université
PARIS-SACLAY

Méthodes homotopiques et algèbres de Lie de dimension infinie

Habilitation à diriger des recherches
de l'Université Paris-Saclay

présentée et soutenue à Orsay,
le lundi 5 décembre 2022, par

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Habilitation à diriger des recherches

Résumé

Dans ce mémoire d'habilitation à diriger des recherches, j'expose quelques résultats récents sur l'étude des variétés algébriques complexes lisses au travers d'algèbres de Lie à la fois de dimension infinie, et de nature homotopique. La plupart des résultats en question sont bien connus dans le cas des courbes, et je mettrai en avant les difficultés rencontrées pour les étendre aux variétés de plus grande dimension. On parlera en particulier de généralisations en dimension supérieure des algèbres de Kac–Moody et de Virasoro, et de leur relation (en particulier dans le cas des algèbres de Kac–Moody) avec des espaces de modules dérivés associés à des variétés lisses. Le cas des algèbres de Virasoro soulèvera le problème plus fondamental du calcul de la cohomologie de l'algèbre de Lie des champs de vecteurs (aussi appelée cohomologie de Gelfand–Fuchs). Je donnerai également quelques pistes de recherche future, plus ou moins avancées, dans la même direction d'une part, mais aussi dans une direction nouvelle, autour des invariants de Donaldson–Thomas.

Abstract

This habilitation thesis gives an overview of recent results, about the study of smooth complex algebraic varieties using infinite dimensional Lie algebras of homotopical nature. Most of said results are well-known in the case of curves, and we will highlight the new difficulties arising in higher dimensions. We will on particular discuss generalizations of Kac–Moody and Virasoro algebras, and their ties to moduli spaces. The Virasoro case will raise the more fundamental question of computing the cohomology of Lie algebras of vector fields of algebraic varieties (also called the Gelfand–Fuks cohomology). A couple of future research leads, in various states of completion, first in the same vein as the work explained here, but also in a new direction, around the categorification of Donaldson–Thomas invariants

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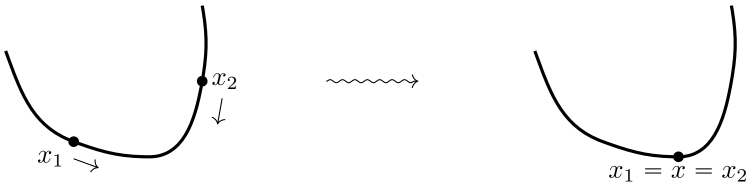
Introduction

Ce mémoire résume mes travaux, réalisés depuis la soutenance de ma thèse de doctorat. Ces travaux consistent essentiellement en l'utilisation de méthodes homotopiques modernes en vue d'étudier divers problèmes de modules algébriques, en lien avec la théorie des cordes et les théories des champs holomorphes[1].

La plupart de ces travaux sont liés, quoique de façon indirecte, à la notion de théorie des champs holomorphes en dimension supérieure. En dimension complexe 1 (on parle alors plutôt de théorie des champs conforme de dimension (réelle) 2), il existe plusieurs formulations mathématiques de cette notion : algèbre vertex (voir [Kac96 ; FB04]), algèbres chirales ou algèbres à factorisations (voir [BD04]). De ces formulations, deux ont été étendues à la dimension supérieure : les algèbres chirales et les algèbres à factorisations, à la fois dans un contexte algébrique (voir [FG11]) et dans un contexte analytique (voir [CG16 ; CG21]).

De manière très informelle, nous pouvons (en dimension 1) penser à ces théories comme à des fibrés sur une courbe complexe, munis d'une opération entre les fibres lorsque deux points de la courbe collisionnent :



$$\mathcal{F}_{x_1} \otimes \mathcal{F}_{x_2} \longrightarrow \mathcal{F}_x. \tag{1}$$

Des exemples de ces théories en dimension supérieure ont été construits à partir de l'étude d'algèbres de Lie de dimension infinie et de nature homotopique (des dg-algèbres de Lie ou des $\mathcal{L}_\infty$-algèbres) liées à l'étude de problèmes de modules impliquant des variétés de dimension (complexe) supérieure ou égale à 2. Une majeure partie de ce mémoire est consacré à la construction et à l'étude de ces algèbres de Lie, sur un corps $\mathbf{k}$ de caractéristique nulle.

Le premier exemple que nous aborderons est celui des algèbres de Kac–Moody (affine) associées à un groupe réductif G, bien connu en dimension 1. Explicitement, l'algèbre de Kac–Moody associée à G est une extension centrale de l'algèbre de Lie $\mathfrak{g}(\!(t)\!)$ des séries de Laurent à coefficients dans l'algèbre de Lie $\mathfrak{g}$ de G. Elle apparaît en lien avec l'espace de modules des G-fibrés sur une courbe complexe lisse. De cet espace de modules, on construit une théorie des champs conformes en dimension 2 (holomorphes en dimension 1), par différentes méthodes, selon le modèle visé. Géométriquement par exemple, la Grassmannienne affine de G (ou sa cohomologie) est munie d'une structure d'algèbre à factorisations. De l'autre côté, algébriquement, la représentation du vide de l'algèbre de Kac–Moody est une algèbre vertex, représentant la même théorie. Pour passer d'une version à l'autre, il faut observer que les séries de Laurent représentent les fonctions sur un voisinage formel épointé $\widehat{\mathrm{D}}_x^\circ$ d'un point x sur notre courbe. C'est en considérant ces voisinages formels épointés lorsque x varie en famille, ou plutôt lorsque deux points x_1 et x_2 collisionnent, que les structures que nous observons apparaissent.

En dimension supérieure, il s'agira d'étudier l'espace de modules des G-fibrés non plus sur une courbe, mais sur une variété lisse de dimension d quelconque. Du côté géométrique, deux subtilités supplémentaires apparaissent. La première, le problème de modules des G-fibrés n'est plus lisse. Afin de conserver de bonnes propriétés de théorie de la déformation, nous devrons munir ce problème de modules d'une structure dérivée. La géométrie dérivée est en effet une généralisation de la géométrie algébrique dans laquelle la théorie de la déformation se comporte particulièrement

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[1] Le rôle de ces théories en physique dépasse, de loin, le contenu de ce mémoire comme les connaissances de l'auteur.

bien. La seconde subtilité concerne l'analogue des séries de Laurent et les différentes versions de notions de factorisations : par exemple en dimension 2, deux possibilités s'offrent à nous :

La version torique : il s'agit de remplacer les séries de Laurent $\mathbf{k}(\!(t)\!)$ par des séries de Laurent à deux variables $\mathbf{k}(\!(x)\!)(\!(y)\!)$. On remplace donc notre voisinage formel épointé $\widehat{\mathrm{D}}^\circ$ par un disque formel de dimension 2 privé de deux axes. En ce qui nous concerne, cette approche, voisine de la notion d'adèles, a un désavantage. Algébriquement, l'anneau $\mathbf{k}(\!(x)\!)(\!(y)\!)$ dépend d'un choix dans l'ordre des variables. Géométriquement, cela engendre un dissymétrie entre les axes que l'on retire. De plus, ce disque formel privé de deux axes, ne peut être associé à un point dans la surface. Il faudra plus de données (typiquement, un drapeau). Il est dès lors difficile de définir les bonnes notions (par exemple d'algèbre à factorisations «stratifiées»), ou de se former de bonnes intuitions correspondantes à l'idée du diagramme (1). Notons cependant que cette direction est l'objet d'un travail en cours, en collaboration avec Valerio Melani et Gabriele Vezzosi. Nous n'en dirons toutefois pas plus dans ce mémoire.

La version sphérique : il s'agit ici de conserver l'intuition du diagramme (1), en remplaçant simplement la courbe par une surface (ou plus généralement une variété lisse). Les points évoluent ainsi librement sur la surface. Cette idée correspond exactement à la notion d'algèbres à factorisations en dimension supérieure, telle qu'étudiée par exemple dans [FG11] ou [HK22]. C'est la version que nous allons choisir, pour l'algèbre de Kac–Moody comme pour les autres généralisations à suivre.

D'un point de vue local, le disque formel (épointé) de dimension 1 est simplement remplacé par son analogue en dimension supérieure $\widehat{\mathrm{D}}^\circ := \operatorname{Spec}(\mathbf{k}[\![t_1, \dots, t_d]\!]) \smallsetminus \{0\}$. Cependant, contrairement au cas $d = 1$, ce schéma n'est plus affine. Le rôle des séries de Laurent devra donc être rempli, non plus par les simples fonctions globales sur $\widehat{\mathrm{D}}^\circ$, mais par toute la cohomologie $\mathrm{H}^\bullet(\widehat{\mathrm{D}}^\circ, \mathcal{O}_{\widehat{\mathrm{D}}^\circ})$, munie de sa structure algébrique à homotopie près (cup produit, mais aussi produits de Massey supérieurs). On pourra alors construire une version homotopique de l'extension de Kac–Moody, semblable à celle apparaissant en dimension 1.

Nous aborderons également une généralisation à la dimension supérieure de l'algèbre de Virasoro. En dimension 1, il s'agit de l'unique (à scalaire près) extension de l'algèbre de Witt, algèbre de Lie $\mathbf{k}(\!(t)\!)\frac{d}{dt}$ des champs de vecteurs sur le disque épointé $\widehat{\mathrm{D}}^\circ$. Comme précédemment, nous aborderons une version sphérique en dimension supérieure. Puisque le disque épointé n'est plus affine en dimension supérieure, il faudra considérer, non pas simplement les champs de vecteurs globaux, mais toute la cohomologie desdits champs de vecteurs, vue comme une $\mathcal{L}_\infty$-algèbre :

$$\mathrm{Witt}_d := \mathrm{H}^\bullet(\widehat{\mathrm{D}}^\circ, \mathrm{T}_{\widehat{\mathrm{D}}^\circ}).$$

La situation est toutefois sensiblement plus complexe que dans le cas des algèbres de Kac–Moody. En effet, la classification des extensions de Witt_d n'est à ce jour pas connue (voir conjecture 3.2.2 ci-dessous).

De manière générale, la classification des extensions centrales d'algèbres de Lie de champs de vecteurs passent par l'étude de la cohomologie de Chevalley–Eilenberg de ces algèbres de Lie, connue sous le nom de cohomologie de Gelfand–Fuchs. Elle fut très étudiée dans le cas des variétés différentiables dans les 70 et 80, après les articles fondateurs de Gelfand et Fuchs [GF68 ; GF69], en particulier pour ses applications aux feuilletages. Avec comme objectif (inatteint) de classifier les extensions de cette $\mathcal{L}_\infty$-algèbre de Witt en dimension supérieure, nous étudierons ensuite la cohomologie de Gelfand–Fuchs des variétés algébriques (lisses).

Cette étude, dont les résultats principaux sont le sujet de [HK22], repose sur les algèbres à factorisations, à la fois dans un contexte algébrique, et dans un contexte différentiable. Elle permet de prouver une conjecture, énoncée par Feigin dans les années 1980, concernant la cohomologie de Gelfand–Fuchs des variétés algébriques affines lisses sur un corps de caractéristique nulle. Ce résultat implique en particulier que la cohomologie de Gelfand–Fuchs d'une variété algébrique affine lisse est de dimension finie en chaque degré, et donne une méthode de calcul de cette cohomologie, en termes du type d'homotopie rationnel de l'analytification de la variété d'origine.

Je donnerai ensuite quelques pistes, encore à concrétiser, visant à classifier les extensions de Witt_d, et à prouver la conjecture 3.2.2, utilisant en particulier la notion de module sur les algèbres à factorisations. Ce mémoire se conclura sur un travail en cours, sans lien direct avec ce qui précède, autour des invariants de Donaldson–Thomas.

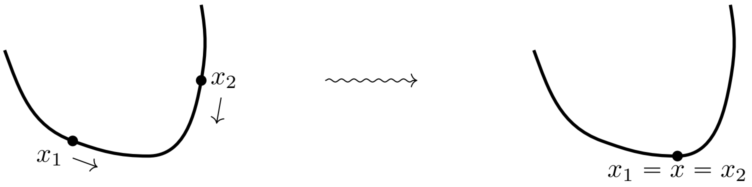
Structure du mémoire : La première partie traitera des algèbres de Kac–Moody en dimension supérieure. Après avoir rappelé quelques fait concernant la dimension 1, nous introduirons les nouveaux outils nécessaires en dimension supérieure, en particulier la géométrie dérivée. Dans la seconde partie, nous ferons un écart en théorie des déformations dérivée, appliquée à la K-théorie. Cela permettra de retrouver un résultat de la première partie, de manière plus conceptuelle. La troisième partie évoquera la question des algèbres de Virasoro en dimension supérieure, soulignera les difficultés rencontrées et énoncera une conjecture à leur sujet. La cohomologie des Gelfand–Fuchs des variétés algébriques et différentiables sera à l'honneur dans une quatrième partie. Enfin, la cinquième et dernière partie parlera de différentes pistes de recherche, à différents stades d'avancement, en lien avec les problématiques abordées dans ce mémoire.

Introduction

This habilitation thesis gives an overview of the results I obtained since the defence of my PhD thesis. My work revolves around the use of homotopical methods in the study of various moduli spaces in algebraic geometry. More specifically, the moduli spaces at hand will have ties to string theory and holomorphic field theories[2].

Most of the work presented here is related, although somewhat indirectly, to the notion of holomorphic field theory in higher dimension. In the case of complex dimension 1 (most often referred to as (real) 2-dimensional conformal field theories), there exist several mathematical formulation of the notion: namely vertex algebras (see [Kac96; FB04]), chiral algebras or factorization algebras (see [BD04]). Of those formulations, two have been extended to the higher dimensional case : chiral algebras and factorization algebras, both in an algebraic context (see [FG11]) and in a complex analytic context (see [CG16; CG21]).

Very informally, we can think of such theories (in dimension 1) as bundles over a complex curve endowed with an operation between the stalks, as two points (vertices) on the curve collide:



$$\mathcal{F}_{x_1} \otimes \mathcal{F}_{x_2} \longrightarrow \mathcal{F}_x. \tag{1}$$

Examples of such theories in higher dimension have been constructed from the study of some Lie algebras, of infinite dimension and of homotopical nature (dg-Lie algebras or $\mathcal{L}_\infty$-algebras), related to the study of moduli problems involving higher dimensional varieties. For the most part, this thesis is devoted to the construction and study of such Lie algebras, over a field $\mathbf{k}$ of characteristic 0.

The first example we will study is that of (affine) Kac–Moody algebras associated to a reductive group G. This example is very well studied in dimension 1. Explicitly, the Kac–Moody algebra associated to G is a central extension of the Lie algebra $\mathfrak{g}((t))$ of Laurent series with coefficients in the Lie algebra $\mathfrak{g}$ of G. It appears naturally in relation with the moduli space of principal G-bundles on a smooth complex curve. From this moduli space, we can construct a $2D$-conformal field theory ($1D$-holomorphic field theory) via various methods, depending on the targeted model. Geometrically for instance, the affine Grassmannian of G (or maybe its (co)homology) is endowed with a factorization algebra structure. On the other hand, algebraically, the vacuum representation of the Kac–Moody Lie algebra is a vertex algebra, representing the same theory. To pass from one version to the other, observe that Laurent series play the role of functions on a punctured formal neighbourhood $\widehat{\mathrm{D}}_x^\circ$ of a point on our curve. The additional structures (factorization or vertex algebras) will appear when considering the collision of the punctured neighbourhoods at two points x_1 and x_2.

In higher dimensions, we will study the moduli space of g-bundles, not on a curve, but on a d-dimensional smooth variety. From the geometric perspective, two subtleties appear. Firstly, the moduli space of G-bundles is no longer smooth in general. In order to keep having good deformation theoretic properties, we need to endow this moduli space with a derived structure (in the sense of derived geometry). Derived (algebraic) geometry is indeed a generalisation of algebraic geometry in which deformation theory behaves particularly well.

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[2] The physics behind those notions goes way beyond the author's understanding, and it will not be included in the thesis.

The second subtlety is the choice of an analog for Laurent series, and different notions of factorization. For instance, in dimension 2, there are (at least) two distinct approaches:

The toric version: replacing the field of Laurent series $\mathbf{k}(\!(t)\!)$ by that of Laurent series in 2-variables $\mathbf{k}(\!(x)\!)(\!(y)\!)$. This amounts to replacing our punctured formal neighbourhood $\widehat{\mathrm{D}}^\circ$ by a 2-dimensional formal disk minus two axes. For us, this approach, which is close to the notion of adèles, has a drawback. Algebraically, the field $\mathbf{k}(\!(x)\!)(\!(y)\!)$ depends on a choice of ordering of the variables. Geometrically, this breaks the symmetry between the to removed axes. Moreover, this formal disk deprived of two axes can no longer be associated to a point in a surface. More data is required (typically, that of a flag). It is then much harder to define the appropriate notions (for instance of "stratified" factorization algebras), or to form good intuitions, similar to the idea represented in diagram (1). Nonetheless, this direction is the object of work in progress with Valerio Melani and Gabriele Vezzosi, although this work will not be described in this thesis.

The spherical version: Trying to keep the intuition of diagram (1), we simply replace the curve with a surface (or more generally a smooth variety). We let points evolve freely on the surface. This idea corresponds precisely to the notion of factorization algebras in higher dimension, as studied for instance in [FG11] or [HK22]. This is the version we will pursue in the work, for the Kac–Moody algebra as for other generalizations to come.

From a local point of view, the punctured formal disk is simply replaced by its higher dimensional analog $\widehat{\mathrm{D}}^\circ := \mathrm{Spec}(\mathbf{k}[\![t_1, \dots, t_d]\!]) \smallsetminus \{0\}$. However, the scheme is no longer affine as soon as $d \geqslant 2$. The role of the field of Laurent series will then have to be taken, not by mere functions on $\widehat{\mathrm{D}}^\circ$, but by the entire cohomology $\mathrm{H}^\bullet(\widehat{\mathrm{D}}^\circ, \mathcal{O}_{\widehat{\mathrm{D}}^\circ})$, endowed with its algebraic structure up to homotopy (i.e. the cup product, but also higher order Massey operations). We will then be able to construct and study a homotopical version of the Kac–Moody central extension, similar to the one existing in dimension 1.

We will also, in this thesis, discuss a higher dimensional generalization of the Virasoro algebra. In dimension 1, it is the unique (up to scalar) central extension of the Witt algebra – the Lie algebra $\mathbf{k}(\!(t)\!)\frac{d}{dt}$ of vector fields on the punctured disk $\widehat{\mathrm{D}}^\circ$. As before, we shall study a spherical version in higher dimension, and since the punctured disk is no longer affine, we must consider the whole cohomology of vector fields, seen as an $\mathcal{L}_\infty$-algebra:

$$\mathrm{Witt}_d := \mathrm{H}^\bullet(\widehat{\mathrm{D}}^\circ, \mathrm{T}_{\widehat{\mathrm{D}}^\circ}).$$

The setting is however considerably more complex than on the case of Kac–Moody algebra. The classification of central extensions of Witt_d is not know to this day and to the author (see conjecture 3.2.2 below).

Generally speaking, the study of central extensions of Lie algebras of vector fields goes through the study of their Chevalley–Eilenberg cohomology, also known as the Gelfand–Fuchs cohomology of the manifold or variety at hand. This cohomology was the focus of a lot of articles in the 70's and 80's, in the differentiable case. This started with the founding papers of Gelfand and Fuchs [GF68; GF69], in particular for the applications to foliations. With the (not yet achieved) objective of classifying extensions of this Witt $\mathcal{L}_\infty$-algebra in higher dimension, we will study the Gelfand–Fuchs cohomology of smooth algebraic varieties.

The most significant results in this direction so far are the topic of [HK22]. They rely heavily on factorization algebras, both in an algebraic context and in a differentiable context. Those results prove in particular a conjecture stated by Feigin in the 80's, about the Gelfand–Fuchs cohomology of smooth affine algebraic varieties over a field of characteristic 0. Namely, we prove that the Gelfand–Fuchs cohomology of smooth affine algebraic varieties is finite dimensional in every degree (but there can be an infinite number of non-vanishing cohomology groups). We further give a computational method for this cohomology, in terms of the rational homotopy type of the analytification of the variety.

I will then discuss some ongoing research and leads, targeted, among other things, at classifying the central extensions of Witt_d (and thus at proving conjecture 3.2.2). A key ingredient will be the notion of factorization modules over factorization algebras. This thesis will end with the description of a work in progress, without direct link to the above, about categorifying the Donaldson–Thomas invariants.

Outline of the thesis: The first part will deal with Kac–Moody algebras in higher dimensions. After recalling a couple of known facts in dimension 1, we will introduce the necessary tools for the higher dimensional case (in particular derived geometry). In a second part, we will make an detour into derived deformation theory, applied to K-theory. This will allow us to recover a crucial result of the first section, in a more conceptual and satisfying way. The third part will layout the difficulties encountered in the case of Virasoro algebras in higher dimensions. We will also state the main conjecture about them. In a fourth section, we will introduce and discuss Gelfand–Fuchs cohomology of algebraic varieties and differentiable manifolds. Lastly, the fifth part will draw out research leads, at various states of progress, related to the problems discussed in this thesis.

Remerciements

Commençons par remercier Myriam qui, depuis quelques temps déjà, supporte toutes choses : moi au quotidien, mon sporadique travail vespéral – en particulier en période de rédaction, mes visios trop bruyantes, mes cours en ligne – heureusement révolus, et j'en passe. Merci !

En second, et dans un contexte plus mathématiques, je souhaite remercier Marco Robalo. Pour des heures de discussion hebdomadaires, pour les projets avortés sur le genre de Witten au projet prometteur sur les factorisations matricielles, pour les discussions mathématiques, mais aussi physiques (et plus que dans d'autres domaines, il parle et je ne comprends pas), professionnelles, sociétales ou personnelles. J'ai indéniablement énormément appris à son contact.

J'ai également une considérable dette mathématique envers Misha Kapranov. Notre collaboration, démarrée au sortir de ma thèse, a grandement élargi mon domaine de compétence et mes intérêts mathématiques, en puisant dans l'incroyable spectre de sa culture scientifique. Ce que je lui dois, mais aussi ce que les mathématiques lui doivent, ne peut qu'être sous-estimé.

Je remercie Damien Calaque, pour son rapport sur cette habilitation et sa participation au jury, bien évidemment, mais aussi et surtout pour les interactions, mathématiques ou autres, que nous avons pu avoir. La multitude des ses domaines de connaissances a toujours permis d'enrichissantes discussions (au moins pour moi).

Merci encore à Mauro Porta et Olivier Schiffmann, pour de nombreuses discussions. Ils m'ont introduit à des mathématiques fantastiques, à la croisée de la théorie des représentations et de la théorie de l'homotopie. Je remercie aussi Valerio Melani et Gabriele Vezzosi, pour nos discussions et collaborations. Ils m'ont chacun beaucoup appris.

Merci aux rapporteurs de cette habilitation, Dima Arinkin, Damien Calaque et Olivier Schiffmann, pour leur travail et leur lecture attentive de ce manuscrit. Leurs trois rapports se complémentent, prenant chacun un point de vue différent sur mes travaux. Je les en remercie !

Je remercie enfin les membres du jury (outre les rapporteurs sus-nommés) : Grégory Ginot, Mikhail Kapranov, Emanuele Macrì, Étienne Mann et Anne Moreau. J'ai bien entendu, avec chacun et chacune, partagé conversations mathématiques, groupes de travail et moments conviviaux, et c'est avec grand plaisir que je leur présenterai mes travaux.

Pour terminer cette liste froide de remerciements chaleureux, mes collègues et amis. Pour les pauses cafés, les complaintes, les conseils et les soutiens, merci Benjamin, Valentin, Anne, Anne, Emanuele, Olivier, Olivier, Maria, Vincent, François, Bac, Kęstutis, Lukas, et j'en passe.

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Autres textes - Other writings

[HPV16] Benjamin Hennion, Mauro Porta, and Gabriele Vezzosi. *Formal gluing along non-linear flags.* 2016. arXiv: `1607.04503`.

[HMV22] Benjamin Hennion, Valerio Melani, and Gabriele Vezzosi. *A flag version of Beilinson-Drinfeld Grassmannian for surfaces.* 2022. arXiv: `2210.04798`.

1 Moduli of G-bundles and Kac–Moody extensions

Our first example of infinite dimensional dg-Lie algebra is the Kac–Moody algebra. This section's content is mostly extracted from [FHK19]. We first recall how the dimension 1 version arises, from a geometric perspective.

1.1 The dimension 1 case

We first give in a nutshell a quite informal picture of the 1-dimensional case. This summary is in no way exhaustive, and shows a clear geometric bias. None of the results exposed here are due to the author. A good introduction to the topic can be found in [FB04]. More details will be given later down this habilitation thesis, when explaining the higher dimensional case.

Actions on rigidified G-bundles: Let X be a smooth projective algebraic curve over $\mathbf{k}$ and G an algebraic group. The moduli of principal G-bundles $\mathrm{Bun}_G(X)$ on X is representable by a smooth Artin stack. Fixing a closed point $x \in X$ and its formal neighbourhood $\widehat{\mathrm{D}}_x$ in X, we can rigidify the situation by considering the moduli $\mathrm{Bun}_G(X, \widehat{\mathrm{D}}_x)$ of rigidified G-bundles at x: G-bundles on X equipped with a trivialisation on $\widehat{\mathrm{D}}_x$:

$$\mathrm{Bun}_G(X, \widehat{\mathrm{D}}_x) := \mathrm{Bun}_G(X) \underset{\mathrm{Bun}_G(\widehat{\mathrm{D}}_x)}{\times} \{\mathrm{Triv}\}.$$

By adding this trivialisation, we simplify the geometric nature of the moduli space and remove its stacky structure: one can now show that $\mathrm{Bun}_G(X, \widehat{\mathrm{D}}_x)$ is representable by an ind-scheme. Moreover, the group (ind-)scheme $G(\widehat{\mathrm{D}}_x)$ (the jets group) naturally acts on $\mathrm{Bun}_G(X, \widehat{\mathrm{D}}_x)$ by changing the trivialisation, making $\mathrm{Bun}_G(X, \widehat{\mathrm{D}}_x)$ a principal $G(\widehat{\mathrm{D}}_x)$-bundle over $\mathrm{Bun}_G(X)$.

This first action can then be extended to a much richer action of a bigger group: the loop group. This group is constructed in a similar fashion as $G(\widehat{\mathrm{D}}_x)$, by replacing the formal disc $\widehat{\mathrm{D}}_x$ by a punctured formal disc $\widehat{\mathrm{D}}_x^\circ \subset \widehat{\mathrm{D}}_x \smallsetminus \{x\}$. Remark that as a formal scheme, $\widehat{\mathrm{D}}_x$ has only one point, so the definition of the complement $\widehat{\mathrm{D}}_x^\circ$ is not really satisfying (we get the empty scheme). Still, there is a sensible definition of the loop group $G(\widehat{\mathrm{D}}_x^\circ)$ (we will come back to it in the higher dimensional case) as the moduli:

$$G(\widehat{\mathrm{D}}_x^\circ) \colon A \mapsto G(A(\!(t)\!)).$$

With G being of finite type, we get a restriction morphism

$$G(\widehat{\mathrm{D}}_x)(A) = \lim G\left(A[t]/_{t^n}\right) \simeq G(A[\![t]\!]) \to G(A(\!(t)\!)) = G(\widehat{\mathrm{D}}_x^\circ)(A).$$

It turn out to give a closed immersion $G(\widehat{\mathrm{D}}_x) \to G(\widehat{\mathrm{D}}_x^\circ)$.

In order to extend the action of the jets group $G(\widehat{\mathrm{D}}_x)$ to an action of the loop group $G(\widehat{\mathrm{D}}_x^\circ)$; we shall describe $\mathrm{Bun}_G(X, \widehat{\mathrm{D}}_x)$ in terms involving $\widehat{\mathrm{D}}_x^\circ$. This is done by first defining a moduli of principal G-bundles on $\widehat{\mathrm{D}}_x^\circ$ (similarly to the definition of the loop group), then using algebraization as in [Bha16] to construct restriction morphisms $\mathrm{Bun}_G(\widehat{\mathrm{D}}_x) \to \mathrm{Bun}_G(\widehat{\mathrm{D}}_x^\circ) \leftarrow \mathrm{Bun}_G(X^\circ)$ (where $X^\circ := X \smallsetminus \{x\}$) and finally by using the Beauville–Laszlo lemma

Lemma 1.1.1 (Beauville–Laszlo [BL95], Bhatt [Bha16])**.** *The natural morphism*

$$\mathrm{Bun}_G(X) \to \mathrm{Bun}_G(X^\circ) \underset{\mathrm{Bun}_G(\widehat{\mathrm{D}}_x^\circ)}{\times} \mathrm{Bun}_G(\widehat{\mathrm{D}}_x)$$

is an isomorphism.

A straightforward corollary is the following description of the moduli of rigidified G-bundles:

$$\mathrm{Bun}_G(X, \widehat{\mathrm{D}}_x) \simeq \left(\mathrm{Bun}_G(X^\circ) \underset{\mathrm{Bun}_G(\widehat{\mathrm{D}}_x^\circ)}{\times} \mathrm{Bun}_G(\widehat{\mathrm{D}}_x)\right) \underset{\mathrm{Bun}_G(\widehat{\mathrm{D}}_x)}{\times} \{\mathrm{Triv}\} \simeq \mathrm{Bun}_G(X^\circ) \underset{\mathrm{Bun}_G(\widehat{\mathrm{D}}_x^\circ)}{\times} \{\mathrm{Triv}\}.$$

In particular, changing the trivialization on the right-hand-side gives a natural action of our loop group $G(\widehat{\mathrm{D}}_x^\circ)$ on $\mathrm{Bun}_G(X, \widehat{\mathrm{D}}_x)$, extending that of $G(\widehat{\mathrm{D}}_x)$.

Remark 1.1.2. One can prove the above action of $G(\widehat{\mathrm{D}}_x^\circ)$ on $\mathrm{Bun}_G(X, \widehat{\mathrm{D}}_x)$ to be transitive. It follows from the absence of infinitesimal deformations of principal G-bundles on X°. This result however has no known generalization to the higher dimensional case.

Even though $G(\widehat{\mathrm{D}}_x)$ and $G(\widehat{\mathrm{D}}_x^\circ)$ are not algebraic group, they do have a tangent Lie algebra. Denoting by $\mathfrak{g}$ the Lie algebra of G, we have

$$\mathrm{T}_{G(\widehat{\mathrm{D}}_x),1} \simeq \mathfrak{g}[\![t]\!] := \mathfrak{g} \otimes \mathbf{k}[\![t]\!] \quad \text{and} \quad \mathrm{T}_{G(\widehat{\mathrm{D}}_x^\circ),1} \simeq \mathfrak{g}(\!(t)\!) := \mathfrak{g} \otimes \mathbf{k}(\!(t)\!),$$

where the bracket on $\mathfrak{g} \otimes A$ is given by $[x \otimes f, y \otimes g] = [x, y] \otimes fg$, for $A = \mathbf{k}[\![t]\!]$ or $\mathbf{k}(\!(t)\!)$. Those are so called current Lie algebras. The geometric actions of $G(\widehat{\mathrm{D}}_x)$ and $G(\widehat{\mathrm{D}}_x^\circ)$ thus induce infinitesimal actions of $\mathfrak{g}[\![t]\!]$ and $\mathfrak{g}(\!(t)\!)$.

Action on the determinantal bundle: Given $\rho \colon G \to \mathbf{GL}_n$ a representation of G, we construct the determinantal[3] line bundle $\det_\rho$ over the stack $\mathrm{Bun}_G(X)$. It is defined as the determinant of the cohomology: the stalk over any principal G-bundle P over X (or family thereof), is the line

$$\det(\mathrm{H}^*(X, P^\rho)) = \bigotimes_n \det(\mathrm{H}^n(X, P^\rho))^{(-1)^n} = \det(\mathrm{H}^0(X, P^\rho)) \otimes \det(\mathrm{H}^1(X, P^\rho))^{-1}.$$

As an example, if ρ is the adjoint representation, then $\det_\rho$ is isomorphic (through the Kodaira–Spencer isomorphism) to the canonical line bundle on $\mathrm{Bun}_G(X)$.

We can easily extend the action of $G(\widehat{\mathrm{D}}_x)$ to the determinantal bundle (so that the projection is equivariant). There is however an interesting obstruction to extending the richer action of $G(\widehat{\mathrm{D}}_x^\circ)$ to $\det_\rho$: the determinantal anomaly. This means there exists a central extension $\widetilde{G}(\widehat{\mathrm{D}}_x^\circ)_\rho$ of $G(\widehat{\mathrm{D}}_x^\circ)$ by $\mathbb{G}_m$ and actions

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] & \widetilde{G}(\widehat{\mathrm{D}}_x^\circ)_\rho \arrow[r] & G(\widehat{\mathrm{D}}_x^\circ) \arrow[r] & 1 \\
 & & \circlearrowright & \circlearrowright & \\
 & & \det_\rho \arrow[r] & \mathrm{Bun}_G(X, \widehat{\mathrm{D}}_x).
\end{tikzcd} \tag{2}$$

The determinantal anomaly is nothing but the class in $\mathrm{H}^2(G(\widehat{\mathrm{D}}_x^\circ), \mathbb{G}_m)$ classifying the aforementioned central extension. In particular, a trivialisation of the determinantal anomaly yields a section $G(\widehat{\mathrm{D}}_x^\circ) \to \widetilde{G}(\widehat{\mathrm{D}}_x^\circ)_\rho$ and thus an action of $G(\widehat{\mathrm{D}}_x^\circ)$ on the determinantal bundle.

On the Lie algebra level and with a choice of local coordinates, the determinantal anomaly becomes a central extension $\widetilde{\mathfrak{g}}(\!(t)\!)_\rho$ of $\mathfrak{g}(\!(t)\!)$ by our base field $\mathbf{k}$.

Definition 1.1.3. The Lie algebra extension $\widetilde{\mathfrak{g}}(\!(t)\!)_\rho$ is called the Kac–Moody Lie algebra associated to G and its representation ρ. Explicitly, it is classified by the 2-cocycle

$$\begin{aligned} \mathfrak{g} \otimes \mathbf{k}(\!(t)\!) \otimes \mathfrak{g} \otimes \mathbf{k}(\!(t)\!) &\longrightarrow \mathbf{k} \\ x \otimes f \otimes y \otimes g &\longmapsto \mathrm{ch}_2^\rho(x, y)\, \mathrm{Res}(f'g), \end{aligned}$$

with $\mathrm{ch}_2^\rho \in \mathrm{Sym}(\mathfrak{g}^\vee)^{\mathfrak{g}}$ the second Chern character of the representation ρ and Res is the residue.

Remark 1.1.4. From the Kac–Moody algebra $\widetilde{\mathfrak{g}}(\!(t)\!)_\rho$, we form vertex algebras by looking at vacuum modules (with various central charges). The picture given above can then be constructed the other way around: starting with the Kac–Moody algebras and the associated vacuum modules, and building up to a geometric interpretation.

From this perspective, the space of sections of the determinantal bundle $\det_\rho$ is identified with some space of conformal blocks of a suitable vacuum module, thus giving a geometric interpretation of said conformal blocks.

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[3] Sometimes also called the *theta* line bundle and denoted θ_ρ.

1.2 (Derived) loop group

Replacing our curve with a smooth variety X of dimension $d \geqslant 2$ introduces various complications. The first one is the definition of the loop group $G(\widehat{\mathrm{D}}_x^\circ)$ at $x \in X$.

To simplify the exposition, we fix formal local coordinates at the point, and doing so, we reduce to the standard formal disc $\widehat{\mathrm{D}} = \mathrm{Spf}(\mathbf{k}[\![t_1, \dots, t_d]\!])$. We can still define without difficulty $G(\widehat{\mathrm{D}})$ as the moduli

$$G(\widehat{\mathrm{D}})\colon A \mapsto G(A[\![t_1, \dots, t_d]\!]).$$

However, the ring of Laurent series is to be replaced by the complement of the origin in the affinization of $\widehat{\mathrm{D}}$:

$$G(\widehat{\mathrm{D}}^\circ)\colon A \mapsto G(\mathrm{Spec}(A[\![t_1, \dots, t_d]\!]) \smallsetminus 0).$$

Now, with G being affine and $d \geqslant 2$, by Hartogs' theorem, any morphism $\mathrm{Spec}(A[\![t_1, \dots, t_d]\!]) \smallsetminus 0 \to G$ extends uniquely into a morphism $\mathrm{Spec}(A[\![t_1, \dots, t_d]\!]) \to G$, so that with the above definition, we get $G(\widehat{\mathrm{D}}^\circ) = G(\widehat{\mathrm{D}})$.

There are two ways of circumventing this issue:

Toric approach: Instead of looking at $\mathrm{Spec}(\mathbf{k}[\![t_1, t_2]\!]) \smallsetminus 0$, we can consider complements of divisors (with functions, e.g. $\mathbf{k}(\!(t_1)\!)[\![t_2]\!]$, $\mathbf{k}[\![t_1]\!](\!(t_2)\!)$, or $\mathbf{k}(\!(t_1)\!)(\!(t_2)\!)$). The cost of this solution is twofold. First, those rings depend on an ordering of the variables: $\mathbf{k}(\!(t_1)\!)(\!(t_2)\!) \neq \mathbf{k}(\!(t_2)\!)(\!(t_1)\!)$. Geometrically, we would need to fix not only a point $x \in X$, but also a (non necessarily linear) complete flag in $\widehat{\mathrm{D}}_x$. This approach is thus somewhat related to Beilinson's theory of adèles. The second drawback is best understood when generalizing the extensions of definition 1.1.3 to this context. Indeed, we can very well form the Lie algebras $\mathfrak{g}(\!(t_1)\!)(\!(t_2)\!)$ but the natural cohomology class that arises lies in degree 3 (or in general $n+1$). There are therefore no Lie algebra extensions per se. We will not pursue this approach, but significant work has been done in this direction. See for instance [OZ16], [BGW21].

Derived (spherical) approach: This approach circumvents the issue by using derived geometry and considering the moduli spaces at hand as derived stacks. This allows to differentiate $G(\widehat{\mathrm{D}}^\circ)$ from $G(\widehat{\mathrm{D}})$ using the cohomology of functions of $\mathrm{Spec}(\mathbf{k}[\![t_1, \dots, t_d]\!]) \smallsetminus 0$, as the following example shows. This is the approach we will follow in this work.

Example 1.2.1. Say $G = \mathbb{G}_a$ (albeit not reductive) and $d = 2$. For any ring A, we have $\mathbb{G}_a(\widehat{\mathrm{D}}^\circ)(A) = \Gamma(\mathrm{Spec}(A[\![t_1, t_2]\!]) \smallsetminus 0, \mathcal{O}) = A[\![t_1, t_2]\!]$. Considering the derived structure here means we allow for A to be commutative dg-algebra in non-positive degrees:

$$A = \left[\cdots \xrightarrow{\partial} A^{-2} \xrightarrow{\partial} A^{-1} \xrightarrow{\partial} A^0 \to 0 \to \cdots\right].$$

Let for instance A be such that $A^0 = A^{-1} = \mathbf{k}$, $A^p = 0$ otherwise and $\partial = 0$. In this case $\mathbb{G}_a(\widehat{\mathrm{D}}^\circ)(A)$ is now an ∞-groupoid with homotopy groups $\pi_i(\mathbb{G}_a(\widehat{\mathrm{D}}^\circ)(A)) = \mathrm{H}^{1-i}(\mathrm{Spec}(A[\![t_1, t_2]\!]) \smallsetminus 0, \mathcal{O})$:

$$\begin{aligned}
\pi_i(\mathbb{G}_a(\widehat{\mathrm{D}}^\circ)(A)) &= \mathrm{H}^{-i}(\mathrm{Spec}(A[\![t_1, t_2]\!]) \smallsetminus 0, \mathcal{O}) \\
&= \mathrm{H}^{-i}(\mathrm{Spec}(\mathbf{k}[\![t_1, t_2]\!]) \smallsetminus 0, \mathcal{O}) \oplus \mathrm{H}^{1-i}(\mathrm{Spec}(\mathbf{k}[\![t_1, t_2]\!]) \smallsetminus 0, \mathcal{O}) \\
&= \begin{cases} \mathbf{k}[\![t_1, t_2]\!] & \text{if } i = 1 \\ \mathbf{k}[\![t_1, t_2]\!] \oplus t_1^{-1} t_2^{-1} \mathbf{k}[t_1^{-1}, t_2^{-1}] & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

This derived enhancement of $\mathbb{G}_a(\widehat{\mathrm{D}}^\circ)$ thus remembers some form of "codimension 2 polar part".

1.3 Interlude: derived geometry

As summarily explained above, derived (algebraic) geometry consists in allowing commutative dg-algebras (cdga's) as test rings of functor of points. For an overview of the subject, see e.g. [Toë14]. See also [Ben+21].

Definitions 1.3.1.

(a) Let $\mathbf{cdga}_{\mathbf{k}}^{\leqslant 0}$ denote the ∞-category of non-positively graded cdga's over $\mathbf{k}$. An object A in $\mathbf{cdga}_{\mathbf{k}}^{\leqslant 0}$ is thus a complex

$$\cdots \to A^{-2} \xrightarrow{\partial} A^{-1} \xrightarrow{\partial} A^{0} \to 0 \to \cdots$$

equipped with a graded commutative multiplicative structure.

(b) A derived prestack (over $\mathbf{k}$) is an ∞-functor $\mathbf{cdga}_{\mathbf{k}}^{\leqslant 0} \to \mathbf{Gpd}_{\infty}$, where $\mathbf{Gpd}_{\infty}$ denotes the ∞-category of ∞-groupoids. A derived prestack S representable by a cdga A is called a derived affine scheme. We write $S = \operatorname{Spec} A$. We denote by $\mathbf{dAff}_{\mathbf{k}}$ the ∞-category of derived affine schemes (so that $\mathbf{dAff}_{\mathbf{k}}^{\mathrm{op}} = \mathbf{cdga}_{\mathbf{k}}^{\leqslant 0}$).

(c) A morphism of cdga's $f \colon A \to B$ is flat (resp. smooth, resp. étale, resp. a Zariski open immersion) if the induced morphism $\mathrm{H}^0 f \colon \mathrm{H}^0 A \to \mathrm{H}^0 B$ is, and if for every n, the canonical morphism $\mathrm{H}^n A \otimes_{\mathrm{H}^0 A} \mathrm{H}^0 B \to \mathrm{H}^n B$ is an isomorphism.

(d) A derived stack is a derived prestack satisfying étale descent. We denote by $\mathbf{dSt}_{\mathbf{k}}$ the ∞-category of derived stacks.

From those definitions, one can straightforwardly define the notion of a derived scheme, derived Artin or Deligne–Mumford stack or derived ind-scheme.

Definition 1.3.2 (Higher dimensional derived loop group)**.** Let G be a reductive group and X a d-dimensional smooth variety over $\mathbf{k}$. Let $x \in X$ be a $\mathbf{k}$-point. We denote by $\widehat{\mathrm{D}}_x^A$ the derived ind-scheme $\widehat{\mathrm{D}}_x \times \operatorname{Spec} A$, and by $\operatorname{Aff}(\widehat{\mathrm{D}}_x^A)$ its affinization:

$$\operatorname{Aff}(\widehat{\mathrm{D}}_x^A) = \operatorname{Spec}\Big(\mathrm{R}\Gamma\Big(\widehat{\mathrm{D}}_x^A, \mathcal{O}_{\widehat{\mathrm{D}}_x^A}\Big)\Big) = \operatorname{Spec}\Big(\widehat{\mathcal{O}}_{X,x} \mathbin{\widehat{\otimes}} A\Big)$$

The derived loop group at x is the derived stack

$$\begin{aligned} \mathbf{cdga}_{\mathbf{k}}^{\leqslant 0} &\to \mathbf{Gpd}_{\infty} \\ A &\mapsto G\big(\operatorname{Aff}(\widehat{\mathrm{D}}_x^A) \smallsetminus \{x_A\}\big) \end{aligned}$$

1.4 Derived stack of rigidified bundles

The second complication arising from replacing our curve with a higher dimensional variety X concerns the moduli of G-bundles $\mathrm{Bun}_G(X)$. Indeed, this moduli space is no longer smooth: its tangent bundle at a family of bundles P computes $\mathrm{H}^1(X, P^{\mathrm{ad}})$ (where ad denotes the adjoint representation), and this needs not be a projective module.

It comes however with a non-trivial derived enhancement $\mathrm{RBun}_G(X)$ that is better behaved. This derived enhancement is (by definition) an extension of the functor of points to an ∞-functor

$$\mathrm{RBun}_G(X) \colon \mathbf{cdga}_{\mathbf{k}}^{\leqslant 0} \to \mathbf{Gpd}_{\infty}.$$

It is most easily defined as a derived mapping stack

$$\mathrm{RBun}_G(X) = \underline{\mathrm{RMap}}(X, \mathrm{B}G) \colon A \mapsto \mathrm{RMap}(\operatorname{Spec} A \times X, \mathrm{B}G).$$

This construction applies as well for the formal disc $\widehat{\mathrm{D}}_x$ at a point $x \in X$, so that $\mathrm{RBun}_G(\widehat{\mathrm{D}}_x) \colon A \mapsto \mathrm{RMap}(\widehat{\mathrm{D}}_x^A, \mathrm{B}G)$. With the use of the trivial bundle, we can thus define a derived stack of rigidified bundles:

$$\mathrm{RBun}_G(X, \widehat{\mathrm{D}}_x) := \mathrm{RBun}_G(X) \underset{\mathrm{RBun}_G(\widehat{\mathrm{D}}_x)}{\times} \{\text{trivial}\}.$$

Like in the 1-dimensional case, we would like the (derived) loop group $G(\widehat{\mathrm{D}}_x^\circ)$ to act on the (derived) stack of rigidified bundles. This action will rely on a more general version of the Beauville–Laszlo lemma.

The first step is to define the derived stack of G-bundles on the punctured neighbourhood $\widehat{\mathrm{D}}_x^\circ$. This definition is very similar to that of $G(\widehat{\mathrm{D}}_x^\circ)$:

$$\mathrm{RBun}_G(\widehat{\mathrm{D}}_x^\circ) \colon A \mapsto \mathrm{RMap}\big(\mathrm{Aff}(\widehat{\mathrm{D}}_x^A) \smallsetminus \{x_A\}, \mathrm{B}G\big).$$

Of course, it should come with a restriction morphism $\mathrm{RBun}_G(\widehat{\mathrm{D}}_x) \to \mathrm{RBun}_G(\widehat{\mathrm{D}}_x^\circ)$, which is not completely straightforward. Indeed, for a fixed $A \in \mathbf{cdga}_{\mathbf{k}}^{\leqslant 0}$, there is no morphism $\mathrm{Aff}(\widehat{\mathrm{D}}_x^A) \smallsetminus \{x_A\} \to \widehat{\mathrm{D}}_x^A$, but only a zigzag

$$\mathrm{Aff}(\widehat{\mathrm{D}}_x^A) \smallsetminus \{x_A\} \to \mathrm{Aff}(\widehat{\mathrm{D}}_x^A) \leftarrow \widehat{\mathrm{D}}_x^A.$$

For this reason, we shall need a derived version of Bhatt's algebraization (see [Bha16; HPV16]):

Proposition 1.4.1. *For any $A \in \mathbf{cdga}_{\mathbf{k}}^{\leqslant 0}$, the restriction morphism induces an equivalence*

$$\mathrm{RMap}\big(\mathrm{Aff}(\widehat{\mathrm{D}}_x^A), \mathrm{B}G\big) \simeq \mathrm{RMap}\big(\widehat{\mathrm{D}}_x^A, \mathrm{B}G\big).$$

Using this equivalence's inverse, we can thus define a restriction morphism $\mathrm{RBun}_G(\widehat{\mathrm{D}}_x) \to \mathrm{RBun}_G(\widehat{\mathrm{D}}_x^\circ)$. It is used in a derived geometric version of the Beauville–Laszlo lemma

Proposition 1.4.2 ([HPV16; FHK19])**.** *There is a canonical equivalence*

$$\mathrm{RBun}_G(X) \simeq \mathrm{RBun}_G(X^\circ) \underset{\mathrm{RBun}_G(\widehat{\mathrm{D}}_x^\circ)}{\times} \mathrm{RBun}_G(\widehat{\mathrm{D}}_x).$$

Informally, that means a principal G-bundle on X amounts to a principal G-bundle on the complement X° of a point, a principal G-bundle on the formal neighbourhood $\widehat{\mathrm{D}}_x$ of the point, and some glueing data on the punctured neighbourhood $\widehat{\mathrm{D}}_x^\circ$.

Corollary 1.4.3. *The derived loop group $G(\widehat{\mathrm{D}}_x^\circ)$ acts on the derived moduli space of rigidified bundles:*

$$\begin{tikzcd} G(\widehat{\mathrm{D}}_x^\circ) \arrow[d, phantom, "\circlearrowleft"] \\ \mathrm{RBun}_G(X, \widehat{\mathrm{D}}_x). \end{tikzcd}$$

Like in the 1-dimensional case, fixing a representation $\rho \colon G \to \mathbf{GL}_n$ yields a determinantal bundle $\det_\rho$ on $\mathrm{RBun}_G(X)$, and thus on $\mathrm{RBun}_D(X, \widehat{\mathrm{D}}_x)$. There is an obstruction to lifting the action of the derived loop group to this bundle, called the determinantal anomaly.

The construction of this anomaly relies on [Hen17b] and the shift in the K-theory of Tate complexes. We will not give a detailed account on the matter, but we will say that this anomaly can be constructed as a group stack morphism

$$G(\widehat{\mathrm{D}}_x^\circ) \xrightarrow{\rho} \mathbf{GL}_n(\widehat{\mathrm{D}}_x^\circ) \longrightarrow \Omega\,\mathrm{K}_{\widehat{\mathrm{D}}_x^\circ} \longrightarrow \mathrm{K} \xrightarrow{\det} \mathrm{B}\mathbb{G}_m, \tag{3}$$

where K is the stack of K-theory (of the point $\operatorname{Spec} \mathbf{k}$), $\Omega\,\mathrm{K}_{\widehat{\mathrm{D}}_x^\circ}$ is the loop stack in the stack of K-theory of the punctured disc $\widehat{\mathrm{D}}_x^\circ$, and $\mathrm{B}\mathbb{G}_m$ is the stack classifying line bundles. In particular, the determinantal anomaly classifies an extension $\widetilde{G}(\widehat{\mathrm{D}}^\circ_x)_\rho$ of the derived loop group by $\mathbb{G}_m$. The action of corollary 1.4.3 then extends, similarly to the 1-dimensional case (see diagram (2)):

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] & \widetilde{G}(\widehat{\mathrm{D}}^\circ_x)_\rho \arrow[r] \arrow[d, phantom, "\circlearrowright"] & G(\widehat{\mathrm{D}}^\circ_x) \arrow[r] \arrow[d, phantom, "\circlearrowright"] & 1 \\
 & & \det_\rho \arrow[r] & \mathrm{RBun}_G(X, \widehat{\mathrm{D}}_x). &
\end{tikzcd} \tag{4}$$

1.5 Kac–Moody extension and local Riemann–Roch theorem

We now turn to the induced infinitesimal actions of the associated Lie algebras. To simplify the notations in this section, we will work with fixed local coordinates:

$$\widehat{\mathrm{D}} = \mathrm{Spf}(\mathbf{k}[\![t_1, \dots, t_d]\!]) \text{ and } \widehat{\mathrm{D}}^\circ = \mathrm{Spec}(\mathbf{k}[\![t_1, \dots, t_d]\!]) \smallsetminus \{0\}.$$

First, let us highlight that the derived nature of the higher dimensional loop group makes us leave the classical theory of Lie algebras. Indeed, working with derived geometry implies (amongst other things) looking at the tangent complex at the unit $\mathfrak{g}(\widehat{\mathrm{D}}^\circ) := \mathbb{T}_{G(\widehat{\mathrm{D}}^\circ),1}$, rather than the tangent vector space. Through a standard computation, we find

$$\mathfrak{g}(\widehat{\mathrm{D}}^\circ) = \mathfrak{g} \otimes \mathrm{R}\Gamma(\widehat{\mathrm{D}}^\circ, \mathcal{O}). \tag{5}$$

This implies

$$\mathrm{H}^i(\mathfrak{g}(\widehat{\mathrm{D}}^\circ)) = \begin{cases} \mathfrak{g} \otimes \mathbf{k}[\![t_1, \dots, t_d]\!] & \text{if } i = 0, \\ \mathfrak{g} \otimes (t_1 \cdots t_d)^{-1} \mathbf{k}[t_1^{-1}, \dots, t_d^{-1}] & \text{if } i = d-1, \\ 0 & \text{else.} \end{cases}$$

This tangent complex $\mathfrak{g}(\widehat{\mathrm{D}}^\circ)$ comes with a dg-Lie algebra structure (it is a Lie algebra in complexes). Equivalently, this structure can be seen as an $\mathcal{L}_\infty$-structure on the above cohomology.

The group extension $\widetilde{G}(\widehat{\mathrm{D}}^\circ)_\rho$ yields a Lie algebraic central extension

$$0 \longrightarrow \mathbf{k} \longrightarrow \widetilde{\mathfrak{g}}(\widehat{\mathrm{D}}^\circ)_\rho \longrightarrow \mathfrak{g}(\widehat{\mathrm{D}}^\circ) \longrightarrow 0.$$

It is classified by a class $\tau_\rho \in \mathrm{H}^2_{\mathrm{Lie}}(\mathfrak{g}(\widehat{\mathrm{D}}^\circ), \mathbf{k})$ in Chevalley–Eilenberg cohomology.

Recall that in dimension 1, the vector space $\mathrm{H}^2_{\mathrm{Lie}}(\mathfrak{g}(\widehat{\mathrm{D}}^\circ), \mathbf{k})$ is naturally isomorphic to the space $(\mathrm{Sym}^2 \mathfrak{g}^\vee)^{\mathfrak{g}}$ of equivariant symmetric bilinear forms. The class τ_ρ is identified, through this isomorphism, with the second Chern character of the representation ρ. In higher dimensions however, the vector space $\mathrm{H}^2_{\mathrm{Lie}}(\mathfrak{g}(\widehat{\mathrm{D}}^\circ), \mathbf{k})$ is infinite-dimensional. Identifying the class τ_ρ will then require significantly more work.

The first step consists in reducing to the general linear group (through the representation ρ). Further, we reduce to the infinite general linear group (albeit not an algebraic group):

$$\mathbf{GL}_\infty := \bigcup_n \mathbf{GL}_n \text{ ; } \mathfrak{gl}_\infty := \bigcup_n \mathfrak{gl}_n.$$

The dg-Lie algebra $\mathfrak{gl}_\infty(\widehat{\mathrm{D}}^\circ)$ carries a central extension classified by some $\tau \in \mathrm{H}^2_{\mathrm{Lie}}(\mathfrak{gl}_\infty(\widehat{\mathrm{D}}^\circ), \mathbf{k})$. This class τ is a universal version of the τ_ρ aforementioned, as each τ_ρ stems from it by pullback.

The cohomology space $\mathrm{H}^2_{\mathrm{Lie}}(\mathfrak{gl}_\infty(\widehat{\mathrm{D}}^\circ), \mathbf{k})$ is of course infinite dimensional. It has however a nice description, through the Loday–Quillen–Tsygan theorem:

Theorem 1.5.1 (See [LQ83; Tsy83]). *For any* $\mathbf{k}$*-algebra* A*, the direct sum of matrices endows* $\mathrm{H}^{\mathrm{Lie}}_\bullet(\mathfrak{gl}_\infty(A), \mathbf{k})$ *with a graded Hopf algebra structure, and there is a canonical isomorphism of graded Hopf algebras*

$$\mathrm{H}^{\mathrm{Lie}}_\bullet(\mathfrak{gl}_\infty(A), \mathbf{k}) \simeq \mathrm{Sym}(\mathrm{HC}^{\mathbf{k}}_{\bullet-1}(A)),$$

where $\mathrm{HC}^{\mathbf{k}}_{\bullet}$ *refers to* $\mathbf{k}$*-linear cyclic homology (see [Lod92] for an introduction). Equivalently, the subspace of primitive elements in the Hopf algebra* $\mathrm{H}^{\mathrm{Lie}}_{\bullet}(\mathfrak{gl}_\infty(A),\mathbf{k})$ *is isomorphic to* $\mathrm{HC}^{\mathbf{k}}_{\bullet-1}(A)$*.*

This isomorphism is obtained from an explicit morphism $\mathrm{H}^{\mathrm{Lie}}_{\bullet}(\mathfrak{gl}_\infty(A),\mathbf{k}) \to \mathrm{HC}^{\mathbf{k}}_{\bullet-1}(A)$ *known as the generalized trace.*

A dual version, for cohomology, of a generalization to dg-algebras (see [Bur86; FHK19]) applies to our case. In particular, there is a notion of primitive classes in $\mathrm{H}^2_{\mathrm{Lie}}(\mathfrak{gl}_\infty(\widehat{\mathrm{D}}^\circ),\mathbf{k})$. Their space is a direct summand isomorphic to the dual space $\mathrm{HC}^{\mathbf{k}}_1(\widehat{\mathrm{D}}^\circ)^\vee$:

$$\mathrm{HC}^{\mathbf{k}}_1(\widehat{\mathrm{D}}^\circ)^\vee \simeq \mathrm{H}^2_{\mathrm{prim}}(\mathfrak{gl}_\infty(\widehat{\mathrm{D}}^\circ),\mathbf{k}) \subset \mathrm{H}^2_{\mathrm{Lie}}(\mathfrak{gl}_\infty(\widehat{\mathrm{D}}^\circ),\mathbf{k})$$

Proposition 1.5.2 ([FHK19, §5.4-C])**.** *The class* τ *is primitive (and thus belongs to the direct summand* $\mathrm{HC}^{\mathbf{k}}_1(\widehat{\mathrm{D}}^\circ)^\vee$*).*

The proof found in [FHK19] relies on additive properties of the assignment $\rho \mapsto \tau_\rho$ with respect to direct sums of representations. We will explain in section 2 below a more conceptual proof of that fact, based on the results of [Hen21].

Now, the Hochschild–Kostant–Rozenberg theorem allows for a Hodge decomposition of the cyclic homology of $\widehat{\mathrm{D}}^\circ$ (see [Wei97] for the case of schemes we use here), in terms of the hypercohomology of truncated de Rham complexes:

$$\mathrm{HC}^{\mathbf{k}}_1(\widehat{\mathrm{D}}^\circ) \simeq \bigoplus_p \mathbb{H}^{2p-1}(\widehat{\mathrm{D}}^\circ,\Omega^{\leqslant p}).$$

Notice that, as soon as the dimension is at least 2, this space is still infinite dimensional. To further understand $\tau\colon \mathrm{HC}^{\mathbf{k}}_1(\widehat{\mathrm{D}}^\circ) \to \mathbf{k}$, we will need the following observation: the morphism τ is invariant under the action of automorphisms of the punctured disc $\widehat{\mathrm{D}}^\circ$. We will actually only need to consider the action of linear automorphisms, i.e. of the group $\mathrm{GL}_d(\mathbf{k})$ for d the dimension of our variety X.

We start with a representation-theoretic analysis on the cohomology groups $\mathrm{H}^i(\widehat{\mathrm{D}}^\circ,\Omega^j)$. Using then the (truncated) Frölicher spectral sequence computing the hypercohomology $\mathbb{H}^{2p-1}(\widehat{\mathrm{D}}^\circ,\Omega^{\leqslant p})$, we show

Proposition 1.5.3. *There is (up to scalar) only one* $\mathrm{GL}_d(\mathbf{k})$*-invariant morphism*

$$\mathrm{Res}\colon \mathrm{HC}^{\mathbf{k}}_1(\widehat{\mathrm{D}}^\circ) \to \mathbf{k}.$$

We shall call it the (higher order) residue.

Let us briefly explain the name. First and foremost, if the dimension d is 1, it actually is the residue. In general, the above representation-theoretic analysis shows Res arises from the one invariant class in $\mathrm{H}^{d-1}(\widehat{\mathrm{D}}^\circ,\Omega^d)$. In the case $\mathbf{k} = \mathbb{C}$, the invariant morphism

$$\mathrm{H}^{d-1}(\widehat{\mathrm{D}}^\circ,\Omega^d) \to \mathbb{C}$$

can then be identified, using a Dolbeault complex, with integrating a form of weight $(d, d-1)$ along the unit sphere (of real dimension $2d-1$) in $C^d \smallsetminus \{0\}$. See [FHK19, Prop. 1.5.8] for a more precise statement. This fact gives, should it be needed, more ground to our approach being called "spherical".

We are at last able to give a formula for the central extension class τ_ρ associated to a representation $\rho\colon G \to \mathbf{GL}_n$, similar to that of definition 1.1.3. For this, recall equation (5): $\mathfrak{g}(\widehat{\mathrm{D}}^\circ) \simeq \mathfrak{g}\otimes \mathrm{R}\Gamma(\widehat{\mathrm{D}}^\circ,\mathcal{O})$.

Theorem 1.5.4 (Local Riemann–Roch theorem, [FHK19])**.** *Up to an invertible multiplicative constant, the class* $\tau_\rho \in \mathrm{H}^2_{\mathrm{Lie}}(\mathfrak{g}(\widehat{\mathrm{D}}^\circ),\mathbf{k})$ *is represented by the* 2*-cocycle*

$$\begin{aligned}\gamma_\rho\colon \big(\mathfrak{g}\otimes \mathrm{R}\Gamma(\widehat{\mathrm{D}}^\circ,\mathcal{O})\big)^{\otimes d+1}[d-1] &\to \mathbf{k}\\ (x_0\otimes f_0)\otimes\cdots\otimes(x_d\otimes f_d) &\mapsto \mathrm{ch}^\rho_{d+1}(x_0,\dots,x_d)\,\mathrm{Res}(f_0\cdot df_1\wedge\cdots\wedge df_d),\end{aligned}$$

where

- *$x_i \in \mathfrak{g}$,*
- *each f_i is homogeneous of degree $|f_i|$ in $\mathrm{R}\Gamma(\widehat{\mathrm{D}}^\circ, \mathcal{O})$ and the sum of their degrees satisfies $\sum |f_i| = d-1$,*
- *$\mathrm{ch}^\rho_{d+1} \in \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^\mathfrak{g}$ is the $(d+1)$-th Chern character of the representation ρ.*

The above formula deserves explanations. First, even though γ_ρ has arity $d+1$, its total degree in the Chevalley–Eilenberg complex of $\mathfrak{g}(\widehat{\mathrm{D}}^\circ)$ is 2. This stems from the cohomological degrees of the "functions" f_i and the equality $\sum |f_i| = d-1$. Moreover the quantity $f_0 \cdot df_1 \wedge \cdots \wedge df_d$ is of Hodge weights $(d, d-1)$, so that $\mathrm{Res}(f_0 \cdot df_1 \wedge \cdots \wedge df_d)$ is well defined. By convention, and for the above formula to be valid in all generality, we can set $\mathrm{Res} = 0$ for other Hodge weights.

This theorem can be seen as a local (infinitesimal) version of the Grothendieck–Riemann–Roch theorem. Indeed, the class τ_ρ arise from the determinantal anomaly, which acts on the determinantal bundle (see diagram (4) above). Recall the determinantal bundle is by definition the "determinant of the cohomology" of a given representation.

The formula $\tau_\rho = [\gamma_\rho]$ thus infinitesimally relates this determinantal bundle (i.e. a first Chern class) with a higher order Chern character. The notable absence of the Todd genus stems from the fact that we are only considering the Kac–Moody gauge Lie algebra. We shall see in section 3 below how considering the Virasoro Lie algebra will factor in the Todd class.

The proof of theorem 1.5.4 goes as follows. First, we reduce to $\mathbf{GL}_\infty$. Second, we use an explicit formula for the generalized trace appearing in theorem 1.5.1 to contemplate that γ_ρ comes from a primitive class. Then the invariance of the formula of γ_ρ under the action of $\mathrm{GL}_d(\mathbf{k})$, together with proposition 1.5.3 ensures the result.

Building of the above results, a further study, with a more physics-oriented point of view, has been carried out by Gwilliam and Williams in [GW21].

2 The tangent complex to K-theory

In this section, we will circle back to a proof of proposition 1.5.2 based on the results of [Hen21]. The original proof of proposition 1.5.2 (found in [FHK19]) is more hands-on than the one we will present here. We give here an original and more conceptual proof.

We start with the determinantal anomaly briefly introduced in diagram (3). For the canonical representation of $\mathbf{GL}_n$, it is a group stack morphism

$$\mathbf{GL}_n(\widehat{\mathrm{D}}^\circ) \longrightarrow \Omega\,\mathrm{K}_{\widehat{\mathrm{D}}^\circ} \longrightarrow \mathrm{K} \xrightarrow{\det} \mathrm{B}\mathbb{G}_m,$$

Taking classifying stacks and replacing $\mathbf{GL}_n$ by $\mathbf{GL}_\infty = \mathrm{colim}_n\, \mathbf{GL}_n$, we get

$$\mathrm{B}\mathbf{GL}_\infty(\widehat{\mathrm{D}}^\circ) \longrightarrow \mathrm{K}_{\widehat{\mathrm{D}}^\circ} \longrightarrow \mathrm{B}\,\mathrm{K} \xrightarrow{\det} \mathrm{K}(\mathbb{G}_m, 2).$$

The first morphism $\mathrm{B}\mathbf{GL}_\infty(\widehat{\mathrm{D}}^\circ) \to \mathrm{K}_{\widehat{\mathrm{D}}^\circ}$ is the universal morphism, sending a vector bundle to its class in K-theory. To construct the second morphism $\mathrm{K}_{\widehat{\mathrm{D}}^\circ} \to \mathrm{B}\,\mathrm{K}$, we first consider the localization sequence of ∞-categorical derived stacks:

$$\mathbf{Perf}_{\{0\}}(\widehat{\mathrm{D}}) \longrightarrow \mathbf{Perf}(\widehat{\mathrm{D}}) \longrightarrow \mathbf{Perf}(\widehat{\mathrm{D}}^\circ), \tag{6}$$

where $\mathbf{Perf}(Y)$ denote the stack of perfect complexes over Y, and $\mathbf{Perf}_{\{0\}}(\widehat{\mathrm{D}})$ is the stack of perfect complexes on $\widehat{\mathrm{D}}$ supported at the origin. Taking *non-connective* K-theory gives a boundary morphism (of derived prestacks in spectra)

$$\mathrm{K}_{\widehat{\mathrm{D}}^\circ} \to \Sigma\,\mathrm{K}(\mathbf{Perf}_{\{0\}}(\widehat{\mathrm{D}})).$$

Composing with the global section functor $R\Gamma\colon \mathbf{Perf}_{\{0\}}(\widehat{\mathrm{D}}) \to \mathbf{Perf}(\mathbf{k})$, we get $\mathrm{K}_{\widehat{\mathrm{D}}^\circ} \to \Sigma\,\mathrm{K}_{\mathbf{k}}$. Passing to the connective cover, and taking the étale stackification, we find the announced morphism

$$\mathrm{K}_{\widehat{\mathrm{D}}^\circ} \to \mathrm{B}\,\mathrm{K}_{\mathbf{k}} = \mathrm{B}\,\mathrm{K}\,.$$

The crucial observation is the following: the above group-theoretic construction has a direct Lie-algebraic analog. Indeed, the localization sequence of diagram (6) induces a fiber-cofiber sequence of cyclic homologies:

$$\mathrm{HC}^{\mathbf{k}}_\bullet(\mathrm{Perf}_{\{0\}}(\widehat{\mathrm{D}})) \longrightarrow \mathrm{HC}^{\mathbf{k}}_\bullet(\widehat{\mathrm{D}}) \longrightarrow \mathrm{HC}^{\mathbf{k}}_\bullet(\widehat{\mathrm{D}}^\circ).$$

The induced boundary operator combined with the global section functor (as above) yields

$$\mathrm{HC}^{\mathbf{k}}_\bullet(\widehat{\mathrm{D}}^\circ) \longrightarrow \mathrm{HC}^{\mathbf{k}}_{\bullet-1}(\mathrm{Perf}_{\{0\}}(\widehat{\mathrm{D}})) \longrightarrow \mathrm{HC}^{\mathbf{k}}_{\bullet-1}(\mathbf{k}).$$

Then, the generalized trace morphism (see e.g. [Lod92, Def.1.2.1]) defines an $\mathcal{L}_\infty$-morphism $\mathfrak{gl}_\infty(\widehat{\mathrm{D}}^\circ) \to \mathrm{HC}^{\mathbf{k}}_\bullet(\widehat{\mathrm{D}}^\circ)$ (the RHS being endowed with its abelian dg-Lie algebra structure). All in all, this gives an $\mathcal{L}_\infty$-morphism

$$\mathfrak{gl}_\infty(\widehat{\mathrm{D}}^\circ) \longrightarrow \mathrm{HC}^{\mathbf{k}}_\bullet(\widehat{\mathrm{D}}^\circ) \longrightarrow \mathrm{HC}^{\mathbf{k}}_{\bullet-1}(\mathbf{k}) \xrightarrow{\substack{\mathrm{HC}^{\mathbf{k}}_0(\mathbf{k})=\mathbf{k}\\ \downarrow}} \mathbf{k}[1].$$

We claim the above two constructions (the group-theoretic one and the Lie-algebraic one) are related by more than an analogy: the Lie-algebraic construction is tangent to the group-theoretic one:

$$\begin{tikzcd}
 & \mathbf{GL}_\infty(\widehat{\mathrm{D}}^\circ) \arrow[r] \arrow[d, dashed, bend right] & \Omega\,\mathrm{K}_{\widehat{\mathrm{D}}^\circ} \arrow[r] & \mathrm{K} \arrow[r, "\det"] & \mathrm{B}\mathbb{G}_m & \text{Group stack maps} \\
\text{Tangent dg-Lie algebra} & \mathfrak{gl}_\infty(\widehat{\mathrm{D}}^\circ) \arrow[r] & \mathrm{HC}^{\mathbf{k}}_\bullet(\widehat{\mathrm{D}}^\circ) \arrow[r] & \mathrm{HC}^{\mathbf{k}}_{\bullet-1}(\mathbf{k}) \arrow[r] & \mathbf{k}[1] & \mathcal{L}_\infty\text{-maps.}
\end{tikzcd} \tag{7}$$

The main issue here is to actually define a well behaved tangent to the K-theory stack, in order to make sense to the above claim. This is the content of [Hen21].

2.1 Infinitesimal behaviour of K-theory

Studying the infinitesimal behaviour of K-theory is by no means a novel idea. The first article [Blo73] on the matter is due to Spencer Bloch and dates back to 1973. It was followed by the celebrated article [Goo86] of Thomas Goodwillie, whose main result can be formulated as follows:

Theorem 2.1.1 (Goodwillie, 1986)**.** *The relative rational* K*-theory of a nilpotent extension of* $\mathbb{Q}$*-algebras is isomorphic to its relative rational cyclic homology: if* $B \to A$ *is a nilpotent extension, then there is a natural isomorphism*

$$\mathrm{hofib}(\mathrm{K}(B) \wedge \mathbb{Q} \to \mathrm{K}(A) \wedge \mathbb{Q}) \simeq \mathrm{hofib}\Big(\mathrm{HC}^{\mathbb{Q}}_{\bullet-1}(B) \to \mathrm{HC}^{\mathbb{Q}}_{\bullet-1}(A)\Big).$$

This powerful theorem allows for a computation of the tangent complex to K-theory, if K-theory had been an algebraic group, or at least followed some geometric behaviours. This naive tangent complex is

$$\begin{aligned}
\mathbb{T}^{\mathrm{naive}}_{\mathrm{K}_A \wedge \mathbb{Q},0} &= \mathrm{hofib}(\mathrm{K}(A[\epsilon]) \wedge \mathbb{Q} \to \mathrm{K}(A) \wedge \mathbb{Q}) \qquad (\text{where } \epsilon^2 = 0)\\
&\simeq \mathrm{hofib}\Big(\mathrm{HC}^{\mathbb{Q}}_{\bullet-1}(A[\epsilon]) \to \mathrm{HC}^{\mathbb{Q}}_{\bullet-1}(A)\Big)\\
&\simeq \bigoplus_{p\geqslant 0} \mathrm{HH}^{\mathbb{Q}}_{\bullet-2p-1}(A).
\end{aligned}$$

Because K-theory is extremely far from being an algebraic group, this naive tangent complex does not quite behave as expected. For example, the group morphism $\mathbf{GL}_\infty(A) \to \Omega\,\mathrm{K}_A \to \Omega\,\mathrm{K}_A \wedge \mathbb{Q}$ does not induce a non-trivial $\mathcal{L}_\infty$-morphism

$$\mathfrak{gl}_\infty(A) \to \mathbb{T}^{\mathrm{naive}}_{\mathrm{K}_A \wedge \mathbb{Q},0}[-1] \simeq \bigoplus_{p \geqslant 0} \mathrm{HH}^{\mathbb{Q}}_{\bullet-2p}(A).$$

Indeed, since the right-hand-side is equipped with its abelian Lie algebra structure, such an $\mathcal{L}_\infty$-morphism would amount to a central extension of $\mathfrak{gl}_\infty$ by the complex $\bigoplus_{p\geqslant 0} \mathrm{HH}^{\mathbb{Q}}_{\bullet-2p+1}(A)$. One can then prove that such a functorial central extension needs to be trivial (even when the base field $\mathbf{k}$ is $\mathbb{Q}$).

To try and fix this issue, one could think of using Goodwillie calculus by stabilizing the construction:

$$\mathbb{T}^{\mathrm{stab}}_{\mathrm{K}_A \wedge \mathbb{Q},0} = \operatorname*{colim}_n \mathrm{hofib}(\mathrm{K}(A[\epsilon_n]) \wedge \mathbb{Q} \to \mathrm{K}(A) \wedge \mathbb{Q})[-n],$$

where ϵ_n lies in cohomological degree $-n$ and $\epsilon_n^2 = 0$. This however does not solve our issue. Indeed, one can compute[4] $\mathbb{T}^{\mathrm{stab}}_{\mathrm{K}_A \wedge \mathbb{Q},0} \simeq \mathrm{HH}^{\mathbb{Q}}_{\bullet-1}(A)$ and as above, there is no non-trivial functorial central extension of $\mathfrak{gl}_\infty(A)$ by this complex (with the proper shift).

In order to give a definition of the tangent complex to K-theory more suited to our needs, we will rather use deformation theory.

2.2 Formal deformation theory

The key reason for wanting to use deformation theory is the following statement: Deformations of algebro-geometric objects (in characteristic 0) are governed by dg-Lie algebras.

The statement has been a principle more than a theorem for a long time, as many authors worked to establish such a correspondence (see notably [Hin01], [Man99; Man09]). This lead to a classification (see below) by Pridham and Lurie independently. We refer to [Toë17] for a more complete history.

Theorem 2.2.1 (Pridham [Pri10], Lurie [Lur11])**.** *Let* $\mathbf{k}$ *be a field of characteristic* 0*. There is an* ∞*-categorical equivalence*

$$\begin{tikzcd} \mathbf{FMP}_{\mathbf{k}} := \{\textit{(Derived) Formal moduli problems over } \mathbf{k}\} \arrow[r, "\sim"] & \mathbf{dgLie}_{\mathbf{k}} \\ F \arrow[r, mapsto] & \mathbb{T}_F[-1]. \end{tikzcd}$$

Obviously, we need to define the left-hand side and the notion of formal moduli problems.

Definition 2.2.2. A (derived) *formal moduli problem* is an ∞-functor $\mathbf{dgArt}_{\mathbf{k}} \to \mathbf{Gpd}_\infty$ defined on Artinian cdga's over $\mathbf{k}$ satisfying a version of Schlessinger's condition. A cdga A over $\mathbf{k}$ is Artinian if

(i) $\mathrm{H}^0 A$ is an Artinian $\mathbf{k}$-algebra, with residue field $\mathbf{k}$,

(ii) $\mathrm{H}^n A = 0$ for $n > 0$ or $n << 0$, and is finite dimensional otherwise.

The Schlessinger condition satisfied by a formal moduli problem F is the following:

(S1) We have $F(\mathbf{k}) \simeq *$ and

(S2) For any fiber product of Artinian cdga's $D \simeq A \times_B C$ where the morphism $A \to B$ is surjective on H^0, the induced morphism (of ∞-groupoids):

$$F(D) \to F(A) \underset{F(B)}{\times} F(C)$$

is an equivalence.

---

[4] Note that this computation is actually used in the proof of theorem 2.1.1.

In general, one can construct a formal moduli problem governing deformations of a $\mathbf{k}$-point in any geometric enough stack. Here are some examples

Examples 2.2.3.

(a) Let X be a (possibly derived) scheme over $\mathbf{k}$ and $x \in X$ a $\mathbf{k}$-point. The functor $\widehat{X}_x \colon A \mapsto X(A) \times_{X(\mathbf{k})} \{x\}$ is a formal moduli problem. The corresponding dg-Lie algebra has underlying complex $\mathbb{T}_{X,x}[-1]$. It is (trivially) abelian whenever X is smooth at x.

(b) Let G be an algebraic group and $\mathrm{B}G$ its classifying stack. The formal moduli problem $\mathrm{B}\widehat{G} \colon A \mapsto \mathrm{B}G(A) \times_{\mathrm{B}G(\mathbf{k})} \{\mathrm{triv}\}$ classifies formal deformations of the trivial principal G bundle on the point $\operatorname{Spec} \mathbf{k}$. The corresponding dg-Lie algebra is nothing but the Lie algebra $\mathfrak{g}$ of G (concentrated in degree 0).

(c) More generally, let X be a scheme and G an algebraic group. Let P be a principal G-bundle on X. The formal deformations of P are classified by the formal moduli problem

$$\mathrm{Def}(P) \colon A \mapsto \mathrm{RBun}_G(X)(A) \underset{\mathrm{RBun}_G(X)(\mathbf{k})}{\times} \{P\}.$$

The associated dg-Lie algebra is $\mathrm{R}\Gamma(X, P)$. If P is trivial, then this dg-Lie algebra is the current dg-Lie algebra $\mathfrak{g} \otimes \mathrm{R}\Gamma(X, \mathcal{O}_X)$.

(d) If **Perf** denotes the stack of perfect complexes over the base field, and E is such a complex, then we can form the formal moduli problem

$$A \mapsto \mathbf{Perf}(A) \underset{\mathbf{Perf}(\mathbf{k})}{\times} \{E\}.$$

The associated dg-Lie algebra is the dg-algebra of endomorphisms $\mathrm{End}(E)$.

Note that some natural deformation functors are not formal moduli problem. They usually come from moduli spaces that are not representable by anything geometric (i.e. schemes, Deligne–Mumford or Artin stacks – possibly derived). Before giving some examples, let us remark that the fully faithful functor $\mathbf{dgLie_k} \simeq \mathbf{FMP_k} \subset \mathrm{Fct}(\mathbf{dgArt_k}, \mathbf{Gpd}_\infty)$ admits a left adjoint, that we will denote by ℓ

$$\begin{tikzcd} \ell \colon \mathrm{Fct}(\mathbf{dgArt}, \mathbf{Gpd}_\infty) \arrow[r] & \mathbf{FMP_k} \simeq \mathbf{dgLie_k} \\ F \arrow[r, mapsto] & \ell_F. \end{tikzcd}$$

In particular, we can associate to any functor $F \colon \mathbf{dgArt_k} \to \mathbf{Gpd}_\infty$ a dg-Lie algebra ℓ_F, regardless of whether F is a formal moduli problem or not.

Counterexamples 2.2.4.

(a) Denote by $\mathcal{D}_{\mathrm{qcoh}}$ the stack of (unbounded) quasi-coherent complexes over $\mathbf{k}$. Let $E \in \mathcal{D}_{\mathrm{qcoh}}(\mathbf{k})$. The functor

$$\mathrm{Def}_E \colon A \mapsto \mathcal{D}_{\mathrm{qcoh}}(A) \underset{\mathcal{D}_{\mathrm{qcoh}}(\mathbf{k})}{\times} \{E\}$$

is not necessarily a formal moduli problem when E is not bounded on the right. One can however show that this functor is a 1-*proximate formal moduli problem* (see [Lur11, §5.1]). This notion is beyond the scope of this memoir, but it is worth mentioning that it allows to compute the associated dg-Lie algebra

$$\ell_{\mathrm{Def}_E} \simeq \mathrm{End}(E).$$

(b) This example is central to our discussion and [Hen21]. The functor

$$\bar{\mathrm{K}} \colon A \mapsto \mathrm{K}(A) \underset{\mathrm{K}(\mathbf{k})}{\times} \{0\}$$

is not a formal moduli problem. We can even show that it is not n-proximate for any n: if it were, the associated dg-Lie algebra $\ell_{\bar{K}}$ would be the stabilized tangent mentioned at the end of section 2.1 (see [Lur11, lem. 5.1.12]), which does not behave it should in terms of Lie algebra extensions.

2.3 The abelian tangent of K-theory

We can now start focusing on the content of [Hen21]. It deal with computing the dg-Lie algebra associated to $\bar{K}$ as in counterexample 2.2.4(b). First, we observe that K-theory (of a scheme or a derived scheme X) is richer than just an ∞-groupoid (or a space): it lifts to the non-connective K-theory spectrum $\mathrm{K}^{\mathrm{nc}}(X) \in \mathbf{Sp}$. As a consequence, we expect its associated dg-Lie algebra to be abelian. This abelianity is however not a consequence of the above, and we will look rather towards the *abelian tangent dg-Lie algebra*, defined as follows.

The ∞-category $\mathbf{FMP}_{\mathbf{k}}^{\mathrm{Ab}}$ of abelian groups in formal moduli problems embeds fully faithfully into the ∞-category of functors $\mathbf{dgArt}_{\mathbf{k}} \to \mathbf{Sp}$. It is actually equivalent to the full subcategory spanned by functors satisfying the same Schlessinger condition as in definition 2.2.2. The equivalence of theorem 2.2.1 induces an equivalence $\mathbf{FMP}_{\mathbf{k}}^{\mathrm{Ab}} \simeq \mathrm{C}(\mathbf{k})$ with the ∞-category of complexes of $\mathbf{k}$-vector spaces (up to quasi-isomorphisms). We get an embedding $\mathrm{C}(\mathbf{k}) \subset \mathrm{Fct}(\mathbf{dgArt}_{\mathbf{k}}, \mathbf{Sp})$, which, for formal reasons, admits a left adjoint ℓ^{Ab}:

$$\begin{tikzcd}
\mathrm{C}(\mathbf{k}) \simeq \mathbf{FMP}_{\mathbf{k}}^{\mathrm{Ab}} \arrow[r, hook] & \mathrm{Fct}(\mathbf{dgArt}_{\mathbf{k}}, \mathbf{Sp}). \arrow[l, bend right, "\ell^{\mathrm{Ab}}"']
\end{tikzcd}$$

We are now able to state the main theorem of [Hen21].

Theorem 2.3.1 ([Hen21])**.** *Let X be a quasi-compact quasi-separated (derived) scheme. Let $\mathrm{K}_X^{\mathrm{nc}}$ denote the ∞-functor $\mathbf{dgArt}_{\mathbf{k}} \to \mathbf{Sp}$ mapping A to the spectrum $\mathrm{K}^{\mathrm{nc}}(X \times \operatorname{Spec} A)$. Then*

$$\ell^{\mathrm{Ab}}_{\mathrm{K}_X} \simeq \mathrm{HC}^{\mathbf{k}}_{\bullet}(X).$$

It is important to notice that $\mathrm{HC}^{\mathbf{k}}_{\bullet}(X)$ stands for the $\mathbf{k}$-*linear* cyclic homology, as opposed to $\mathbb{Q}$-linear (relative) cyclic homology or Hochschild homology mentioned in section 2.1.

Corollary 2.3.2 ([Hen21])**.** *Let X be a quasi-compact quasi-separated (derived) scheme. The canonical stack morphism*

$$\mathrm{B}\mathbf{GL}_{\infty}(X) \to \mathrm{K}_X$$

induces an $\mathcal{L}_\infty$-morphism $\mathfrak{gl}_\infty(\mathrm{R}\Gamma(X, \mathcal{O}_X)) \to \mathrm{HC}^{\mathbf{k}}_{\bullet}(X)$, where the right-hand-side is endowed with its abelian Lie structure. This gives a morphism between their Chevalley–Eilenberg homology complexes

$$\mathrm{CE}^{\mathbf{k}}_{\bullet}(\mathfrak{gl}_\infty(\mathrm{R}\Gamma(X, \mathcal{O}_X))) \to \mathrm{Sym}_{\mathbf{k}}(\mathrm{HC}^{\mathbf{k}}_{\bullet-1}(X))$$

which identifies the generalized trace of Loday–Quillen–Tsygan and is a quasi-isomorphism.

The proof of theorem 2.3.1 relies on the following steps. First, we reduce to the affine case $X = \operatorname{Spec} B$. A formal adjunction game then allows us to replace non-connective K-theory with connective K-theory, and further with rational K-theory:

$$\begin{aligned}
\mathrm{K}_B^{\mathbb{Q}} \colon \mathbf{dgArt}_{\mathbf{k}} &\longrightarrow \mathrm{C}(\mathbb{Q})^{\leqslant 0} \\
A &\longmapsto \mathrm{K}(A \otimes_{\mathbf{k}} B) \wedge \mathbb{Q}.
\end{aligned}$$

In this process, we also replace the functor ℓ^{Ab} by the left adjoint $\ell^{\mathbb{Q}}$ of the inclusion

$$\begin{aligned}
\mathrm{C}(\mathbf{k}) &\lhook\joinrel\longrightarrow \mathrm{Fct}(\mathbf{dgArt}_{\mathbf{k}}, \mathrm{C}(\mathbb{Q})^{\leqslant 0}) \\
V &\longmapsto \Big(A \mapsto (\mathrm{Aug}(A) \otimes_{\mathbf{k}} V)^{\leqslant 0}\Big),
\end{aligned}$$

where $\mathrm{Aug}(A)$ denotes the augmentation ideal of A. All in all, we get $\ell^{\mathrm{Ab}}_{\mathrm{K}_B} \simeq \ell^{\mathbb{Q}}_{\mathrm{K}^{\mathbb{Q}}_B}$. Because of Schlessinger's condition condition (S1), we can further replace rational K-theory with relative rational K-theory. Using Goodwillie's theorem 2.1.1, we can even work with relative rational cyclic homology:

$$\ell^{\mathrm{Ab}}_{\mathrm{K}_B} \simeq \ell^{\mathbb{Q}}_{\overline{\mathrm{HC}}^{\mathbb{Q}}_{\bullet-1}(B\otimes_{\mathbf{k}}-)}, \tag{8}$$

where $\overline{\mathrm{HC}}^{\mathbb{Q}}_{\bullet}(B \otimes_{\mathbf{k}} -) \colon A \mapsto \mathrm{fib}\Big(\mathrm{HC}^{\mathbb{Q}}_{\bullet}(B \otimes_{\mathbf{k}} A) \to \mathrm{HC}^{\mathbb{Q}}_{\bullet}(B)\Big)$.

Now, to relate this *rational* cyclic homology to $\mathbf{k}$-linear cyclic homology, we will need the following proposition, cornerstone to the proof of theorem 2.3.1.

Proposition 2.3.3 ([Hen21])**.** *The restriction of $\ell^{\mathbb{Q}}$ to functors satisfying condition* (S1) *(i.e. such that $F(\mathbf{k}) \simeq 0$) is* non-unitally *symmetric monoidal:*

$$\begin{tikzcd}
\mathrm{Fct}^{(\mathrm{S1})}(\mathbf{dgArt}_{\mathbf{k}}, \mathrm{C}(\mathbb{Q})^{\leqslant 0}) \subset \mathrm{Fct}(\mathbf{dgArt}_{\mathbf{k}}, \mathrm{C}(\mathbb{Q})^{\leqslant 0}) \arrow[r, "\ell^{\mathbb{Q}}"] & \mathrm{C}(\mathbf{k}) \\
\otimes_{\mathbb{Q}} \arrow[r, mapsto] & \otimes_{\mathbf{k}},
\end{tikzcd}$$

where the tensor product on the left-hand-side is computed pointwise. Observe that this tensor product has no unit, for the constant functor with value $\mathbb{Q}$ does not satisfy condition (S1)*.*

A formal consequence of this proposition concerns the behaviour of $\ell^{\mathbb{Q}}$ when applied to cyclic homology of non-unital algebras. The first observation is that there is a definition of cyclic homology of non-unital algebras that does not rely on the existence of a monoidal unit. We can thus form the functor $\mathrm{HC}^{\mathbb{Q}}_{\bullet}(B \otimes_{\mathbf{k}} \mathrm{Aug}(-)) \colon A \mapsto \mathrm{HC}^{\mathbb{Q}}_{\bullet}(B \otimes_{\mathbf{k}} \mathrm{Aug}(A))$ internally in $\mathrm{Fct}^{(\mathrm{S1})}(\mathbf{dgArt}_{\mathbf{k}}, \mathrm{C}(\mathbb{Q})^{\leqslant 0})$, using only the non-unital monoidal structure and colimits in this category and starting from the functor $A \mapsto B \otimes_{\mathbf{k}} \mathrm{Aug}(A)$. This implies

$$\ell^{\mathbb{Q}}_{\mathrm{HC}^{\mathbb{Q}}_{\bullet-1}(B\otimes_{\mathbf{k}}\mathrm{Aug}(-))} \simeq \mathrm{HC}^{\mathbf{k}}_{\bullet-1}\Big(\ell^{\mathbb{Q}}_{B\otimes_{\mathbf{k}}\mathrm{Aug}(-)}\Big) \simeq \mathrm{HC}^{\mathbf{k}}_{\bullet-1}(B).$$

However, the above left-hand-side is a priori not the same as the right-hand-side of equation (8). There is a canonical natural transformation

$$\alpha_B \colon \mathrm{HC}^{\mathbb{Q}}_{\bullet-1}(B \otimes_{\mathbf{k}} \mathrm{Aug}(-)) \xrightarrow{\not\simeq} \overline{\mathrm{HC}}^{\mathbb{Q}}_{\bullet-1}(B \otimes_{\mathbf{k}} -),$$

which is never an equivalence (except if $B = \mathbf{k} = \mathbb{Q}$). Determining whether morphisms like $\alpha_B(A)$ are equivalences amounts to Wodzicki's excision theorem in cyclic homology:

Theorem 2.3.4 (Wodzicki, 1989 [Wod89])**.** *Let $C \to D$ be a surjective morphism of connective $\mathbb{Q}$-dg-algebras and I its kernel (seen as a non-unital dg-algebra). If I is H-unital (meaning that its bar complex is contractible), then*

$$\mathrm{HC}^{\mathbb{Q}}_{\bullet}(I) \simeq \mathrm{fib}\Big(\mathrm{HC}^{\mathbb{Q}}_{\bullet}(C) \to \mathrm{HC}^{\mathbb{Q}}_{\bullet}(D)\Big).$$

Conversely, if I is a connective non-unital $\mathbb{Q}$-dg-algebra and if for every $C \to D$ as above such that $\ker(C \to D) \simeq I$, *we have* $\mathrm{HC}^{\mathbb{Q}}_{\bullet}(I) \simeq \mathrm{fib}\Big(\mathrm{HC}^{\mathbb{Q}}_{\bullet}(C) \to \mathrm{HC}^{\mathbb{Q}}_{\bullet}(D)\Big)$*, then I is H-unital.*

For α_B above to be an equivalence, we would then need $B \otimes_{\mathbf{k}} \mathrm{Aug}(A)$ to be H-unital for any Artinian A. This can easily be disproved. However, we do know that $\ell^{\mathbb{Q}}_{B\otimes_{\mathbf{k}}\mathrm{Aug}(-)} \simeq B$ is H-unital (it is unital by assumption). The idea is then to adapt a proof of Wodzicki's theorem due to Guccione and Guccione [GG96] to prove the following:

Theorem 2.3.5 ([Hen21]). *Let* $\mathcal{A}\colon \mathbf{dgArt}_{\mathbf{k}} \to \mathbf{dgAlg}_{\mathbb{Q}}^{\leqslant 0,\mathrm{nu}}$ *be an* ∞*-functor. Denote by* $\mathcal{I}\colon \mathbf{dgArt}_{\mathbf{k}} \to \mathbf{dgAlg}_{\mathbb{Q}}^{\leqslant 0,\mathrm{nu}}$ *the* ∞*-functor* $A \mapsto \mathrm{fib}(\mathcal{A}(A) \to \mathcal{A}(\mathbf{k}))$. *If* $\ell_{\mathcal{A}}^{\mathbb{Q}} \in \mathbf{dgAlg}_{\mathbf{k}}^{\mathrm{nu}}$ *is H-unital, then the canonical natural transformation*

$$\mathrm{HC}_{\bullet}^{\mathbb{Q}}(\mathcal{I}) \to \overline{\mathrm{HC}}_{\bullet}^{\mathbb{Q}}(\mathcal{A}) := \mathrm{fib}\Big(\mathrm{HC}_{\bullet}^{\mathbb{Q}}(\mathcal{A}) \to \mathrm{HC}_{\bullet}^{\mathbb{Q}}(\mathcal{A}(\mathbf{k}))\Big)$$

induces an equivalence $\ell_{\mathrm{HC}_{\bullet}^{\mathbb{Q}}(\mathcal{I})}^{\mathbb{Q}} \simeq \ell_{\overline{\mathrm{HC}}_{\bullet}^{\mathbb{Q}}(\mathcal{A})}^{\mathbb{Q}}$.

Applying this theorem to $\mathcal{A} = B \otimes_{\mathbf{k}} -$ then shows

$$\ell_{\mathrm{K}_B}^{\mathrm{Ab}} \simeq \ell_{\overline{\mathrm{HC}}_{\bullet-1}^{\mathbb{Q}}(B\otimes_{\mathbf{k}}-)}^{\mathbb{Q}} \simeq \ell_{\mathrm{HC}_{\bullet-1}^{\mathbb{Q}}(B\otimes_{\mathbf{k}}\mathrm{Aug}(-))}^{\mathbb{Q}} \simeq \mathrm{HC}_{\bullet-1}^{\mathbf{k}}\Big(\ell_{B\otimes_{\mathbf{k}}\mathrm{Aug}(-)}^{\mathbb{Q}}\Big) \simeq \mathrm{HC}_{\bullet-1}^{\mathbf{k}}(B),$$

thus proving theorem 2.3.1.

2.4 Deducing the primitivity of τ

We will now explain how proposition 1.5.2 can be deduced from theorem 2.3.1. Recall diagram (7) above. After having replaced $\mathrm{B}\mathbb{G}_m$ with the graded Picard group $\mathrm{B}\mathbb{G}_m^{\mathrm{gr}}$ and the determinant with its graded version, the morphisms of group stacks

$$\Omega\,\mathrm{K}_{\widehat{\mathrm{D}}^\circ} \longrightarrow \mathrm{K} \xrightarrow{\det^{\mathrm{gr}}} \mathrm{B}\mathbb{G}_m^{\mathrm{gr}}$$

are morphisms of abelian group stacks. We get after taking (formal) classifying spaces

$$\begin{tikzcd}
 & & \overbrace{\hphantom{\mathrm{K}_{\widehat{\mathrm{D}}^\circ} \longrightarrow \Sigma\,\mathrm{K} \longrightarrow \mathrm{BB}\mathbb{G}_m^{\mathrm{gr}}}}^{\textbf{Abelian}\text{ formal moduli problems}} & \\
\mathrm{B}\mathbf{GL}_\infty(\widehat{\mathrm{D}}^\circ) \arrow[r] \arrow[d, mapsto, "\ell"'] & \mathrm{K}_{\widehat{\mathrm{D}}^\circ} \arrow[r] \arrow[d, mapsto, "\ell^{\mathrm{Ab}}"'] & \Sigma\,\mathrm{K} \arrow[r, "\det^{\mathrm{gr}}"'] \arrow[d, mapsto, "\ell^{\mathrm{Ab}}"'] & \mathrm{BB}\mathbb{G}_m^{\mathrm{gr}} \arrow[d, mapsto, "\ell^{\mathrm{Ab}}"'] \\
\mathfrak{gl}_\infty(\widehat{\mathrm{D}}^\circ) \arrow[r] & \mathrm{HC}_{\bullet}^{\mathbf{k}}(\widehat{\mathrm{D}}^\circ) \arrow[r] & \mathrm{HC}_{\bullet-1}^{\mathbf{k}}(\mathbf{k}) \arrow[r] & \mathbf{k}[1] \\
\begin{array}{c}\mathcal{L}_\infty\text{-morphism}\\ \text{(generalized trace)}\end{array} & & & \textbf{Abelian } \mathcal{L}_\infty\text{-morphisms.}
\end{tikzcd}$$

By construction, the class τ of proposition 1.5.2 classifies the composite morphism $\mathfrak{gl}_\infty(\widehat{\mathrm{D}}^\circ) \to \mathbf{k}[1]$ above. The primitivity of τ amounts to the abelianity of the morphism $\mathrm{HC}_{\bullet}^{\mathbf{k}}(\widehat{\mathrm{D}}^\circ) \to \mathbf{k}[1]$.

3 Virasoro extensions

After the higher dimensional version of the Kac–Moody, we turn towards a second example of infinite-dimensional dg-Lie algebras: the Virasoro algebras. The 1-dimensional picture is somewhat similar to that section 1.1, where, instead of considering the moduli of G-bundles on a fixed curve, we consider the moduli space of curves itself. See for instance [FB04] for the details of what is summed up below.

3.1 The case of curves

Denote by $\mathcal{M}_{g,1}$ the moduli space of curves of genus $g \geqslant 2$ with one marked point, and by $\pi\colon \mathcal{M}_{g,1,\mathrm{rig}} \to \mathcal{M}_{g,1}$ the bundle of infinitesimal coordinates at the marked point. More explicitly, $\mathcal{M}_{g,1,\mathrm{rig}}$ classifies triplets (X, x, ν) where (X, x) is a genus g marked curve, and $\nu\colon \widehat{\mathrm{D}}_x \simeq \widehat{\mathrm{D}}$ is an isomorphism, where, as in section 1, $\widehat{\mathrm{D}}_x$ denotes the formal neighbourhood of x in X, and $\widehat{\mathrm{D}}$ that of 0 in $\mathbb{A}^1$.

Clearly $\mathcal{M}_{g,1,\mathrm{rig}}$ is a principal $\mathrm{Aut}(\widehat{\mathrm{D}})$-bundle over $\mathcal{M}_{g,1}$. Remarkably, the (fiber-wise) $\mathrm{Aut}(\widehat{\mathrm{D}})$-action extends to a *transitive* action on the total space $\mathcal{M}_{g,1,\mathrm{rig}}$ (see [Kon87; BS88]). To describe informally this action, we fix $(X, x, \nu) \in \mathcal{M}_{g,1,\mathrm{rig}}$ and $\alpha \in \mathrm{Aut}(\widehat{\mathrm{D}}^\circ)$. Intuitively, we can see the curve X as the glueing of $X \smallsetminus x$ and $\widehat{\mathrm{D}}_x$ along the punctured formal neighbourhood $\widehat{\mathrm{D}}^\circ_x$. Altering the glueing data using $\nu^{-1}\alpha\nu \in \mathrm{Aut}(\widehat{\mathrm{D}}^\circ_x)$ gives the new curve X_α, image of the action by α. In practice, X_α has the same underlying space as X, with an altered structure sheaf around x.

Denote by $W_1^{\mathbf{k}} = \mathrm{Der}(\widehat{\mathrm{D}}) = \mathbf{k}[\![t]\!]\partial_t$ the Lie algebra of formal vector fields (in dimension 1), and by $\mathrm{Witt}_1 = \mathrm{Der}(\widehat{\mathrm{D}}^\circ) = \mathbf{k}(\!(t)\!)\partial_t$ the Witt algebra:

$$W_1^{\mathbf{k}} \subset \mathrm{Witt}_1.$$

The above group actions yield an infinitesimal action of $W_1^{\mathbf{k}}$ on the fibers of the projection $\pi\colon \mathcal{M}_{g,1,\mathrm{rig}} \to \mathcal{M}_{g,1}$ and a transitive action of Witt_1 on the total space $\mathcal{M}_{g,1,\mathrm{rig}}$.

Denote by $\mathcal{M}_g$ the moduli spaces of genus g curves (as before, $g \geqslant 2$) and by ω the Hodge line bundle on $\mathcal{M}_g$. Forgetting the marked point yields a projection $p\colon \mathcal{M}_{g,1,\mathrm{rig}} \to \mathcal{M}_{g,1} \to \mathcal{M}_g$. The line bundle $p^*\omega$ plays the role of the determinantal bundle (similarly to what was described in section 1.1 above). Namely, the action of Witt_1 does not extend to $p^*\omega$. The obstruction to this extension is a central extension class of Witt_1 by $\mathbf{k}$ called the Virasoro class. It is represented by the cocycle

$$\begin{aligned} \mathrm{Witt}_1 \otimes \mathrm{Witt}_1 = \mathbf{k}(\!(t)\!)\partial_t \otimes \mathbf{k}(\!(t)\!)\partial_t &\longrightarrow \mathbf{k} \\ f\partial_t \otimes g\partial_t &\longmapsto -\tfrac{1}{12}\,\mathrm{Res}(fg'''dt). \end{aligned}$$

The corresponding extension is the Virasoro algebra Vir:

$$0 \to \mathbf{k} \to \mathrm{Vir} \to \mathrm{Witt}_1 \to 0.$$

The action of Witt_1 on $\mathcal{M}_{g,1,\mathrm{rig}}$ then extends to an action of Vir on the Hodge bundle $p^*\omega$.

3.2 The higher dimensional case

The global geometric picture available in dimension 1, albeit appealing, is not quite within reach (yet) in the higher dimensional setting. Let us describe the state of progress on this question. Most of the content of this section is work in progress, with M. Kapranov and A. Khoroshkin.

First, we can form the automorphism derived group stacks $\mathrm{Aut}(\widehat{\mathrm{D}})$ and

$$\mathrm{Aut}(\widehat{\mathrm{D}}^\circ)\colon A \mapsto \mathrm{Map}^{\simeq}_{/\operatorname{Spec} A}(\widehat{\mathrm{D}}^\circ_A, \widehat{\mathrm{D}}^\circ_A),$$

where $\widehat{\mathrm{D}}$ now denotes the formal disc of dimension d, and $\widehat{\mathrm{D}}^\circ$ is a placeholder for a wannabe d-dimensional punctured formal disc. A construction, similar to that of the determinantal anomaly (see diagram (3) and section 2) gives a derived group stack extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \widetilde{\mathrm{Aut}}(\widehat{\mathrm{D}}^\circ) \longrightarrow \mathrm{Aut}(\widehat{\mathrm{D}}^\circ) \longrightarrow 1. \tag{9}$$

One possible construction relies on the following argument: the group stack $\mathrm{Aut}(\widehat{\mathrm{D}}^\circ)$ acts canonically on the boundary operator $\mathrm{K}_{\widehat{\mathrm{D}}^\circ} \to \mathrm{B}\,\mathrm{K}$. The class of $\mathcal{O}_{\widehat{\mathrm{D}}^\circ}$ is obviously invariant under this action. It gives an $\mathrm{Aut}(\widehat{\mathrm{D}}^\circ)$-invariant morphism

$$* \xrightarrow{[\mathcal{O}]} \mathrm{K}_{\widehat{\mathrm{D}}^\circ} \longrightarrow \mathrm{B}\,\mathrm{K} \xrightarrow{\det} \mathrm{K}(\mathbb{G}_m, 2). \tag{10}$$

The resulting quotient morphism $\mathrm{B}\,\mathrm{Aut}(\widehat{\mathrm{D}}^\circ) \to \mathrm{K}(\mathbb{G}_m, 2)$ determines the above extension (9):

$$\widetilde{\mathrm{Aut}}(\widehat{\mathrm{D}}^\circ) = \Omega\big(\mathrm{fib}\big(\mathrm{B}\,\mathrm{Aut}(\widehat{\mathrm{D}}^\circ) \to \mathrm{K}(\mathbb{G}_m, 2)\big)\big).$$

This extension has a Lie algebra counterpart, generalizing the 1-dimensional picture. First, the group $\operatorname{Aut}(\widehat{\mathrm{D}})$ has Lie algebra $W_d^{\mathbf{k}}$, the Lie algebra of d-dimensional formal vector fields:

$$W_d^{\mathbf{k}} = \bigoplus_{i=1}^{d} \mathbf{k}[\![t_1, \dots, t_d]\!]\partial_{t_i}$$

with $[f\partial_{t_i}, g\partial_{t_j}] = f\frac{dg}{dt_i}\partial_{t_j} - g\frac{df}{dt_j}\partial_{t_i}$ (See [GF70b]. See also section 4 below for the importance of this Lie algebra).

The automorphism group stack $\operatorname{Aut}(\widehat{\mathrm{D}}^\circ)$ being derived in nature, its Lie algebra is really the dg-Lie algebra of derived derivations on $\widehat{\mathrm{D}}^\circ$:

$$\mathrm{Witt}_d := \mathrm{R}\Gamma(\widehat{\mathrm{D}}^\circ, \mathrm{T}_{\widehat{\mathrm{D}}^\circ}).$$

As with the Kac–Moody case, the cohomology of the underlying complex is easily computed:

$$\mathrm{H}^p(\widehat{\mathrm{D}}^\circ, \mathrm{T}_{\widehat{\mathrm{D}}^\circ}) = \begin{cases} W_d^{\mathbf{k}} & \text{if } p = 0, \\ \bigoplus_{i=1}^{d} \frac{1}{t_1 \dots t_d}\mathbf{k}[t_1^{-1}, \dots, t_d^{-1}]\partial_{t_i} & \text{if } p = d-1, \\ 0 & \text{otherwise.} \end{cases}$$

The group stack extension $\widetilde{\operatorname{Aut}}(\widehat{\mathrm{D}}^\circ)$ (cf. (9) above) induces a dg-Lie algebra extension

$$0 \longrightarrow \mathbf{k} \longrightarrow \mathrm{Vir}^d \longrightarrow \mathrm{Witt}_d \longrightarrow 0 \tag{11}$$

called the Virasoro (dg-)algebra in dimension d. This extension class has a direct Lie-algebraic construction. The action of $\operatorname{Aut}(\widehat{\mathrm{D}}^\circ)$ on $\mathcal{O}_{\widehat{\mathrm{D}}^\circ}$ gives an infinitesimal action of Witt_d onto $\mathrm{R}\Gamma(\widehat{\mathrm{D}}^\circ, \mathcal{O})$:

$$\mathrm{Witt}_d \longrightarrow \mathrm{REnd}\big(\mathrm{R}\Gamma(\widehat{\mathrm{D}}^\circ, \mathcal{O})\big).$$

Note that on the right-hand side, $\mathrm{R}\Gamma(\widehat{\mathrm{D}}^\circ, \mathcal{O})$ is to be considered with its "topology". In practice, it is a Tate complex: obtained as an extension of a complete (i.e. pro-perfect) complex by a discrete (i.e. ind-perfect) complex. The endomorphisms considered are assumed to be continuous in the proper sense. See [Hen17b].

Following [Tat68] (see [FHK19] for the details), we can construct a canonical morphism $\tau \colon \mathrm{HC}_1\big(\mathrm{REnd}\big(\mathrm{R}\Gamma(\widehat{\mathrm{D}}^\circ, \mathcal{O})\big)\big) \to \mathbf{k}$ called the trace anomaly. Its restriction along the universal morphism $\mathrm{R}\Gamma(\widehat{\mathrm{D}}^\circ, \mathcal{O}) \to \mathrm{REnd}\big(\mathrm{R}\Gamma(\widehat{\mathrm{D}}^\circ, \mathcal{O})\big)$ agrees (up to an invertible scalar) with the residue mentioned in proposition 1.5.3.

As a consequence, we get a central extension class $\tau_{\mathrm{Witt}} \in \mathrm{H}^2_{\mathrm{Lie}}(\mathrm{Witt}_d, \mathbf{k})$, corresponding to the morphism

$$\begin{tikzcd} \mathrm{H}_2^{\mathrm{Lie}}(\mathrm{Witt}_d, \mathbf{k}) \arrow[r] & \mathrm{H}_2^{\mathrm{Lie}}\big(\mathrm{REnd}\big(\mathrm{R}\Gamma(\widehat{\mathrm{D}}^\circ, \mathcal{O})\big), \mathbf{k}\big) \arrow[r, "\text{Universal trace}^5"'] & \mathrm{HC}_1\big(\mathrm{REnd}\big(\mathrm{R}\Gamma(\widehat{\mathrm{D}}^\circ, \mathcal{O})\big)\big) \arrow[r] & \mathbf{k}. \end{tikzcd}$$

Using an argument similar to that of section 2.4, we can prove that τ_{Witt} classifies the Virasoro extension (11) coming from the group theoretic side.

We next try to better understand the extension class $\tau_{\mathrm{Witt}} \in \mathrm{H}^2_{\mathrm{Lie}}(\mathrm{Witt}_d, \mathbf{k})$. To do so, we look at the space $\mathrm{H}^2_{\mathrm{Lie}}(\mathrm{Witt}_d, \mathbf{k})$ itself. First, the dg-Lie algebra Witt_d comes as the derived global section of a sheaf of local Lie algebras: the bracket is given by bi-differential operators. In

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[5] The universal trace is a morphism $\mathrm{H}_\bullet^{\mathrm{Lie}}(A, \mathbf{k}) \to \mathrm{HC}_{\bullet-1}(A)$ functorial in A any associating dg-algebra. It relates to the generalized trace of Loday–Quillen–Tsygan from theorem 1.5.1.

particular, it can be seen as the derived solutions of a Lie algebra in D-modules. In general, if $\mathcal{L}$ is a sheaf of Lie algebras in D-modules over a scheme X, we can form a D-module version $\mathrm{CE}^D_\bullet(\mathcal{L})$ of its Chevalley–Eilenberg homology (simply using the tensor product of D-modules). Taking its (global) de Rham cohomology gives notion of "diagonal homology":

$$\mathrm{CE}^\Delta_\bullet(\mathrm{R}\Gamma_{\mathrm{dR}}(X,\mathcal{L})) := \mathrm{R}\Gamma_{\mathrm{dR}}(X, \mathrm{CE}^D_\bullet(\mathcal{L})). \tag{12}$$

It can be thought as an homological analog (i.e. a predual) of Gelfand and Fuks' diagonal cohomology (see [GF68; Fuk86], see also section 4 below), in algebraic geometry. It comes with a canonical morphism

$$\mathrm{CE}_\bullet(\mathrm{R}\Gamma_{\mathrm{dR}}(X,\mathcal{L})) \to \mathrm{CE}^\Delta_\bullet(\mathrm{R}\Gamma_{\mathrm{dR}}(X,\mathcal{L}))$$

which is in general not an equivalence.

In the case of $\widehat{\mathrm{D}}^\circ$ (or rather a polynomial version $\mathrm{D}^\circ := \mathbb{A}^d \smallsetminus \{0\}$) and $\mathcal{L} := \mathrm{T} \otimes \mathcal{D}_{\mathrm{D}^\circ}$ the Lie algebra of vector fields, we can prove

Lemma 3.2.1. *The diagonal homology* $\mathrm{CE}^\Delta_\bullet(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathrm{D}^\circ,\mathcal{L}))$ *sits in a fiber-cofiber sequence*

$$\mathrm{CE}_\bullet(W^{\mathbf{k}}_d) \to \mathrm{CE}^\Delta_\bullet(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathrm{D}^\circ,\mathcal{L})) \to \mathrm{CE}_{\bullet-2d+1}(W^{\mathbf{k}}_d),$$

where $W^{\mathbf{k}}_d$ *is (as above) the Lie algebra of formal vector fields in dimension* d.

A lot is known of the (co)homology of $W^{\mathbf{k}}_d$ (see [Fuk86, p.89]):

$$\mathrm{H}^n_{\mathrm{Lie}}(W^{\mathbf{k}}_d, \mathbf{k}) = \begin{cases} 0 & \text{for } 0 < n < 2d+1 \text{ and } n > d(d+2) \\ \mathbf{k}[x_1,\dots,x_d]^{S_d}_{d+1} & \text{for } n = 2d+1, \end{cases}$$

where $\mathbf{k}[x_1,\dots,x_d]^{S_d}_{d+1}$ denotes the vector space of symmetric polynomials of total degree $d+1$. As a consequence, the degree 2 diagonal cohomology of vector fields on D° is isomorphic to $\mathbf{k}[x_1,\dots,x_d]^{S_d}_{d+1}$. The canonical morphism between diagonal and non-diagonal cohomology thus induces a morphism

$$\mathbf{k}[x_1,\dots,x_d]^{S_d}_{d+1} \to \mathrm{H}^2_{\mathrm{Lie}}(\mathrm{R}\Gamma(\mathrm{D}^\circ,\mathrm{T}),\mathbf{k}) \xleftarrow{\sim} \mathrm{H}^2_{\mathrm{Lie}}(\mathrm{Witt}_d,\mathbf{k}).$$

Conjecture 3.2.2. *The morphism* $\mathbf{k}[x_1,\dots,x_d]^{S_d}_{d+1} \to \mathrm{H}^2_{\mathrm{Lie}}(\mathrm{R}\Gamma(\mathrm{D}^\circ,\mathrm{T}),\mathbf{k}) \xleftarrow{\sim} \mathrm{H}^2_{\mathrm{Lie}}(\mathrm{Witt}_d,\mathbf{k})$ *is an isomorphism. Moreover, the class* τ_{Witt_d} *corresponds, in* $\mathbf{k}[x_1,\dots,x_d]^{S_d}_{d+1}$, *to the* $(2d+2)$*-th Todd character* Td_{2d+2}*:*

- $\mathrm{Td} = \prod_{i=1}^d \frac{x_i}{1-e^{-x_i}} \in \mathbf{k}[\![x_1,\dots,x_d]\!]^{S_d}$ *is the Todd genus,*
- Td_{2d+2} *is its degree* $d+1$ *component.*

This conjecture is a local Riemann–Roch kind of statement, comparable to theorem 1.5.4. For a fully fledged local Riemann–Roch theory, mixing both Chern and Todd characters, one should look at a higher dimensional version of the Atiyah algebra, by acting on the tangent sheaf rather than on the structure sheaf in the above construction.

We will describe, in section 4 below, an attempt at proving this conjecture. However unsuccessful, this attempt still lead to interesting results in [HK22]. Further ideas towards proving conjecture 3.2.2 will be discussed in section 5. Before that, let us conclude this section with a remark about the geometric picture. Trying to generalize the statements of section 3.1, an issue arises with the moduli space of varieties itself: deformations of algebraic varieties need not be algebraic themselves. In particular, one needs to work in the setting of derived analytic geometry, as developed in [Por19; Por15; PY20], adding technical difficulties to the mix. This direction will (hopefully) be pursued in future work.

4 Gelfand–Fuchs cohomology in algebraic geometry

The conjecture 3.2.2 fits in a more global question:

> Given a smooth variety X, can we compute the Chevalley–Eilenberg cohomology of it Lie algebra of vector fields?

In differential geometry, the cohomology of the Lie algebra of vector fields of a manifold M is known as the Gelfand–Fuchs cohomology of M. It was the subject of intense research in the 1970', starting with the founding articles of Gelfand and Fuchs [GF68; GF70b], followed by [Gui73; Hae76; BS77] among others. See below or in [Fuk86, Chap. 2 §4] for an account on the matter.

In this section, we will describe the main results of [HK22]. It deals with Gelfand–Fuchs cohomology of algebraic varieties:

Theorem 4.0.1 ([HK22])**.** *Let X be a d-dimensional smooth algebraic variety*[6] *over $\mathbb{C}$.*

(a) *The Gelfand–Fuchs cohomology $\mathrm{H}^\bullet_{\mathrm{GF}}(X) := \mathrm{H}^\bullet_{\mathrm{Lie}}(\mathrm{R}\Gamma(X, \mathrm{T}_X), \mathbb{C})$ of X comes equipped with a so-called diagonal filtration $\mathrm{H}^\bullet_{\mathrm{GF},\leqslant n}(X)$. We call the total space of this filtration the topological Gelfand–Fuchs cohomology $\mathrm{H}^\bullet_{\mathrm{GF,top}}(X)$ (see definition 4.3.1). It is equipped with a canonical morphism*

$$\tau_X \colon \mathrm{H}^\bullet_{\mathrm{GF,top}}(X) := \operatorname*{colim}_n \mathrm{H}^\bullet_{\mathrm{GF},\leqslant n}(X) \to \mathrm{H}^\bullet_{\mathrm{GF}}(X).$$

(b) *There is space Y_d (depending only on the dimension d), a bundle $Y^{\mathrm{an}}_X \to X_{\mathrm{an}}$ with fiber Y_d (over the analytification X_{an} of X) and a canonical equivalence*

$$\mathrm{H}^\bullet_{\mathrm{GF,top}}(X) \simeq \mathrm{H}^\bullet_{\mathrm{Sing}}(\mathrm{Sect}(Y^{\mathrm{an}}_X \to X_{\mathrm{an}}), \mathbb{C})$$

with the singular cohomology of the space of sections of the bundle $Y^{\mathrm{an}}_X \to X_{\mathrm{an}}$.

(c) *If X is affine, then the diagonal filtration is exhaustive: i.e. the morphism τ_X is an equivalence. As a consequence, the Gelfand–Fuchs cohomology $\mathrm{H}^\bullet_{\mathrm{GF}}(X)$ of X is finite dimensional in every degree (but there may be infinitely many non-zero cohomology groups).*

A differentiable analog of assertion (c) was known for a while [Hae76; BS77]. See theorem 4.1.1 below. In the algebraic case, is was conjecture by Feigin in the 1980', and the question stayed open since. In the case of complex analytic vector fields, Kawazumi proved a similar statement for open curves (see [Kaw93]).

As we will explain in section 4.1 below, the space Y_d and the idea of bundles Y^{an}_X is by no means new, and can be traced back to the early work of Gelfand and Fuchs [GF70b] in the differentiable case. The space Y_d has the crucial property that its (real or complex) singular cohomology is the Chevalley–Eilenberg cohomology of W_d, the Lie algebra of (real or complex) formal vector fields.

Seeing how the bundle Y_X is constructed (see below), we can deduce that $\mathrm{H}^\bullet_{\mathrm{GF,top}}(X)$ (and thus also $\mathrm{H}^\bullet_{\mathrm{GF}}(X)$ in the affine case) depends only on the rational homotopy type of the analytification X^{an} of X, on its dimension and of its tangent sheaf's Chern classes.

We give here some interesting examples. The rest of the section will be dedicated to sketching the proof of theorem 4.0.1.

Examples 4.0.2.

(a) If $X = \mathbb{A}^1_{\mathbf{k}} \smallsetminus \{0\}$ is the punctured affine line. Reducing to $\mathbf{k} = \mathbb{C}$, we get

$$\mathrm{H}^\bullet_{\mathrm{Lie}}(\mathbf{k}[t, t^{-1}]\partial_t, \mathbf{k}) \simeq \mathbf{k}[\alpha, \beta],$$

with α of degree 2 and β of degree 3. Indeed, assuming $\mathbf{k} = \mathbb{C}$, then Y^{an}_X is a trivial bundle with fiber Y_1 homotopic to S^3. The theorem allows to compute

$$\mathrm{H}^\bullet_{\mathrm{Lie}}(\mathbb{C}[t, t^{-1}]\partial_t, \mathbb{C}) \simeq \mathrm{H}^\bullet_{\mathrm{Sing}}(\mathrm{Map}(S^1, S^3), \mathbb{C}) \simeq \mathrm{H}^\bullet_{\mathrm{Sing}}(\Omega S^3 \times S^3, \mathbb{C}) \simeq \mathbb{C}[\beta, \gamma]$$

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[6] The more general case of smooth varieties over any field $\mathbf{k}$ of characteristic 0 can be deduced from that of $\mathbb{C}$. See [HK22].

with β of degree 2 and γ of degree 3. This computation relates to the Gelfand–Fuchs cohomology of the circle [GF68], and explicit formulas can be spelled out for the classes β and γ. The class β leads to the Virasoro extension in dimension 1.

(b) If $X = \mathbb{P}^1_{\mathbf{k}}$ then we have $\mathrm{H}^\bullet_{\mathrm{GF}}(\mathbb{P}^1_{\mathbf{k}}) = \mathrm{H}^\bullet_{\mathrm{Lie}}(\mathfrak{sl}_2(\mathbf{k}), \mathbf{k}) \simeq \mathbf{k}[\gamma]$ with γ in degree 3. On the other hand, the bundle Y^{an}_X is also trivial and we find (after reducing to the complex case):

$$\mathrm{H}^\bullet_{\mathrm{GF,top}}(\mathbb{P}^1_{\mathbf{k}}) \simeq \mathrm{H}^\bullet_{\mathrm{Sing}}(\mathrm{Map}(\mathbb{CP}^1, S^3), \mathbf{k}) \simeq \mathrm{H}^\bullet_{\mathrm{Sing}}(\Omega^2 S^3 \times S^3, \mathbf{k}) \simeq \mathbf{k}[\alpha, \gamma]$$

with α of degree 1 and γ in degree 3. The graded ring $\mathbf{k}[\alpha, \gamma]$ is isomorphic to $\mathrm{H}^\bullet_{\mathrm{Lie}}(\mathfrak{gl}_2(\mathbf{k}), \mathbf{k})$, and the canonical morphism

$$\tau_{\mathbb{P}^1} : \mathrm{H}^\bullet_{\mathrm{Lie}}(\mathfrak{gl}_2(\mathbf{k}), \mathbf{k}) \simeq \mathrm{H}^\bullet_{\mathrm{GF,top}}(\mathbb{P}^1_{\mathbf{k}}) \to \mathrm{H}^\bullet_{\mathrm{GF}}(\mathbb{P}^1_{\mathbf{k}}) = \mathrm{H}^\bullet_{\mathrm{Lie}}(\mathfrak{sl}_2(\mathbf{k}), \mathbf{k})$$

identifies with the restriction along the inclusion $\mathfrak{sl}_2 \subset \mathfrak{gl}_2$.
A similar statement for higher dimensional projective spaces is not yet known. It would provide an interesting insight on the topological cohomology and its behaviour.

(c) If $X = E$ is an elliptic curve, then

$$\mathrm{H}^\bullet_{\mathrm{GF,top}}(E) \simeq \mathrm{H}^\bullet_{\mathrm{Sing}}(\mathrm{Map}(S^1 \times S^1, S^3), \mathbf{k}) \simeq \mathbf{k}[\alpha, \beta_1, \beta_2, \gamma]$$

with α in degree 1, β_1 and β_2 in degree 2 and γ in degree 3. On the other hand, $\mathrm{R}\Gamma(E, \mathrm{T}) = \mathbf{k} \oplus \mathbf{k}[-1]$ is an abelian dg-Lie algebra. In particular

$$\mathrm{H}^n_{\mathrm{GF}}(E) \simeq \begin{cases} \mathbf{k}[\![x]\!] & \text{if } n = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

In this example we thus have $\mathrm{H}^\bullet_{\mathrm{GF,top}}(E)$ finite dimensional in each degree, but spread out on an infinite number of cohomology groups on one side, and $\mathrm{H}^\bullet_{\mathrm{GF}}(E)$ infinite dimensional in two degrees, and zero elsewhere on the other side.

(d) Let $X = \mathbb{P}^d_{\mathbf{k}} \smallsetminus Z$ with Z a smooth quadric hypersurface. A more evolved computation (see [HK22, §6.5]) shows

$$\mathrm{H}^\bullet_{\mathrm{GF}}(\mathbb{P}^d_{\mathbf{k}} \smallsetminus Z) \simeq \begin{cases} \mathrm{H}^\bullet_{\mathrm{Sing}}(Y_d, \mathbf{k}) \simeq \mathrm{H}^\bullet_{\mathrm{Lie}}(W^{\mathbf{k}}_d, \mathbf{k}) & \text{if } d \text{ is even,} \\ \mathrm{H}^\bullet_{\mathrm{Sing}}(\mathrm{Map}(S^d, Y_d), \mathbf{k}) & \text{if } d \text{ is odd.} \end{cases}$$

(e) If $X = \mathrm{D}^\circ := \mathbb{A}^d \smallsetminus \{0\}$, then $Y^{\mathrm{an}}_X \simeq \mathbb{C}^d \smallsetminus \{0\} \times Y_d$ and we get

$$\mathrm{H}^\bullet_{\mathrm{GF,top}}(\mathrm{D}^\circ) \simeq \mathrm{H}^\bullet_{\mathrm{Sing}}(\mathrm{Map}(\mathbb{C}^d \smallsetminus \{0\}, Y_d), \mathbf{k}) \simeq \mathrm{H}^\bullet_{\mathrm{Sing}}(\mathrm{Map}(S^{2d-1}, Y_d), \mathbf{k})$$

Using Serre's spectral sequence, one can compute $\mathrm{H}^2_{\mathrm{Sing}}(\mathrm{Map}(S^{2d-1}, Y_d), \mathbf{k}) \simeq \mathrm{H}^{2d+1}_{\mathrm{Sing}}(Y_d, \mathbf{k}) \simeq \mathrm{H}^{2d+1}_{\mathrm{Lie}}(W^{\mathbf{k}}_d, \mathbf{k}) \simeq \mathbf{k}[x_1, \ldots, x_d]^{S_d}_{d+1}$. In particular, the morphism τ_{D° induces a morphism

$$\mathbf{k}[x_1, \ldots, x_d]^{S_d}_{d+1} \simeq \mathrm{H}^2_{\mathrm{GF,top}}(\mathrm{D}^\circ) \to \mathrm{H}^2_{\mathrm{GF}}(\mathrm{D}^\circ)$$

to be identified with the morphism of conjecture 3.2.2. In particular, conjecture 3.2.2 would follow from an extension of assertion (c) of theorem 4.0.1 to the quasi-affine case, or more humbly to $\mathbb{A}^d \smallsetminus \{0\}$ alone. See section 5 for possible research in this direction.

4.1 The case of differentiable manifolds

Let us first give a modern overview of the differentiable case. The results explained in this section are due to Gelfand–Fuchs [GF68; GF70b], Guillemin [Gui73], Haefliger [Hae76] and Bott–Segal [BS77].

Let us fix a smooth manifold M, of dimension d and denote by $\mathrm{H}^\bullet_{\mathrm{GF}}(M)$ its Gelfand–Fuchs cohomology, that is the Chevalley–Eilenberg cohomology of its Lie algebra of global vector fields $\mathrm{Vect}(M)$:

$$\mathrm{H}^\bullet_{\mathrm{GF}}(M) := \mathrm{H}^\bullet(\mathrm{Vect}(M), \mathbb{R}).$$

The goal is to compute $\mathrm{H}^\bullet_{\mathrm{GF}}(M)$.

The local case: Consider first $M = \mathbb{R}^d$. Expansion at $0 \in \mathbb{R}^d$ gives a morphism $\mathrm{Vect}(\mathbb{R}^d) \to W_d^{\mathbb{R}}$, and thus a restriction map

$$\mathrm{H}^\bullet_{\mathrm{Lie}}(W_d^{\mathbb{R}}, \mathbb{R}) \xrightarrow{\sim} \mathrm{H}^\bullet_{\mathrm{GF}}(\mathbb{R}^d)$$

which turns out to be an isomorphism (see [GF70b]). As a consequence, for any immersion of a disc D in $\mathbb{R}^d$, the induced morphism $\mathrm{H}^\bullet_{\mathrm{GF}}(D) \to \mathrm{H}^\bullet_{\mathrm{GF}}(\mathbb{R}^d)$ is an isomorphism. From this, we get the (informal and not quite true) intuition that Gelfand–Fuchs cohomology should appear as the global sections of a locally constant cosheaf, with costalk $\mathrm{H}^\bullet_{\mathrm{Lie}}(W_d^{\mathbb{R}}, \mathbb{R})$. Before explaining why this intuition is incorrect, and how to fix it, let us describe the tentative costalk in more details. This is the content of [GF70b].

The main ingredient here is the Gelfand–Fuchs skeleton Y_d (mentioned above), satisfying

$$\mathrm{H}^\bullet_{\mathrm{Lie}}(W_d^{\mathbb{R}}, \mathbb{R}) \simeq \mathrm{H}^\bullet_{\mathrm{Sing}}(Y_d, \mathbb{R}). \tag{13}$$

The approach depicted here is not quite the original (and more computational) approach of Gelfand and Fuchs. We only sketch the construction and the proof of (13). For further details, see [HK22, Sec. 1].

The first step is to reduce to complex coefficients. Then, the space Y_d will come with an action of the Lie group $\mathrm{GL}_d(\mathbb{C})$, and the above isomorphism will be compatible with this action. To construct Y_d, we will first construct the quotient ${Y_d}/{\mathrm{GL}_d(\mathbb{C})}$. We start by considering the natural $\mathrm{GL_d}(\mathbb{C})$-action on $W_d^{\mathbb{C}}$. It induces, on the Chevalley–Eilenberg complex of $W_d^{\mathbb{C}}$, an action à la Bernstein–Lunts: the Lie group acts and the induced infinitesimal action of the Lie algebra is null-homotopic. The (homotopy) fixed points under this action are then given by the relative Chevalley–Eilenberg cohomology that Gelfand and Fuchs compute

$$\mathrm{H}^\bullet_{\mathrm{Lie}}(W_d^{\mathbb{C}}, \mathfrak{gl}_d(\mathbb{C}), \mathbb{C}) \simeq {\mathbb{C}[e_1, \dots, e_d]}/{I}$$

where e_i has cohomological degree $2i$ and I is the ideal of polynomials of cohomological degree greater than $2d$.

The wannabe quotient ${Y_d}/{\mathrm{GL}_d(\mathbb{C})}$ should thus satisfy

$$\mathrm{H}^\bullet_{\mathrm{Sing}}\left({Y_d}/{\mathrm{GL}_d(\mathbb{C})}, \mathbb{C}\right) \simeq \mathrm{H}^\bullet_{\mathrm{Lie}}(W_d^{\mathbb{C}}, \mathfrak{gl}_d(\mathbb{C}), \mathbb{C}) \simeq {\mathbb{C}[e_1, \dots, e_d]}/{I}.$$

Observe then that the cohomology of the classifying space $\mathrm{BGL}_d(\mathbb{C})$ is the graded ring $\mathbb{C}[e_1, \dots, e_d]$ with e_i as above. The space ${Y_d}/{\mathrm{GL}_d(\mathbb{C})}$ should thus be a $2d$-skeleton in $\mathrm{BGL}_d(\mathbb{C})$. To make proper sense of this skeleton, we rely on the description of (the homotopy type of) $\mathrm{BGL}_d(\mathbb{C})$ as an infinite Grassmannian $\mathrm{Gr}(d, \mathbb{C}^\infty)$ of d-dimensional subspaces of $\mathbb{C}^\infty$. We then define $\mathrm{sk}_{2d} \subset \mathrm{Gr}(\mathbb{C}^d, \mathbb{C}^\infty)$ as the $2d$-skeleton with respect to the Schubert cell decomposition. The space sk_{2d} is our candidate for ${Y_d}/{\mathrm{GL}_d(\mathbb{C})}$, as it satisfies by construction

$$\mathrm{H}^\bullet_{\mathrm{Sing}}(\mathrm{sk}_{2d}, \mathbb{C}) \simeq {\mathbb{C}[e_1, \dots, e_d]}/{I} \simeq \mathrm{H}^\bullet_{\mathrm{Lie}}(W_d^{\mathbb{C}}, \mathfrak{gl}_d(\mathbb{C}), \mathbb{C}). \tag{14}$$

The space Y_d is the homotopy fiber product

$$\begin{tikzcd}
Y_d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
\mathrm{sk}_{2d} \arrow[r] & \mathrm{BGL}_d(\mathbb{C}).
\end{tikzcd}$$

Because the equivalence (14) is compatible with the action of $\mathbb{C}[e_1, \ldots, e_d] = \mathrm{H}^\bullet_{\mathrm{Sing}}(\mathrm{BGL}_d(\mathbb{C}), \mathbb{C})$, this implies an equivalence

$$\mathrm{H}^\bullet_{\mathrm{Lie}}(W_d^{\mathbb{C}}, \mathbb{C}) \simeq \mathrm{H}^\bullet_{\mathrm{Sing}}(Y_d, \mathbb{C}), \tag{15}$$

which is furthermore compatible with the Bernstein–Lunts actions of $\mathrm{GL}_d(\mathbb{C})$. Further computations lead to the properties of $\mathrm{H}^\bullet_{\mathrm{Lie}}(W_d^{\mathbb{C}}, \mathbb{C})$ (or more generally of $\mathrm{H}^\bullet_{\mathrm{Lie}}(W_d^{\mathbf{k}}, \mathbf{k})$ for any field $\mathbf{k}$ of characteristic 0) mentioned above conjecture 3.2.2:

$$\mathrm{H}^n_{\mathrm{Lie}}(W_d^{\mathbf{k}}, \mathbf{k}) = \begin{cases} 0 & \text{for } 0 < n < 2d+1 \text{ and } n > d(d+2) \\ \mathbf{k}[x_1, \ldots, x_d]^{S_d}_{d+1} & \text{for } n = 2d+1. \end{cases}$$

As a corollary, we get that the Gelfand–Fuchs cohomology of $\mathbb{R}^d$ is finite dimensional.

The global case: To state the global result, we first need a version of Y_d over a d-dimensional manifold. Consider the complexified tangent vector bundle $\mathrm{T}_M^{\mathbb{C}} := \mathrm{T}_M \otimes_{\mathbb{R}} \mathbb{C}$ of M. It is classified by a morphism $M \to \mathrm{BGL}_d(\mathbb{C})$. We denote by Y_M the homotopy fiber product

$$\begin{tikzcd}
Y_M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "\mathrm{T}_M^{\mathbb{C}}"] \\
\mathrm{sk}_{2d} \arrow[r] & \mathrm{BGL}_d(\mathbb{C}).
\end{tikzcd}$$

In other words, $Y_M := \mathrm{T}_M^{\mathbb{C}} \times_{\mathrm{GL}_d(\mathbb{C})} Y_d$ is the bundle over M with fiber Y_d associated to $\mathrm{T}_M^{\mathbb{C}}$.

Theorem 4.1.1 (Haefliger [Hae76], Bott–Segal [BS77])**.** *Let M be a smooth manifold of dimension d that admits a finite cover by convex discs (e.g. M is compact or is the interior of a compact manifold with boundary). There is an equivalence*

$$\mathrm{H}^\bullet_{\mathrm{GF}}(M) \simeq \mathrm{H}^\bullet_{\mathrm{Sing}}(\mathrm{Sect}(Y_M \to M), \mathbb{R}).$$

As a consequence, the Gelfand–Fuchs cohomology $\mathrm{H}^\bullet_{\mathrm{GF}}(M)$ of M is finite dimensional in every degree (although there may be infinitely many non-vanishing cohomology groups).

This theorem is a global version of equation (13) above. In order to globalize such an isomorphism, we would like to say that Gelfand–Fuchs cohomology is some kind of cosheaf. Taken to naively, this statement is false. Consider the very simple example of a manifold M consisting of two connected components M_1 and M_2. If $\mathrm{H}^\bullet_{\mathrm{GF}}$ were a cosheaf (in complexes), we'd have $\mathrm{H}^\bullet_{\mathrm{GF}}(M) \simeq \mathrm{H}^\bullet_{\mathrm{GF}}(M_1) \oplus \mathrm{H}^\bullet_{\mathrm{GF}}(M_2)$, but instead we have

$$\mathrm{H}^\bullet_{\mathrm{GF}}(M) \simeq \mathrm{H}^\bullet_{\mathrm{GF}}(M_1) \otimes \mathrm{H}^\bullet_{\mathrm{GF}}(M_2).$$

This sort of multiplicative cosheaf structure is precisely what factorization algebras are encoding. We refer to [Gin15] for a survey of factorization algebras in this topological context.

Informally, a pre-factorization algebra (say, in complexes of $\mathbf{k}$-vector spaces) over M is a pre-cosheaf $\mathcal{F} \colon \mathrm{Open}(M) \to \mathrm{C}(\mathbf{k})$ endowed with (compatible) additional structural equivalences

$$\mu_{U_1, \ldots, U_n} \colon \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \simeq \mathcal{F}(U_1 \amalg \cdots \amalg U_n) \tag{16}$$

for $U_1, \dots, U_n$ pairwise disjoint in M. Using this structure, we can form for any $N \subset M$ open and any finite covering $\mathfrak{U} = \{U_1, \dots, U_n\}$ of N a factorizing Čech diagram

$$\check{\mathrm{C}}^{\otimes}(\mathcal{F}, \mathfrak{U}) := \left(\cdots \Rrightarrow \bigoplus_{\alpha,\beta \in \mathrm{P}\mathfrak{U}} \bigotimes_{\substack{U \in \alpha \\ V \in \beta}} \mathcal{F}(U \cap V) \rightrightarrows \bigoplus_{\alpha \in \mathrm{P}\mathfrak{U}} \bigotimes_{U \in \alpha} \mathcal{F}(U) \right),$$

where $\mathrm{P}\mathfrak{U}$ is the set of sets of mutually disjoint elements of $\mathfrak{U}$: i.e. $\alpha \in \mathrm{P}\mathfrak{U}$ if $\alpha = \{U_{i_1}, \dots, U_{i_p}\}$ with $U_{i_r} \cap U_{i_s} = \varnothing$ if $r \neq s$. A pre-factorization algebra $\mathcal{F}$ is a factorization algebra if for any open subset $N \subset M$ and any finite covering $\mathfrak{U}$ of N, the canonical morphism $\operatorname{colim} \check{\mathrm{C}}^{\otimes}(\mathcal{F}, \mathfrak{U}) \to \mathcal{F}(N)$ is an equivalence.

The factorization homology of a factorization algebra $\mathcal{F}$ is, by definition, its global cosections:

$$\int_M \mathcal{F} := \mathcal{F}(M).$$

Theorem 4.1.2 (Bott–Segal, [BS77, Cor. 5.8 and Prop. 6.2])**.** *The following functors are factorization algebras on M:*

$$\begin{aligned} \mathcal{F}_{\mathrm{GF}} &: U \mapsto \mathrm{GF}^\bullet(U) \\ \mathcal{F}_{\mathrm{Sect}} &: U \mapsto \mathrm{Sing}^\bullet_{\mathbb{R}}(\mathrm{Sect}(Y_U \to U)), \end{aligned}$$

where $\mathrm{GF}^\bullet(U) := \mathrm{CE}^\bullet(\mathrm{Vect}(U), \mathbb{R})$ *is the Gelfand–Fuchs complex of U and* $\mathrm{Sing}^\bullet_R(\mathrm{Sect}(Y_U \to U))$ *is the singular complex of* $\mathrm{Sect}(Y_U \to U)$*, so that we have*

$$\mathrm{H}^n(\mathrm{GF}^\bullet(U)) \simeq \mathrm{H}^n_{\mathrm{GF}}(U) \quad \textit{and} \quad \mathrm{H}^n(\mathrm{Sing}^\bullet_{\mathbb{R}}(\mathrm{Sect}(Y_U \to U))) \simeq \mathrm{H}^\bullet_{\mathrm{Sing}}(\mathrm{Sect}(Y_U \to U), \mathbb{R}).$$

The case of $\mathcal{F}_{\mathrm{GF}}$ relies on Gelfand and Fuchs' *diagonal filtration* on the Gelfand–Fuchs complex $\mathrm{GF}^\bullet(M)$ (introduced in [GF69; GF70a]). We will describe in more details its algebraic analog in section 4.3.

The case of $\mathcal{F}_{\mathrm{Sect}}$ can be seen as a precursor to Lurie's non-abelian Poincaré duality theorem [Lur17, Thm. 5.5.6.6]. It relies on the crucial fact that the space Y_d is d-connected (actually, it is $2d$-connected). Using said fact, we can then use a generalization of the Eilenberg–Moore spectral sequence, due to Anderson [And72].

Those factorization algebras are moreover locally constant, in the sense that if $U \subset V$ is an homotopy equivalence, then $\mathcal{F}_{\mathrm{FG}}(U) \to \mathcal{F}_{\mathrm{FG}}(V)$ is an equivalence (and similarly for $\mathcal{F}_{\mathrm{Sect}}$). The key point is then the following: such factorization algebras are determined by their costalk at a point, equipped with an $\mathbb{E}_d$-algebra structure (here, actually a commutative algebra structure), and an action of $\mathrm{GL}_d(\mathbb{R})$. In this case, the costalks are (respectively) $\mathrm{H}^\bullet_{\mathrm{Lie}}(W^{\mathbb{R}}_n, \mathbb{R})$ and $\mathrm{H}^\bullet_{\mathrm{Sing}}(Y_d, \mathbb{R})$. The equivalence (15) identifies said stalks (with all their structure!) and thus provides an equivalence $\mathcal{F}_{\mathrm{GF}} \simeq \mathcal{F}_{\mathrm{Sect}}$. This implies theorem 4.1.1

$$\mathrm{H}^\bullet_{\mathrm{GF}}(M) \simeq \int_M \mathcal{F}_{\mathrm{FG}} \simeq \int_M \mathcal{F}_{\mathrm{Sect}} \simeq \mathrm{H}^\bullet_{\mathrm{Sing}}(\mathrm{Sect}(Y_M \to M)).$$

4.2 Factorization algebras in the algebraic context

Because of the lack of open subsets for the Zariski topology, factorization algebras defined as cosheaves (like in the differentiable context) do not work so well. Like in Verdier duality, the trick is to mimic cosheaf-behaviours using exceptional pullbacks and pushforward of sheaves. Algebraic factorization algebras appeared first in [BD04]. Like therein, we will work with factorization algebras in $\mathcal{D}$-modules.

To get to define structure morphisms like $\mu_{U,V}$ in (16), for disjoint opens subsets U and V, we see the pair U, V as a neighbourhood of a configuration $\{x, y\}$ of two distinct points $x \in U$ and

$y \in V$. In plain terms, this means we shall work with a space of configurations of points (i.e. of finite subsets, without fixing a cardinal): the Ran space. First introduced by Borsuk and Ulam [BU31] in the topological setting, the Ran space was later studied in the algebraic context by Ran [Ran93; Ran00]. Interestingly, it also appears in Haefliger's work on Gelfand–Fuchs cohomology [Hae76], a sign that ideas behind factorization algebras were already present at the time.

The algebro-geometric definition of the Ran space is the following. Denote by $\overrightarrow{\mathrm{Fin}}$ the category whose objects are non-empty finite sets, and morphisms are surjections. We then set, for X a variety

$$\operatorname{Ran} X := \operatorname*{colim}_{I \in \overrightarrow{\mathrm{Fin}}} X^I.$$

$\mathcal{D}^!$-factorization algebra: A $\mathcal{D}^!$-module over $\operatorname{Ran} X$ is a family $(E^{(I)})_I$ of $\mathcal{D}$-modules over each X^I, such that for any diagonal embedding $\delta \colon X^J \to X^I$, we have $\delta^! E^{(I)} \simeq E^{(J)}$. It factorization homology (we do not yet require a factorization structure) is the global (co)sections:

$$\int_X E := \operatorname*{colim}_{I \in \overrightarrow{\mathrm{Fin}}} \mathrm{R}\Gamma_{\mathrm{dR}}\Big(X^I, E^{(I)}\Big).$$

There is also a compactly supported factorization homology, defined as

$$\int_X^{\mathrm{c}} E := \operatorname*{colim}_{I \in \overrightarrow{\mathrm{Fin}}} \mathrm{R}\Gamma^{\mathrm{c}}_{\mathrm{dR}}\Big(X^I, E^{(I)}\Big).$$

The factorization structure (corresponding to the morphisms μ of (16)) is defined as follows. Given a surjection $\alpha \colon I \twoheadrightarrow J$, we define $U(\alpha)$ the open subset of X^I given by

$$U(\alpha) = \{(x_i)_{i \in I} \mid \text{the subsets } I_j := \{x_i, i \in \alpha^{-1}(j)\},\ j \in J, \text{ are pairwise disjoint}\}.$$

A factorization structure on $E = (E^{(I)})_I$ is the datum of compatible equivalences

$$E^{(I)}_{|U(\alpha)} \simeq \left(\boxed{\times}_{j \in J} E^{(I_j)}\right)_{|U(\alpha)}.$$

For example, with a factorization structure, the stalk of $E^{(\{1,2\})}$ at a configuration $x \neq y \in X$ is equivalent to the tensor product of the stalks of $E^{(*)} = E^{(\{1\})} = E^{(\{2\})}$ at x and y. A $\mathcal{D}^!$-module over $\operatorname{Ran} X$ equipped with a factorization structure is called a $\mathcal{D}^!$-factorization algebra[7].

It is rather easy to construct a $\mathcal{D}^!$-factorization algebra computing Chevalley–Eilenberg homology of a sheaf of Lie algebras. Fix L a sheaf of $\mathbf{k}$-linear Lie algebras on X. The Chevalley–Eilenberg homology of the dg-Lie algebra $\mathrm{R}\Gamma(X, L)$ is computed by the complex

$$\begin{tikzcd}
\cdots \arrow[r] & \Lambda^3 \mathrm{R}\Gamma(X,L) \arrow[r] \arrow[d, phantom, "\wr"] & \Lambda^2 \mathrm{R}\Gamma(X,L) \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{R}\Gamma(X,L) \arrow[r] & \mathbf{k} \arrow[r] & 0 \\
 & \mathrm{R}\Gamma(X^3, L^{\boxtimes 3})_{\mathfrak{S}_3} & \mathrm{R}\Gamma(X^2, L \boxtimes L)_{\mathfrak{S}_2} & & &
\end{tikzcd} \tag{17}$$

where the symmetric group $\mathfrak{S}_n$ acts on $\mathrm{R}\Gamma(X^n, L^{\boxtimes n})$ by permuting the factors and multiplying by the signature. From this perspective, it becomes quite natural to try and define a factorization algebra E such that $E^{(I)} = L^{\boxtimes I}$. This does not quite work, for three reasons: first, we should be working with $\mathcal{D}$-modules ; second, the antisymmetric action and the Lie structure need to be factored in somehow ; and third, and more importantly, we cannot hope for an equivalence $\Delta^!(L \boxtimes L) \simeq L$. Fixing those three points lead to the following construction.

---

[7] This notion is often simply called a factorization algebra in the literature. We add the $\mathcal{D}^!$ to emphasize the difference with a dual notion to be introduced below.

Example 4.2.1. Fix a Lie algebra $\mathcal{L}$ in $\mathcal{D}$-modules over X (for instance $\mathcal{L} = \mathrm{T}_X \otimes \mathcal{D}_X$ induced from the tangent sheaf, with the bracket of vector fields). For I a finite set a q an integer, we denote by $\mathrm{Eq}_q(I)$ the (finite) set of equivalence relations on elements of I with exactly q equivalence classes. We set

$$\mathcal{C}_q^{(I)} := \bigoplus_{\sim \in \mathrm{Eq_q(I)}} \delta_*\big(\mathcal{L}^{\boxtimes I/\sim} \otimes \det(\mathbf{k}^{I/\sim})\big)$$

where $\delta\colon X^{I/\sim} \to X^I$ is the diagonal embedding. In particular,

$$\mathcal{C}_q^{(I)} = \begin{cases} \mathcal{L}^{\boxtimes I} \otimes \det(\mathbf{k}^I) & \text{if } q = \mathrm{card}(I) \text{ is the cardinal of } I, \\ \mathcal{L} & \text{if } q = 1, \\ 0 & \text{if } q \leqslant 0 \text{ or } q > \mathrm{card}(I). \end{cases}$$

The Lie bracket induces a square-zero differential $\mathcal{C}_q^{(I)} \to \mathcal{C}_{q-1}^{(I)}$. We define $\mathcal{C}_\bullet(X, \mathcal{L})$ by

$$\mathcal{C}_\bullet(X, \mathcal{L})^{(I)} := \Big(0 \to \mathcal{C}_{\mathrm{card}(I)}^{(I)} \to \cdots \to \mathcal{C}_q^{(I)} \to \cdots \to \mathcal{C}_1^{(I)} \to 0\Big)$$

This is not quite a $\mathcal{D}^!$-module over the Ran space. Indeed, the natural morphisms $\mathcal{C}_\bullet(X, \mathcal{L})^{(J)} \to \delta^! \mathcal{C}_\bullet(X, \mathcal{L})^{(I)}$ (associated to diagonal embeddings $\delta\colon X^J \to X^I$) are not equivalences. Nevertheless, this can formally be made into a factorization algebra by a strictification process (see [HK22, Def. 5.1.4] for more details). Its factorization homology computes the (reduced) Chevalley–Eilenberg homology of the dg-Lie algebra of global solutions of $\mathcal{L}$:

$$\int_X \mathcal{C}_\bullet(X, \mathcal{L}) \simeq \overline{\mathrm{CE}}_\bullet(\mathrm{R}\Gamma_{\mathrm{dR}}(X, \mathcal{L}), \mathbf{k}).$$

When $\mathcal{L} = \mathrm{T}_X \otimes \mathcal{D}_X$, this computes the (reduced) Gelfand–Fuchs homology of X.

$[\![\mathcal{D}]\!]$-factorization algebras: Since we are mostly interested in Chevalley–Eilenberg cohomology, we will need a dual notion (in the sense of Verdier duality), that we call $[\![\mathcal{D}]\!]$-factorization algebras. Because we do not have any finiteness assumption on our $\mathcal{D}$-modules (such as holonomicity), this Verdier duality takes values in complete $\mathcal{D}$-modules (represented as pro-perfect $\mathcal{D}$-modules). For $f\colon X \to Y$ and E a complete $\mathcal{D}$-module on Y, we denote by $f^{[\![*]\!]}(E)$ the complete $\mathcal{D}$-module on X:

$$f^{[\![*]\!]}(E) := \big(f^!(E^\vee)\big)^\vee.$$

This allows us to define $[\![\mathcal{D}]\!]$-modules over $\mathrm{Ran}\, X$, as a collection $(F_{(I)})$ of complete $\mathcal{D}$-modules over the X^I's, with equivalences $\delta^{[\![*]\!]}(F_{(I)}) \simeq F_{(J)}$ for any diagonal embedding $\delta\colon X^J \to X^I$. The factorization structure is defined similarly. The (compactly supported) factorization cohomology of a $[\![\mathcal{D}]\!]$-factorization algebra F is then the complete complex of $\mathbf{k}$-vector spaces

$$\oint_X F := \lim_{I \in \overset{\twoheadrightarrow}{\mathrm{Fin}}} \mathrm{R}\Gamma_{\mathrm{dR}}(X^I, F_{(I)}) \quad \text{and} \quad \oint_X^{\mathrm{c}} F := \lim_{I \in \overset{\twoheadrightarrow}{\mathrm{Fin}}} \mathrm{R}\Gamma_{\mathrm{dR}}^{\mathrm{c}}(X^I, F_{(I)}).$$

The Verdier duality provides an equivalence between $\mathcal{D}^!$- and $[\![\mathcal{D}]\!]$-factorization algebra. We have moreover

$$\left(\oint_X F\right)^* \simeq \int_X^{\mathrm{c}} F^\vee \quad \text{and} \quad \left(\oint_X^{\mathrm{c}} F\right)^* \simeq \int_X F^\vee.$$

Example 4.2.2. The $[\![\mathcal{D}]\!]$-factorization algebra $\check{\mathcal{C}}^\bullet(X, \mathcal{L})$ is defined as the Verdier dual of $\mathcal{C}_\bullet(X, \mathcal{L})$ from example 4.2.1. Its compactly supported factorization cohomology computes the (reduced) Chevalley–Eilenberg cohomology:

$$\oint_X^{\mathrm{c}} \check{\mathcal{C}}^\bullet(X, \mathcal{L}) \simeq \overline{\mathrm{CE}}^\bullet(\mathrm{R}\Gamma_{\mathrm{dR}}(X, \mathcal{L}), \mathbf{k}).$$

In particular, if $\mathcal{L} = \mathrm{T}_X \otimes \mathcal{D}_X$, the right-hand-side is nothing else than the (reduced) Gelfand–Fuchs cohomology of X:

$$\oint_X^{\mathrm{c}} \check{\mathcal{C}}^\bullet(X, \mathrm{T}_X \otimes \mathcal{D}_X) \simeq \overline{\mathrm{CE}}^\bullet(\mathrm{R}\Gamma(X, \mathrm{T}_X), \mathbf{k}) = \overline{\mathrm{GF}}^\bullet(X).$$

4.3 Diagonal filtration and covariant Verdier duality

Introduced by Gelfand and Fuchs in [GF69; GF70a], the diagonal filtration on Gelfand–Fuchs cohomology is central to the differentiable case. It induces an important spectral sequence, allowing some computations. We will use neither the diagonal filtration nor the induced spectral sequence directly. Still, let us describe it in terms of factorization algebras, as it shed light on the similarities between the arguments we give, and the ones in differential geometry.

Fix a positive integer n and, for I a finite set, denote by $X^I_{\leqslant n} \subset X^I$ the closed subvariety of I-tuples composed of at most n different points. For F a $[\![\mathcal{D}]\!]$-factorization algebra, we set

$$\oint_X^{\mathrm{c},\leqslant n} F := \lim_I \mathrm{R}\Gamma^{\mathrm{dR,c}}_{X^I_{\leqslant n}}\big(X^I, F_{(I)}\big),$$

where $\mathrm{R}\Gamma^{\mathrm{dR,c}}_Z$ computes the compactly-supported de Rham cohomology supported on Z. This defines a filtration[8]

$$0 \to \oint_X^{\mathrm{c},\leqslant 1} F \to \cdots \to \oint_X^{\mathrm{c},\leqslant n} F \to \cdots \to \oint_X^{\mathrm{c}} F.$$

Crucially, and contrary to the differentiable situation, this filtration does not need to be exhaustive: the morphism

$$\operatorname*{colim}_n \oint_X^{\mathrm{c},\leqslant n} F \to \oint_X^{\mathrm{c}} F$$

does not need to be an equivalence. Specializing to the Gelfand–Fuchs factorization algebra $\check{\mathcal{C}}^\bullet(X, \mathrm{T}_X \otimes \mathcal{D}_X)$ gives the diagonal filtration in the algebraic setting:

$$0 \to \mathrm{GF}^\bullet_{\leqslant 1}(X) \to \cdots \to \mathrm{GF}^\bullet_{\leqslant n}(X) \to \cdots \to \mathrm{GF}^\bullet(X).$$

The first term, $\mathrm{GF}^\bullet_{\leqslant 1}(X)$ is often referred to as the diagonal complex, and its cohomology as the diagonal cohomology. It is dual to the diagonal homology introduced in equation (12) above. Like for diagonal homology, diagonal cohomology comes as the (derived and compactly supported) global solutions of a $\mathcal{D}$-module $\mathcal{C}^\bullet_\Delta(X)$ on X.

We are now ready to define the topological Gelfand–Fuchs cohomology used in the statement of theorem 4.0.1:

Definition 4.3.1. The topological Gelfand–Fuchs cohomology of X is computed by the total space of the diagonal filtration

$$\mathrm{GF}^\bullet_{\mathrm{top}}(X) := \operatorname*{colim}_n \mathrm{GF}^\bullet_{\leqslant n}(X), \qquad \mathrm{H}^\bullet_{\mathrm{GF,top}}(X) := \mathrm{H}^\bullet(\mathrm{GF}^\bullet_{\mathrm{top}}(X)).$$

This concludes the construction of assertion (a) of theorem 4.0.1.

We can give a more explicit description of this diagonal filtration. Set $\mathcal{L} = \mathrm{T} \otimes \mathcal{D}_X$. Recall the (bi)complex (17), or rather its dual, computing Gelfand–Fuchs cohomology:

$$0 \longrightarrow \mathrm{R}\Gamma^{\mathrm{c}}_{\mathrm{dR}}(X, \mathcal{L}^\vee) \longrightarrow \mathrm{R}\Gamma^{\mathrm{c}}_{\mathrm{dR}}\big(X^2, (\mathcal{L}^\vee)^{\boxtimes 2}\big)^{\mathfrak{S}_2} \longrightarrow \mathrm{R}\Gamma^{\mathrm{c}}_{\mathrm{dR}}\big(X^3, (\mathcal{L}^\vee)^{\boxtimes 3}\big)^{\mathfrak{S}_3} \longrightarrow \cdots$$

---

[8] A dual version of the filtration is also available for $\mathcal{D}^!$-factorization algebras.

In each X^q, we can consider the closed subvariety $X^q_{\leqslant n}$. The filtration part $\mathrm{H}^\bullet_{\mathrm{GF},\leqslant n}(X)$ is then computed by the above complex, where de Rham cohomology is replaced with de Rham cohomology with support in $X^q_{\leqslant n}$:

$$0 \longrightarrow \mathrm{R}\Gamma^{\mathrm{c}}_{\mathrm{dR}}(X, \mathcal{L}^\vee) \longrightarrow \cdots \longrightarrow \mathrm{R}\Gamma^{\mathrm{dR,c}}_{X^q_{\leqslant n}}(X^q, (\mathcal{L}^\vee)^{\boxtimes q})^{\mathfrak{S}_q} \longrightarrow \cdots$$

Since $X^q_{\leqslant n} = X^q$ if $n \geqslant q$, the filtration is trivially exhaustive on every graded part. However, in the resulting total complex, a fixed cohomological degree may sees contributions of every $\mathrm{R}\Gamma^{\mathrm{dR,c}}_{X^q_{\leqslant n}}(X^q, (\mathcal{L}^\vee)^{\boxtimes q})^{\mathfrak{S}_q}$ (since this complex is not necessarily coconnective).

In particular, the filtration does not need to be exhaustive in the general case (and it is not, see examples 4.0.2). In the affine case however:

Proposition 4.3.2. *Assuming X is affine, the complex*

$$\mathrm{R}\Gamma^{\mathrm{dR,c}}_{X^q_{\leqslant n}}(X^q, (\mathcal{L}^\vee)^{\boxtimes q}) \simeq \mathrm{R}\Gamma\Big(\widehat{X}^q_{\leqslant n}, \mathrm{T}_X^{\boxtimes q}\Big)^*$$

is concentrated is degree 0. *This implies the diagonal filtration is exhaustive in this case, and thus proves assertion* (c) *of theorem 4.0.1.*

It remains to proof assertion (b) in theorem 4.0.1. This will be proven by using analytification of factorization algebras and theorem 4.1.2. We will see in section 4.4 that topological factorization algebras can be obtained from (nice enough) $\mathcal{D}^!$-factorization algebras. The proof will thus rely on expressing $\mathrm{GF}^\bullet_{\mathrm{top}}(X)$ as the factorization homology of a $\mathcal{D}^!$-factorization algebra associated to $\check{\mathcal{C}}^\bullet(X, \mathrm{T}_X \otimes \mathcal{D}_X)$. This process, introduced by Gaitsgory and Lurie [GL14], is called *covariant Verdier duality.*

Covariant Verdier duality: From a $[\![\mathcal{D}]\!]$-factorization algebra F, we build a $\mathcal{D}^!$-factorization algebra as follows. Fix I a non-empty finite set and consider the canonical morphism $f_I \colon X^I \to \operatorname{Ran} X$.

Imagine for a moment that F was an honest sheaf (and not a $[\![\mathcal{D}]\!]$-module) on $\operatorname{Ran} X$. Constructing a $\mathcal{D}^!$-module becomes fairly straightforward: simply set $E^{(I)} := f_I^! F$. In practice however, F is a $[\![\mathcal{D}]\!]$-module and thus a family of $\mathcal{D}$-modules $(F_{(S)})$, $S \in \overrightarrow{\mathrm{Fin}}$, with additional structural equivalences.

We could then consider $E^{(I)}_S := f_I^! f_{S*} F_{(I)}$. Again the formal meaning of this is not completely clear, so we rely on a putative base change formula to properly define $E^{(I)}_S$. Let $\Delta(I, S)$ be the intersection of X^I and X^S in $\operatorname{Ran} X$. In practice, $\Delta(I, S)$ is the closed subvariety of $X^I \times X^S$ defined by

$$\Delta(I, S) := \{(x_i)_{i \in I}, (y_s)_{s \in S} \mid \{x_i, i \in I\} = \{y_s, s \in S\}\}.$$

It is thus made of pairs of tuples spanning the same set. Denote by $p \colon : \Delta(I, S) \to X^I$ and $q \colon \Delta(I, S) \to X^S$ the projection. We define[9]:

$$E^{(I)}_S := p_* q^! F_{(S)} \qquad \text{and} \quad E^{(I)} := \lim_S E^{(I)}_S.$$

The family $\psi(F) := E = (E^{(I)})$ assembles into a $\mathcal{D}^!$-module on $\operatorname{Ran} X$. A highly technical argument allows further to carry the factorization structure on F to a factorization structure on $\psi(F)$. The $\mathcal{D}^!$-factorization algebra $\psi(F)$ is called the covariant Verdier dual[10] of F. It comes with canonical morphisms

$$\int_X \psi(F) \to \oint_X F \qquad \text{and} \qquad \int_X^{\mathrm{c}} \psi(F) \to \oint_X^{\mathrm{c}} F$$

---

[9] The definition relies on some finiteness assumption (coherence) on F, a technical detail we choose to omit. See [HK22] for details

[10] A dual construction carries $\mathcal{D}^!$-factorization algebras to $[\![\mathcal{D}]\!]$-factorization algebras. Those constructions are not necessarily inverse to each other.

Moreover, a formal computation identifies $\int_X^{\mathrm{c}} \psi(F)$ with the total space of the diagonal filtration. Applying this to $\check{\mathcal{C}}^\bullet(X, \mathrm{T}_X \otimes \mathcal{D}_X)$ yield the following proposition.

Proposition 4.3.3.

$$\int_X^{\mathrm{c}} \psi\big(\check{\mathcal{C}}^\bullet(X, \mathrm{T}_X \otimes \mathcal{D}_X)\big) \simeq \operatorname*{colim}_n \mathrm{GF}^\bullet_{\leqslant n}(X) =: \mathrm{GF}^\bullet_{\mathrm{top}}(X).$$

4.4 Analytification and the role of factorization structures

Let us denote by X_{an} the analytification of X. The Riemann–Hilbert correspondence implies that if $E^{(I)}$ is regular holonomic over X^I, then there is an analytic $\mathcal{D}$-module $E^{(I)}_{\mathrm{an}}$ over X^I_{an}; and $E^{(I)}$ and $E^{(I)}_{\mathrm{an}}$ have the same solutions (i.e. analytic solutions are algebraic)

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X^I, E^{(I)}) \simeq \mathrm{R}\Gamma(X^I_{\mathrm{an}}, \mathrm{DR}(E^{(I)}_{\mathrm{an}})) \qquad \Big(\text{and } \mathrm{R}\Gamma^{\mathrm{c}}_{\mathrm{dR}}(X^I, E^{(I)}) \simeq \mathrm{R}\Gamma^{\mathrm{c}}(X^I_{\mathrm{an}}, \mathrm{DR}(E^{(I)}_{\mathrm{an}}))\Big)$$

We refer to [Bor87] for a good account on algebraic $\mathcal{D}$-modules and the Riemann–Hilbert correspondence. This allows us to analytify regular holonomic $\mathcal{D}^!$-modules over $\operatorname{Ran} X$:

Definition 4.4.1. Let E be a regular holonomic $\mathcal{D}^!$-module over $\operatorname{Ran} X$ (meaning that each $E^{(I)}$ is regular holonomic). For any $V \subset X_{\mathrm{an}}$ open subset, we set

$$\mathcal{A}_E(V) := \operatorname*{colim}_I \mathrm{R}\Gamma^{\mathrm{c}}(V^I, \mathrm{DR}(E^{(I)}_{\mathrm{an}})).$$

The assignment $V \mapsto \mathcal{A}_E(V)$ defines a precosheaf on X_{an}, which, by the Riemann–Hilbert correspondence, satisfies:

$$\mathcal{A}_E(X_{\mathrm{an}}) =: \int_{X_{\mathrm{an}}} \mathcal{A}_E \simeq \int_X^{\mathrm{c}} E.$$

In the previous section, and up until this point, the (algebraic) factorization structures were not paramount to the discussion. Most of what was said concerns $\mathcal{D}^!$- and $[\![\mathcal{D}]\!]$-modules over the Ran space. The role of the factorization structures appears here:

Proposition 4.4.2. *If E is a $\mathcal{D}^!$-factorization algebra then*

(a) *If $E^{(*)}$ is regular holonomic over $X^* = X$ then E is regular holonomic (the converse tautologically holds);*

(b) *If E is regular holonomic, then $\mathcal{A}_E$ is a factorization algebra.*

We will apply this to $E = \psi\big(\check{\mathcal{C}}^\bullet(X, \mathrm{T}_X \otimes \mathcal{D}_X)\big)$. It starts with computing the stalk of $E^{(*)}$ at any closed point $x \in X$. By construction, the $\mathcal{D}$-module $E^{(*)}$ is nothing but the diagonal cohomology $\mathcal{D}$-module $\mathcal{C}^\bullet_\Delta(X)$ mentioned just above definition 4.3.1. A rather straightforward computation shows that the stalk of $\mathcal{C}^\bullet_\Delta(X)$ at x is the Chevalley–Eilenberg cohomology of the Lie algebra of formal vector fields at x:

$$i_x^!(\mathcal{C}^\bullet_\Delta(X)) \simeq \mathrm{H}^\bullet_{\mathrm{Lie}}(W_{X,x}) \simeq \mathrm{H}^\bullet_{\mathrm{Lie}}(W_d^{\mathbb{C}}).$$

This is in particular finite dimensional, which implies that $\mathcal{C}^\bullet_\Delta(X)$ is holonomic. It is also regular, since $\mathcal{C}^\bullet_\Delta(X)$ (and actually E itself) extend naturally to any smooth compactification of X.

Definition 4.4.3. We denote by $\mathcal{A}_X$ the topological factorization algebra $\mathcal{A}_{\psi(\check{\mathcal{C}}^\bullet(X, \mathrm{T}_X \otimes \mathcal{D}_X))}$.

From all of the above, we get

$$\mathrm{GF}^\bullet_{\mathrm{top}}(X) \simeq \int_X^{\mathrm{c}} \psi\big(\check{\mathcal{C}}^\bullet(X, \mathrm{T}_X \otimes \mathcal{D}_X)\big) \simeq \int_{X_{\mathrm{an}}} \mathcal{A}_X.$$

To conclude the proof of theorem 4.0.1, it remains to compute the right-hand-side. This is the content of the next subsection.

4.5 Finishing the proof: the topological side

The remaining arguments will closely follow the (sketch of) proof of theorem 4.1.1 given above. A couple of adaptations need to be made. First, to construct the bundle $Y_X^{\mathrm{an}} \to X_{\mathrm{an}}$. Its fiber Y_d is the same as in section 4.1 except d is now the complex dimension of X or X_{an} (and no the real dimension). It is in particular a $\mathrm{GL}_d(\mathbb{C})$-space. Using the principal $\mathrm{GL}_d(\mathbb{C})$-bundle associated to the tangent of X_{an}, we form the fibration $Y_X^{\mathrm{an}} \to X_{\mathrm{an}}$:

$$Y_X^{\mathrm{an}} := Y_d \underset{\mathrm{GL}_d(\mathbb{C})}{\times} \mathrm{T}_{X_{\mathrm{an}}} \to X_{\mathrm{an}}.$$

Since Y_d is $2d$-connected, and $2d$ is the real dimension of X_{an}, the proof of theorem 4.1.2 extends to show that the functor

$$\mathcal{F}_{\mathrm{Sect}}^{X_{\mathrm{an}}} : U \mapsto \mathrm{Sing}_{\mathbb{C}}^{\bullet}(\mathrm{Sect}(Y_X^{\mathrm{an}} \to X_{\mathrm{an}}))$$

is a topological factorization algebra over the smooth real manifold X_{an}. By definition, we thus have

$$\mathrm{H}_{\mathrm{Sing}}^{\bullet}(\mathrm{Sect}(Y_X^{\mathrm{an}} \to X_{\mathrm{an}}), \mathbb{C}) \simeq \int_{X_{\mathrm{an}}} \mathcal{F}_{\mathrm{Sect}}^{X_{\mathrm{an}}}.$$

Like in the $\mathcal{C}^\infty$ case, the factorization algebras $\mathcal{F}_{\mathrm{Sect}}^{X_{\mathrm{an}}}$ and $\mathcal{A}_X$ are both locally constant. They are thus determined by they stalks (with their $\mathrm{GL}_d(\mathbb{C})$-equivariant $\mathbb{E}_{2d}$-algebra structures). Those stalks are respectively $\mathrm{H}_{\mathrm{Sing}}^{\bullet}(Y_d, \mathbb{C})$ and $\mathrm{H}_{\mathrm{Lie}}^{\bullet}(W_d^{\mathbb{C}}, \mathbb{C})$. They are equivalent (with all their structure) by (15). We get $\mathcal{F}_{\mathrm{Sect}}^{X_{\mathrm{an}}} \simeq \mathcal{A}_X$, and thus

$$\mathrm{H}_{\mathrm{GF,top}}^{\bullet}(X) \simeq \int_{X_{\mathrm{an}}} \mathcal{A}_X \simeq \int_{X_{\mathrm{an}}} \mathcal{F}_{\mathrm{Sect}}^{X_{\mathrm{an}}} \simeq \mathrm{H}_{\mathrm{Sing}}^{\bullet}(\mathrm{Sect}(Y_X^{\mathrm{an}} \to X_{\mathrm{an}}), \mathbb{C}).$$

This concludes our survey of the proof of theorem 4.0.1.

5 Further work

5.1 Gelfand–Fuchs cohomology with coefficients

The differentiable case: Gelfand–Fuchs cohomology is not restricted to constant field coefficients, and given, say, a manifold M, we could for instance consider the cohomology

$$\mathrm{H}_{\mathrm{Lie}}^{\bullet}(\mathrm{Vect}(M), \mathcal{C}^\infty(M, \mathbb{R})).$$

It is known as the Gelfand–Fuchs cohomology with coefficients in functions (in this case). Of course, other coefficient sheaves (or complexes thereof) can be used: e.g. the de Rham complex, the sheaf of forms, or of vector fields. In this differentiable context, those cohomology groups where studied by Tsujishita in [Tsu81] (see [Fuk86, Thm. 2.4.10] for an account of this work). First observe that the total space $\mathcal{T}_M$ of the principal $\mathrm{GL}_d(\mathbb{C})$-bundle associated to the complexified tangent bundle $\mathrm{T}_M^{\mathbb{C}}$ of M embeds in the space Y_M. Tsujishita then considers the space

$$Z_M := \{(x, s) \in M \times \mathrm{Sect}(Y_M \to M) \mid s(x) \in \mathcal{T}_M\}.$$

Theorem 5.1.1 (Tsujishita [Tsu81]). *Under the assumption that M is simply connected and can be covered by a finite number of discs:*

$$\mathrm{H}_{\mathrm{Lie}}^{\bullet}(\mathrm{Vect}(M), \mathcal{C}^\infty(M, \mathbb{R})) \simeq \mathrm{H}_{\mathrm{Sing}}^{\bullet}(Z_M, \mathbb{R}).$$

His proof relies on the following ideas:

(i) The Chevalley–Eilenberg complex $\mathrm{CE}^\bullet(\mathrm{Vect}(M), \mathcal{C}^\infty(M, \mathbb{R}))$ comes with a diagonal filtration, similar to the one existing for constant coefficients. We write $\mathrm{CE}_\Delta^\bullet(\cdots)$ for its diagonal part.

(ii) The Chevalley–Eilenberg complex $\mathrm{CE}^\bullet(\mathrm{Vect}(M), \Omega^\bullet_M)$ with coefficients in the de Rham complex[11] acts on $\mathrm{CE}^\bullet(\mathrm{Vect}(M), \mathcal{C}^\infty(M, \mathbb{R}))$ (in a way compatible with the diagonal filtrations);

(iii) There is an equivalence

$$\mathrm{CE}^\bullet(\mathrm{Vect}(M), \mathcal{C}^\infty(M, \mathbb{R})) \simeq \mathrm{CE}^\bullet_\Delta(\mathrm{Vect}(M), \mathcal{C}^\infty(M, \mathbb{R})) \underset{\mathrm{CE}^\bullet_\Delta(\mathrm{Vect}(M), \Omega^\bullet_M)}{\otimes} \mathrm{CE}^\bullet(\mathrm{Vect}(M), \Omega^\bullet_M);$$

(iv) There are compatible equivalences

$$\begin{tikzcd}
\mathrm{CE}^\bullet_\Delta(\mathrm{Vect}(M), \mathcal{C}^\infty(M, \mathbb{R})) \arrow[d, phantom, "\wr"] & \mathrm{CE}^\bullet_\Delta(\mathrm{Vect}(M), \Omega^\bullet_M) \arrow[l] \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{CE}^\bullet(\mathrm{Vect}(M), \Omega^\bullet_M) \arrow[d, phantom, "\wr"] \\
\mathrm{Sing}^\bullet_{\mathbb{R}}(\mathcal{T}_M) & \mathrm{Sing}^\bullet_{\mathbb{R}}(Y_M) \arrow[l] \arrow[r] & \mathrm{Sing}^\bullet_{\mathbb{R}}(\mathrm{Sect}(Y_M \to M) \times M);
\end{tikzcd}$$

(v) Under the extra assumption that M (and thus Y_M) is simply connected, the Eilenberg–Moore spectral sequence and items *(iii)* and *(iv)* imply the result, since

$$Z_M \simeq \mathcal{T}_M \underset{Y_M}{\times} (\mathrm{Sect}(Y_M \to M) \times M).$$

In the algebraic setting: An algebraic version of Tsujishita's theorem is available, in a work in progress with Anton Khoroshkin and Mikhail Kapranov. It relies, not only on factorization algebras, but on factorization modules in the algebraic setting (see [BD04, §3.4.18] or [Roz10]).

We will abstain from a formal definition of factorization modules and will settle for the following informal idea. Fix a $\mathcal{D}^!$-factorization algebra $\mathcal{A}$ over a smooth variety X of dimension d. A $\mathcal{D}^!$-factorization module over $\mathcal{A}$ is a $\mathcal{D}^!$-module $\mathcal{M}$ over the pointed Ran space: i.e. the space of pointed configuration of points in X:

$$\mathrm{Ran}^* X := \{\{x_0, \dots, x_n\} \subset X, \text{ pointed at } x_0\}.$$

At a configuration $\{x_0, \dots, x_n\}$ as above, with $x_i \neq x_j$ for $i \neq j$, the stalk of $\mathcal{M}$ should satisfy

$$\mathcal{M}_{\{x_0, \dots, x_n\}} \simeq \mathcal{M}_{x_0} \otimes \bigotimes_{i=1}^n \mathcal{A}_{x_i}.$$

As the configuration varies, and (say) x_0 and x_1 collide, we get an operation

$$\mathcal{M}_{x_0} \otimes \mathcal{A}_{x_1} \to \mathcal{M}_{x_0}.$$

Whenever x_i and x_j collide, with both i and j different from 0, the corresponding operation comes from the factorization structure on $\mathcal{A}$. Global sections of a $\mathcal{D}^!$-factorization module define a very well-behaved notion of factorization homology of a module:

$$\int_X \mathcal{M} := \mathrm{R}\Gamma_{\mathrm{dR}}(\mathrm{Ran}^* X, \mathcal{M}).$$

There are moreover variations, involving compact support in the algebra direction, in the module direction, or in both. For us, the most useful will be the factorization homology with compact support in the algebra direction. It can be defined as follows. Consider the projection $\pi\colon \mathrm{Ran}^* X \to X$ at the marked point. We set

$$\int_X^{\mathrm{c}_A} \mathcal{M} := \mathrm{R}\Gamma_{\mathrm{dR}}(X, \pi_! \mathcal{M}).$$

---

[11] There are (at least) two natural actions of vector fields on the de Rham complex: the trivial one and the one acting on each Ω^p_M. It turns out the Chevalley–Eilenberg complexes with coefficients in those two representations are quasi-isomorphic.

Dually, there is a notion of $[\![\mathcal{D}]\!]$-factorization comodules over a $[\![\mathcal{D}]\!]$-factorization algebra, with suitably defined factorization cohomology $\oint_X$ or factorization cohomology compactly supported in the algebra direction $\oint_X^{\mathrm{c}_A}$.

With the natural stratification $\mathrm{Ran}^*_{\leqslant n} X \subset \mathrm{Ran}^* X$ of $\mathrm{Ran}^* X$ by the cardinal of configurations comes a diagonal filtration on said factorization homology:

$$\oint_X^{\leqslant n} \mathcal{N} := \mathrm{R}\Gamma_{\mathrm{Ran}^*_{\leqslant n} X}(\mathrm{Ran}^* X, \mathcal{N}), \qquad 0 \to \oint_X^{\leqslant 1} \mathcal{N} \to \cdots \to \oint_X^{\leqslant n} \mathcal{N} \to \cdots \to \oint_X \mathcal{N},$$

and similarly for factorization homology with compact support in the algebra direction:

$$0 \to \oint_X^{\mathrm{c}_A, \leqslant 1} \mathcal{N} \to \cdots \to \oint_X^{\mathrm{c}_A, \leqslant n} \mathcal{N} \to \cdots \to \oint_X^{\mathrm{c}_A} \mathcal{N},$$

This filtration is again related to a covariant Verdier duality functor ψ from $[\![\mathcal{D}]\!]$-factorization comodules to $\mathcal{D}^!$-factorization modules by $\mathrm{colim}_n \oint_X^{\leqslant n} \mathcal{N} \simeq \int_X \psi\mathcal{N}$ (and similarly with compact support conditions).

Proposition 5.1.2. *For any coherent $\mathcal{D}_X$-module (or bounded complex thereof) N with an action of the Lie algebra of vector fields, there is a $[\![\mathcal{D}]\!]$-factorization comodule $\mathcal{N}(X, N)$ over $\check{\mathcal{C}} := \check{\mathcal{C}}(X, \mathrm{T}_X \otimes \mathcal{D}_X)$ such that*

$$\mathrm{GF}^\bullet(X, N) := \mathrm{CE}^\bullet(\mathrm{R}\Gamma(X, \mathrm{T}_X), \mathrm{R}\Gamma_{\mathrm{dR}}(X, N)) \simeq \oint_X^{\mathrm{c}_A} \mathcal{N}(X, N).$$

We denote by $\mathrm{GF}^\bullet_\Delta(X, N)$ the corresponding diagonal complex

$$\mathrm{GF}^\bullet_\Delta(X, N) := \mathrm{CE}^\bullet_\Delta(\mathrm{R}\Gamma(X, \mathrm{T}_X), \mathrm{R}\Gamma_{\mathrm{dR}}(X, N)) := \oint_X^{\mathrm{c}_A, \leqslant 1} \mathcal{N}(X, N).$$

and by $\mathrm{GF}^\bullet_{\mathrm{top}}(X, N)$ the "topological" cohomology:

$$\mathrm{GF}^\bullet_{\mathrm{top}}(X, N) := \underset{n}{\mathrm{colim}} \oint_X^{\mathrm{c}_A, \leqslant 1} \mathcal{N}(X, N) \quad \left(\simeq \int_X^{\mathrm{c}_A} \psi(\mathcal{N}(X, N)) \right).$$

The above accounts for an algebraic analog of Tsujishita's argument *(i)*. For item *(ii)*, the action of

$$\mathrm{CE}^\bullet(\mathrm{R}\Gamma(X, \mathrm{T}_X), \mathrm{R}\Gamma(X, \Omega^\bullet_X)) \simeq \mathrm{GF}^\bullet(X, \mathcal{O}_X)$$

comes from the action of $\mathcal{O}_X$ on any $\mathcal{D}_X$-module. As an analog of item *(iii)*, we have

$$\mathrm{GF}^\bullet_{\mathrm{top}}(X, N) \simeq \mathrm{GF}^\bullet_{\mathrm{top}}(X, \mathcal{O}_X) \underset{\mathrm{GF}^\bullet_\Delta(X, \mathcal{O}_X)}{\otimes} \mathrm{GF}^\bullet_\Delta(X, N). \tag{18}$$

The equivalence (18) arises from a very general base-change formula for factorization modules. Like in section 4, the diagonal filtration is exhaustive in the affine case, and we get in that case

$$\mathrm{GF}^\bullet(X, N) \simeq \mathrm{GF}^\bullet(X, \mathcal{O}_X) \underset{\mathrm{GF}^\bullet_\Delta(X, \mathcal{O}_X)}{\otimes} \mathrm{GF}^\bullet_\Delta(X, N).$$

The remaining items *(iv)* and *(v)* then follow from the computation $\mathrm{GF}^\bullet(X, \mathcal{O}_X) \simeq \mathrm{GF}^\bullet(X) \otimes_{\mathbf{k}} \mathrm{H}^\bullet_{\mathrm{dR}}(X)$ and from theorem 4.0.1. All in all, we get

Theorem 5.1.3. *If X is a simply connected smooth affine variety, then we have*

$$\mathrm{CE}^\bullet(\mathrm{R}\Gamma(X, \mathrm{T}_X), \mathrm{R}\Gamma(X, \mathcal{O}_X)) \simeq \mathrm{GF}^\bullet(X, \mathcal{D}_X) \simeq \mathrm{Sing}^\bullet_{\mathbf{k}}(Z_X^{\mathrm{an}})$$

where Z_X^{an} is defined as

$$Z_X^{\mathrm{an}} \simeq \mathcal{T}_X^{\mathrm{an}} \underset{Y_X^{\mathrm{an}}}{\times} (\mathrm{Sect}(Y_X^{\mathrm{an}} \to X_{\mathrm{an}}) \times X_{\mathrm{an}}).$$

In order to drop the simple connectedness assumption (in both the differentiable and algebraic cases), a possible approach could be to prove the above theorem through analytification (similarly to our proof of theorem 4.0.1). The main missing technical ingredient here is the notion of factorization modules in the differentiable context[12] and the corresponding non-abelian Poincaré duality.

5.2 Geometrizing the Hochschild–Serre filtration

In order to compute the cohomology $\mathrm{H}^\bullet_{\mathrm{Lie}}(\mathrm{Witt}_d)$ and thus prove a part of conjecture 3.2.2, we can try to extend theorem 4.0.1, assertion (c) to the case of the punctured disc, or more generally, to the quasi-affine case.

Let us fix X a smooth affine variety and $U \subset X$ an open subvariety. The basic idea is to use the Hochschild–Serre filtration (yielding the Hochschild–Serre spectral sequence, see [HS53]) relative to the dg-Lie sub-algebra $\mathfrak{h} = \Gamma(X, \mathrm{T}_X)$ in $\mathfrak{g} = \mathrm{R}\Gamma(U, \mathrm{T}_U)$, in order the compute $\mathrm{H}^\bullet_{\mathrm{GF}}(U) := \mathrm{H}^\bullet_{\mathrm{Lie}}(\mathrm{R}\Gamma(U, \mathrm{T}_U), \mathbf{k})$.

Recall that the Hochschild–Serre filtration is a filtration

$$\cdots \to \mathrm{CE}^\bullet_{(n)}(\mathfrak{g}, \mathfrak{h}, \mathbf{k}) \to \cdots \to \mathrm{CE}^\bullet_{(0)}(\mathfrak{g}, \mathfrak{h}, \mathbf{k}) = \mathrm{CE}^\bullet(\mathfrak{g}, \mathbf{k})$$

such that

- $\lim_n \mathrm{CE}^\bullet_{(n)}(\mathfrak{g}, \mathfrak{h}, \mathbf{k}) = 0$, and
- $\mathrm{hofib}\Big(\mathrm{CE}^\bullet_{(n)}(\mathfrak{g}, \mathfrak{h}, \mathbf{k}) \to \mathrm{CE}^\bullet_{(n-1)}(\mathfrak{g}, \mathfrak{h}, \mathbf{k})\Big) \simeq \mathrm{CE}^\bullet(\mathfrak{h}, (\Lambda^n \mathfrak{g}/\mathfrak{h})^*)[-n]$.

Theorem 5.2.1. *Consider the $[\![\mathcal{D}]\!]$-factorization algebra $\check{\mathcal{C}} := \check{\mathcal{C}}(U, \mathrm{T}_U \otimes \mathcal{D}_U)$ over U. As a $[\![\mathcal{D}]\!]$-module over $\mathrm{Ran}\, U$, it admits a filtration:*

$$\cdots \to \check{\mathcal{C}}(n) \to \cdots \to \check{\mathcal{C}}(0) \simeq \check{\mathcal{C}}$$

such that

- $\lim_n \check{\mathcal{C}}(n) \simeq 0$,
- $\oint_U \check{\mathcal{C}}(n) \simeq \mathrm{CE}^\bullet_{(n)}(\mathfrak{g}, \mathfrak{h}, \mathbf{k})$.

Notice that in our example, $\mathfrak{g}/\mathfrak{h} \simeq \mathrm{R}\Gamma_Z(X, \mathrm{T}_X)[1]$, with $Z = X \smallsetminus U$.

Denoting by $\check{\mathcal{D}}(n)$ the homotopy fiber $\mathrm{hofib}(\check{\mathcal{C}}(n) \to \check{\mathcal{C}}(n-1))$, we also have $\oint_U \check{\mathcal{D}}(n) \simeq \mathrm{CE}^\bullet(\mathfrak{h}, (\Lambda^n \mathfrak{g}/\mathfrak{h})^*)[-n]$. Moreover, with X being affine, we can show the diagonal filtration on $\oint_U \check{\mathcal{D}}(n)$ to be exhaustive. This is, however, not enough to conclude that the diagonal filtration on $\mathrm{GF}^\bullet(U) \simeq \oint_U \check{\mathcal{C}}$ is exhaustive as well. To circumvent this issue, we will try and use equation (18). To do so, we shall

- Extend equation (18) to the case where N is not longer coherent (here, powers of the $\mathcal{D}_X$-module $N = i_* i^! (\mathrm{T}_X \otimes \mathcal{D}_X)[1]$ for $i \colon Z \to X$ the closed immersion; so that $\mathrm{R}\Gamma_{\mathrm{dR}}(X, N) \simeq \mathrm{R}\Gamma_Z(X, \mathrm{T}_X)[1] \simeq \mathfrak{g}/\mathfrak{h}$),
- Compare the diagonal filtration on $\oint_X \check{\mathcal{D}}(n)$ over $\mathrm{Ran}\, X$ with that on $\oint_X \mathcal{N}(X, N^{\otimes n})$,
- Apply to the case $U = \mathbb{A}^d \smallsetminus \{0\} \subset \mathbb{A}^d = X$, and prove using equation (18) that the diagonal filtration on $\oint_X \mathcal{N}(X, N^{\otimes n})$ stabilizes at weight 1, independently of n.
- Conclude that the diagonal filtration on $\mathrm{GF}^\bullet(\mathbb{A}^d \smallsetminus \{0\})$ stabilizes at weight 1 and is exhaustive, thus proving conjecture 3.2.2.

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[12] Note that this notion is different from the notion of stratified factorization algebras from, e.g., [AFT17].

5.3 Matrix factorizations and categorical Donaldson–Thomas invariants

Donaldson–Thomas invariants: The Donaldson–Thomas invariants are integral invariants with roots in theoretical physics (counting strings in a space-time). For the sake of concision, we will abstain from a lengthy introduction to the domain and and settle with the following short sketch. We will be interested in the complex dimension 3 case, after [Tho00] and [Beh09]. We choose, however, to take the point of view of derived algebraic geometry that only appeared later on.

Fix Y a Calabi–Yau 3-fold. In order to count strings (=curves) in Y, we study the compact moduli space $\overline{\mathcal{M}}_{\mathrm{DT}}(Y)$ of stable coherent sheaves on Y (corresponding to the definition ideal of curves) with fixed numerical invariants (Chern classes). This moduli stack is in general not smooth, and its deformation theory is governed by a perfect obstruction theory, which is moreover symmetric (i.e. self dual, up to a shift).

In terms of derived geometry, the moduli space $\overline{\mathcal{M}}_{\mathrm{DT}}(Y)$ admits a canonical derived enhancement $\mathrm{R}\overline{\mathcal{M}}_{\mathrm{DT}}(Y)$ (whose cotangent complex gives the above perfect obstruction theory). The Calabi–Yau structure on Y allows, through Serre duality, to construct a duality

$$\omega_2 \colon \mathbb{L}_{\mathrm{R}\overline{\mathcal{M}}_{\mathrm{DT}}(Y)} \simeq \mathbb{T}_{\mathrm{R}\overline{\mathcal{M}}_{\mathrm{DT}}(Y)}[1]$$

corresponding to the symmetry of the perfect obstruction theory. In [Pan+13], Pantev, Toën, Vaquié and Vezzosi show that ω_2 actually arises from a (-1)-shifted symplectic form ω on the derived moduli space $\mathrm{R}\overline{\mathcal{M}}_{\mathrm{DT}}(Y)$. This is the fundamental structure of study in this section.

Given a (nice enough) proper (-1)-shifted symplectic derived stack X (such as the moduli space $\mathrm{R}\overline{\mathcal{M}}_{\mathrm{DT}}(Y)$), we can construct a so-called *virtual class* $[X]^{\mathrm{vir}}$ in the Chow group $A^*(X)$ of X (this is by definition the Chow group of the underlying non-derived stack), using Behrend–Fantechi's deformation to the normal cone (see [BF97]) associated to the induced perfect obstruction theory. The Donaldson–Thomas invariant of X is the volume

$$\int_{[X]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

If Y is a Calabi–Yau 3-fold, its Donaldson–Thomas invariants is the family, indexed by all possible choices of fixed numerical invariants, of the invariant of $\mathrm{R}\overline{\mathcal{M}}_{\mathrm{DT}}(Y)$.

In [Beh09], Behrend gives an equivalent construction of the Donaldson–Thomas invariant of a symplectic derived stack X as above. He constructs a locally constant integer-valued function ν_X on X, the Euler characteristic of which computes the Donaldson–Thomas invariant of X:

$$\int_{[X]^{\mathrm{vir}}} 1 = \chi(\nu_X) := \sum_n n\, \chi(\nu_X^{-1}(n)).$$

Amongst other things, this result allows to extend the definition to non-proper derived stacks, as the right-hand side is then well-defined (albeit possibly rational instead of integral).

Sheafification and vanishing cycles: In a series of papers [Joy15; BBJ19; Bra+15], Joyce and his collaborators construct a sheafification of Behrend's function ν_X, at least in the case of a schematic X (see also [Ben+15; JU20] for the case of stacks).

For X a (-1)-shifted symplectic scheme equipped with a square root of the canonical bundle, they construct a perverse sheaf $\mathcal{P}_X$ on X such that

$$\chi(\mathcal{P}_X) = \nu_X.$$

This sheaf $\mathcal{P}_X$ is a globalization of the sheaf of vanishing cycles, in the following sense. Consider a smooth scheme U equipped with a function $f \colon U \to \mathbb{A}^1$. Its derived critical locus is the derived intersection

$$\begin{tikzcd}
\mathrm{dCrit}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U \arrow[d, "df"] \\
U \arrow[r, "0"'] & \mathrm{T}^*U
\end{tikzcd}$$

The (0-shifted) symplectic structure on T^*U induces a (-1)-shifted symplectic structure on $\mathrm{dCrit}(f)$. By construction, the perverse sheaf $\mathcal{P}_{U,f} := \mathcal{P}_{\mathrm{dCrit}(f)}$ is the perverse sheaf of vanishing cycles of the Landau–Ginzburg model (U, f).

Theorem 5.3.1.

(a) *(Darboux lemma, [BBJ19]) Any (-1)-shifted symplectic scheme is Zariski locally equivalent to a derived critical locus.*

(b) *([Bra+15]) Given X a (-1)-shifted symplectic scheme, any choice of Landau–Ginzburg model (U, f) such that $X \simeq \mathrm{dCrit}(f)$ yields an isomorphic perverse sheaf $\mathcal{P}_{U,f}$, up to a twist by the principal $\mathbb{Z}/2$-bundle of square roots of the canonical bundle ω_X.*

(c) *([Bra+15]) Provided a square root of ω_X and a covering X by a family of derived critical loci $\mathrm{dCrit}(f_i)$, $f_i \colon U_i \to \mathbb{A}^1$, the sheaves $\mathcal{P}_{U_i,f_i}$ glue to a perverse sheaf $\mathcal{P}_X$, independent of the choice of the covering.*

Strictly speaking, the articles of Joyce and his collaborators does not deal with (-1)-shifted symplectic schemes, but with d-critical loci. A d-critical locus is the shadow of a (-1)-shifted derived scheme, in non-derived terms (see [Joy15]).

We will sketch here a proof of the above theorem 5.3.1(c) relying on derived geometry. It is joint work in progress with Marco Robalo and Julian Holstein. This proof is more conceptual than the original and we find it gives a better understanding of the phenomena behind the above theorem. It also opens a way to prove a categorification of this statement (see below).

First, we consider the derived stack $\mathsf{DarbCharts}_X$ (on the small étale site of X) of *Darboux charts* on X. It classifies smooth formal thickenings[13] $\widehat{U}$ of X, equipped with a function $f \colon \widehat{U} \to \mathbb{A}^1$, and a symplectic equivalence $\mathrm{dCrit}(f) \simeq X$.

Consider also the stack Quad_X^∇ of non-degenerate quadratic bundles with a flat connection. We further require the sections of $\mathsf{Quad}_X^\nabla(S)$ to be étale locally equivalent[14] to $\mathbb{A}^d \times S$ with the trivial flat connection and with the quadratic form $q = \sum x_i^2$.

Theorem 5.3.2. *The stack Quad_X^∇ is a monoid (for the sum), and it acts on $\mathsf{DarbCharts}_X$. The action is moreover transitive, so that the quotient stack $\mathsf{DarbCharts}_X/\mathsf{Quad}_X^\nabla$ is connected.*

The action is as follows. Fix $S \to X$ an étale morphism, and $(\widehat{U}, f) \in \mathsf{DarbCharts}_X(S)$ and $(M, q) \in \mathsf{Quad}_X^\nabla(S)$. The flat connection on M implies that the bundle $M \to S$ descends to a bundle on $\pi \colon M_{\widehat{U}} \to \widehat{U}$. Moreover, with the quadratic form q being flat for the connection, it descends to a form $q_{\widehat{U}} \colon M_{\widehat{U}} \to \mathbb{A}^1$. Denote by $\widehat{M_{\widehat{U}}}$ the formal neighbourhood of the zero section in $M_{\widehat{U}}$. We set

$$f_q := f \circ \pi + q_{\widehat{U}} \colon \widehat{M_{\widehat{U}}} \to \mathbb{A}^1.$$

The derived critical locus $\mathrm{dCrit}(f_q)$ then identifies with the fiber product $\mathrm{dCrit}(f) \times_{\widehat{U}} \mathrm{dCrit}_{\widehat{U}}(q_{\widehat{U}}) \simeq \mathrm{dCrit}(f) \simeq S$. The assignment

$$\Big((\widehat{U}, f), (M, q)\Big) \mapsto (\widehat{M_{\widehat{U}}}, f_q)$$

---

[13] We use formal thickenings, rather than plain smooth schemes, because the latter notion is not functorial: if $X \simeq \mathrm{dCrit}(g)$ with (U, g) a Landau–Ginzburg model, and $Y \subset X$ is an open subscheme, there is no *canonical* restriction of (U, g) that would give a Darboux chart on Y.

[14] Any quadratic form is locally of the form $\sum x_i^2$, and the flat connection is always locally trivial. However, we cannot in general trivialize both those data simultaneously.

defines an action of Quad_X^∇ on $\mathsf{DarbCharts}_X$. To understand the quotient stack $\mathsf{DarbCharts}_X/\mathsf{Quad}_X^\nabla$, we start with an adaptation of [Joy15, Thm. 2.20]:

Proposition 5.3.3. *Let $(\widehat{U}, f)$ and $(\widehat{V}, g)$ be elements of $\mathsf{DarbCharts}_X(S)$, for some S étale over X. Locally on S, there exist (M, q) and (N, p) in $\mathsf{Quad}_X^\nabla(S)$ such that*

$$(\widehat{M}_{\widehat{U}}, f_q) \simeq (\widehat{N}_{\widehat{V}}, g_p).$$

In particular, $\pi_0\Big(\mathsf{DarbCharts}_X/\mathsf{Quad}_X^\nabla\Big) = *$.

Let us briefly describe M. Consider the formal linear stack

$$M = \operatorname{Spec}_S\big(\operatorname{Sym}_S(\Omega_{\widehat{V}} \oplus \mathrm{T}_{\widehat{V}})\big).$$

where $\Omega_{\widehat{V}}$ and $\mathrm{T}_{\widehat{V}}$ are implicitly restricted to S. Since it does not have a flat connection a priori, we need to shrink S enough to trivialize those bundles and use the trivial connection. The quadratic form q is the canonical pairing.

On the other hand, the quadratic bundle N is more complicated to describe. As a bundle, it is by construction $\Omega_{\widehat{V}} \oplus \mathrm{T}_{\widehat{U}}$ (those sheaves are implicitly restricted to S). The non-degenerate quadratic form on this bundle is neither obvious nor canonical. We will not, however, give a description here.

Notice that the above argument is not symmetric in $\widehat{U}$ and $\widehat{V}$, so swapping the roles of $\widehat{U}$ and $\widehat{V}$ gives a different pair of quadratic bundles.

We can now focus on theorem 5.3.1(c). To any Darboux chart $(\widehat{U}, f)$ over S étale over X, we can associate the perverse sheaf of vanishing cycles $\mathcal{P}_{\widehat{U},f}$. It is a priori a perverse sheaf over $\widehat{U}$, but perverse sheaves on $\widehat{U}$ are simply perverse sheaves on S, since $\widehat{U}$ is a formal thickening of S. This construction assembles into a morphism of (derived) stacks

$$\mathcal{P} \colon \mathsf{DarbCharts}_X \to \mathsf{Perv}_X\,.$$

Similarly, to any Morse thickening $(\widehat{M}, q)$ over S, we associate the sheaf of vanishing cycles $\mathcal{P}_{\widehat{M},q} \in \mathsf{Perv}_X(S)$. Moreover, the vanishing cycles of $(\widehat{M} \times_S \widehat{M}, q \boxplus q)$ form the constant unit sheaf. Using the Thom–Sebastiani isomorphism [Mas01], we deduce the functor $(\widehat{M}, q) \mapsto \mathcal{P}_{\widehat{M},q}$ has values in tensor 2-torsion perverse sheaves, i.e. $\mathbb{Z}/2$-bundles. We get a stack morphism

$$\mathcal{P} \colon \mathsf{Quad}_X^\nabla \to \mathrm{B}\mathbb{Z}/2.$$

A second use of the Thom–Sebastiani isomorphism shows

$$\mathcal{P}_{\widehat{M}_{\widehat{U}}, f_q} \simeq \mathcal{P}_{\widehat{U},f} \otimes \mathcal{P}_{\widehat{M},q}.$$

As a consequence, we get a quotient morphism $\mathcal{P} \colon \mathsf{DarbCharts}_X/\mathsf{Quad}_X^\nabla \to \mathsf{Perv}_X/\big(\mathrm{B}\mathbb{Z}/2\big)$. The key point now is the surprising existence of a natural factorization of $\mathcal{P}$ through the projection p:

$$\begin{tikzcd} \mathsf{DarbCharts}_X/\mathsf{Quad}_X^\nabla \arrow[r, "p"'] \arrow[rr, bend left, "\mathcal{P}"] & X \arrow[r, "\exists"'] & \mathsf{Perv}_X/\big(\mathrm{B}\mathbb{Z}/2\big). \end{tikzcd} \tag{19}$$

This factorization $X \to \mathsf{Perv}_X/\big(\mathrm{B}\mathbb{Z}/2\big)$ can be thought as a twisted version of $\mathcal{P}_X$, that exists over any (-1)-shifted symplectic scheme. We will come back to it in a second. Before that, let us explain how the datum of an orientation yields an actual perverse sheaf.

The above factorization induces a (plain) commutative diagram

$$\begin{tikzcd}
 & \mathsf{Perv}_X \arrow[dr] \arrow[dd] & \\
 & & \mathsf{Perv}_X/\big(\mathrm{B}\mathbb{Z}/2\big) \arrow[dd] \\
 & * \arrow[dr] & \\
X \arrow[uuur, dashed] \arrow[uurr] \arrow[ur, dashed] \arrow[rr] & & \mathrm{K}\big(\mathbb{Z}/2, 2\big).
\end{tikzcd}$$

As is turns out, the morphism $X \to \mathrm{K}\big(\mathbb{Z}/2, 2\big)$ classifies the $\mathbb{Z}/2$-gerbe of square roots of the canonical bundle. The choice of such a square root therefore trivializes this morphism (so provides us with a dashed factorization in the diagram above). This in turn induces a lift $X \to \mathsf{Perv}_X$, classifying the sheaf $\mathcal{P}_X$ of theorem 5.3.1(c).

We double back on the factorization of (19). Although we shall not give explicitly its construction or properties in this thesis, let us highlight one of the steps in the proof:

Lemma 5.3.4. *The morphism*

$$\mathcal{P}\colon\ \mathsf{DarbCharts}_X/\mathsf{Quad}_X^{\nabla} \xrightarrow{\mathcal{P}} \mathsf{Perv}_X/\big(\mathrm{B}\mathbb{Z}/2\big),$$

seen as a map of stacks on the small étale site of S, is trivial on all homotopy sheaves.

From theorem 5.3.2, the case of π_0 is trivial. Since the target stack $\mathsf{Perv}_X/\big(\mathrm{B}\mathbb{Z}/2\big)$ is in fact a 1-stack, the case of π_n, $n \geqslant 2$ is also trivial.

To understand this morphism π_1, we first need to describe the homotopy sheaf of the source. The sheaf $\pi_1(\mathsf{DarbCharts}_X/\mathsf{Quad}_X^{\nabla})$ in fact receives an epimorphism from $\pi_1(\mathsf{DarbCharts}_X)$, and is thus locally generated by automorphisms of Darboux charts. There may be (and there are) complicated such automorphisms, but one can show that the image of such an automorphism φ by $\mathcal{P}$ is determined by the determinant of φ, seen as a principal $\mathbb{Z}/2$-bundle. The lemma then follows.

Categorification and matrix factorization: The main benefit of the new proof of theorem 5.3.1(c) described above is how easily it can be adapted to other invariants. We will later down be interested here in categorical invariants, namely the categories of matrix factorizations.

Given $f\colon U \to \mathbb{A}^1$ a function on a smooth scheme U, a matrix factorization of f is a pair of vector bundles E_0 and E_1, with morphisms $\partial_0\colon E_0 \to E_1$ and $\partial_1\colon E_1 \to E_0$, such that $\partial_0 \circ \partial_1 = f$ and $\partial_1 \circ \partial_0 = f$. The category $\mathsf{MF}(U, f)$ of matrix factorizations is, by a theorem of Orlov [Orl04], equivalent to the category of singularities of the zero-locus of f. As an invariant of Landau–Ginzburg loci, it can be seen as a categorification of vanishing cycles, for ($\mathbb{Z}/2$-graded) vanishing cycles can be recovered from the periodic homology of matrix factorization (see [Efi17]).

The goal of this ongoing work, joint with Julian Holstein and Marco Robalo, is to prove the following statement:

Statement 5.3.5. *Given X a (-1)-shifted symplectic scheme equipped with some orientation data (square root of ω_X and something else, see below), there is a sheaf of $\mathbb{Z}/2$-graded dg-categories MF_X such that, if $X \simeq \mathrm{dCrit}(f)$, then $\mathsf{MF}_X(X) = \mathsf{MF}(U, f)$.*

To understand the need for orientation data, consider the following example. Take $U = \mathbb{A}^1_{\mathbb{C}}$ and $f = x^2$. The (derived) critical locus of x^2 is a single point $\operatorname{Spec} \mathbb{C}$. In particular, $\mathrm{dCrit}(x^2)$ is also the derived critical locus of the 0-function on the point $\mathbb{A}^0_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}$. On the other hand, $\mathsf{MF}(\mathbb{A}^1_{\mathbb{C}}, x^2)$ is the $\mathbb{Z}/2$-graded dg-category of 1-periodic complexes, while $\mathsf{MF}(\mathbb{A}^0_{\mathbb{C}}, 0)$ is the $\mathbb{Z}/2$-graded dg-category of 2-periodic complexes. This difference stems from the orientation structure, that $\mathrm{dCrit}(\mathbb{A}^1_{\mathbb{C}}, x^2)$ and $\mathrm{dCrit}(\mathbb{A}^0_{\mathbb{C}}, 0)$ do not share.

To understand the orientation, we follow the same procedure as for perverse sheaves: study the matrix factorization of a Morse thickening. By Preygel's Thom–Sebastiani equivalence [Pre11], for any Morse thickening $(\widehat{M}, q)$ over S, the $\mathbb{Z}/2$-graded dg-category $\mathsf{MF}(\widehat{M}, q)$ squares to the monoidal unit (i.e. the category of 2-periodic complexes). As a consequence, $\mathsf{MF}(\widehat{M}, q)$ is naturally a $\mathbb{Z}/2$-graded Azumaya algebra of 2-torsion. The group stack $\mathsf{Az}_{\mathbb{Z}/2}^{\mathbb{Z}/2}$ of such Azumaya algebras can be computed through Hochschild cohomology, and we get

$$\pi_n\left(\mathsf{Az}_{\mathbb{Z}/2}^{\mathbb{Z}/2}\right) \simeq \begin{cases} \mathbb{Z}/2 \times \mathbb{Z}/2 & \text{if } n = 0, \\ \mathbb{Z}/2 & \text{if } n = 1 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$

In a picture similar to the one above, we have

$$\begin{tikzcd} \mathsf{DarbCharts}_X/\mathsf{Quad}_X^{\nabla} \arrow[r, "\mathsf{MF}"] \arrow[d, "p"'] & \mathbf{dgCat}_X^{\mathbb{Z}/2}/\mathsf{Az}_{\mathbb{Z}/2}^{\mathbb{Z}/2} \arrow[d] \\ X \arrow[ur, dashed, "\overline{\mathsf{MF}}"] \arrow[r, dashed, "\tau"'] & \mathsf{BAz}_{\mathbb{Z}/2}^{\mathbb{Z}/2} \end{tikzcd}$$

With this picture in mind, we would need two things to prove statement 5.3.5:

- To show the morphism MF factors as $\overline{\mathsf{MF}} \circ p$ (similarly to diagram (19)) and
- To start with a trivialization of the induced morphism τ (the orientation data).

The existence of a factorization $\overline{\mathsf{MF}}$ is not yet known. There are however positive signs. For instance, a version of lemma 5.3.4 also holds.

The orientation data itself can be understood using the description of $\mathsf{Az}_{\mathbb{Z}/2}^{\mathbb{Z}/2}$ given above. The morphism τ corresponds to obstruction classes

$$\alpha_1, \alpha_2 \in \mathrm{H}^1\left(X, \mathbb{Z}/2\right), \beta \in \mathrm{H}^2\left(X, \mathbb{Z}/2\right), \gamma \in \mathrm{H}^3\left(X, \mathbb{Z}/2\right).$$

Of the classes α_i, one is canonically trivialized. The other classifies some sort of dimension-parity torsor, a trivialization of which allows to lift the aforementioned uncertainty between $\mathsf{MF}(*, 0)$ and $\mathsf{MF}(\mathbb{A}^1, x^2)$.

The class β classifies the gerbe of square roots of the canonical bundle ω_X. The datum of a trivialization of β thus amounts to Joyce's orientation data, as in theorem 5.3.1(c). Intuitively, the vanishing of the class allows to lift the uncertainty in the choice of automorphisms of $\mathsf{MF}(*, 0)$ when glueing (there are exactly two such automorphisms, namely the identity and the shift by 1).

The last class γ corresponds intuitively to twisting the glueing data by a 2-torsion line bundle. It is not yet clear what higher gerbe it classifies.

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