Positive harmonic functions on the Heisenberg group I

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Abstract

We present the classification of positive harmonic functions on the Heisenberg group in the case of the southwest measure.

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Contents

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1 Introduction

In this self-contained paper, we present the classification of the positive harmonic functions on the Heisenberg group $H_3(\mathbb{Z})$ in the special case of the *southwest measure*. This example is striking because the famous partition functions occur as positive harmonic functions. In this case, our main result tells us that roughly all positive harmonic functions are combinations of characters and partition functions (Theorem 1.1).

We will also explain with no proof how this result can be extended to finite positive measures on $H_3(\mathbb{Z})$ (Theorem 3.8). The proof of this extension can be found in [2].

1.1 The partition function p(x,y,z) as a potential

We first introduce the "partition function" $p(x, y, z)$ for any integers x, y, z in $\mathbb{Z}$.

1.1.1 The partition function

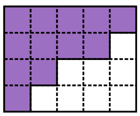

Figure 1: The partition $12=5+4+2+1$ is included in a 5×4 rectangle.

This function counts the "number of Young diagrams of area z", also called "partitions of z", included in a rectangle with side lengths x and y. More precisely, when x, y and z are non-negative, one has

$$\begin{aligned} p(x,y,z) \;\; = \;\; & |\{(n_1,\ldots,n_y)\in\mathbb{Z}^y \mid x\geq n_1\geq\cdots\geq n_y\geq 0 \\ & \text{and} \;\; n_1+\cdots+n_y = z\}|, \end{aligned} \tag{1.1}$$

and $p(x, y, z) = 0$ otherwise. The integers n_i are the lengths of the rows of the partition. By convention, for $x \geq 0$, one has $p(x, 0, z) = 0$ when $z \neq 0$, and $p(x, 0, 0) = 1$. This partition function satisfies the functional equation, for all $g = (x, y, z)$ in $\mathbb{Z}^3$, $g \neq (0, 0, 0)$,

$$p(x, y, z) = p(x-1, y, z-y) + p(x, y-1, z). \tag{1.2}$$

One checks it by splitting this set of partitions according to the colour of the lower-left case of the rectangle as in Figure 2.

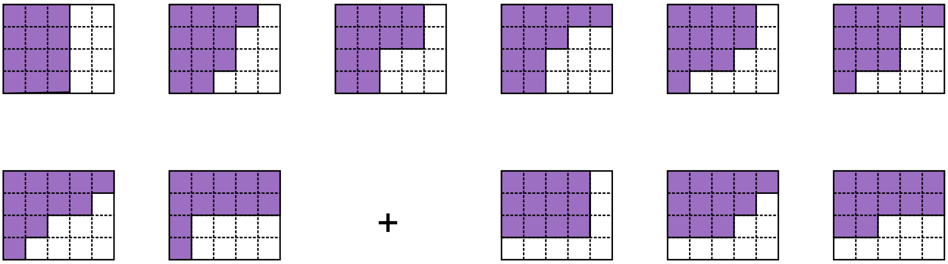

Figure 2: The 11 partitions in the equality $p(5,4,12) = p(4,4,8) + p(5,3,12)$.

1.1.2 The Heisenberg group

Recall that the Heisenberg group $G := H_3(\mathbb{Z})$ is the set $\mathbb{Z}^3$ of triples seen as matrices $(x, y, z) := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$. It is endowed with the product

$$(x_0, y_0, z_0)\,(x, y, z) = (x_0 + x, y_0 + y, z_0 + z + x_0 y)\,. \tag{1.3}$$

Let μ_0 be the southwest measure on G. It is given by

$$\mu_0 = \delta_{a^{-1}} + \delta_{b^{-1}} \quad \text{where } a := (1,0,0) \text{ and } b = (0,1,0). \tag{1.4}$$

Let $e := (0,0,0)$ be the unity of G and $\mathbf{1}_{\{e\}}$ be the characteristic function of $\{e\}$. Equation (1.2) can be rewritten as, for all $g = (x, y, z)$ in G,

$$p(g) = p(a^{-1}g) + p(b^{-1}g) + \mathbf{1}_{\{e\}}(g). \tag{1.5}$$

In particular, the function $f = p$ satisfies

$$f(g) \geq P_{\mu_0} f(g) \quad \text{where} \quad P_{\mu_0} f(g) := f(a^{-1}g) + f(b^{-1}g). \tag{1.6}$$

This inequality (1.6) tells us that the function f is a μ_0-superharmonic function on the Heisenberg group G.

1.1.3 The potential

More precisely, *the partition function* $p(g)$ *is the potential of* μ_0 *at* e. This means that one has the equality

$$p = \sum_{n \geq 0} P_{\mu_0}^n \mathbf{1}_{\{e\}}$$

Indeed, as can be seen in Figure 3, for g in G,

$$\textit{p(g) is the number of ways to write g as a word in a and b.} \tag{1.7}$$

A function h on G is said to be μ_0-harmonic if it satisfies

$$h(g) = P_{\mu_0} h(g), \text{ for all } g \text{ in } G, \text{ or equivalently} \tag{1.8}$$

$$h(x, y, z) = h(x-1, y, z-y) + h(x, y-1, z), \quad \text{for all } (x, y, z) \text{ in } \mathbb{Z}^3. \tag{1.9}$$

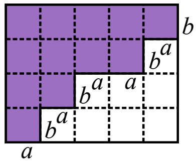

Figure 3: The partition $12 = 5+4+2+1$ associated to the word $w = ababaabab$ gives the element $g = g_w = ababaabab = (5, 4, 12) \in H_3(\mathbb{Z})$.

1.2 Construction of positive harmonic functions

We want to classify all the positive[2] solutions of (1.6), i.e. all the positive μ_0-superharmonic functions h on G. We begin with five remarks.

1.2.1 Choquet Theorem

By a theorem of Choquet in [5], every positive superharmonic function h is an average of extremal[3] positive superharmonic functions h_α. Moreover when h is harmonic the h_α are harmonic. By Riesz decomposition theorem [13, Thm 2.1.4], every positive μ_0-superharmonic function can be written in a unique way as the sum of a potential[4] and a positive μ_0-harmonic function. Therefore it is enough to describe the extremal positive μ_0-harmonic functions on G.

1.2.2 Choquet-Deny Theorem

If we look for a μ_0-harmonic function h which does not depend on z, Equation (1.9) becomes

$$h(x, y) = h(x-1, y) + h(x, y-1)\,, \quad \text{for all } (x, y) \text{ in } \mathbb{Z}^2. \tag{1.10}$$

This equation tells us that the function h is μ_0-harmonic on the abelian quotient $\mathbb{Z}^2$ of G. According to a theorem of Choquet and Deny in [6], since the support of the measure μ_0 spans the group $\mathbb{Z}^2$, every extremal positive μ_0-harmonic function on this abelian group is proportional to a character[5]:

$$\chi(x, y, z) = r^x s^y \quad \text{with } r, s > 0 \text{ and } 1/r + 1/s = 1. \tag{1.11}$$

[2]A function f on G is said to be positive if $f(g) \geq 0$ for all g in G and $f \neq 0$.

[3]A positive (super)harmonic function is said to be extremal if it cannot be written as the sum of two non-proportional positive (super)harmonic functions.

[4]A potential is a function of the form $f = \sum_{n\geq 0} P_{\mu_0}^n F$ for a positive function F on G.

[5]The proof is very short. One notices that Equality (1.10) gives a decomposition of h as a sum of two positive harmonic functions and hence both of them are proportional to h

1.2.3 The partition function as a harmonic function

If we look for a μ_0-harmonic function h which does not depend on x, Equation (1.9) becomes

$$h(y,z) = h(y,z-y) + h(y-1,z)\,, \quad \text{for all } (y,z) \text{ in } \mathbb{Z}^2. \tag{1.12}$$

A nice example is given by the partition function $(y,z) \mapsto p_y(z)$ where

$$\begin{aligned} p_y(z) &= \sup_{x\in\mathbb{Z}} p(x,y,z) = \lim_{x\to\infty} p(x,y,z) = p(z,y,z) \\ &= \text{the number of partitions of } z \text{ with at most } y \text{ rows.} \end{aligned} \tag{1.13}$$

$$\begin{array}{cccccccccccc}
 & \uparrow z & & & & & & & & & & \\
0 & \mathbf{0} & 1 & 5 & 10 & 15 & 18 & 20 & 21 & 22 & & \\
0 & \mathbf{0} & 1 & 4 & 8 & 11 & 13 & 14 & 15 & 15 & & \\
0 & \mathbf{0} & 1 & 4 & 7 & 9 & 10 & 11 & 11 & 11 & & \\
0 & \mathbf{0} & 1 & 3 & 5 & 6 & 7 & 7 & 7 & 7 & & \\
0 & \mathbf{0} & 1 & 3 & 4 & 5 & 5 & 5 & 5 & 5 & & \\
0 & \mathbf{0} & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & & \\
0 & \mathbf{0} & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & & \\
0 & \mathbf{0} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
\mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \rightarrow & y \\
0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & &
\end{array}$$

Figure 4: The function $p_y(z)$ satisfies $p_y(z) = p_y(z-y) + p_{y-1}(z)$.

Hence the function $h_0(x,y,z) := p_y(z)$ is a μ_0-harmonic function on G.

1.2.4 Margulis First Theorem

According to the first theorem of Margulis, a theorem he proved in [10] when he was not yet twenty, the Choquet-Deny Theorem is still true on a finitely generated nilpotent group G as soon as the support of the measure spans G as a semigroup (See Fact 3.7). This is why it might look surprising at first glance, that there exists a positive μ_0-harmonic function h_0 on $H_3(\mathbb{Z})$ which is not invariant by the center. The reason it exists is that the support of μ_0 spans G as a group but not as a semigroup. What is more surprising is that this "new" positive harmonic function h_0 is given by the famous partition function $p_y(z)$.

1.2.5 Switching and translating harmonic functions

We denote by σ the automorphism of G exchanging a and b. It is given by

$$\sigma(x,y,z) = (y,x,xy-z)\,.$$

Since the function h_0 is μ_0-harmonic, the function

$$h_1 := h_0 \circ \sigma : (x, y, z) \mapsto p_x(xy - z)$$

is also μ_0-harmonic. For g_0 in G, we denote by $\rho_{g_0} : g \mapsto g g_0$ the right translation by g_0 on G. The translated functions $h_0 \circ \rho_{g_0} : g \mapsto h_0(g g_0)$ and $h_1 \circ \rho_{g_0} : g \mapsto h_1(g g_0)$ are also μ_0-harmonic.

1.3 Classification of positive harmonic functions

We can now state our main result for the southwest measure μ_0 introduced in (1.4).

1.3.1 Main result and strategy

Theorem 1.1. *Let h be an extremal positive μ_0-harmonic function on the Heisenberg group $G := H_3(\mathbb{Z})$. Then, up to a multiplicative scalar,*
- either $h = \chi$ is a μ_0-harmonic character $\chi(x, y, z) = r^x s^y$ as in (1.11),
- or $h = h_0 \circ \rho_{g_0}$ is a translate of the function $h_0(x, y, z) := p_y(z)$,
- or $h = h_1 \circ \rho_{g_0}$ is a translate of the function $h_1(x, y, z) := p_x(xy - z)$.

This classification has been annouced on May 28^{th} 2019 in a short informal videotaped speech at the Cetraro Conference "Dynamics of group actions". This video can be found on the author's web page.

As we will see the partition function $p(x, y, z)$ will play a crucial role in the proof of Theorem 1.1. Indeed, in Chapter 2, we will prove a ratio limit theorem for the partition function $p(x, y, z)$. In Chapter 3, we will deduce from this ratio limit theorem the proof of Theorem 1.1.

Notice that the positive μ_0-harmonic function h_0 vanishes. In particular, it does not satisfy the Harnack inequality. This contrasts with the case studied in [10] where the support of μ spans G as a semigroup.

In the last Section 3.4, we will present the classification of the positive μ-harmonic functions, for all finitely supported measures μ on G.

1.3.2 Dealing with a probability measure

At first glance it might look a little bit weird to deal with μ_0-harmonic function for a measure μ_0 which is not a probability measure. We could have worked instead with the probability measure

$$\widetilde{\mu}_0 = \tfrac{1}{2}(\delta_{a^{-1}} + \delta_{b^{-1}}) \quad \text{where } a := (1, 0, 0) \text{ and } b = (0, 1, 0)$$

which is the law for the *southwest random walk on* $\mathbb{H}_3(\mathbb{Z})$. The $\widetilde{\mu}_0$-harmonic functions $\widetilde{h}$ on G are the functions satisfying

$$\widetilde{h} = P_{\widetilde{\mu}_0} h \quad \text{where} \quad P_{\widetilde{\mu}_0} h(x, y, z) = \frac{1}{2}\left(\widetilde{h}(x-1, y, z-y) + \widetilde{h}(x, y-1, z)\right),$$

is the expected value of the function h after one step of the random walk.

It is easy to see that

$$h(x,y,z) \text{ is } \mu_0\text{-harmonic if and only } 2^{-x-y}\, h(x,y,z) \text{ is } \widetilde{\mu}_0\text{-harmonic} \ .$$

Therefore, classifying positive μ_0-harmonic functions is equivalent to classifying positive $\widetilde{\mu}_0$-harmonic functions. The main reason we are using μ_0 instead of $\widetilde{\mu}_0$ is to get rid of all these factors 2^{-x-y}.

1.3.3 Extremal superharmonic functions

We have seen in (1.5) that the partition function p is μ_0-superharmonic and more precisely that it is the potential of μ_0 at e. For every g_0 in G, the function $p \circ \rho_{g_0}$ is also a potential of μ_0 at g_0^{-1}. By Riesz Decomposition Theorem, those potentials are exactly the extremal positive μ_0-superharmonic functions on G which are not harmonic. Therefore,

every extremal positive μ_0-superharmonic functions f on G which is not harmonic is a translate $f = p \circ \rho_{g_0}$ of the function $p(x,y,z)$.

We would like to end this introduction by pointing out other limit theorems for random walks on the Heisenberg group and other nilpotent groups as [8], [3], [4],[7] even though we will not use them here.

2 The partition function

The aim of this chapter is to prove the ratio limit theorem (Proposition 2.2) for the partition function $p(x,y,z)$.

2.1 The unimodality of the partition functions

We recall that, for $x, y, z \geq 0$, the partition function $p(x,y,z)$ counts the number of partitions of z included in a rectangle with side lengths x and y. See Definition (1.1) and Figure 1.

This function is non-zero for $0 \leq z \leq xy$ and satisfies the equalities

$$p(x,y,z) = p(y,x,z) = p(x,y,xy-z). \tag{2.1}$$

This function is well-studied. For instance one has

Fact 2.1. (Cayley, Sylvester 1850) *The sequence $z \mapsto p(x,y,z)$ is unimodal, i.e. it is increasing for $z \leq xy/2$.*

The proof below relies on the theory of finite dimensional representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. This proof is due to Hughes in [9]. See [12] for an elementary proof and [14, p. 522] for a survey of various generalizations.

Sketch of proof of Fact 2.1. Let $n := x + y$ and (Y, H, X) be the principal $\mathfrak{sl}_2$-triple in the Lie algebra $\mathfrak{g} := \mathfrak{sl}(n, \mathbb{R})$ so that $H = \mathrm{diag}(n-1, n-3, \ldots, -n+1)$. This Lie algebra $\mathfrak{g}$ has a natural representation in the space $V := \Lambda^x(\mathbb{R}^n)$. One checks that $p(x, y, z) = \dim V_{xy-2z}$ where V_λ denotes the eigenspace of H in V for the eigenvalue λ. The theory of representations of $\mathfrak{sl}(2, \mathbb{R})$ tells us that for $\lambda > 0$, one always has $\dim V_\lambda \leq \dim V_{\lambda-2}$. □

2.2 The ratio limit theorem

Here is the *Ratio Limit Theorem for* $p(x, y, z)$:

Proposition 2.2. *One has* $\displaystyle\lim_{\substack{z\to\infty \\ xy-z\to\infty}} \frac{p(x, y, z-1)}{p(x, y, z)} = 1.$

This limit is taken along sequences of positive triples (x, y, z) such that $z \to \infty$ and $xy - z \to \infty$.

With this generality this theorem seems to be new, even though there already exist very precise estimates of $p(x, y, z)$ in certain ranges. For instance, when $x, y \geq z$, the partition function $p(x, y, z) = p(z, z, z)$ depends only on z. It is the classical partition function $p(z)$ which admits a famous asymptotic expansion due to Hardy and Ramanujan in 1920 (See [1, Ch. 5]). These estimates have been extended to larger ranges of (x, y, z) as in [15] and [11]. We will not use them.

The proof of Proposition 2.2 is tricky but elementary. The rough idea is to introduce a relation between the set of partitions w of z and the set of partitions w' of $z - 1$ such that "most of the time" when w and w' are related, they are related to approximately the same number of partitions (see Lemma 2.5).

Because of (2.1), we can assume that $y \leq x$ and $z \leq xy/2$.

2.3 When the height of the rectangles is bounded

In this section, we deal with the easy case when the height y remains bounded.

Lemma 2.3. *For all* $y \geq 1$*, one has* $\displaystyle\lim_{\substack{x,z\to\infty \\ z \leq xy/2}} \frac{p(x, y, z-1)}{p(x, y, z)} = 1.$

Note that in this limit y is fixed, and x, z go to ∞ with $z \leq xy/2$.

Proof of Lemma 2.3. This follows from Lemma 2.4 and the inequalities

$$0 \leq p(x, y, z) - p(x, y, z-1) \leq p(x, y-1, z).$$

The first inequality is the unimodality of the partition function.

For the second inequality, just notice that one can inject the set of partitions of z of height exactly y inside the set of partitions of $z - 1$ of height at most y by removing the last square in the bottom row of each partition. □

We have used the following Lemma.

Lemma 2.4. *a) For all $x, y, z \geq 1$, one has* $p(x, y, z) \leq z^{y-1}$.
b) For all $y \geq 1$, there exists $\alpha_y > 0$ such that, for all $x, z \geq 1$ with $z \leq xy/2$, one has $p(x, y, z) \geq \alpha_y\, z^{y-1}$.

Proof of Lemma 2.4. *a*) The lengths of the last $y-1$ rows of the partition are bounded by $z-1$ and the first row is deduced from the others.

b) Choose $y-1$ integers $m_1, .., m_{y-1}$ in the interval $[0, \frac{z}{y^2}]$. and keep only those for which the system

$$n_1 - n_2 = m_1\ ,\ \ldots\ ,\ n_{y-1} - n_y = m_{y-1} \ \text{ and } \ n_1 + \cdots + n_y = z$$

has a solution $(n_1, \ldots, n_y)$ in $\mathbb{Z}^y$. But then one has

$$n_y = \frac{1}{y}(z - m_1 - 2m_2 - \cdots - (y-1)m_{y-1}) \geq 0 \quad \text{and}$$

$$n_1 = n_y + m_1 + \cdots + m_{y-1} \leq \frac{z}{y} + \frac{z}{y} \leq x.$$

This gives about $\frac{1}{y}(\frac{z}{y^2})^{y-1}$ partitions of z with $x \geq n_1 \geq \cdots \geq n_y \geq 0$. □

2.4 Inner and outer corner of a partition

We now introduce notations that will stengthen the connection between partitions and words in the Heisenberg group.

We recall that $a = (1, 0, 0)$ and $b = (0, 1, 0)$ are the generators of the Heisenberg group $G = H_3(\mathbb{Z})$. Let

$$G^+ := \{g = (x, y, z) \in G \mid x, y \geq 0 \ \text{ and } \ 0 \leq z \leq xy\}$$

be the semigroup generated by a and b and let

$$c = aba^{-1}b^{-1} = (0, 0, 1) \tag{2.2}$$

be the generator of the center Z of G.

Let $B_n := \{a, b\}^n$ be the set of finite words w in a, b of length $\ell_w = n$ and let $B := \cup_{n\geq 0} B_n$. Using the product law in G, to each word $w \in B$, we can associate an element g_w in G^+. The partition function gives the size of the fibers of this map :

$$p(g) = |B_g| \ \text{ where } \ B_g := \{w \in B \mid g_w = g\}. \tag{2.3}$$

Indeed, as explained in Figure 3, when $g = (x, y, z)$, each word w in B_g corresponds uniquely to a partition of z included in a rectangle with side lengths x and y. We introduce now the following relation $\mathcal{R}$ on B,

$$\mathcal{R} \ := \ \{(w, w') \in B \times B \mid w = w_0 abw_1 \text{ and } w' = w_0 baw_1 \text{ for some } w_0, w_1 \text{ in } B\}.$$

Let $\pi : \mathcal{R} \to B$ and $\pi' : \mathcal{R} \to B$ be the two projections

$$\pi(w, w') = w \quad \text{and} \quad \pi(w, w') = w'.$$

For w, w' in B, the cardinality of the fiber $f_w := |\pi^{-1}(w)|$ is the number of pairs ab occuring in the word w. The size f_w is also the number of *inner corners* of the partition associated to w. See Figure 5. Similarly the cardinality of the fiber $f'_{w'} := |\pi'^{-1}(w')|$ is the number of pairs ba occuring in the word w'. It is equal to the number of *outer corners* of the partition associated to w'.

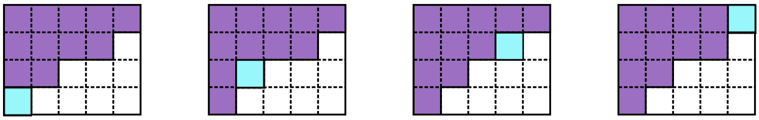

Figure 5: The fiber $\pi^{-1}(w)$ of the word $w = ababaabab$ has size $f_w = 4$.

The following lemma compares the size of these fibers.

Lemma 2.5. *a) For all $(w, w') \in \mathcal{R}$, one has $g_w = g_{w'} c$.*
b) For all $(w, w') \in \mathcal{R}$, one has $|f_w - f'_{w'}| \leq 2$.
In particular, one also has $f_w \leq 3 f'_{w'}$.

Proof of Lemma 2.5. *a*) This follows from the equality $c = aba^{-1}b^{-1}$.

b) Comparing the number of pairs ab and pairs ba occuring in w and in w', one gets $|f_w - f_{w'}| \leq 1$ and $|f_{w'} - f'_{w'}| \leq 1$. □

2.5 Partitions with bounded number of corners

We will need to control the number $p_{\leq i}(x, y, z)$ of partitions of z included in a rectangle with side length x, y that have at most i inner corner.

The following Lemma 2.6 tells us that $p_{\leq i}(x, y, z)$ is negligeable compared to the total number of partitions $p(x, y, z)$.

Lemma 2.6. *For all $i \geq 0$, one has* $\displaystyle \lim_{\substack{x,y,z \to \infty \\ z \leq xy/2}} \frac{p_{\leq i}(x, y, z)}{p(x, y, z)} = 0.$

The limit is taken along sequences where all coordinates x, y, z go to ∞ and $z \leq xy/2$.

Proof of Lemma 2.6. Use the following slight upgrade of Lemma 2.4. □

Lemma 2.7. *a) For all $x, y, z, i \geq 1$, one has $p_{\leq i}(x, y, z) \leq (2z)^{2i}$.*
b) For all $j > 1$, there exists $z_0 = z_0(j) \geq 1$ such that, for all $x, y, z \geq 1$ with $4j \leq y \leq x$ and $z_0 \leq z \leq xy/2$, one has $p(x, y, z) \geq z^j$.

Proof of Lemma 2.7. It is similar to Lemma 2.4.

a) We can assume $x = y = z$. We want to choose integers $a_1, \dots, a_i \geq 1$ and $m_1, \dots, m_i \geq 0$, bounded by z such that $a_1 m_1 + \dots + a_i m_i = z$. There are at most $(2z)^{2i}$ possibilities.

b) We give a rough count. Choose $L_y \leq y$ as large as possible such that, setting $\ell_y = [L_y/2]$ and $\ell_x = [z/L_y]$, one has $\ell_y \leq \ell_x \leq x/2$. There exists a partition w_0 of z with L_y rows and all of whose rows have length ℓ_x or $\ell_x + 1$. For every sequence $\ell_x > m_1 \geq \dots \geq m_{\ell_y} \geq 0$, we can modify this partition w_0 by adding m_j spots to the j^{th} highest row of w_0 and removing m_j spots to the j^{th} lowest row of w_0, for all $j \leq \ell_y$. This gives N different partitions of z where $N := \binom{\ell_x + \ell_y - 1}{\ell_y} \geq \max(2, \ell_x/\ell_y)^{\ell_y}$. Hence, one has $p(x, y, z) \geq N$.

First case : when $z \leq y^2/2$. In this case, we have $L_y = [\sqrt{2z}]$.
One gets $N \geq 2^{\ell_y} \geq 2^{\sqrt{z}/2} \geq z^j$.

Second case : when $z \geq y^2/2$. In this case, we have $L_y = y$.
If $z \leq y^4$, one gets $N \geq 2^{\ell_y} \geq 2^{\sqrt[4]{z}/4} \geq z^j$.
If $z \geq y^4$, one gets $N \geq (\frac{\ell_x}{\ell_y})^{\ell_y} \geq (\frac{z}{y^2})^{\ell_y} \geq \sqrt{z}^{\,\ell_y} \geq z^{y/4} \geq z^j$. □

2.6 When the height of the rectangles is unbounded

We can now explain the proof of the ratio limit theorem.

Proof of Proposition 2.2. By (2.1) and Lemma 2.3, we can assume that the three positive integers x, y, z are going to ∞ with $y \leq x$ and $z \leq xy/2$. For $g = (x, y, z)$ in G^+, one sets $\mathcal{R}_g := \{(w, w') \in \mathcal{R} \mid g_w = g\}$, and one computes

$$p(g) = |B_g| = \varepsilon_g + \sum_{(w,w') \in \mathcal{R}_g} \frac{1}{f_w} \tag{2.4}$$

where $\varepsilon_g = 1$ if $\mathcal{R}_g = \emptyset$ and $\varepsilon_g = 0$ otherwise. Similarly, by Lemma 2.5.*a*, one has

$$p(gc^{-1}) = |B_{gc^{-1}}| = \varepsilon'_g + \sum_{(w,w') \in \mathcal{R}_g} \frac{1}{f'_{w'}} \tag{2.5}$$

where $\varepsilon'_g = 0$ or 1. Combining (2.4), (2.5) and Lemma 2.5.*b*, one gets

$$|p(g) - p(gc^{-1})| \quad \leq \quad 2 + \sum_{(w,w') \in \mathcal{R}_g} \frac{2}{f_w f'_{w'}} \quad \leq \quad 2 + \sum_{(w,w') \in \mathcal{R}_g} \frac{6}{f_w^2} \quad \leq \quad 2 + \sum_{\substack{w \in B_g \\ f_w \neq 0}} \frac{6}{f_w}$$

We recall that $p_{\leq i}(g)$ is the number of w with $f_w \leq i$. Therefore, one has

$$|p(g) - p(gc^{-1})| \quad \leq \quad 2 + 6\, p_{\leq i}(g) + \frac{6}{i}\, p(g) \quad \text{ for all } i \geq 1.$$

We let x, y, z go to infinity with $z \leq xy/2$. According to Lemma 2.6, for all $i \geq 1$, the ratios $p_{\leq i}(g)/p(g)$ converge to 0. Therefore

$$\limsup \left| \frac{p(gc^{-1})}{p(g)} - 1 \right| \quad \leq \quad \frac{6}{i}.$$

and therefore the sequence $\dfrac{p(gc^{-1})}{p(g)}$ converges to 1 as required. □

3 Positive harmonic functions

We now start the classification of extremal positive μ_0-harmonic functions h. In Section 3.1, we deal with the case where h has a non-zero limit along an orbit of a^{-1} or b^{-1}. In Sections 3.2 and 3.3, we deal with the case where h goes to zero along all orbits of a^{-1} and b^{-1}. In Section 3.4 we present the generalization of this classification to any finitely supported measure μ on G.

3.1 The partition function as an harmonic function

In this section we characterize the functions $h_0 \circ \rho_{g_0}$ and $h_1 \circ \rho_{g_0}$ among extremal positive μ_0-harmonic functions by their behavior along the orbits $a^{-\mathbb{N}} g_0$ and $b^{-\mathbb{N}} g_0$ of G.

We recall that $a = (1, 0, 0)$ and $b = (0, 1, 0)$ are the generators of the Heisenberg group $G = H_3(\mathbb{Z})$, that $\mu_0 = \delta_{a^{-1}} + \delta_{b^{-1}}$, and that h_0 and h_1 are the μ_0-harmonic functions $h_0(x, y, z) = p_y(z)$ and $h_1(x, y, z) = p_x(xy - z)$.

We first begin by an alternative construction of the function h_0. Let H_0 be the abelian subgroup of G generated by a and let $\psi_0 := \mathbf{1}_{H_0}$ be the characteristic function of H_0. One has

$$\begin{aligned} \psi_0(x, y, z) &= 1 \quad \text{when } y = z = 0 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Lemma 3.1. *One has the equality* $\quad h_0 \;=\; \lim_{n\to\infty} P^n_{\mu_0} \psi_0.$

Remark. Since the function ψ_0 is μ_0-subharmonic, i.e. $\psi_0 \leq P_{\mu_0}\psi_0$, the sequence $n \mapsto P^n_{\mu_0}\psi_0$ is increasing.

Proof of Lemma 3.1. One can compute explicitly this function $P^n_{\mu_0}\psi_0$. It does not depend on x. Indeed $P^n_{\mu_0}\psi_0(x, y, z)$ is the number of ways of writing the element $(n-y, y, z)$ as a word w of length n in a and b. This proves the equality, involving the partition function,

$$P^n_{\mu_0}\psi_0(x, y, z) = p(n - y, y, z)\,.$$

Letting n go to ∞, we conclude using (1.13). □

Lemma 3.2. *Let $g_0 \in G$ and h be an extremal positive μ_0-harmonic function on G such that* $\limsup_{n\to\infty} h(a^{-n} g_0) > 0$. *Then one has $h = \lambda\, h_0 \circ \rho_{g_0}$ with $\lambda > 0$.*

In particular, the positive μ_0-harmonic function $h_0 \circ \rho_{g_0}$ is extremal.

Proof of Lemma 3.2. We can assume that $g_0 = e$. Since the function h is positive and μ_0-harmonic, the sequence $n \mapsto h(a^{-n})$ is decreasing. Hence it has a limit λ. By assumption, this limit λ is positive. By construction, one has the equality $h \geq \lambda\psi_0$. Since h is μ_0-harmonic, one also has the inequality $h \geq \lambda P^n_{\mu_0}\psi_0$ for all $n \geq 0$. Therefore, by Lemma 3.1, one gets $h \geq \lambda h_0$. Since h is extremal, it has to be proportional to h_0 and therefore one has $h = \lambda h_0$.

It remains to check that h_0 is extremal. If one can write $h_0 = h_0' + h_0''$ with both h_0' and h_0'' positive μ_0-harmonic, for at least one of them, say h_0', the sequence $h_0'(a^{-n})$ does not converge to 0 for $n \to \infty$. Hence, by the previous discussion, h_0' is proportional to h_0. This proves that h_0 is extremal. □

Exchanging the role of a and b we get

Corollary 3.3. *Let h be an extremal positive μ_0-harmonic function on G such that* $\limsup_{n\to\infty} h(b^{-n} g_0) > 0$. *Then one has $h = \lambda\, h_1 \circ \rho_{g_0}$ for some $\lambda > 0$.*

In particular, the positive μ_0-harmonic function $h_1 \circ \rho_{g_0}$ is extremal.

3.2 Harmonic functions that decay on cosets

We now discuss positive harmonic functions on G that decay to 0 along the orbits $a^{-\mathbb{N}} g_0$ and $b^{-\mathbb{N}} g_0$.

Let G_n be the subset of G consisting of elements of "degree" n,

$$G_n \quad = \quad \{g = (x, y, z) \in G \mid x+y=n\}.$$

By definition and by (1.7), a positive μ_0-harmonic function h on G satisfies the equality, for all $n \geq 1$,

$$h(g_0) = \sum_{w\in B_n} h(g_w^{-1} g_0) = \sum_{g\in G_n} p(g)\, h(g^{-1} g_0)\,. \tag{3.1}$$

For an integer $A > 0$, we set

$$\begin{aligned} G_{n,A} &= \{g = (x, y, z) \in G_n \mid z \leq A\}, \\ G^{\sigma}_{n,A} &= \{g = (x, y, z) \in G_n \mid xy - z \leq A\}. \end{aligned} \tag{3.2}$$

The following lemma tells us when the contributions of $G_{n,A}$ and $G^{\sigma}_{n,A}$ in Formula (3.1) is negligeable.

Lemma 3.4. *Let h be a positive μ_0-harmonic function on G such that,*

$$\lim_{n\to\infty} h(a^{-n}g_0) = 0 \quad \text{and} \quad \lim_{n\to\infty} h(b^{-n}g_0) = 0 \quad \text{for all } g_0 \text{ in } G. \tag{3.3}$$

Then, for all $A > 0$ and g_0 in G, one has

$$\lim_{n\to\infty} \sum_{g\in G_{n,A}\cup G^{\sigma}_{n,A}} p(g)\, h(g^{-1}g_0) = 0\,. \tag{3.4}$$

Proof of Lemma 3.4. It is enough to prove (3.4) with $g_0 = e$. Moreover, since $G^{\sigma}_{n,A}$ is the image of $G_{n,A}$ by the involution σ which exchanges a and b, it is enough to prove (3.4) with $g \in G_{n,A}$. Equivalently, it is enough to prove

$$\lim_{n\to\infty} \sum_{w\in B_{n,A}} h(g_w^{-1}) = 0 \quad \text{where} \quad B_{n,A} := \{w \in B_n \mid g_w^{-1} \in G_{n,A}\}. \tag{3.5}$$

Note that, when $n > A$, every word $w \in B_{n,A}$ can be written as

$$w = b^m s a^k$$

with $s \in B_{A+1}$ a word of length $A+1$. See Figure 6. One splits the set $B_{n,A}$ according

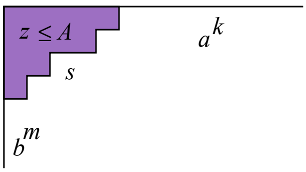

Figure 6: The decomposition $w = b^m s a^k$ for a word $w \in B_{n,A}$.

to $m \geq A$ or $m < A$. Therefore, for $n \geq 2A$, one has the inclusion

$$B_{n,A} \subset b^A\, B_{n-A} \;\cup\; B_{2A}\, a^{n-2A}.$$

Therefore, using (3.1), one gets the inequalities

$$\begin{aligned}\sum_{w\in B_{n,A}} h(g_w^{-1}) \;&\leq\; \sum_{w\in B_{n-A}} h(g_w^{-1}b^{-A}) \;+\; \sum_{w\in B_{2A}} h(a^{-(n-2A)}g_w^{-1})\\ &=\; h(b^{-A}) \;+\; \sum_{w\in B_{2A}} h(a^{-(n-2A)}g_w^{-1})\end{aligned}$$

For all $\varepsilon > 0$, we choose A large enough so that, by the second assumption (3.3), one has $h(b^{-A}) \leq \varepsilon$. Then the last sum is a sum over the fixed finite set B_{2A}, and, by the first assumption (3.3), this last sum converges to 0 when n goes to infinity. This proves (3.5) as required. □

3.3 Using the ratio limit theorem

Combining Lemma 3.4 with the ratio limit theorem we can finish the last case of the proof of Theorem 1.1.

Lemma 3.5. *Let h be a positive μ_0-harmonic function on G such that, for all g_0 in G, $\lim_{n\to\infty} h(a^{-n}g_0) = \lim_{n\to\infty} h(b^{-n}g_0) = 0$. Then h is invariant by the center $Z = c^{\mathbb{Z}}$ of G.*

Proof of Lemma 3.5. Using (3.1) with g_0 and g_0c, we compute,

$$h(g_0) - h(g_0c) \quad = \quad \sum_{g\in G_n} (p(g) - p(gc))\, h(g^{-1}g_0). \tag{3.6}$$

We fix $\varepsilon > 0$. According to the ratio limit theorem (Proposition 2.2), there exists an integer $A > 0$ such that, for all $g = (x, y, z)$ in G^+ with $z \geq A$ and $xy - z \geq A$, one has

$$|p(g) - p(gc)| \leq \varepsilon\, p(g)\,. \tag{3.7}$$

Therefore, using (3.6), (3.7) and Definition (3.2), one gets

$$|h(g_0) - h(g_0c)| \leq \sum_{g\in G_n} \varepsilon\, p(g)\, h(g^{-1}g_0) + \sum_{g\in G_{n,A}\cup G^{\sigma}_{n,A}} p(g)(h(g^{-1}g_0) + h(g^{-1}g_0c))$$

By (3.1), the first term is equal to $\varepsilon h(g_0)$. Therefore using twice Lemma 3.4 and letting n go to infinity, one gets $|h(g_0) - h(g_0c)| \leq \varepsilon\, h(g_0)$. Since ε is arbitrary small, this proves that $h(g_0) = h(g_0c)$ as required. □

Corollary 3.6. *Let h be an extremal positive μ_0-harmonic function on G such that, for all g_0 in G, $\lim_{n\to\infty} h(a^{-n}g_0) = \lim_{n\to\infty} h(b^{-n}g_0) = 0$. Then h is a character of G.*

In particular, every μ_0-harmonic character of G is an extremal positive μ_0-harmonic function.

Proof of Corollary 3.6. By Lemma 3.5, the function h is μ_0-harmonic on the abelian group G/Z. By Choquet-Deny Theorem, it is a character.

It remains to check that a μ_0-harmonic character χ is extremal. Assume that $\chi = h' + h''$ with both h' and h'' positive μ_0-harmonic. For all g_0 in G, the sequences $h'(a^{-n}g_0)$ and $h'(b^{-n}g_0)$ converge to 0 for $n \to \infty$. Hence, by the previous discussion and by Choquet Theorem, the function h' is an integral $h' = \int_C \chi' \mathrm{d}\sigma(\chi')$ where σ is a finite positive measure on the set C of (harmonic) character χ' of G. Since $h' \leq \chi$, the measure σ must be supported by χ. This proves that χ is extremal. □

This ends the proof of Theorem 1.1.

3.4 Extension to finitely supported measures

In this section we give the classification of the positive μ-harmonic functions on the Heisenberg group for all finitely supported measure μ.

Let $G = H_3(\mathbb{Z})$ be the Heisenberg group and S be a finite subset of G. We denote by G_S the subgroup of G generated by S. Let $\mu = \sum_{s\in S} \mu_s \delta_s$ be a positive measure on G with support S.

We recall that a function h on G is said to be μ-harmonic if

$$h = P_\mu h \quad \text{where} \quad P_\mu h(g) := \textstyle\sum_{s\in S} \mu_s\, h(sg). \tag{3.8}$$

We want to describe the cone $\mathcal{H}^+$ of positive μ-harmonic functions h on G. By Choquet Theorem, it is enough to describe the extremal rays of this cone $\mathcal{H}^+$.

There are two constructions of extremal positive μ-harmonic functions.

3.4.1 The harmonic characters χ

By definition, the μ-harmonic characters are the characters $\chi : G \to \mathbb{R}_{>0}$ of G such that $\sum_{s\in S} \mu_s\, \chi(s) = 1$. Such a function $h = \chi$ is an extremal positive μ-harmonic function on G which is invariant by the center Z of G.

We now recall Margulis Theorem which tells us that this first construction is the only possible when $G_\mu^+ = G$.

Fact 3.7. (Margulis) *Let μ be a finite positive measure on a finitely generated nilpotent group G. If the semigroup G_μ^+ generated by the support of μ is equal to G, then every extremal positive μ-harmonic function h on G is a character.*

Sketch of proof of Fact 3.7 for $G = H_3(\mathbb{Z})$. Because of the assumption $G_\mu^+ = G$, we can assume that $\mu_c > 0$ and $\mu_a > 0$. The first part of the argument is as in the abelian case : since $h(x, y, z) \geq \mu_c\, h(x, y, z + 1)$, these two μ-harmonic functions are proportional and we get that, for some $t > 0$, one has $h(x, y, z) = h(x, y, 0)t^z$. We now want to prove that $t = 1$.

Let K_t be the set of positive harmonic functions $h_0(x, y, z) = \psi_0(x, y)t^z$ with $h_0(e) = 1$. Since $G_\mu^+ = G$, the convex set K_t is compact for the pointwise convergence. The element $a \in G$ acts continuously by "right-translation and renormalization" on K_t. By Schauder fixed point theorem, this action has a fixed point h_0 in K_t. It can be written as $h_0(x, y, z) = r^x \varphi_0(y)t^z$ with $r > 0$. But then one writes $h_0(g) \geq \mu_a h_0(ag)$ for all g in G, or equivalently $\varphi_0(y) \geq \mu_a r \varphi_0(y)t^y$ for all $y \in \mathbb{Z}$. This proves that $t = 1$. □

When $G_\mu^+ \neq G$, a second construction is possible.

3.4.2 The functions h_{S_0,χ_0} induced from a harmonic character

Let $S_0 \subset S$ be an abelian subset. Denote by $\mu_{S_0} := \sum_{s\in S_0} \mu_s\,\delta_s$ the measure restriction of μ to S_0. Let χ_0 be a μ_{S_0}-harmonic character of G_{S_0}. We extend χ_0 as a function

$$\psi_0 := \chi_0\,\mathbf{1}_{G_{S_0}}$$

on G which is 0 outside G_{S_0}. This function ψ_0 is μ-subharmonic, so that the sequence $P_\mu^n\psi_0$ is increasing. We set

$$h_{S_0,\chi_0} = \lim_{n\to\infty} P_\mu^n\psi_0.$$

We can tell exactly for which pairs (S_0, χ_0) the function h_{S_0,χ_0} is finite (see [2]). In this case the function h_{S_0,χ_0} is an extremal positive μ-harmonic function on G.

We can now state the extension of Theorem 1.1 to a more general finitely supported measure μ on G.

Theorem 3.8. *Let $G = H_3(\mathbb{Z})$ and μ be a positive measure on G whose finite support S generates the group G. Then every extremal positive μ-harmonic function h on G is proportional either to a character χ of G or to a translate $h_{S_0,\chi_0} \circ \rho_{g_0}$ of a function induced from a harmonic character.*

Corollary 3.9. *Let $G = H_3(\mathbb{Z})$, Z its center and μ a probability measure on G whose finite support S generates the group G. The following are equivalent:*
(i) Every positive μ-harmonic function on G is Z-invariant.
(ii) G_μ^+ contains two non-central elements whose product is in $Z \smallsetminus \{0\}$.

Theorem 3.8 and Corollary 3.9 are proven in the sequel paper [2].

We will also see in [2] that on the nilpotent group of rank 4 with cyclic center, there exist extremal positive harmonic functions which are neither an harmonic character nor a function induced from a harmonic character.

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