otimes^{-1}(c) \subset \mathcal{C} \times \mathcal{C}/c$ is cofinal and the category $\otimes^{-1}(c)$ is equivalent to a finite set $S(c)$. In particular, if it exists, the right Day convolution of two functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is given, for every $c \in \mathcal{C}$, by the finite product

$$\prod_{[d_1, d_2] \in S(c)} F(d_1) \otimes G(d_2).$$

The result follows. □

Finally, we recall the following key property of the Day convolution.

Theorem 4.1.5 (see [Gla, Prop. 2.12])**.** *An algebra for the left Day convolution product is tantamount to a lax monoidal functor. A coalgebra for the right Day convolution product is tantamount to a colax monoidal functor.*

C. Monoidal functors and Grothendieck constructions:

Proposition 4.1.6. *Let $\mathcal{F}\colon \mathcal{C} \to \mathrm{Cat}_\infty$ be a non-unital lax monoidal functor. The coCartesian Grothendieck construction $\overline{\mathscr{G}}\mathcal{F}$ is endowed with a natural non-unital symmetric monoidal structure, such that the projection $\overline{\mathscr{G}}\mathcal{F} \to \mathcal{C}$ is a symmetric monoidal functor.*

The Cartesian Grothendieck construction $\mathscr{G}\mathcal{F}$ is also endowed with a natural non-unital symmetric monoidal structure compatible with the projection $\mathscr{G}\mathcal{F} \to \mathcal{C}^{\mathrm{op}}$.

Proof. We first deal with the coCartesian case. The ∞-category Cat_∞ is endowed with its cartesian symmetric monoidal structure. In particular, the ∞-category of lax monoidal functors $\mathcal{C} \to \mathrm{Cat}_\infty$ embeds fully faithfully into the category $\mathrm{Fun}(\mathcal{C}^{\otimes}, \mathrm{Cat}_\infty)$ (see [Lu-HA, Prop. 2.4.1.7]). Let $\tilde{\mathcal{F}}$ denote the image of $\mathcal{F}$ under this inclusion.

We can now apply the coCartesian Grothendieck construction and obtain a coCartesian fibration

$$\overline{\mathscr{G}}\tilde{\mathcal{F}} \to \mathcal{C}^{\otimes} \to \mathscr{S}^*.$$

The fact that it defines a non-unital symmetric monoidal structure on $\overline{\mathscr{G}}\mathcal{F}$ compatible with the projection follows directly from [Lu-HA, Prop. 2.4.1.7].

We can now focus on the Cartesian case. We consider the functor $\mathcal{F}^{\mathrm{op}}$ mapping $c \in \mathcal{C}$ to $\mathcal{F}(c)^{\mathrm{op}}$. It inherits from $\mathcal{F}$ the lax monoidal structure and we can thus apply the above. We get on $\overline{\mathscr{G}}\mathcal{F}^{\mathrm{op}}$ a monoidal structure compatible with the projection $\overline{\mathscr{G}}\mathcal{F}^{\mathrm{op}} \to \mathcal{C}$. We now observe the equivalence

$$\mathscr{G}\mathcal{F} \simeq \left(\overline{\mathscr{G}}\mathcal{F}^{\mathrm{op}}\right)^{\mathrm{op}}.$$

and conclude. □

Corollary 4.1.7. *Let $\mathcal{F}\colon \mathcal{C} \to \mathrm{Cat}_\infty$ be a non-unital lax-monoidal functor. Assume that $\mathcal{C}$ is of finite decompositions and that for any $c \in \mathcal{C}$, the category $\mathcal{F}(c)$ admits finite direct sums. Then the Day convolution (either left or right, equivalently) defines a non-unital monoidal structure on both $\underleftarrow{\mathrm{lax}}\,\mathcal{F}$ and $\underleftarrow{\mathrm{lax}}^{\circ}\,\mathcal{F}$.*

Proof. Follows directly from Proposition 4.1.4. □

Proposition 4.1.8. *Given two lax monoidal functors $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathrm{Cat}_\infty$ and a lax monoidal transformation $\mathcal{F} \Rightarrow \mathcal{G}$, the induced functors*

$$\overline{\mathscr{G}}\mathcal{F} \to \overline{\mathscr{G}}\mathcal{G} \qquad \text{and} \qquad \mathscr{G}\mathcal{F} \to \mathscr{G}\mathcal{G}$$

are lax monoidal. If the natural transformation was (strictly) monoidal, then the induced functors are too.

Sketch of proof. The lax monoidal functors $\mathcal{F}$ and $\mathcal{G}$ correspond to functors $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}} \colon \mathcal{C}^{\otimes} \to \mathrm{Cat}_\infty$. The lax monoidal transformation $\mathcal{F} \Rightarrow \mathcal{G}$ then correspond to a lax natural transformation $\tilde{\mathcal{F}} \Rightarrow \tilde{\mathcal{G}}$ and therefore to a commutative diagram

$$\begin{array}{ccc} \overline{\mathscr{G}}\tilde{\mathcal{F}} & \longrightarrow & \overline{\mathscr{G}}\tilde{\mathcal{G}} \\ & \searrow \quad \swarrow & \\ & \mathcal{C}^{\otimes}. & \end{array}$$

The result follows. The case of the Cartesian Grothendieck construction is done using the duality between the Cartesian and the coCartesian constructions described in [BGS]. □

Corollary 4.1.9. *Let $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathrm{Cat}_\infty$ be lax monoidal functors such that for any $c \in \mathcal{C}$ the categories $\mathcal{F}(c)$ and $\mathcal{G}(c)$ admit finite limits. Assume that $\mathcal{C}$ is of finite decomposition. For any lax monoidal transformation $\mathcal{F} \Rightarrow \mathcal{G}$, the induced functors*

$$\underleftarrow{\mathrm{lax}}^{\circ}\,\mathcal{F} \to \underleftarrow{\mathrm{lax}}^{\circ}\,\mathcal{G} \qquad \text{and} \qquad \underleftarrow{\mathrm{lax}}\,\mathcal{F} \to \underleftarrow{\mathrm{lax}}\,\mathcal{G}$$

are lax monoidal.

Proposition 4.1.10. *Consider two functors $\mathcal{F} \colon \mathcal{C}_1 \to \mathcal{C}_2$ and $\mathcal{G} \colon \mathcal{C}_2 \to \mathrm{Cat}_\infty$. Assume that $\mathcal{F}$ is monoidal, that $\mathcal{G}$ is lax monoidal and that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are of finite decomposition. The restriction functor $\underleftarrow{\mathrm{lax}}\,\mathcal{G} \longrightarrow \underleftarrow{\mathrm{lax}}\,\mathcal{G} \circ \mathcal{F}$ is lax monoidal.*

Proof. Recall the pullback diagram

$$\begin{array}{ccc} \mathscr{G}(\mathcal{G} \circ \mathcal{F}) & \xrightarrow{\underline{\mathcal{F}}} & \mathscr{G}(\mathcal{G}) \\ \downarrow & & \downarrow \\ \mathcal{C}_1 & \xrightarrow[\mathcal{F}]{} & \mathcal{C}_2. \end{array}$$

Since $\mathcal{F}$ is monoidal, so is the projection $\underline{\mathcal{F}}$ and the above diagram is a pullback of symmetric monoidal categories. The result follows. □

4.2 Definition as modules over $X^{\mathscr{S}}$

Introduced for X a curve by Beilinson and Drinfeld [BD], factorization algebras are structured $\mathcal{D}^!$-modules over $\mathrm{Ran}(X)$. Intuitively, a factorization structure on $E = (E^{(I)})$ is the data of compatible equivalences between $E^{(I)}$ and $\boxtimes_{i\in I} E^{(\{i\})}$ once restricted to the complement of the big diagonal in X^I.

In [FG], Francis and Gaitsgory generalized the definition to X of any dimension.

Definition 4.2.1. Let $\alpha\colon I \twoheadrightarrow J$ be a surjection, seen as a partition $I = \coprod_j I_j$ where $I_j = \alpha^{-1}(j)$. We denote by $U(\alpha)$ the open subvariety of X^I whose points are families (x_i) such that the sets $\{x_i, i \in I_j\}$ indexed by $j \in J$ are pairwise disjoint. Equivalently:

$$U(\alpha) = \{(x_i)_{i\in I} \mid \forall i_1, i_2 \in I,\ \alpha(i_1) \neq \alpha(i_2) \implies x_{i_1} \neq x_{i_2}\}.$$

We denote by $j_\alpha\colon U(\alpha) \to X^I$ the open immersion.

Given two finite sets I_1 and I_2, we denote by γ_{I_1,I_2} the surjection $I_1 \amalg I_2 \to \{1,2\}$ mapping points of I_j to j, for $j = 1, 2$.

Definition 4.2.2. (Beilinson-Drinfeld, Francis-Gaitsgory)
The category $\underline{\underline{\mathrm{Mod}}}^!_{\mathcal{D}}(X^{\mathscr{S}})$ admits a tensor product called the chiral tensor product and denoted by $\otimes^{\mathrm{ch}}$, such that

$$\left(E \overset{\mathrm{ch}}{\otimes} F\right)^{(I)} = \bigoplus_{I = I_1 \amalg I_2} j_* j^* (E^{(I_1)} \boxtimes E^{(I_2)})$$

where $j = j_{\gamma_{I_1,I_2}}$.

A factorization algebra E on X is a strict $\mathcal{D}^!$-module endowed with a coalgebra structure $E \to E \otimes^{\mathrm{ch}} E$ such that for any surjection $\alpha\colon I \twoheadrightarrow J$, the morphism induced by the iterated comultiplication

$$j_\alpha^* E^{(I)} \to j_\alpha^* \left(\boxtimes_{j\in J} E^{(I_j)} \right)$$

is an equivalence.

We will first extend the above definition to lax $\mathcal{D}^!$-modules (and to lax $[[\mathcal{D}]]$-modules). Recall the functor $\mathcal{D}^!_{X^{\mathscr{S}}} : \mathscr{S} \to \mathrm{Cat}_\infty$ mapping I to $D(\mathrm{QCoh}_{\mathcal{D},X^I})$

and a map to the associated !-pullback functor. It follows from [FG, §2.2.3] that $\mathcal{D}^!_{X^{\mathscr{S}}}$ is endowed with a lax monoidal structure given by

$$j_*j^*(-\boxtimes -)\colon D(\mathrm{QCoh}_{\mathcal{D},X^{I_1}}) \times D(\mathrm{QCoh}_{\mathcal{D},X^{I_2}}) \longrightarrow D(\mathrm{QCoh}_{\mathcal{D},X^{I_1 \amalg I_2}})$$

for $j = j_{\gamma_{I_1,I_2}}$.

Using Proposition 4.1.6, we get a symmetric monoidal structure on the Grothendieck construction $\mathscr{G}(\mathcal{D}^!_{X^{\mathscr{S}}})$ compatible with the projection

$$p\colon \mathscr{G}(\mathcal{D}^!_{X^{\mathscr{S}}}) \to \mathscr{S}^{\mathrm{op}}.$$

It is given on two objects (I_1, E_1) and (I_2, E_2) in $\mathscr{G}(\mathcal{D}^!_{X^{\mathscr{S}}})$ by the formula

$$(I_1, E_1) \otimes (I_2, E_2) = (I_1 \amalg I_2, j_*j^*(E_1 \boxtimes E_2)).$$

Definition 4.2.3. A lax $\mathcal{D}^!$-factorization algebra (over X) is a lax $\mathcal{D}^!$-module E over $X^{\mathscr{S}}$ seen as a section of p, endowed with a colax monoidal structure, such that for any surjection $\alpha\colon I \twoheadrightarrow J$, the induced morphism

$$j_\alpha^* E^{(I)} \to j_\alpha^* \left(\boxtimes_{j \in J} E^{(I_j)} \right)$$

is an equivalence. We denote by $\mathrm{FA}^!_{\mathcal{D}}(X)$ the category of $\mathcal{D}^!$-factorization algebras over X. We denote by $\underline{\underline{\mathrm{FA}}}^!_{\mathcal{D}}(X)$ the full subcategory of $\mathrm{FA}^!_{\mathcal{D}}(X)$ spanned by lax !-factorization algebras on X whose underlying lax $\mathcal{D}^!$-module is strict.

Remark 4.2.4. The monoidal category $\mathscr{S}$ is of finite decomposition (see Definition 4.1.3 and the categories of $\mathcal{D}$-modules admit finite direct sums. It follows from Corollary 4.1.7 that the category $\mathrm{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}})$ carries a Day convolution product called the chiral tensor structure and denoted by $\otimes^{\mathrm{ch}}$. In particular, a colax monoidal structure on a section E of p as above is tantamount to a coalgebra structure on E for the chiral tensor structure.

Remark 4.2.5. The box-product

$$-\boxtimes -\colon D(\mathrm{QCoh}_{\mathcal{D},X^{I_1}}) \times D(\mathrm{QCoh}_{\mathcal{D},X^{I_2}}) \longrightarrow D(\mathrm{QCoh}_{\mathcal{D},X^{I_1 \amalg I_2}})$$

defines (by the same procedure) another tensor structure on $\mathrm{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}})$ called the $*$-product and denoted by $\otimes^*$.

Lemma 4.2.6. *Let E be a strict $\mathcal{D}^!$-module over $X^{\mathscr{S}}$. The datum of a factorization structure on E in the sense of Definition 4.2.2 is equivalent to the datum of a factorization structure in the sense of Definition 4.2.3.*

Proof. The chiral tensor structure on $\underline{\underline{\mathrm{Mod}}}^!_{\mathcal{D}}(X^{\mathscr{S}})$ corresponds to the Day convolution product on the category of Cartesian sections of p. The result follows. □

Dually, we define (lax) $[[\mathcal{D}]]$-factorization algebras. The functor $\mathcal{D}^{[[*]]}_{X^{\mathscr{S}}}$ mapping I to $\mathrm{Pro}(D^b\mathrm{Coh}_{\mathcal{D},X^I})$ admits a lax monoidal structure given by the formula $j_{[[!]]}j^{[[*]]}(- \boxtimes -)$:

$$\mathrm{Pro}(D^b\mathrm{Coh}_{\mathcal{D},X^{I_1}}) \times \mathrm{Pro}(D^b\mathrm{Coh}_{\mathcal{D},X^{I_2}}) \longrightarrow \mathrm{Pro}(D^b\mathrm{Coh}_{\mathcal{D},X^{I_1\amalg I_2}})$$

with $j = j_{I_1\amalg I_2 \twoheadrightarrow \{1,2\}}$. It follows from Proposition 4.1.6 that the (coCartesian) Grothendieck construction $\overline{\mathscr{G}}(\mathcal{D}^{[[*]]}_{X^{\mathscr{S}}})$ admits a symmetric monoidal structure compatible with the projection

$$q\colon \overline{\mathscr{G}}(\mathcal{D}^{[[*]]}_{X^{\mathscr{S}}}) \to \mathscr{S}$$

Definition 4.2.7. A lax $[[\mathcal{D}]]$-factorization algebra is a lax $[[\mathcal{D}]]$-module, seen as a section of q, endowed with a lax monoidal structure, such that for any surjection $\alpha\colon I \twoheadrightarrow J$, the induced morphism

$$j^{[[*]]}_{\alpha}\left(\boxtimes_{j\in J} E^{(I_j)}\right) \to j^{[[*]]}_{\alpha} E^{(I)}$$

is an equivalence.

We denote by $\mathrm{FA}_{[[\mathcal{D}]]}(X)$ (resp. $\underline{\underline{\mathrm{FA}}}_{[[\mathcal{D}]]}(X)$) the category of lax (resp. strict) $[[\mathcal{D}]]$-factorization algebras.

The following is obvious:

Proposition 4.2.8. *Verdier duality induces equivalences*

$$\mathrm{FA}_{[[\mathcal{D}]]}(X)^{\mathrm{op}} \simeq \mathrm{FA}^!_{\mathcal{D}}(X) \qquad \textit{and} \qquad \underline{\underline{\mathrm{FA}}}_{[[\mathcal{D}]]}(X)^{\mathrm{op}} \simeq \underline{\underline{\mathrm{FA}}}^!_{\mathcal{D}}(X)$$

compatible with the equivalence of Proposition 3.2.12.

4.3 Definition in terms of arrow categories

In the above definitions, factorization structures for $\mathcal{D}^!$- and $[[\mathcal{D}]]$-modules are dual to one another. The factorization structure is in one case a coalgebra structure, and in the other case an algebra structure. In order to prove that covariant Verdier duality preserves the factorization structures, we will need alternative models for factorization algebras that do not make a choice between algebras and coalgebras.

Definition 4.3.1. We denote by $\mathscr{S}^1$ the category of arrows in $\mathscr{S}$. A morphism σ from $\alpha\colon I \twoheadrightarrow J$ to $\beta\colon S \twoheadrightarrow T$ in $\mathscr{S}^1$ is a commutative diagram

$$\begin{array}{ccc} I & \xrightarrow{\overrightarrow{\sigma}} & S \\ {\scriptstyle\alpha}\downarrow & \sigma & \downarrow{\scriptstyle\beta} \\ J & \xrightarrow[\underrightarrow{\sigma}]{} & T. \end{array}$$

Disjoint union makes $\mathscr{S}^1$ into a symmetric monoidal category.

A morphism σ as above is an open (resp. a closed) morphism if $\overrightarrow{\sigma}$ (resp. $\underrightarrow{\sigma}$) is a bijection.

Fix a commutative diagram σ as above. We get an open immersion $U(\alpha) \to U(\underrightarrow{\sigma}\alpha)$ and a closed immersion $U(\beta) \to U(\underrightarrow{\sigma}\alpha)$. We denote by $\widehat{U}(f)$ the pullback

(4.3.2)
$$\begin{array}{ccc} \widehat{U}(f) & \xrightarrow{\widehat{\jmath}(\sigma)} & U(\beta) \\ {\scriptstyle\widehat{\imath}(\sigma)}\downarrow & \lrcorner & \downarrow \\ U(\alpha) & \longrightarrow & U(\underrightarrow{\sigma}\alpha). \end{array}$$

Note that the horizontal maps are open immersions and the vertical ones are closed immersions. If σ is an open (resp. a closed) morphism, then $\widehat{\imath}(\sigma)$ (resp. $\widehat{\jmath}(\sigma)$) is an isomorphism.

If it exists, let γ be a surjection $S \twoheadrightarrow J$ such that $\gamma\overrightarrow{\sigma} = \alpha$. Since $\overrightarrow{\sigma}$ is surjective, such a map γ is unique if it exists, and automatically satisfies $\underrightarrow{\sigma}\gamma = \beta$. We then have

$$\widehat{U}(f) := \begin{cases} U(\gamma) & \text{if } \gamma \text{ exists} \\ \varnothing & \text{else.} \end{cases}$$

The assignement $\alpha \mapsto U(\alpha)$, $\sigma \mapsto (U(\beta) \leftarrow \widehat{U}(\sigma) \to U(\alpha))$ defines a functor

$$\widehat{U}\colon (\mathscr{S}^1)^{\mathrm{op}} \to \mathrm{Var}_{\mathbf{k}}^{\mathrm{corr}}.$$

Recall the functor $\mathcal{D}^{\mathrm{corr}}\colon \mathrm{Var}_{\mathbf{k}}^{\mathrm{corr}} \to \mathrm{Cat}_\infty$ mapping a variety Y to $D(\mathrm{QCoh}_{\mathcal{D},Y})$ and a correspondence $Y_1 \leftarrow^a Z \to^b Y_2$ to $b_* a^!$.

Definition 4.3.3. We denote by $\mathcal{D}^{\mathrm{corr}}_{\widehat{U}}$ the composite functor

$$\mathcal{D}^{\mathrm{corr}}_{\widehat{U}} := \mathcal{D}^{\mathrm{corr}} \circ \widehat{U}\colon (\mathscr{S}^1)^{\mathrm{op}} \to \mathrm{Cat}_\infty.$$

The functor $\widehat{U}$ admits a lax-monoidal structure, given by the open immersions $U(\alpha_1 \amalg \alpha_2) \to U(\alpha_1) \times U(\alpha_2)$. Composed with the lax monoidal structure on $\mathcal{D}^{\mathrm{corr}}$, we get a lax monoidal structure on $\mathcal{D}^{\mathrm{corr}}_{\widehat{U}}$ and hence a monoidal structure on $\overline{\mathscr{G}}(\mathcal{D}^{\mathrm{corr}}_{\widehat{U}})$.

Definition 4.3.4. We say that a section of $\overline{\mathscr{G}}(\mathcal{D}^{\mathrm{corr}}_{\widehat{U}}) \to (\mathscr{S}^1)^{\mathrm{op}}$ is openly coCartesian if it sends every open morphisms in $\mathscr{S}^1$ to a coCartesian morphism. We denote by $\mathrm{Mod}^{\mathrm{corr}}_{\mathcal{D}}(\widehat{U})$ the full subcategory of $\underleftarrow{\mathrm{lax}}^\circ\, \mathcal{D}^{\mathrm{corr}}_{\widehat{U}}$ spanned by openly coCartesian sections.

Since $\mathscr{S}^1$ is of finite decomposition, Day convolution endows $\underleftarrow{\mathrm{lax}}^\circ\, \mathcal{D}^{\mathrm{corr}}_{\widehat{U}}$ with a symmetric monoidal structure (see Corollary 4.1.7). This tensor structure preserves openly coCartesian sections and thus defines a monoidal structure on $\mathrm{Mod}^{\mathrm{corr}}_{\mathcal{D}}(\widehat{U})$.

Proposition 4.3.5. *Restriction along the functor $\eta\colon \mathscr{S} \to \mathscr{S}^1$ given by $I \mapsto (I \twoheadrightarrow *)$ induces a symmetric monoidal equivalence*

$$B\colon \mathrm{Mod}^{\mathrm{corr}}_{\mathcal{D}}(\widehat{U}) \simeq \mathrm{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}}).$$

Proof. Denote by $\mathcal{D}^{X^{\mathscr{S}}}_*$ the functor $\mathscr{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ mapping a finite set I to $D(\mathrm{QCoh}_{\mathcal{D},X^I})$ and a surjection $I \to S$ to the associated $*$-pushforward functor

$$D(\mathrm{QCoh}_{\mathcal{D},X^S}) \longrightarrow D(\mathrm{QCoh}_{\mathcal{D},X^I}).$$

Since for a closed immersion δ, the functor $\delta^!$ is right adjoint to δ_*, we have $\mathscr{G}(\mathcal{D}^!_{X^{\mathscr{S}}}) \simeq \overline{\mathscr{G}}(\mathcal{D}^{X^{\mathscr{S}}}_*)$. In particular

$$\mathrm{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}}) := \underleftarrow{\mathrm{lax}}\, \mathcal{D}^!_{X^{\mathscr{S}}} \simeq \underleftarrow{\mathrm{lax}}^\circ\, \mathcal{D}^{X^{\mathscr{S}}}_*.$$

Let s be the source functor $\mathscr{S}^1 \to \mathscr{S}$, mapping a surjection α to its domain. It is a monoidal functor, and thus induces a monoidal functor (by Corollary 4.1.10)

$$\underleftarrow{\mathrm{lax}}^\circ \mathcal{D}_*^{X^{\mathscr{S}}} \to \underleftarrow{\mathrm{lax}}^\circ(\mathcal{D}_*^{X^{\mathscr{S}}} \circ s).$$

The canonical immersions $j_\alpha \colon U(\alpha) \to X^I$ define a monoidal natural transformation $j_\bullet^* \colon \mathcal{D}_*^{X^{\mathscr{S}}} \circ s \to \mathcal{D}_{\widehat{U}}^{\mathrm{corr}}$. In particular, it defines a monoidal functor (see Corollary 4.1.9)

$$\underleftarrow{\mathrm{lax}}^\circ(\mathcal{D}_*^{X^{\mathscr{S}}} \circ s) \to \underleftarrow{\mathrm{lax}}^\circ \mathcal{D}_{\widehat{U}}^{\mathrm{corr}}$$

whose image lies in $\mathrm{Mod}_{\mathcal{D}}^{\mathrm{corr}}(\widehat{U})$. We find a monoidal functor

$$A \colon \mathrm{Mod}_{\mathcal{D}}^!(X^{\mathscr{S}}) \simeq \underleftarrow{\mathrm{lax}}^\circ \mathcal{D}_*^{X^{\mathscr{S}}} \to \underleftarrow{\mathrm{lax}}^\circ(\mathcal{D}_*^{X^{\mathscr{S}}} \circ s) \to \mathrm{Mod}_{\mathcal{D}}^{\mathrm{corr}}(\widehat{U}).$$

The restriction along $\eta \colon \mathscr{S} \to \mathscr{S}^1$ gives a functor

$$B \colon \mathrm{Mod}_{\mathcal{D}}^{\mathrm{corr}}(\widehat{U}) \to \mathrm{Mod}_{\mathcal{D}}^!(X^{\mathscr{S}})$$

which is inverse to A. □

Definition 4.3.6. Let $\mathrm{FA}_{\mathcal{D}}^{\mathrm{corr}}(\widehat{U})$ denote the category of sections in $\mathrm{Mod}_{\mathcal{D}}^{\mathrm{corr}}(\widehat{U})$ endowed with symmetric monoidal structure.

Corollary 4.3.7. *The functor $\mathscr{S} \to \mathscr{S}^1$ mapping I to $I \twoheadrightarrow *$ induces an equivalence*

$$B \colon \mathrm{FA}_{\mathcal{D}}^{\mathrm{corr}}(\widehat{U}) \simeq \mathrm{FA}_{\mathcal{D}}^!(X).$$

Proof. The equivalence B of Proposition 4.3.5 being monoidal, it preserves coalgebras. Coalgebras for the (right) Day convolution are colax monoidal functor (see [Gla, Prop. 2.12]). Let $\mathcal{A} \in \mathrm{FA}_{\mathcal{D}}^{\mathrm{corr}}(\widehat{U})$. It is a section with a monoidal structure, and thus $B(\mathcal{A})$ is is a colax monoidal section. The factorizing property of $B(\mathcal{A})$ corresponds to the fact that $\mathcal{A}$ is (strictly) monoidal. □

The dual statements obviously hold for $[[\mathcal{D}]]$-factorization algebras. Consider the functor $\mathcal{D}_V^{[[\mathrm{corr}]]} \colon (\mathscr{S}^1)^{\mathrm{op}} \to \mathrm{Cat}_\infty$ mapping α to $\mathrm{Pro}\, D^b(\mathrm{Coh}_{\mathcal{D},U(\alpha)})$ and $\sigma \colon \alpha \to \beta$ to $\widehat{\imath}(\sigma)_{[[*]]} \circ \widehat{\jmath}(\sigma)^{[[*]]}$ (using the notations of Eq. (4.3.2)). It is naturally lax monoidal and $\mathscr{G}(\mathcal{D}_V^{[[\mathrm{corr}]]})$ inherits a symmetric monoidal structure.

Definition 4.3.8. A section of $\mathscr{G}(\mathcal{D}_V^{[[\mathrm{corr}]]}) \to \mathscr{S}^1$ is called openly Cartesian if it maps open morphisms of $\mathscr{S}^1$ to Cartesian morphisms.

Denote by $\mathrm{FA}^{\mathrm{corr}}_{[[\mathcal{D}]]}(\widehat{U})$ the category of openly Cartesian symmetric monoidal sections of $\mathscr{G}(\mathcal{D}_V^{[[\mathrm{corr}]]}) \to \mathscr{S}^1$.

Proposition 4.3.9. *Restriction along η induces an equivalence*

$$\mathrm{FA}^{\mathrm{corr}}_{[[\mathcal{D}]]}(\widehat{U}) \simeq \mathrm{FA}_{[[\mathcal{D}]]}(X)$$

compatible through Verdier duality with the equivalence of Corollary 4.3.7.

4.4 Definition in terms of twisted arrows

Definition 4.4.1. We denote by $\mathrm{Tw}(\mathscr{S})$ the category of twisted arrows in $\mathscr{S}$. Its objects are morphisms in $\mathscr{S}$, and its morphisms from $\alpha\colon I \twoheadrightarrow J$ to $\beta\colon S \twoheadrightarrow T$ are commutative diagrams

$$\begin{array}{ccc} I & \xleftarrow{\overleftarrow{\tau}} & S \\ {\scriptstyle\alpha}\downarrow & \tau & \downarrow{\scriptstyle\beta} \\ J & \xrightarrow[\overrightarrow{\tau}]{} & T \end{array}$$

in $\mathscr{S}$. For any surjections $\alpha_1\colon I_1 \to J_1$ and $\alpha_2\colon I_2 \to J_2$, we set

$$\alpha_1 \amalg \alpha_2\colon I_1 \amalg I_2 \twoheadrightarrow J_1 \amalg J_2.$$

It induces on $\mathrm{Tw}(\mathscr{S})$ a symmetric monoidal structure.

Definition 4.4.2. Let $\tau\colon \alpha \to \beta$ be a morphism in $\mathrm{Tw}(\mathscr{S})$ corresponding to a diagram as above. We say that τ is open if the map $\overleftarrow{\tau}$ is a bijection. We say that τ is closed if the map $\overrightarrow{\tau}$ is a bijection.

Consider the diagram $U_{\mathrm{Tw}}\colon \mathrm{Tw}(\mathscr{S}) \to \mathrm{Var}_{\mathbf{k}}$ mapping $\alpha\colon I \to J$ to $U(\alpha)$. It maps a commutative diagram τ as above to the natural immersion $U(\alpha) \to U(\beta)$. Note that it maps open (resp. closed) morphisms in $\mathrm{Tw}(\mathscr{S})$ to open (resp. closed) immersions of varieties. The functor $\mathcal{D}^!_{U_{\mathrm{Tw}}}\colon \mathrm{Tw}(\mathscr{S})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ (recall the notation from Section 3.2 §B.) has a lax monoidal structure, given by

$$j^*(- \boxtimes -)\colon D(\mathrm{QCoh}_{\mathcal{D},U(\alpha_1)}) \times D(\mathrm{QCoh}_{\mathcal{D},U(\alpha_2)}) \to D(\mathrm{QCoh}_{\mathcal{D},U(\alpha_1 \amalg \alpha_2)}).$$

with $j\colon U(\alpha_1 \amalg \alpha_2) \to U(\alpha_1) \times U(\alpha_2)$ the open embedding. We get from Proposition 4.1.6 a symmetric monoidal structure on $\mathscr{G}(\mathcal{D}^!_{U_{\mathrm{Tw}}})$.

Definition 4.4.3. A section of $\mathscr{G}(\mathcal{D}^!_{U_{\mathrm{Tw}}}) \to \mathrm{Tw}(\mathscr{S})^{\mathrm{op}}$ is called openly Cartesian if it maps open morphisms in $\mathrm{Tw}(\mathscr{S})$ to Cartesian morphisms. We denote by $\mathrm{Mod}^{!o}_{\mathcal{D}}(U_{\mathrm{Tw}})$ the full subcategory of $\underleftarrow{\mathrm{lax}}\,\mathcal{D}^!_{U_{\mathrm{Tw}}}$ spanned by openly Cartesian sections. Since $\mathrm{Tw}(\mathscr{S})$ is of finite decomposition, Corollary 4.1.7 defines on $\underleftarrow{\mathrm{lax}}\,\mathcal{D}^!_{U_{\mathrm{Tw}}}$ a Day convolution. The full subcategory $\mathrm{Mod}^{!o}_{\mathcal{D}}(U_{\mathrm{Tw}})$ is stable by this tensor product, and thus inherits a symmetric monoidal structure.

Definition 4.4.4. We denote by $\mathrm{FA}^!_{\mathcal{D}}(U_{\mathrm{Tw}})$ the category of openly Cartesian symmetric monoidal sections of $\mathscr{G}(\mathcal{D}^!_{U_{\mathrm{Tw}}})$. Forgetting the monoidal structure defines a functor

$$\mathrm{FA}^!_{\mathcal{D}}(U_{\mathrm{Tw}}) \longrightarrow \mathrm{Mod}^{!o}_{\mathcal{D}}(U_{\mathrm{Tw}}).$$

Proposition 4.4.5. *Restriction along the functor $\bar{\eta}\colon I \mapsto (I \twoheadrightarrow *)$ induces a symmetric monoidal equivalence of categories*

$$\mathrm{Mod}^{!o}_{\mathcal{D}}(U_{\mathrm{Tw}}) \simeq \mathrm{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}}).$$

It induces an equivalence

$$\mathrm{FA}^!_{\mathcal{D}}(U_{\mathrm{Tw}}) \simeq \mathrm{FA}^!_{\mathcal{D}}(X^{\mathscr{S}}).$$

Proof. Let us denote by $\bar{s}\colon \mathrm{Tw}(\mathscr{S}) \to \mathscr{S}^{\mathrm{op}}$ the monoidal functor mapping a surjection to its domain. The canonical open immersions $j_\alpha\colon U(\alpha) \to X^I$ (for $\alpha\colon I \twoheadrightarrow J$) induce a monoidal natural transformation $j^!_\bullet\colon \mathcal{D}^!_{X^{\mathscr{S}}} \circ s \to \mathcal{D}^!_U$. We find a monoidal functor

$$\mathrm{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}}) := \underleftarrow{\mathrm{lax}}\,\mathcal{D}^!_{X^{\mathscr{S}}} \to \underleftarrow{\mathrm{lax}}\,\mathcal{D}^!_{X^{\mathscr{S}}} \circ \bar{s} \to \underleftarrow{\mathrm{lax}}\,\mathcal{D}^!_{U_{\mathrm{Tw}}}$$

whose image lie in $\mathrm{Mod}^{!o}_{\mathcal{D}}(U_{\mathrm{Tw}})$:

$$C\colon \mathrm{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}}) \longrightarrow \mathrm{Mod}^{!o}_{\mathcal{D}}(U_{\mathrm{Tw}}).$$

Restriction along $\bar{\eta}$ gives an inverse functor $D\colon \mathrm{Mod}^{!o}_{\mathcal{D}}(U_{\mathrm{Tw}}) \to \mathrm{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}})$ to C. This equivalence preserves the factorization structures. □

4.5 Coherent factorization algebras

Recall that a coherent (lax) [[$\mathcal{D}$]]-modules is a lax [[$\mathcal{D}$]]-module $(E^{[I)})$ such that $E^{(I)}$ is a coherent $\mathcal{D}$-module over X^I, for any I.

Equivalently, a coherent $[[\mathcal{D}]]$-module can be seen as an object in the oplax-limit of the functor $\mathrm{Coh}_*^{X^{\mathscr{S}}}$ mapping I to $\mathrm{Coh}_{\mathcal{D},X^I}$ and a surjection $\alpha\colon I \to J$ to the associated pushforward functor (which preserves coherent $\mathcal{D}$-modules).

Definition 4.5.1. A coherent lax $[[\mathcal{D}]]$-factorization algebra is a lax $[[\mathcal{D}]]$-factorization algebra whose underlying lax $[[\mathcal{D}]]$-module is coherent. We denote by $\mathrm{FA}^{\mathrm{Coh}}_{[[\mathcal{D}]]}(X)$ the full subcategory of $\mathrm{FA}_{[[\mathcal{D}]]}(X)$ spanned by coherent factorization algebras.

Fix a morphism $g\colon Y \to Z$ of varieties.

- If g is an open immersions, then the functors g^* and $g^{[[*]]}$ both preserve coherent $\mathcal{D}$-modules and they coincide on coherent $\mathcal{D}$-modules.
- If g is proper, then the functors g_* and $g_{[[*]]}$ both preserve coherent $\mathcal{D}$-modules and coincide on coherent $\mathcal{D}$-modules.

In particular, the lax monoidal functors $\mathcal{D}^{\mathrm{corr}}_{\widehat{U}}$ and $\mathcal{D}^{[[\mathrm{corr}]]}_{\widehat{U}}$ admit a common lax monoidal full subfunctor $\mathrm{Coh}^{\mathrm{corr}}_{\widehat{U}}$ mapping a surjection α to $D^b(\mathrm{Coh}_{\mathcal{D},U(\alpha)})$ and a morphism $\sigma\colon \alpha \to \beta$ in $\mathscr{S}^1$ to the functor $\widehat{\imath}(\sigma)_* \circ \widehat{\jmath}(\sigma)^* \simeq \widehat{\imath}(\sigma)_{[[*]]} \circ \widehat{\jmath}(\sigma)^{[[*]]}$:

$$
\begin{array}{ccccc}
\mathcal{D}^{\mathrm{corr}}_{\widehat{U}} & \longleftarrow & \mathrm{Coh}^{\mathrm{corr}}_{\widehat{U}} & \longrightarrow & \mathcal{D}^{[[\mathrm{corr}]]}_{\widehat{U}} \\
D(\mathrm{QCoh}_{\mathcal{D},U(\beta)}) & \longleftarrow & D^b(\mathrm{Coh}_{\mathcal{D},U(\beta)}) & \longrightarrow & \mathrm{Pro}\, D^b(\mathrm{Coh}_{\mathcal{D},U(\beta)}) \\
\downarrow {\scriptstyle \widehat{\imath}(\sigma)_*\circ\widehat{\jmath}(\sigma)^*} & & \downarrow & & \downarrow {\scriptstyle \widehat{\imath}(\sigma)_{[[*]]}\circ\widehat{\jmath}(\sigma)^{[[*]]}} \\
D(\mathrm{QCoh}_{\mathcal{D},U(\alpha)}) & \longleftarrow & D^b(\mathrm{Coh}_{\mathcal{D},U(\alpha)}) & \longrightarrow & \mathrm{Pro}\, D^b(\mathrm{Coh}_{\mathcal{D},U(\alpha)}).
\end{array}
$$

Applying the Grothendieck construction, we get symmetric monoidal and fully faithful functors

$$\mathscr{G}(\mathcal{D}^{\mathrm{corr}}_{\widehat{U}}) \longleftarrow \mathscr{G}(\mathrm{Coh}^{\mathrm{corr}}_{\widehat{U}}) \longrightarrow \mathscr{G}(\mathcal{D}^{[[\mathrm{corr}]]}_{\widehat{U}})$$

over $\mathscr{S}^1$. We find using Proposition 4.3.9:

Proposition 4.5.2. *A coherent lax $[[\mathcal{D}]]$-factorization algebra is tantamount to any of the following equivalent datum.*

1. *An openly Cartesian symmetric monoidal section of* $\mathscr{G}(\mathcal{D}_{\widehat{U}}^{[[\mathrm{corr}]]}) \to \mathscr{S}^1$ *mapping any* $\alpha \in \mathscr{S}^1$ *to a coherent* $\mathcal{D}$*-module.*
2. *An openly Cartesian symmetric monoidal section of* $\mathscr{G}(\mathrm{Coh}_{\widehat{U}}^{\mathrm{corr}}) \to \mathscr{S}^1$*.*
3. *An openly Cartesian symmetric monoidal section of* $\mathscr{G}(\mathcal{D}_{\widehat{U}}^{\mathrm{corr}}) \to \mathscr{S}^1$ *mapping any* $\alpha \in \mathscr{S}^1$ *to a coherent* $\mathcal{D}$*-module.*

Definition 4.5.3. We denote by $\overline{\mathrm{FA}}_{\mathrm{corr}}^{\mathcal{D}}(\widehat{U})$ the category of openly Cartesian symmetric monoidal sections of $\mathscr{G}(\mathcal{D}_{\widehat{U}}^{\mathrm{corr}})$. The above proposition gives a fully faithful functor

$$\mathrm{FA}_{[[\mathcal{D}]]}^{\mathrm{Coh}}(X) \longrightarrow \overline{\mathrm{FA}}_{\mathrm{corr}}^{\mathcal{D}}(\widehat{U}).$$

We shall now give another model for coherent lax $[[\mathcal{D}]]$-factorization algebras. Consider the functor $\mathcal{D}_*^{U_{\mathrm{Tw}}}\colon \mathrm{Tw}(\mathscr{S}) \to \mathrm{Cat}_\infty$ mapping a surjection α to $D(\mathrm{QCoh}_{\mathcal{D},U(\alpha)})$ and a morphism of twisted arrows $\tau\colon \alpha \to \beta$ to the push-forward functor $D(\mathrm{QCoh}_{\mathcal{D},U(\alpha)}) \longrightarrow D(\mathrm{QCoh}_{\mathcal{D},U(\beta)})$. It has a lax monoidal structure, given by

$$j^*(- \boxtimes -)\colon D(\mathrm{QCoh}_{\mathcal{D},U(\alpha_1)}) \times D(\mathrm{QCoh}_{\mathcal{D},U(\alpha_2)}) \to D(\mathrm{QCoh}_{\mathcal{D},U(\alpha_1 \amalg \alpha_2)}).$$

with $j\colon U(\alpha_1 \amalg \alpha_2) \to U(\alpha_1) \times U(\alpha_2)$ the open embedding. Its Cartesian Grothendieck construction thus admits a symmetric monoidal structure.

Fix a section $E\colon \mathrm{Tw}(\mathscr{S})^{\mathrm{op}} \to \mathscr{G}(\mathcal{D}_*^{U_{\mathrm{Tw}}})$ and a morphism $\tau\colon \alpha \to \beta$ in $\mathrm{Tw}(\mathscr{S})$. The transition morphism $E(\beta) \to U_{\mathrm{Tw}}(\tau)_* E(\alpha)$ induces by adjunction a morphism

$$U_{\mathrm{Tw}}(\tau)^* E(\beta) \to E(\alpha).$$

Definition 4.5.4. Let $\overline{\mathrm{Mod}}_*^{\mathcal{D}}(U_{\mathrm{Tw}})$ denote the full subcategory of $\underleftarrow{\mathrm{lax}}\, \mathcal{D}_*^{U_{\mathrm{Tw}}}$ spanned by sections E such that for any open morphism $\tau\colon \alpha \to \beta$ in $\mathrm{Tw}(\mathscr{S})$, the induced morphism $U_{\mathrm{Tw}}(\tau)^* E(\beta) \to E(\alpha)$ is an equivalence.

Let $\overline{\mathrm{FA}}_*^{\mathcal{D}}(U_{\mathrm{Tw}})$ denote the category of symmetric monoidal sections of $\mathscr{G}(\mathcal{D}_*^{U_{\mathrm{Tw}}})$ that belong to $\overline{\mathrm{Mod}}_*^{\mathcal{D}}(U_{\mathrm{Tw}})$.

Arguments similar to those used in sections 4.3 and 4.4 give:

Proposition 4.5.5. *The categories* $\overline{\mathrm{FA}}_*^{\mathcal{D}}(U_{\mathrm{Tw}})$ *and* $\overline{\mathrm{FA}}_{\mathrm{corr}}^{\mathcal{D}}(\widehat{U})$ *are equivalent. In particular, there is a fully faithful functor*

$$\mu\colon \mathrm{FA}_{[[\mathcal{D}]]}^{\mathrm{Coh}}(X) \longrightarrow \overline{\mathrm{FA}}_*^{\mathcal{D}}(U_{\mathrm{Tw}})$$

whose image consists of sections mapping every $\alpha \in \mathrm{Tw}(\mathscr{S})$ *to a coherent* $\mathcal{D}$*-module.*

4.6 Covariant Verdier duality

We can now prove the following

Theorem 4.6.1. *The covariant Verdier duality functor*

$$\psi\colon \mathrm{Coh}_{[[\mathcal{D}]]}(X^{\mathscr{S}}) \to \underline{\underline{\mathrm{Mod}}}^!_{\mathcal{D}}(X^{\mathscr{S}})$$

preserves factorization structures. In other words, it extends to a functor

$$\psi\colon \mathrm{FA}^{\mathrm{Coh}}_{[[\mathcal{D}]]}(X) \to \mathrm{FA}^!_{\mathcal{D}}(X).$$

By Propositions 4.5.5 and 4.4.5, we are reduced to building a functor

$$\overline{\mathrm{FA}}^{\mathcal{D}}_*(U_{\mathrm{Tw}}) \longrightarrow \mathrm{FA}^!_{\mathcal{D}}(U_{\mathrm{Tw}})$$

that coincides with ψ once restricted to coherent factorization algebras. We will start with constructing a lax monoidal functor

$$\underleftarrow{\mathrm{lax}}\,\mathcal{D}^{U_{\mathrm{Tw}}}_* \longrightarrow \underleftarrow{\mathrm{lax}}\,\mathcal{D}^!_{U_{\mathrm{Tw}}}$$

whose image lies in $\underline{\underline{\mathrm{Mod}}}^!_{\mathcal{D}}(U_{\mathrm{Tw}})$.

Recall the 2-category $\mathrm{Var}^{\mathrm{corr}}_{\mathbf{k}}$ of correspondences between **k**-varieties. It naturally contains (as a non-full subcategory) a copy of (the 1-category) $\mathrm{Var}_{\mathbf{k}}$ and a copy of $\mathrm{Var}^{\mathrm{op}}_{\mathbf{k}}$, through the functors:

$$\begin{aligned} \mathrm{Var}_{\mathbf{k}} &\to \mathrm{Var}^{\mathrm{corr}}_{\mathbf{k}} \\ X &\mapsto X \\ (X \to Y) &\mapsto (X \underset{=}{\leftarrow} X \to Y) \end{aligned} \qquad \begin{aligned} \mathrm{Var}^{\mathrm{op}}_{\mathbf{k}} &\to \mathrm{Var}^{\mathrm{corr}}_{\mathbf{k}} \\ X &\mapsto X \\ (X \leftarrow Y) &\mapsto (X \leftarrow Y \underset{=}{\to} Y). \end{aligned}$$

Definition 4.6.2. We denote by $U^{\mathrm{c}}_{\mathrm{Tw}}$ and $\overline{U^{\mathrm{c}}_{\mathrm{Tw}}}$ the composite functors

$$\begin{aligned} U^{\mathrm{c}}_{\mathrm{Tw}}&\colon\ \mathrm{Tw}(\mathscr{S}) \xrightarrow{U_{\mathrm{Tw}}} \mathrm{Var}_{\mathbf{k}} \longrightarrow \mathrm{Var}^{\mathrm{corr}}_{\mathbf{k}} \\ \overline{U^{\mathrm{c}}_{\mathrm{Tw}}}&\colon\ \mathrm{Tw}(\mathscr{S})^{\mathrm{op}} \underset{U_{\mathrm{Tw}}}{\longrightarrow} \mathrm{Var}^{\mathrm{op}}_{\mathbf{k}} \longrightarrow \mathrm{Var}^{\mathrm{corr}}_{\mathbf{k}}. \end{aligned}$$

They both admit a lax monoidal structure given by the correspondences

$$U(\alpha_1) \times U(\alpha_2) \longleftarrow U(\alpha_1 \amalg \alpha_2) \xrightarrow{=} U(\alpha_1 \amalg \alpha_2).$$

For any two surjections $\alpha\colon I \to J$ and $\beta\colon S \to T$, we denote by $\Delta(\alpha, \beta)$ the subvariety of $X^I \times X^S$ obtained by intersecting $U(\alpha) \times U(\beta)$ with $\Delta(I, S)$ (recall that $\Delta(I, S) \subset X^I \times X^S$ is the closed subvariety spanned by those families $((x_i), (x_s))$ such that $\{x_i, i \in I\} = \{x_s, s \in S\}$).

Proposition 4.6.3. *The data of the $\Delta(\alpha, \beta)$'s define a lax monoidal natural transformation*

$$\mathrm{Tw}(\mathscr{S}) \times \mathrm{Tw}(\mathscr{S})^{\mathrm{op}} \underset{\overline{U^{\mathrm{c}}_{\mathrm{Tw}}}\circ\rho_2}{\overset{U^{\mathrm{c}}_{\mathrm{Tw}}\circ\rho_1}{\rightrightarrows}} \mathrm{Var}^{\mathrm{corr}}_{\mathbf{k}} \qquad (T\colon U^{\mathrm{c}}_{\mathrm{Tw}}\circ\rho_1 \Rightarrow \overline{U^{\mathrm{c}}_{\mathrm{Tw}}}\circ\rho_2)$$

where ρ_1 and ρ_2 are the projections.

Proof. To any pair (α, β) we associate the correspondence

$$U^{\mathrm{c}}_{\mathrm{Tw}}(\alpha) = U(\alpha) \leftarrow \Delta(\alpha, \beta) \to U(\beta) = \overline{U^{\mathrm{c}}_{\mathrm{Tw}}}(\beta).$$

We start by showing it defines a natural transformation. Let $\tau\colon \alpha \to \alpha'$ and $\xi\colon \beta \to \beta'$ be morphisms in $\mathrm{Tw}(\mathscr{S})$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
U(\alpha) & \longleftarrow & \Delta(\alpha, \beta') & \longrightarrow & U(\beta') \\
{\scriptstyle =}\uparrow & & \uparrow & (2) & \uparrow \\
U(\alpha) & \longleftarrow & \Delta(\alpha, \beta) & \longrightarrow & U(\beta) \\
\downarrow & (1) & \downarrow & & \downarrow{\scriptstyle =} \\
U(\alpha') & \longleftarrow & \Delta(\alpha', \beta) & \longrightarrow & U(\beta).
\end{array}$$

It follows from Proposition 3.1.7 that the squares (1) and (2) are pullbacks. We have thus indeed defined a natural transformation.

To show it is lax monoidal, we have to provide, for any $\alpha_1, \alpha_2, \beta_1, \beta_2$ in $\mathrm{Tw}(\mathscr{S})$, a transformation in the 2-category $\mathrm{Var}^{\mathrm{corr}}_{\mathbf{k}}$

$$\begin{array}{ccc}
U(\alpha_1) \times U(\alpha_2) & \longrightarrow & U(\beta_1) \times U(\beta_2) \\
\downarrow & \swarrow\!\!\!\swarrow & \downarrow \\
U(\alpha_1 \amalg \alpha_2) & \longrightarrow & U(\beta_1 \amalg \beta_2).
\end{array}$$

Unfolding the definition of this 2-category, we have to find a commutative diagram

$$\begin{array}{ccccc}
U(\alpha_1) \times U(\alpha_2) & \longleftarrow & \Delta(\alpha_1, \beta_1) \times \Delta(\alpha_2, \beta_2) & \longrightarrow & U(\beta_1) \times U(\beta_2) \\
\uparrow & & \uparrow & (1) & \uparrow \\
U(\alpha_1 \amalg \alpha_2) & \xleftarrow[f]{} & Z & \longrightarrow & U(\beta_1 \amalg \beta_2) \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle g} & & {\scriptstyle =}\downarrow \\
U(\alpha_1 \amalg \alpha_2) & \longleftarrow & \Delta(\alpha_1 \amalg \alpha_2, \beta_1 \amalg \beta_2) & \longrightarrow & U(\beta_1 \amalg \beta_2)
\end{array}$$

in which the square (1) is a pullback and g is proper. We therefore pick Z such that (1) is a pullback. The image of Z into $U(\alpha_1) \times U(\alpha_2)$ lies in $U(\alpha_1 \amalg \alpha_2)$ and the map f is thus canonically defined. Finally, the map g is the natural closed immersion: Z sits in $\Delta(\alpha_1 \amalg \alpha_2, \beta_1 \amalg \beta_2)$, as a closed subvariety of $U(\alpha_1 \amalg \alpha_2) \times U(\beta_1 \amalg \beta_2)$. In particular, g is proper and the above diagram indeed defines a transformation in $\mathrm{Var}_{\mathbf{k}}^{\mathrm{corr}}$.

We then check those transformations behave coherently so that they define a lax symmetric monoidal structure on the transformation. □

Proposition 4.6.4. *There is a lax monoidal functor*

$$\psi_{\mathrm{Tw}} \colon \underleftarrow{\mathrm{lax}}\, \mathcal{D}_*^{U_{\mathrm{Tw}}} \longrightarrow \underleftarrow{\mathrm{lax}}\, \mathcal{D}^!_{U_{\mathrm{Tw}}}$$

that maps a section E to a section $\psi_{\mathrm{Tw}}(E)$ such that for any $\alpha \in \mathrm{Tw}(\mathscr{S})$

$$\psi_{\mathrm{Tw}}(E)^{(\alpha)} = \underleftarrow{\mathrm{holim}}_{\beta} (p_{\alpha\beta})_* q^!_{\alpha\beta} E^{(\beta)}$$

where $U(\alpha) \xleftarrow{p_{\alpha\beta}} \Delta(\alpha, \beta) \xrightarrow{q_{\alpha\beta}} U(\beta)$ are the projections. Moreover the image of ψ_{Tw} lies in $\underline{\underline{\mathrm{Mod}}}^!_{\mathcal{D}}(U_{\mathrm{Tw}})$.

We will need the following straightforward lemma.

Lemma 4.6.5. *Let $\mathcal{P} \colon \mathscr{A} \to \mathrm{Cat}_\infty$ be a lax monoidal functor. Let $\mathscr{B}$ be a symmetric monoidal category and denote by π the projection $\mathscr{A} \times \mathscr{B} \to \mathscr{A}$. Assume that for any $a \in \mathscr{A}$, the category $\mathcal{P}(a)$ admits all limits indexed by $\mathscr{B}$. Then the (monoidal) projection*

$$\underleftarrow{\mathrm{lax}}\, \mathcal{P} \to \underleftarrow{\mathrm{lax}}\, \mathcal{P} \circ \pi$$

admits a lax monoidal right adjoint mapping a section $s\colon \mathscr{A}^{\mathrm{op}} \times \mathscr{B}^{\mathrm{op}} \to \mathscr{G}(\mathcal{P} \circ \pi)$ *to the section* $F(s)\colon \mathscr{A}^{\mathrm{op}} \to \mathscr{G}(\mathcal{P})$ *given by*

$$F(s)(a) = \underset{b}{\underleftarrow{\mathrm{holim}}}\, s(a,b).$$

Proof of Proposition 4.6.4. We first observe the equalities $\mathcal{D}_*^{U_{\mathrm{Tw}}} = \mathcal{D}^{\mathrm{corr}}_{U^{\mathrm{c}}_{\mathrm{Tw}}}$ and $\mathcal{D}^!_{U_{\mathrm{Tw}}} = \mathcal{D}^{\mathrm{corr}}_{\overline{U^{\mathrm{c}}_{\mathrm{Tw}}}}$. By composing the transformation T with the lax monoidal functor $\mathcal{D}^{\mathrm{corr}}$, we thus get a lax monoidal natural transformation

$$\mathcal{D}_*^{U_{\mathrm{Tw}}} \circ \rho_1 \implies \mathcal{D}^!_{U_{\mathrm{Tw}}} \circ \rho_2.$$

By Corollary 4.1.9, it induces a lax monoidal functor $\underleftarrow{\mathrm{lax}}(\mathcal{D}_*^{U_{\mathrm{Tw}}} \circ \rho_1) \to \underleftarrow{\mathrm{lax}}(\mathcal{D}^!_{U_{\mathrm{Tw}}} \circ \rho_2)$. The projection $\rho_1\colon \mathrm{Tw}(\mathscr{S}) \times \mathrm{Tw}(\mathscr{S})^{\mathrm{op}} \to \mathrm{Tw}(\mathscr{S})$ is monoidal, and thus induces a monoidal functor $\underleftarrow{\mathrm{lax}}\,\mathcal{D}_*^{U_{\mathrm{Tw}}} \to \underleftarrow{\mathrm{lax}}(\mathcal{D}_*^{U_{\mathrm{Tw}}} \circ \rho_1)$. Finally, we also have lax monoidal functor $\underleftarrow{\mathrm{lax}}(\mathcal{D}^!_{U_{\mathrm{Tw}}} \circ \rho_2) \to \underleftarrow{\mathrm{lax}}\,\mathcal{D}^!_{U_{\mathrm{Tw}}}$ given by Lemma 4.6.5. We define ψ_{Tw} as the composite functor

$$\psi_{\mathrm{Tw}}\colon \underleftarrow{\mathrm{lax}}\,\mathcal{D}_*^{U_{\mathrm{Tw}}} \to \underleftarrow{\mathrm{lax}}(\mathcal{D}_*^{U_{\mathrm{Tw}}} \circ \rho_1) \to \underleftarrow{\mathrm{lax}}(\mathcal{D}^!_{U_{\mathrm{Tw}}} \circ \rho_2) \to \underleftarrow{\mathrm{lax}}\,\mathcal{D}^!_{U_{\mathrm{Tw}}}.$$

It is by construction lax monoidal and given by the announced formula. Arguments similar to those of section 3.3 prove that the image of ψ_{Tw} lies in $\underline{\underline{\mathrm{Mod}}}^!_{\mathcal{D}}(U_{\mathrm{Tw}})$. ∎

Proof of Theorem 4.6.1. On the categories of sections $\underleftarrow{\mathrm{lax}}\,\mathcal{D}_*^{U_{\mathrm{Tw}}}$ and $\underleftarrow{\mathrm{lax}}\,\mathcal{D}^!_{U_{\mathrm{Tw}}}$, the left and the right Day convolution products coincide (it is given by finite sums). Therefore, any $E \in \overline{\mathrm{FA}}^{\mathcal{D}}_*(U_{\mathrm{Tw}})$ is in particular a commutative algebra in $\underleftarrow{\mathrm{lax}}\,\mathcal{D}_*^{U_{\mathrm{Tw}}}$. It follows by Proposition 4.6.4 that $\psi_{\mathrm{Tw}}(E)$ is a commutative algebra in $\underleftarrow{\mathrm{lax}}\,\mathcal{D}^!_{U_{\mathrm{Tw}}}$ and thus corresponds to a lax-monoidal section. We only have to check it is actually monoidal.

Let $\alpha_1\colon I_1 \to J_1$ and $\alpha_2\colon I_2 \to J_2$ be surjections. We set $\alpha := \alpha_1 \amalg \alpha_2$. We denote by $j\colon U(\alpha) \to U(\alpha_1) \times U(\alpha_2)$ the open immersion. For any surjection $\beta\colon S \to T$, the variety $\Delta(\alpha,\beta)$ is the disjoint union, over all decompositions $\beta = \beta_1 \amalg \beta_2$, of the product $\Delta(\alpha_1,\beta_1) \times \Delta(\alpha_2,\beta_2)$ (pulled back along j). We get

$$\begin{aligned}\psi_{\mathrm{Tw}}(E)^{(\alpha)} = \lim_{\beta} p_* q^! E^{(\beta)} &\simeq j^* \lim_{\beta_1,\beta_2} (p_{1*} q_1^! E^{(\beta_1)} \boxtimes p_{2*} q_2^! E^{(\beta_2)}) \\ &\simeq j^* (\psi_{\mathrm{Tw}}(E)^{(\alpha_1)} \boxtimes \psi_{\mathrm{Tw}}(E)^{(\alpha_2)})\end{aligned}$$

where

$$U(\alpha) \xleftarrow{p} \Delta(\alpha,\beta) \xrightarrow{q} U(\beta)$$
$$U(\alpha_1) \xleftarrow{p_1} \Delta(\alpha_1,\beta_1) \xrightarrow{q_1} U(\beta_1)$$
$$U(\alpha_2) \xleftarrow{p_2} \Delta(\alpha_2,\beta_2) \xrightarrow{q_2} U(\beta_2)$$

are the projections. It follows that the lax monoidal structure is actually monoidal. □

5 Gelfand-Fuchs cohomology in algebraic geometry

Recall that X is a fixed smooth algebraic variety over $\mathbf{k}$.

5.1 Chevalley-Eilenberg factorization algebras

A. Homological $\mathcal{D}^!$-module. Let L be a local Lie algebra on X, i.e., a vector bundle with a Lie algebra structure on the sheaf of sections given by a bi-differential operator. Then the right $\mathcal{D}_X$-module $\mathcal{L} = L \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is a Lie*-algebra, see [BD] §2.5. Using the determinantal factor $\det(\mathbf{k}^2)$, we can write the antisymmetric Lie*-bracket in $\mathcal{L}$ as a permutation equivariant morphism of $\mathcal{D}$-modules on $X \times X$

(5.1.1) $$\eta : (\mathcal{L} \boxtimes \mathcal{L}) \otimes_{\mathbf{k}} \det(\mathbf{k}^2) \longrightarrow \delta_* \mathcal{L}.$$

Here $\delta : X \to X \times X$ is the diagonal embedding. Let us list the most important examples.

Examples 5.1.2. (a) $L = T_X$ is the tangent bundle of X.

(b) Let G be an algebraic group over $\mathbf{k}$ with Lie algebra $\mathfrak{g}$ and P be a principal G-bundle on X. The data of P gives rise to two local Lie algebras on X. First, we have the $\mathcal{O}_X$-linear Lie algebra P^{Ad} (infinitesimal symmetries of P). Second, we have the *Atiyah Lie algebroid* $\mathrm{At}(P)$ (infinitesimal symmetries of the pair (X, P)) fitting into a short exact sequence

$$0 \to P^{\mathrm{Ad}} \longrightarrow \mathrm{At}(P) \longrightarrow T_X \to 0.$$

For $P = X\times G$ the trivial bundle, $P^{\mathrm{Ad}} = \mathfrak{g}\otimes\mathcal{O}_X$ and $\mathrm{At}(P)$ is the semi-direct product of T_X acting on $\mathfrak{g}\otimes\mathcal{O}_X$ via the second factor.

Given a local Lie algebra L, we have the dg-Lie algebra $\mathfrak{l} = R\Gamma(X, L)$ and we are interested in its Lie algebra cohomology with constant coefficients. It is calculated by the (reduced) Chevalley-Eilenberg chain complex of $\mathfrak{l}$ which we denote by

$$\mathrm{CE}_\bullet(\mathfrak{l}) \;=\; \big(\mathrm{Sym}^{\bullet\geqslant 1}(\mathfrak{l}[1]), d_{\mathrm{CE}}\big).$$

Applying the Künneth formula, we see that

$$\mathrm{CE}_\bullet(\mathfrak{l}) \;=\; \mathrm{Tot}\Big\{\cdots\to R\Gamma(X^3, L^{\boxtimes 3})_{-S_3}\to R\Gamma(X^2, L^{\boxtimes 2})_{-S_2}\to R\Gamma(X, L)\Big\}$$

is the total complex of the obvious double complex with horizontal grading ending in degree (-1). Here S_p is the symmetric group and the subscript "$-S_p$" means the space of anti-coinvariants of S_p.

Following [BD] we represent $\mathrm{CE}_\bullet(\mathfrak{l})$ as the factorization homology of an appropriate lax $\mathcal{D}^!$-module $\mathcal{C}_\bullet$ on $X^{\mathscr{S}}$. We first define a lax !-$\mathcal{D}$-module $\mathcal{C}_1$ by putting $\mathcal{C}_1^{(I)} = (\delta_I)_*\mathcal{L}$ for any nonempty finite set I. Here $\delta_I : X\to X^I$ is the diagonal embedding. Given a surjection $g : I\to J$ with the corresponding diagonal embedding $\delta_g : X^J\to X^I$, we define structure map (in the dual form) $(\delta_g)_*(\delta_J)_*\mathcal{L}\to(\delta_I)_*\mathcal{L}$ to be the canonical isomorphism arising from the equality $\delta_g\circ\delta_J = \delta_I$.

Remarks 5.1.3. (a) After passing to the colimit, $\mathcal{C}_1$ becomes the pushforward of $\mathcal{L}$ under the embedding of X into $\mathrm{Ran}(X)$.

(b) Note that the structure maps for $\mathcal{C}_1$ in the form $\mathcal{C}_{1,J}\to\delta_g^!\,\mathcal{C}_1^{(I)}$ are not, in general, isomorphisms, so $\mathcal{C}_1$ is not a strict $\mathcal{D}^!$-module.

Recall that the category $\mathrm{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}})$ has a symmetric monoidal structure $\otimes^*$ (see Remark 4.2.5 above and [BD, 4.2.5]). The fact that $\mathcal{L}$ is a Lie*-algebra means that $\mathcal{C}_1$ is a Lie algebra with respect to $\otimes^*$.

Definition 5.1.4. We define

$$\mathcal{C}_\bullet \;=\; \big(\mathrm{Sym}^{\bullet\geqslant 1}_{\otimes^*}(\mathcal{C}_1[1]), d_{\mathrm{CE}}\big)$$

to be the intrinsic Chevalley-Eilenberg complex of $\mathcal{C}_1$ as a Lie algebra in $(\mathrm{Mod}^!_{\mathcal{D}}(X^{\mathscr{S}}),\otimes^*)$.

For convenience of the reader let us describe $\mathcal{C}_\bullet$ more explicitly. First, for each $q \geqslant 1$, the lax $\mathcal{D}^!$-module $\mathcal{C}_q = \Lambda^q_{\otimes^*}(\mathcal{C}_1)$ has

$$\mathcal{C}_q^{(I)} = \underset{\substack{f\colon I \twoheadrightarrow Q \\ |Q|=q}}{\underrightarrow{\mathrm{holim}}} (\delta_f)_* \left(\mathcal{L}^{\boxtimes Q} \otimes \det(\mathbf{k}^Q)\right).$$

This homotopy colimit can be simplified to

$$\mathcal{C}_q^{(I)} = \bigoplus_{R \in \mathrm{Eq}_q(I)} (\delta_R)_* \big(\mathcal{L}^{\boxtimes (I/R)} \otimes \det(\mathbf{k}^{I/R})\big). \tag{5.1.5}$$

Here $\mathrm{Eq}_q(I)$ is the set of equivalence relations R on I with exactly q equivalence classes, i.e., such that $|I/R| = q$, and $\delta_R : X^{I/R} \to X^I$ is the diagonal embedding. In particular, $\mathcal{C}_q$ is concentrated in degree 0.

Given a surjection $g : I \to J$ and any surjection $f : J \to Q$ with $|Q| = q$, we have the surjection $fg : I \to Q$ with $\delta_g \circ \delta_f = \delta_{fg}$, and so we have an identification

$$(\delta_g)_*(\delta_f)_*(\mathcal{L}^{\boxtimes Q} \otimes \det(\mathbf{k}^Q)) \longrightarrow (\delta_{fg})_* (\mathcal{L}^{\boxtimes Q} \otimes \det(\mathbf{k}^Q))$$

of (an arbitrary) term of the colimit for $(\delta_g)_* \mathcal{C}_q^{(J)}$ with a certain term of the colimit for $\mathcal{C}_q^{(I)}$. The structure map (in the dual form) $(\delta_g)_* \mathcal{C}_q^{(J)} \to \mathcal{C}_q^{(I)}$ is induced by these identifications.

Next, we have the differential $d = d_{\mathrm{CE}} : \mathcal{C}_q \to \mathcal{C}_{q-1}$ defined as follows. Let $g\colon Q \twoheadrightarrow S$ be a surjection between finite sets such that that $|Q| = |S|+1 = q$, so g has exactly one fiber of cardinality 2, all other fibers being of cardinality 1. Applying the bracket (5.1.1) to this fiber and substituting copies of the identity for the other fibers, we get a map

$$\eta^g\colon \mathcal{L}^{\boxtimes Q} \otimes \det(\mathbf{k}^Q) \longrightarrow (\delta_g)_* \left(\mathcal{L}^{\boxtimes S} \otimes \det(\mathbf{k}^S)\right).$$

We define $d\colon \mathcal{C}_q \to \mathcal{C}_{q-1}$ by summing, as g varies, the induced maps

$$(\delta_f)_* \left((\mathcal{L}^{\boxtimes Q} \otimes \det(\mathbf{k}^Q)\right) \to (\delta_{gf})_* \left(\mathcal{L}^{\boxtimes S} \otimes \det(\mathbf{k}^S)\right).$$

The differential squares to zero by the Jacobi identity.

Definition 5.1.6. Let $\mathcal{C}^{(I)}$ be the complex of $\mathcal{D}$-modules on X^I given by $\mathcal{C}_q^{(I)}$ in homological degree q and the above differential. We denote by $\mathcal{C}_{\leqslant q}^{(I)}$ its truncation $\bigoplus_{p \leqslant q} \mathcal{C}_p^{(I)}[p]$ (with the same differential). The differential is compatible with the transition maps, and we get lax $\mathcal{D}^!$-modules $\mathcal{C}_{\leqslant q}$ on $X^{\mathscr{S}}$.

Proposition 5.1.7. *We have*

$$\int_X \mathcal{C}_q \simeq \Lambda^q(\mathfrak{l}), \quad \int_X \mathcal{C}_{\leqslant q} \simeq \mathrm{CE}_{\leqslant q}(\mathfrak{l}), \quad \int_X \mathcal{C} \simeq \mathrm{CE}_\bullet(\mathfrak{l}).$$

Proof: We show the first identification, the compatibility with the differentials will be clear. For any non-empty finite set I, we have, by (5.1.5):

$$R\Gamma_{\mathrm{DR}}(X^I, \mathcal{C}_q^{(I)}) = \bigoplus_{R\in \mathrm{Eq}_q(I)} R\Gamma_{\mathrm{DR}}\big(X^{I/R}, \mathcal{L}^{\boxtimes(I/R)} \otimes \det(\mathbf{k}^{I/R})\big) =$$

$$= \bigoplus_{R\in \mathrm{Eq}_q(I)} R\Gamma\big(X^{I/R}, L^{\boxtimes(I/R)} \otimes \det(\mathbf{k}^{I/R})\big) = \bigoplus_{R\in \mathrm{Eq}_q(I)} \Lambda^{|I/R|}(\mathfrak{l}).$$

Now, $\int_X \mathcal{C}_q$ is the $\underrightarrow{\mathrm{holim}}$ of this over I in $\mathscr{S}$, and so is identified with $\Lambda^q(\mathfrak{l})$. □

Proposition 5.1.8. *The lax $\mathcal{D}^!$-module $\mathcal{C}_\bullet$ is factorizing (ie admits a factorization structure).*

Proof: By construction, $\mathcal{C}_\bullet$ is the symmetric algebra generated by $\mathcal{C}_1[1]$ under $\otimes^*$, with an additional Chevalley-Eilenberg differential. So it is factorizing. □

B. Strictification of $\mathcal{C}_\bullet$ and chiral envelopes. While the components $C_q^{(I)}$ of the lax $\mathcal{D}^!$-module $\mathcal{C}_\bullet$ are very simple $\mathcal{D}_{X^I}$-modules constructed out of $\mathcal{L}^{\boxtimes q}$, the strictification $\underline{\underline{\mathcal{C}_\bullet}}$ is highly non-trivial. More precisely, let pt denote the 1-element set, so $X^{\mathrm{pt}} = X$. For $x \in X$ let $\widehat{x} = \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ be the formal disk around x and $\widehat{x}^\circ = \widehat{x} - \{x\}$ the punctured formal disk.

Consider the left $\mathcal{D}_X$-module $\omega_X^{-1} \otimes_{\mathcal{O}_X} \underline{\underline{\mathcal{C}_\bullet}}^{(\mathrm{pt})}$ and its $\mathcal{O}_X$-module fiber

$$\big(\omega_X^{-1} \otimes_{\mathcal{O}_X} \underline{\underline{\mathcal{C}_\bullet}}^{(\mathrm{pt})}\big)_x = \big(\omega_X^{-1} \otimes_{\mathcal{O}_X} \underline{\underline{\mathcal{C}_\bullet}}^{(\mathrm{pt})}\big) \otimes_{\mathcal{O}_X} \mathbf{k}_x.$$

Proposition 5.1.9. *One has a canonical identification*

$$\big(\omega_X^{-1} \otimes_{\mathcal{O}_X} \underline{\underline{\mathcal{C}_\bullet}}^{(\mathrm{pt})}\big)_x \simeq \mathrm{Ind}_{\Gamma(\widehat{x},L)}^{R\Gamma(\widehat{x}^\circ,L)} \mathbf{k}$$

where on the right hand side we have the vacuum (dg-)module of the dg-Lie algebra $R\Gamma(\widehat{x}^\circ, L)$. In other words, $\omega_X^{-1} \otimes_{\mathcal{O}_X} \underline{\underline{\mathcal{C}_\bullet}}^{(\mathrm{pt})}$ is the chiral envelope of the Lie*-algebra $\mathcal{L}$, see [G, §1.2.4] and [BD, §3.7.1].

Since we will not need this result in the present paper, we leave its proof to the reader. In this way one can see the validity of Theorem 4.8.1.1 of [BD] (the chiral homology of the chiral envelope is the same as $H_\bullet^{\mathrm{Lie}}(\mathfrak{l})$) for any smooth variety X, not necessarily 1-dimensional or proper.

C. Cohomological [[$\mathcal{D}$]]-module and diagonal filtration. Using Verdier duality, we get [[$\mathcal{D}$]]-modules

$$\check{\mathcal{C}}^q = (\mathcal{C}_q)^\vee, \quad \check{\mathcal{C}}^{\leqslant q} = (\mathcal{C}_{\leqslant q})^\vee, \quad \check{\mathcal{C}}^\bullet = (\mathcal{C}_\bullet)^\vee$$

on $X^{\mathscr{S}}$ such that

$$\oint_X^{[[c]]} \check{\mathcal{C}}^\bullet \;\simeq\; \mathrm{CE}^\bullet(\mathfrak{l}).$$

This complex comes with the *diagonal filtration* (cf. section 3.3 or [Fu, Ch.2 §4]) which is a sequence of complexes and morphisms given by

$$R\Gamma^{[[c]]}_{X_1^{\mathscr{S}},\mathrm{DR}}(X^{\mathscr{S}}, \check{\mathcal{C}}^\bullet) \to \cdots \to R\Gamma^{[[c]]}_{X_d^{\mathscr{S}},\mathrm{DR}}(X^{\mathscr{S}}, \check{\mathcal{C}}^\bullet) \to \cdots \quad \to \oint_X^{[[c]]} \check{\mathcal{C}}^\bullet,$$

where

$$R\Gamma^{[[c]]}_{X_d^{\mathscr{S}},\mathrm{DR}}(X^{\mathscr{S}}, \check{\mathcal{C}}^\bullet) \;=\; \oint_X^{[[c]]} (i_d)_* i_d^! \check{\mathcal{C}}^\bullet \;:=\; \underleftarrow{\mathrm{holim}}_I \, R\Gamma^{[[c]]}_{\mathrm{DR}}(X_d^I, i_d^! \check{\mathcal{C}}^{(I)})$$

with $i_d \colon X_d^{\mathscr{S}} \to X^{\mathscr{S}}$ the pointwise closed immersion.

For future use we introduce the following.

Definition 5.1.10. We call the *diagonal $\mathcal{D}$-module* associated to L the complex of $\mathcal{D}$-modules $\check{\mathcal{C}}^\bullet_\Delta = i_1^! \check{\mathcal{C}}^\bullet$ on X:

$$\check{\mathcal{C}}^\bullet_\Delta \;=\; i_1^! \check{\mathcal{C}}^\bullet \;:=\; \underleftarrow{\mathrm{holim}}_J (i_1^{(J)})^! \, \check{\mathcal{C}}^\bullet_J \simeq \psi(\check{\mathcal{C}}^\bullet)^{(\mathrm{pt})},$$

where $i_1^{(J)} \colon X \to X^J$ is the diagonal embedding.

The de Rham complex of $\check{\mathcal{C}}^\bullet_\Delta$ will be denoted by $\mathcal{F}^\bullet_\Delta$ and called *the diagonal complex* of L. We will denote its compactly supported cohomology

$$H^\bullet_\Delta(L) \;=\; H^\bullet_{[[c]]}(X, \mathcal{F}^\bullet_\Delta) \;=\; R\Gamma^{[[c]]}_{X_1^{\mathscr{S}}}(\check{\mathcal{C}}^\bullet)$$

and call it the *diagonal cohomology* of $\mathfrak{l}$. It comes with a canonical map $H^\bullet_\Delta(L) \to H^\bullet_{\mathrm{Lie}}(\mathfrak{l})$.

Explicitly,

$$\mathcal{F}^\bullet_\Delta \;=\; \Big\{ L^\vee \to \underline{H}^n_X((L^\vee)^{\boxtimes 2})^{\Sigma_2} \to \underline{H}^{2n}_X((L^\vee)^{\boxtimes 3})^{\Sigma_3} \to \cdots \Big\}, \tag{5.1.11}$$

with grading normalized so that $L^\vee$ is in degree $1-n$.

Remark 5.1.12. One can see $\mathcal{F}^\bullet_\Delta$ as a sheafified, algebro-geometric version of the diagonal complex of Gelfand-Fuchs [GF] [Fu], see also [CoG2] §4.2. Instead of distributions supported on the diagonal $X \subset X^p$, as in the C^∞-case, our construction involves coherent cohomology of X^p with support in X which is a well known analog of the space of distributions ("holomorphic hyperfunctions").

5.2 The diagonal filtration in the affine case

In this section, we assume that X is a smooth affine variety. As before, L is a local Lie algebra on X and $\mathfrak{l} = \Gamma(X, L)$. Recall that $\mathcal{C} = \mathcal{C}_L$ is the factorizing !-sheaf computing $\mathrm{CE}_\bullet(\mathfrak{l})$. We will prove the following theorem.

Theorem 5.2.1. *The canonical map* $\int_X \mathcal{C} \to \oint_X \phi(\mathcal{C}) \simeq \underleftarrow{\mathrm{holim}}_d \int_X^{\leqslant d} \mathcal{C}$ *is an equivalence.*

Recall (see equation 5.1.5) that $\mathcal{C}$ comes with a natural filtration $\mathcal{C}_{\leqslant q}$ where $\mathcal{C}_q = \mathrm{hocofib}(\mathcal{C}_{\leqslant q-1} \to \mathcal{C}_{\leqslant q})[-q]$ is given by

$$\mathcal{C}_q^{(I)} = \bigoplus_{R \in \mathrm{Eq}_q(I)} (\delta_R)_* \big(\mathcal{L}^{\boxtimes (I/R)} \otimes \det(\mathbf{k}^{I/R})\big).$$

Lemma 5.2.2. *The canonical map* $\int_X \mathcal{C}_q \to \oint_X \phi(\mathcal{C}_q)$ *is an equivalence.*

Proof. We compute explicitly both sides and find $\Lambda^q \Gamma(X, L) = \Lambda^q \mathfrak{l}$. □

From this, we deduce by induction on q:

Lemma 5.2.3. *The canonical map* $\int_X \mathcal{C}_{\leqslant q} \to \oint_X \phi(\mathcal{C}_{\leqslant q})$ *is an equivalence.*

Proof of Theorem 5.2.1. Since $\underrightarrow{\mathrm{holim}}_q \int_X \mathcal{C}_{\leqslant q} \simeq \int_X \mathcal{C}$, it is now enough to prove that the map

$$\underset{q}{\underrightarrow{\mathrm{holim}}} \oint_X \phi(\mathcal{C}_{\leqslant q}) \to \oint_X \phi(\mathcal{C})$$

is an equivalence. Rephrasing with the diagonal filtration, we get

$$\underset{q}{\underrightarrow{\mathrm{holim}}}\, \underset{d}{\underleftarrow{\mathrm{holim}}} \int_X^{\leqslant d} \mathcal{C}_{\leqslant q} \to \underset{d}{\underleftarrow{\mathrm{holim}}}\, \underset{q}{\underrightarrow{\mathrm{holim}}} \int_X^{\leqslant d} \mathcal{C}_{\leqslant q}.$$

Fix an integer p. It is enough to prove that for q big enough (independently of d), the map $H^p_{\mathrm{DR}}(\int_X^{\leqslant d} \mathcal{C}_{\leqslant q}) \to H^p_{\mathrm{DR}}(\int_X^{\leqslant d} \mathcal{C}_{\leqslant q+1})$ is an isomorphism. This amounts to proving that $H^{p+q}_{\mathrm{DR}}(\int_X^{\leqslant d} \mathcal{C}_q)$ vanishes for q big enough.

By definition, we have $\int_X^{\leqslant d} \mathcal{C}_q = \underrightarrow{\mathrm{holim}}_I\, R\Gamma_{\mathrm{DR}}(X^I_d, i_d^{[[*]]}\mathcal{C}_q^{(I)})$. Let us fix I. For any positive integer s, we denote by $Y^{(s)}$ the s^{th} infinitesimal neighborhood of X^I_d in X^I and by $i^{(s)}\colon Y^{(s)} \to X^I$ the canonical inclusion. We get

$$R\Gamma_{\mathrm{DR}}(X^I_d, i_d^{[[*]]}\mathcal{C}_q^{(I)}) \simeq \underleftarrow{\mathrm{holim}}_s\, R\Gamma_{\mathrm{DR}}(Y^{(s)}, (i^{(s)})^*\mathcal{C}_q^{(I)}).$$

Since $Y^{(s)}$ is affine (because X is) and $\mathcal{C}_q^{(I)}$ is induced from a quasicoherent sheaf concentrated in degree 0, the complex $R\Gamma_{\mathrm{DR}}(Y^{(s)}, (i^{(s)})^*\mathcal{C}_q^{(I)})$ only has cohomology in degrees lower or equal to 0. The homotopy limit indexed by s satisfies the Mittag-Leffler condition. We deduce that the cohomology of $R\Gamma_{\mathrm{DR}}(X^I_d, i_d^{[[*]]}\mathcal{C}_q^{(I)})$ is concentrated in degree lower or equal to 0. It follows that $H^{p+q}_{\mathrm{DR}}(\int_X^{\leqslant d} \mathcal{C}_q)$ vanishes for $q \geqslant 1-p$. This concludes the proof of Theorem 5.2.1. □

Applying Verdier duality to Theorem 5.2.1, we get:

Corollary 5.2.4. *For a smooth affine variety X, the diagonal filtration on Chevalley-Eilenberg cohomology is complete, ie:*

$$\mathrm{CE}^\bullet(\mathfrak{l}) \simeq \oint_X^{[[c]]} \check{\mathcal{C}}^\bullet \simeq \int_X^{[[c]]} \psi(\check{\mathcal{C}}^\bullet) \simeq \underrightarrow{\mathrm{holim}}_d\, R\Gamma^{[[c]]}_{X_d^{\mathscr{S}},\mathrm{DR}}(X^{\mathscr{S}}, \check{\mathcal{C}}^\bullet).$$

6 Relation to the topological picture

6.1 From factorizing $\mathcal{D}^!$-modules to C^∞ factorization algebras

In this section we assume $\mathbf{k} = \mathbb{C}$. Thus $X_{\mathrm{an}} := X(\mathbb{C})$ is a complex analytic manifold.

For any complex analytic manifold M we denote by $M^{\mathscr{S}}$ the Ran diagram of complex manifolds M^I and diagonal embeddings $\delta_g : M^J \to M^I$.

Let $\mathcal{E} = (\mathcal{E}^{(I)}, \beta_g : \mathcal{E}^{(J)} \longrightarrow \delta_g^! \mathcal{E}^{(I)})$ be a (right) $\mathcal{D}^!$-module on $X^{\mathscr{S}}$. In particular, for $I = \mathrm{pt}$ a 1-element set, we get a $\mathcal{D}$-module $\mathcal{E}^{(\mathrm{pt})}$ on $X^{\mathrm{pt}} = X$.

We say that $\mathcal{E}$ is h.r. (holonomic regular) if each $\mathcal{E}^{(I)}$ is h.r. on X^I (that is, each cohomology $\mathcal{D}$-module of the complex of $\mathcal{D}$-modules $\mathcal{E}^{(I)}$ is h.r.).

Proposition 6.1.1. *Suppose $\mathcal{E}$ is factorizable. Then, $\mathcal{E}$ is h.r. if and only if $\mathcal{E}^{(\mathrm{pt})}$ is h.r. on X.*

Proof: Suppose $\mathcal{E}^{(\mathrm{pt})}$ is h.r. on X. We prove, by induction on $|I|$, that $\mathcal{E}^{(I)}$ is h.r. on X^I. First of all, the restriction of $\mathcal{E}^{(I)}$ to the open subset $X^I_{\neq}$ (complement to all the diagonals) is, by the factorization structure, identified with the restriction to $X^I_{\neq}$ of the h.r. module $(\mathcal{E}^{(\mathrm{pt})})^{\boxtimes I}$. Further, the complement $X^I - X^I_{\neq}$ is stratified into locally closed subvarieties isomorphic to $X^J_{\neq}$ with $2 \leqslant |J| < |I|$, or to $X^{\mathrm{pt}} = X$, if $|J| = 1$. So our statement follows by stability of h.r. modules under extensions. □

Let $\mathcal{E}$ be a factorizable $\mathcal{D}^!$-module on $X^{\mathscr{S}}$. Fix a partition $I = I_1 \amalg \cdots \amalg I_m$ of a finite set by non-empty finite subsets. We see it as a surjection $\alpha\colon I \twoheadrightarrow J := \{1, \ldots, m\}$. Recall the open immersion

$$U(\alpha) \xrightarrow{\ j\ } X^{I_1} \times \cdots \times X^{I_m} \xrightarrow{\ a\ } X^I$$

The factorization structure gives quasi-isomorphisms

$$\nu_{I_1,\ldots,I_m}\colon j^!(\mathcal{E}^{(I_1)} \boxtimes \cdots \boxtimes \mathcal{E}^{(I_m)}) \longrightarrow (aj)^!\, \mathcal{E}^{(I)}$$

We now associate to $\mathcal{E}$ a pre-cosheaf $\mathcal{A} = \mathcal{A}_{\mathcal{E}}$ on the complex topology of X_{an} as follows. Let $U \subset X_{\mathrm{an}}$ be any open set. We define

$$\mathcal{A}(U) \;=\; \mathcal{A}_{\mathcal{E}}(U) \;=\; \underset{I\in\mathscr{S}}{\underrightarrow{\mathrm{holim}}}\, R\Gamma_{\mathrm{c}}(U^I, \mathrm{DR}(\mathcal{E}^{(I)})_{\mathrm{an}}),$$

where $R\Gamma_{\mathrm{c}}$ is the usual topological cohomology with compact support of the constructible complex $\mathrm{DR}(\mathcal{E}^{(I)})_{\mathrm{an}}$.

Further, let $U_0, U_1, \ldots, U_m \subset X_{\mathrm{an}}$ be open sets such that $U_1, \ldots, U_m$ are pairwise disjoint and contained in U_0. For any partition $I = I_1 \amalg \cdots \amalg I_m$ corresponding to a surjection α as above, we have the embeddings

$$\begin{aligned} k\colon U_1^{I_1} \times \cdots \times U_m^{I_m} &\longrightarrow U(\alpha)_{\mathrm{an}}, \\ l\colon U_1^{I_1} \times \cdots \times U_m^{I_m} &\longrightarrow U_0^I. \end{aligned}$$

We define the morphism

$$\mu^{U_0}_{U_1,\ldots,U_m} :\ \mathcal{A}(U_1) \otimes \cdots \otimes \mathcal{A}(U_m) \longrightarrow \mathcal{A}(U_0)$$

as follows. The source is the homotopy colimit

$$\begin{aligned}
&\underset{I_1,\dots,I_m\in\mathscr{S}}{\underrightarrow{\mathrm{holim}}}\ R\Gamma_{\mathrm{c}}\big(U_1^{I_1}, \mathrm{DR}(\mathcal{E}^{(I_1)})_{\mathrm{an}}\big)\otimes\cdots\otimes R\Gamma_{\mathrm{c}}\big(U_m^{I_m}, \mathrm{DR}(\mathcal{E}^{(I_m)})_{\mathrm{an}}\big)\\
&\simeq \underset{I_1,\dots,I_m\in\mathscr{S}}{\underrightarrow{\mathrm{holim}}}\ R\Gamma_{\mathrm{c}}\big(U_1^{I_1}\times\cdots\times U_m^{I_m},\ k^! j^! \mathrm{DR}\big(\mathcal{E}^{(I_1)}\boxtimes\cdots\boxtimes\mathcal{E}^{(I_m)}\big)_{\mathrm{an}}\big)\\
&\simeq \underset{I_1,\dots,I_m\in\mathscr{S}}{\underrightarrow{\mathrm{holim}}}\ R\Gamma_{\mathrm{c}}\big(U_1^{I_1}\times\cdots\times U_m^{I_m}, \mathrm{DR}\big(k^! j^!(\mathcal{E}^{(I_1)}\boxtimes\cdots\boxtimes\mathcal{E}^{(I_m)})\big)_{\mathrm{an}}\big),
\end{aligned}$$

and the $(I_1,\dots,I_m)$th term of the last colimit diagram maps by $\nu_{I_1,\dots,I_m}$ into

$$R\Gamma_{\mathrm{c}}\big(U_1^{I_1}\times\cdots\times U_m^{I_m}, \mathrm{DR}((aj)^!\mathcal{E}^{(I)})_{\mathrm{an}}\big)$$

for $I := I_1 \amalg\cdots\amalg I_m$. This last complex then maps by l to

$$R\Gamma_{\mathrm{c}}\big(U_0^I, \mathrm{DR}(\mathcal{E}^{(I)})_{\mathrm{an}}\big).$$

These maps induce the desired map of the homotopy colimits. This makes $\mathcal{A}$ into a pre-factorization algebra over X_{an}.

Proposition 6.1.2. *Let $\mathcal{E}$ be a h.r. factorizing $\mathcal{D}^!$-module on $X^{\mathscr{S}}$. Suppose that, in addition, each $\mathcal{E}^{(I)}$ is constructible w.r.t. the diagonal stratification of X^I. Then:*

(a) $\mathcal{A}_{\mathcal{E}}$ is a locally constant topological algebra on X_{an} in the C^∞ sense.

(b) We have

$$\int_X^{[[\mathrm{c}]]}\mathcal{E}\ \simeq\ \int_{X_{\mathrm{an}}}\mathcal{A}_{\mathcal{E}},$$

where in the RHS we have the factorization homology of a C^∞ factorization algebra.

Proof: (a) We prove the required properties.

(a1) $\mathcal{A}$ is a factorization algebra. By definition [CoG1], this means that:

(a11) $\mathcal{A}$ satisfied co-descent with respect to Weiss coverings.

(a12) If $U = U_1\sqcup\cdots\sqcup U_m$ is a disjoint union of several opens, then the map $\mu^U_{U_1,\dots,U_m}$ is a quasi-isomorphism.

To prove (a11), let $\{U_\alpha\}_{\alpha\in A}$ be a Weiss covering of an open $U \subset X_{\rm an}$, that is, for any finite set I the family $\{U_\alpha^I\}_{\alpha\in A}$ is an open covering of U^I. We can assume that $\{U_\alpha\}$ is closed under finite intersections. The indexing set A can be then assumed to be a poset (sub-poset in the poset of all opens in U) and considered as a category in a standard way. Under these assumptions, the co-descent condition means that the canonical morphism

$$\underset{\alpha\in A}{\underrightarrow{\rm holim}}\, \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}(U)$$

is a quasi-isomorphism. By definition of $\mathcal{A}$ this morphism can be written as

$$\underset{\alpha\in A}{\underrightarrow{\rm holim}}\ \underset{I\in\mathscr{S}}{\underrightarrow{\rm holim}}\, R\Gamma_{\rm c}(U_\alpha^I, {\rm DR}(\mathcal{E}^{(I)})_{\rm an}) \longrightarrow \underset{I\in\mathscr{S}}{\underrightarrow{\rm holim}}\, R\Gamma_{\rm c}(U^I, {\rm DR}(\mathcal{E}^{(I)})_{\rm an}).$$

We can interchange the colimits in the source. After this our statement follows from the fact that for each I the canonical arrow

$$\underset{\alpha\in A}{\underrightarrow{\rm holim}}\, R\Gamma_{\rm c}(U_\alpha^I, {\rm DR}(\mathcal{E}^{(I)})_{\rm an}) \longrightarrow R\Gamma_{\rm c}(U^I, {\rm DR}(\mathcal{E}^{(I)})_{\rm an})$$

is a quasi-isomorphism. This last fact is just a reflection of our assumption that $\{U_\alpha^I\}_{\alpha\in A}$ is a covering of U^I.

The property (a12) follows directly from the fact that $\mathcal{E}$ is a factorizing $\mathcal{D}^!$-module.

(a2) $\mathcal{A}$ is locally constant. Let $U_1 \hookrightarrow U_0$ be an embedding of disks in $X_{\rm an}$. Then for each I the embedding $U_1^I \hookrightarrow U_0^I$ is a homotopy equivalence compatible with respect to the diagonal stratification (that is, the embeddings of the corresponding strata are homotopy equivalences). By our assumptions, ${\rm DR}(\mathcal{E}^{(I)})_{\rm an}$ is a constructible complex on X^I with respect to the diagonal stratification. Therefore the natural arrow

$$R\Gamma_{\rm c}(U_1^I, DR(\mathcal{E}^{(I)})_{\rm an}) \longrightarrow R\Gamma_{\rm c}(U_0^I, DR(\mathcal{E}^{(I)})_{\rm an})$$

is a quasi-isomorphism. This means that $\mathcal{A}$ is locally constant. Part (a) of the proposition is proved.

(b) Since $\mathcal{E}^{(I)}$ is a holonomic regular $\mathcal{D}$-module on X^I, we have, for each I, a natural quasi-isomorphism

$$R\Gamma_{[[{\rm c}]]}(X^I, {\rm DR}(\mathcal{E}^{(I)})) \ \xrightarrow{u_I}\ R\Gamma_{\rm c}(X_{\rm an}^I, {\rm DR}(\mathcal{E}^{(I)})_{\rm an}).$$

The u_I combine into an arrow

$$\int_X^{[[c]]} \mathcal{E} \;=\; \varinjlim_{I\in\mathscr{S}}\operatorname{holim} R\Gamma_{[[c]]}(X^I, \mathrm{DR}(\mathcal{E}^{(I)}) \;\xrightarrow{u}$$

$$\xrightarrow{u} \; \varinjlim_{I\in\mathscr{S}}\operatorname{holim} R\Gamma_{\mathrm{c}}(X^I_{\mathrm{an}}, \mathrm{DR}(\mathcal{E}^{(I)})_{\mathrm{an}}) \;=\; \int_{X_{\mathrm{an}}} \mathcal{A}$$

which is a quasi-isomorphism since each u_I is. The proposition is proved.

6.2 The case of the tangent bundle

We now specialize the considerations of Ch. 5 and §6.1 to $\mathcal{E} = \psi(\check{\mathcal{C}}^\bullet)$ being the covariant Verdier dual to the cohomological Chevalley-Eilenberg $*$-sheaf $\check{\mathcal{C}}^\bullet = \check{\mathcal{C}}^\bullet_L$. We further specialize the local Lie algebra to $L = T_X$, the tangent bundle of X.

A. The diagonal $\mathcal{D}$-module. Recall from Definition 5.1.10 the diagonal $\mathcal{D}$-module $\check{\mathcal{C}}^\bullet_\Delta$ on X.

Lemma 6.2.1. *$\check{\mathcal{C}}^\bullet_\Delta$ is regular holonomic.*

Proof: We first show that $\check{\mathcal{C}}^\bullet_\Delta$ is holonomic. Let $x \in X$ be any point with the embedding $i_x : \{x\} \to X$. It suffices to show that for any x the !-fiber $i_x^! \check{\mathcal{C}}^\bullet_\Delta$ (a complex of $\mathcal{D}$-modules on $\{x\}$, i.e., of vector spaces) has bounded and finite-dimensional cohomology.

Consider the Verdier dual complex to $\check{\mathcal{C}}^\bullet_\Delta$ and denote it $\mathcal{C}^\Delta_\bullet$. Then the $[[*]]$-fiber $i_x^{[[*]]}\mathcal{C}^\Delta_\bullet$ is dual to $i_x^!\check{\mathcal{C}}^\bullet_\Delta$ and so it suffices to prove finite-dimensionality of the cohomology of all such fibers. Now, unravelling the definitions shows that $i_x^{[[*]]}\mathcal{C}^\Delta_\bullet = \mathrm{CE}_\bullet(W_x)$ is the homological Chevalley-Eilenberg complex of the topological Lie algebra $W_x = \mathrm{Der}(\widehat{\mathcal{O}}_{X,x})$ of formal vector fields near x. Here the Chevalley-Eilenberg complex is understood in the completed sense.

Since the homology of W_x is finite-dimensional by Gelfand-Fuchs [Fu], the holonomicity follows.

Next, we show that $\check{\mathcal{C}}^\bullet_\Delta$ is regular. For this, we denote this $\mathcal{D}_X$-module by N_X and study its dependence on X. That is, if $j : X \to X'$ is an open embedding of smooth algebraic varieties, then $N_X = j^* N_{X'}$. Now, if X' is compact, then $N_{X'}$, being a local system on X', is regular. Embedding any

X into a smooth proper X' we see that N_X is also regular, being the pullback of a regular $\mathcal{D}$-module. The lemma is proved. □

Remark 6.2.2. Lemma 6.2.1 shows that the diagonal complex $\mathcal{F}^\bullet_\Delta = \mathrm{DR}(\check{\mathcal{C}}^\bullet_\Delta)$ is the *local system of the Gelfand-Fuchs cohomology.*

B. Comparison with analytification. Recall that $\mathbf{k} = \mathbb{C}$.

Theorem 6.2.3. *(a) The factorization algebra $\mathcal{A}_{\psi(\check{\mathcal{C}}^\bullet)}$ on X_{an} is locally constant.*

(b) The canonical map

$$\int_X^{[[\mathrm{c}]]} \psi(\check{\mathcal{C}}^\bullet) \longrightarrow R\Gamma^{\mathrm{c}}(X^{\mathscr{S}}_{\mathrm{an}}, \mathrm{DR}(\psi(\check{\mathcal{C}}^\bullet))_{\mathrm{an}}) \;=\; \int_{X_{\mathrm{an}}} \mathcal{A}_{\psi(\check{\mathcal{C}}^\bullet)}$$

is an equivalence.

Proof: We know that $\check{\mathcal{C}}^\bullet$ is a factorizing coherent $[[\mathcal{D}]]$-module on $X^{\mathscr{S}}$. By Theorem 4.6.1, $\psi(\check{\mathcal{C}}^\bullet)$ is factorizable. Now Lemma 6.2.1 implies that $\check{\mathcal{C}}^\bullet_\Delta \simeq \psi(\check{\mathcal{C}}^\bullet)^{(\mathrm{pt})}$ is holonomic regular. Proposition 6.1.1 implies that $\psi(\check{\mathcal{C}}^\bullet)$ is a h.r. factorizing $\mathcal{D}^!$-module on $X^{\mathscr{S}}$. After this the theorem becomes an application of Proposition 6.1.2. □

Combining Corollary 5.2.4 with Theorem 6.2.3, we find:

Corollary 6.2.4. *For X a smooth affine variety over $\mathbf{k} = \mathbb{C}$, we have*

$$\mathrm{CE}^\bullet(T(X)) \;\simeq\; \int_{X_{\mathrm{an}}} \mathcal{A}_{\psi(\check{\mathcal{C}}^\bullet)}.$$

C. The structure of the factorization algebra $\mathcal{A}$. Denote $\mathcal{A} = \mathcal{A}_X$ the locally constant factorization algebra $\mathcal{A}_{\psi(\check{\mathcal{C}}^\bullet)}$. We note, first of all, that $\mathcal{A}$ is naturally a factorization algebra in the category CDGA. This is because all the steps in constructing $\mathcal{A}$ can be done in CDGA. So by Proposition 1.2.5, $\mathcal{A}$ is a cosheaf of cdga's on X_{an}.

The complex manifold X_{an} can be seen as a C^∞-manifold of dimension $2n$ with $GL_n(\mathbb{C})$-structure in the sense of Definition 1.3.12. Let $\mathbb{GL}_{n,\mathbb{C}}$ be the algebraic group GL_n with field of definiiton $\mathbb{C}$. Any cdga A with a $\mathbb{GL}^*_{n,\mathbb{C}}$-action (Definition 1.3.7) has a BL-action of the Lie group $GL_n(\mathbb{C})$

and so gives rise to a locally constant cosheaf of cdga's $\underline{A} = \underline{A}_{X_{\mathrm{an}}}$ on X_{an}, see Proposition 1.3.13.

We recall from §1.4C that the cdga $\mathrm{CE}^\bullet(W_n(\mathbb{C}))$ has a natural $\mathbb{GL}^*_{n,\mathbb{C}}$-action and so we have the cosheaf of cdga's $\underline{\mathrm{CE}^\bullet(W_n(\mathbb{C}))}$ on X_{an}.

Proposition 6.2.5. *The cosheaf of cdga's $\mathcal{A}_X$ on X_{an} is identified with $\underline{\mathrm{CE}^\bullet(W_n(\mathbb{C}))}$. In particular,*

$$\int_{X_{\mathrm{an}}} \mathcal{A}_X \simeq \int_{X_{\mathrm{an}}} (\mathrm{CE}^\bullet(W_n(\mathbb{C})))$$

is the factorization homology of the complex manifold X_{an} with coefficients in $\mathrm{CE}^\bullet(W_n(\mathbb{C}))$.

Proof: Let $U \subset X_{\mathrm{an}}$ be a disk. Applying Theorem 5.5.4.14 of [Lu-HA], we see that the natural arrow

$$R\Gamma_c\big(U, \mathrm{DR}(\psi(\check{\mathcal{C}}^\bullet)^{(\mathrm{pt})}_{\mathrm{an}})\big) \longrightarrow \mathcal{A}(U) := \underset{I\in\mathscr{S}}{\underrightarrow{\mathrm{holim}}}\, R\Gamma_c\big(U^I, \mathrm{DR}(\psi(\check{\mathcal{C}}^\bullet)^{(I)}_{\mathrm{an}})\big)$$

is a quasi-isomorphism. Further, by definition,

$$R\Gamma_c\big(U, \mathrm{DR}(\psi(\check{\mathcal{C}}^\bullet)^{(\mathrm{pt})}_{\mathrm{an}})\big) = R\Gamma_c(U, \mathcal{F}^\bullet_\Delta)$$

is the compactly supported cohomology with coefficients in the diagonal complex, see (5.1.11).

From this point on the proof proceeds similarly to that of Proposition 1.4.6. Our cosheaf $\underline{\mathrm{CE}^\bullet(W_n)}$ is the inverse of the locally constant sheaf of cdga's $[\mathrm{CE}^\bullet(W_n)]_{X_{\mathrm{an}}}$. So we construct, for each disk U, a family (parametrized by a contractible space T_U) of quasi-isomorphisms

$$q_U : [\mathrm{CE}^\bullet(W_n]_{X_{\mathrm{an}}}(U) \longrightarrow R\Gamma_c(U, \mathcal{F}^\bullet_\Delta)$$

so that for any inclusion of disks $U_1 \subset U_0$ we have a commutative diagram analogous to (1.4.7).

To do this, for any point $x \in X_{\mathrm{an}}$ we define the Lie $\mathbb{C}$-algebras W_x and W_{T_xX} of formal vector fields on X near x and on T_xX near 0 respectively. We have a contractible space of identifications $W_x \to W_{T_xX}$ parametrized by formal isomorpisms $\phi : (T_xX, 0) \to (X, x)$ identical on the tangent spaces.

Recall that $\mathcal{F}^\bullet_\Delta$ is the (shifted) local system of Gelfand-Fuchs cohomology. Therefore for any disk $U \subset X_{\mathrm{an}}$ and any $x \in U$ the pullback map

$$\mathrm{CE}^\bullet(W_x) \to R\Gamma_c(U, \mathcal{F}^\bullet_\Delta) \tag{6.2.6}$$

is a quasi-isomorphism. Further, just like in Proposition 1.4.8(a), the sheaf $[\mathrm{CE}^\bullet(W_n)]_{X_{\mathrm{an}}}$ has, as the stalk at $x \in X$, the complex $\mathrm{CE}^\bullet(W_{T_xX})$. So our maps q_U are constructed in the same way as in (1.4.9), with $R\Gamma_c(U, \mathcal{F}^\bullet_\Delta)$ instead of $\mathrm{CE}^\bullet(\mathrm{Vect}(U))$ and the quasi-isomorphisms (6.2.6) instead of the maps r_x of Proposition 1.4.8(b). □

6.3 Main result

Let now X be a smooth variety over $\mathbb{C}$ of complex dimension n. Using the $GL_n(\mathbb{C})$-action on Y_n, we form the *holomorphic Gelfand-Fuchs fibration* $\underline{Y}_X \to X_{\mathrm{an}}$ with fiber Y_n. Note that Y_n is $2n$-connected, and so the non-Abelian Poincaré duality Theorem 1.3.17 applies to $\underline{Y}_X \to X_{\mathrm{an}}$ (the real dimension of X_{an} is also $2n$).

Combining Theorem 1.4.10 with Proposition 6.2.5 and Theorem 1.3.17, we obtain:

Theorem 6.3.1. *(a) Let X be any smooth algebraic variety over $\mathbb{C}$. Then we have identifications*

$$\int_X^{[[c]]} \psi(\check{\mathcal{C}}^\bullet) \simeq \int_{X_{\mathrm{an}}} \mathcal{A}_{\psi(\check{\mathcal{C}}^\bullet)} \simeq H^\bullet_{\mathrm{top}}(\mathrm{Sect}(\underline{Y}_X/X_{\mathrm{an}}), \mathbb{C}),$$

where on the right we have the space of continuous sections of $\underline{Y}_X$ over X_{an} (considered as just a topological space).

(b) In particular, the canonical arrow $\int_x^{[[c]]} \psi(\check{\mathcal{C}}^\bullet) \to \int_x^{[[c]]} \check{\mathcal{C}}^\bullet$ gives rise to a natural morphism of commutative algebras

$$\tau_X : H^\bullet_{\mathrm{top}}(\mathrm{Sect}(\underline{Y}_X/X_{\mathrm{an}}), \mathbb{C}) \longrightarrow H^\bullet_{\mathrm{Lie}} R\Gamma(X, T_X),$$

compatible with pullbacks under étale maps $X' \to X$.

Applying now Corollary 6.2.4, we obtain our main result:

Theorem 6.3.2. *(a) Let X be a smooth affine variety over $\mathbb{C}$. Then τ_X is an isomorphism, i.e., we have a commutative algebra isomorphism*

$$H^\bullet_{\mathrm{Lie}}(T(X)) \simeq H^\bullet_{\mathrm{top}}(\mathrm{Sect}(\underline{Y}_X/X_{\mathrm{an}}), \mathbb{C}).$$

(b) In particular, $H^\bullet_{\mathrm{Lie}}(T(X))$ is finite-dimensional in each degree and is an invariant of n, of the rational homotopy type of X_{an} and of the Chern classes $c_i(T_X) \in H^{2i}(X, \mathbb{Q})$. □

Example 6.3.3. Let X be an elliptic curve. The tangent bundle T_X is trivial, so $\underline{Y}_X = X_{\mathrm{an}} \times Y_1$. Further, Y_1 is is homotopy equivalent to the 3-sphere S^3, while X_{an} is homeomorphic to $S^1 \times S^1$. So $\mathrm{Sect}(\underline{Y}_X/X_{\mathrm{an}})$ is homotopy equivalent to $\mathrm{Map}(S^1 \times S^1, S^3)$ and has cohomology in infinitely many degrees but finite-dimensional in each given degree. On the other hand,

$$R\Gamma(X, T_X) \simeq \mathbb{C} \oplus \mathbb{C}[-1]$$

is an abelian dg-Lie algebra and so $H^0_{\mathrm{Lie}} R\Gamma(X, T_X) = \mathbb{C}[[q]]$ is infinite-dimensional pro-finite. This shows that τ_X cannot be an isomorphism in general.

6.4 Examples of explicit calculations of $H^\bullet_{\mathrm{Lie}}(T(X))$

A. Curves: Krichever-Novikov algebras. Let X be a smooth affine curve. Assume that X is of genus $g \geqslant 0$ with $m \geqslant 1$ punctures. The Lie algebra $T(X)$ is known as a *Krichever-Novikov algebra*, see [KN] [S].

Theorem 6.3.2 in this case gives the following. The Gelfand-Fuchs skeleton Y_1 is homotopy equivalent to the 3-sphere S^3. The space X_{an} is homotopy equivalent to a bouquet of $\nu = 2g + m - 1$ circles, and so the complex tangent bundle T_X is topologically trivial. Therefore the fibration $\underline{Y}_X \to X$ is trivial, identified, up to homotopy equivalence, with $X \times S^3$. The space of sections $\mathrm{Sect}(\underline{Y}_X/X)$ is therefore identified with the mapping space $\mathrm{Map}(X, S^3)$ and we obtain:

$$H^\bullet_{\mathrm{Lie}}(T(X)) \simeq H^\bullet_{\mathrm{top}}\Big(\mathrm{Map}\big(\bigvee_{i=1}^{\nu} S^1, S^3\big), \mathbb{C}\Big). \tag{6.4.1}$$

The analytic version of this statement (involving all analytic vector fields and their continuous cohomology) has been proved earlier in [Ka].

B. Complexifications. An interesting class of examples is obtained by considering n-dimensional complex affine varieties X which are in fact defined over $\mathbb{R}$ so that the space of $\mathbb{R}$-points $M = X(\mathbb{R})$ is a smooth compact C^∞-manifold of dimension n, homotopy equivalent to X_{an}. In such cases the algebro-geometric cohomology $H^\bullet_{\text{Lie}}(T(X))$ is, by Theorem 6.3.2, identified with the C^∞ cohomology $H^\bullet_{\text{Lie}}(\text{Vect}(M)) \otimes_{\mathbb{R}} \mathbb{C}$. Examples include:

(a) $X = \mathbb{A}^1 - \{0\}$, $M = S^1$.

(b) $X = GL_n$, $M = U(n)$.

(c) X is the affine quadric $\sum_{i=0}^n z_i^2 = 1$, M is the sphere S^n.

C. $\mathbb{P}^n$ minus a hypersurface. Suppose $X = \mathbb{P}^n - Z$ where Z is a smooth hypersurface of degree d. In this case we have, first of all:

Proposition 6.4.2. *The Chern classes of T_X vanish rationally.*

Proof: Indeed, they are the restrictions of the Chern classes of T_{P^n} which lie in

$$H^*(\mathbb{P}^n_{\text{an}}, \mathbb{C}) = \mathbb{C}[h]/h^{n+1}, h = c_1(\mathcal{O}(1)).$$

Now, $dh = c_1(\mathcal{O}(d))$ vanishes on X since Z is the zero locus of a section of $\mathcal{O}(d)$. Therefore $h|_X = 0$ as well, and similarly for all powers of h. □

Now, all the information about the fibration $\underline{Y}_X \to X_{\text{an}}$ which we use, is contained in the Chern classes of T_X, as we are dealing with rational homotopy types. Therefore Theorem 6.3.2 gives that

$$H^\bullet_{\text{Lie}}(T(X)) \ = \ H^\bullet_{\text{top}}(\text{Map}(X_{\text{an}}, Y_n), \mathbb{C}).$$

Let us now identify the rational homotopy type of X_{an}. Let us think of $\mathbb{P}^n$ as the projectivization of $\mathbb{C}^{n+1}$ and let $f(x_0, \dots, x_n)$ be the homogeneous polynomial of degree d defining Z. Without loss of generality, we can take $f = x_0^d + \cdots + x_n^d$. Let $W \subset \mathbb{C}^{n+1}$ be given by $f = 1$. We then have the Galois covering $p : W \to X_{\text{an}}$ with Galois group $\mathbb{Z}/d$ of dth roots of 1 acting diagonally on $\mathbb{C}^{n+1}$.

Now W is the "Milnor fiber" for the isolated singularity $f = 0$. (We could define W by $f = \varepsilon$ for any small ε with the same effect). Therefore by Milnor's theorem [Mi], W is homotopy equivalent to a bouquet of μ spheres S^n.

Here $\mu = (d-1)^{n+1}$ is the Milnor number of the singularity. So topological cohomology of W is $\mathbb{C}^\mu$ in degree n and 0 elsewhere (except $H^0 = \mathbb{C}$).

Further, the vanishing of the higher cohomology of the group $\mathbb{Z}/d$ with coefficients in any $\mathbb{C}$-module and the Leray spectral sequence of the Galois covering p, combined with the theory of rational homotopy type, imply:

Proposition 6.4.3. *The rational homotopy type of X_{an} is that of a bouquet of ν spheres, where ν is the dimension of the invariant subspace $H^n(W)^{\mathbb{Z}/d}$.* □

The number ν can be found explicitly by using the fact, standard in the theory of singularities [Wal], that the space $H^n(W)$ (the space of vanishing cycles for f) has the same $\mathbb{Z}/d$-character as the Jacobian quotient of the module of volume forms

$$\Omega^{n+1}(\mathbb{A}^{n+1})\big/\big(\partial f/\partial x_i\big)_{i=0}^{n} \;=\; \det(\mathbb{C}^{n+1})\otimes\mathbb{C}[x_0,\dots,x_n]\big/(x_0^{d-1},\dots,x_n^{d-1}).$$

So ν is equal to the number of monomials

$$x_0^{i_0}\cdots x_n^{i_n}, \quad 0\leqslant i_k\leqslant d-2, \quad \sum i_k \equiv -n-1 \mod d.$$

We get a statement of the form similar to (6.4.1):

Corollary 6.4.4.

$$H^\bullet_{\mathrm{Lie}}(T(X)) \;\simeq\; H^\bullet\left(\mathrm{Map}\big(\bigvee_{i=1}^{\nu} S^n, Y_n\big), \mathbb{C}\right).$$

Example 6.4.5. Let $d=2$, i.e., Z is a smooth quadric hypersurface. Then W (complex affine quadric) is homotopy equivalent to S^n and so $X_{\mathrm{an}} = \mathbb{CP}^n - Z_{\mathrm{an}}$ is homotopy equivalent to $\mathbb{RP}^n$. The rational homotopy type of $\mathbb{RP}^n$ is that of a point, if n is even and is that of S^n, if n is odd. This means that ν is equal to 0 or 1 in the correponding cases. Accordingly

$$H^\bullet_{\mathrm{Lie}}(T(\mathbb{P}^n - Z)) = \begin{cases} H^\bullet(Y_n,\mathbb{C}), & \text{if } n \text{ is even;}\\ H^\bullet(\mathrm{Map}(S^n, Y_n),\mathbb{C}), & \text{if } n \text{ is odd.}\end{cases}$$

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