Magnetic wells in dimensions 2 and 3. (Memorial Conference Louis Boutet de Monvel)

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(After Helffer, Kordyukov, Raymond, and Vu Ngoc ...)

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The magnetic Schrödinger Operator

Our main object of interest is the Laplacian with magnetic field. In this talk we mainly consider, except for specific toy models, a magnetic field

$$\beta = \text{ curl } \mathbf{A}$$

on a regular domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) associated with a magnetic potential $\mathbf{A}$ (vector field on Ω).
We start from the closed quadratic form Q_h

$$W_0^{1,2}(\Omega) \ni u \mapsto Q_h(u) := \int_\Omega |(-ih\nabla + \mathbf{A})u(x)|^2 \, dx. \qquad (1)$$

where $h > 0$ is a small parameter playing the role of the semi-classical parameter.

Let $\mathcal{H}^D(\mathbf{A}, h, \Omega)$ be the self-adjoint operator associated with Q_h and let $\lambda_j^D(\mathbf{A}, h, \Omega)$ the infinite sequence of eigenvalues (in the case Ω bounded).
Motivated by various questions we consider the connected problems in the asymptotic $h \to +0$.

Pb 1 Determine the structure of the bottom of the spectrum : gaps, typically between the first and second eigenvalue.

Pb2 Find an effective Hamiltonian which through standard semi-classical analysis can explain the complete spectral picture.

The case when the magnetic field is constant

The first results are known from Landau at the beginning of the Quantum Mechanics. They concern the analysis of models with constant magnetic field β.
In the case in $\mathbb{R}^d$ ($d = 2, 3$), the models are more explicitly

$$h^2 D_x^2 + (hD_y - x)^2 \,,$$

($\beta(x, y) = 1$) and

$$h^2 D_x^2 + (hD_y - x)^2 + h^2 D_z^2 \,,$$

($\beta(x, y, z) = (0, 0, 1)$) and we have:

$$\inf \sigma(\mathcal{H}(\mathbf{A}, h, \mathbb{R}^d)) = h \,.$$

h is the so-called first Landau Level.

Pauli operator in $2D$

$$\mathcal{P}_{\pm}{}^{D}(\mathbf{A}, h, \Omega) := \mathcal{H}^{D}(\mathbf{A}, h, \Omega) \pm hB(x)\,.$$

In the constant magnetic field this corresponds with a translation of the spectrum.

$$h^2 D_x^2 + (hD_y - x)^2 \pm h\,.$$

If $h > 0$, the bottom is 0.

Purely magnetic effects in the case of a variable magnetic field

We introduce

$$b = \inf_{x \in \overline{\Omega}} |\beta(x)| \,, \tag{2}$$

Theorem 1 : Rough asymptotics for h small

$$\lambda_1^D(\mathbf{A}, h, \Omega) = hb + o(h) \tag{3}$$

Very poor if $b = 0$.

The consequences are that a ground state is localized as $h \to +0$ for Dirichlet, at the points of $\overline{\Omega}$ where $|\beta(x)|$ is minimum,
All the results of localization are obtained through semi-classical Agmon estimates (as Helffer-Sjöstrand [HS1, HS2], or Simon [Si] have done in the eighties for $-h^2\Delta + V$ or for the Witten Laplacians (Witten, Helffer-Sjöstrand, Helffer-Klein-Nier, Helffer-Nier, Le Peutrec,...) .

There are also Agmon estimates in the magnetic case (Helffer-Sjöstrand, Outassourt, Helffer-Mohamed, Helffer-Raymond, Helffer-Pan, Fournais-Helffer, Bonnaillie, N. Raymond, F. Hérau...). These estimates are not always optimal (see L. Erdös, S. Nakamura, A. Martinez, V. Sordoni,. . .).

A walk through the litterature on the semi-classical analysis of the purely magnetic Laplacians

We only discuss cases where the boundary does not play a role and the semi-classical analysis of the bottom of the spectrum.

1. R. Montgomery (1995) : the magnetic field vanishes along a curve + $\mathbb{S}^1$-group action. Very rough estimate.
2. B. Helffer and A. Mohamed (1996). Vanishing magnetic fields. Nilpotent approximation.
3. B. Helffer and A. Morame (2001) (Morame=Mohamed) Non vanishing magnetic field.

In the period 2001-2010, intensive works for the Neumann problem in connexion with superconductivity (Lu-Pan, Pan-Kwek, Sternberg, Helffer-Mohamed, Fournais-Helffer, Helffer-Pan, Bonnaillie-Noel, Fournais-Persson, Raymond, Kachmar,...)
Two books : Fournais-Helffer (2009), N. Raymond (2016).

A walk continued

1. Helffer-Kordyukov 2008–2012, revisiting Montgomery example: miniwell case, degenerate Montgomery model.

$$h^2 D_x^2 + (hD_y - x^{k+1})^2 \,.$$

2. Helffer-Kordyukov 2009, Non vanishing fields: semi-classical analysis near the minimum, discrete wells or manifold wells.
3. Helffer-Kordyukov 2013, Non vanishing wells in 3D (upper bounds)
4. N. Raymond-San Vu Ngoc (2013) Non vanishing wells in 2D (fine asymptotics through Birkhoff normal forms) and B. Helffer-Y. Kordyukov (2013) (Grushin problem approach).
5. Non vanishing wells in 3D (lower and upper bounds) B. Helffer-Y.Kordyukov-N. Raymond-S.Vu Ngoc (through Birkhoff normal forms).

We will concentrate our presentation on the three last items.

The case of $\mathbb{R}^n$ or the interior case (with Dirichlet condition)

2D case

If

$$0 < b < \inf_{x \in \partial\Omega} |\beta(x)| \,,$$

the asymptotics are the same (modulo an exponentially small error) as in the case of $\mathbb{R}^d$: no boundary effect.
In the case of $\mathbb{R}^d$, we assume

$$0 < b < \liminf_{|x| \to +\infty} |\beta(x)| \,.$$

We assume in addition (generic)

Assumption A

- There exists a unique point $x_{min} \in \Omega$ such that $b = |\beta(x_{min})|$.
- $b > 0$
- This minimum is non degenerate.

We get in $2D$ (Helffer-Morame (2001), Helffer-Kordyukov (2009))

Theorem 2

$$\lambda_1^D(\mathbf{A}, h) = bh + \Theta_1 h^2 + o(h^2)\,. \tag{4}$$

where $\Theta_1 = a^2/2b$.

Here

$$a = \mathrm{Tr}\left(\frac{1}{2}\mathrm{Hess}\,\beta(x_{min})\right)^{1/2}\,.$$

The previous statement can be completed in the following way.

$$\lambda_j^D(\mathbf{A}, h) \sim h \sum_{\ell \geq 0} \alpha_{j,\ell} h^{\frac{\ell}{2}} , \tag{5}$$

with

- $\alpha_{j,0} = b$,
- $\alpha_{j,1} = 0$,
- $\alpha_{j,2} = \frac{2d^{1/2}}{b}(j-1) + \frac{a^2}{2b}$,
- $d = \det\left(\frac{1}{2}\mathrm{Hess}\,\beta(x_{min})\right)^{1/2}$.

In particular, we get the control of the splitting $\sim \frac{2d^{1/2}}{b}$.
Note that behind these asymptotics, two harmonic oscillators are present. Recent improvments (Helffer-Kordyukov and Raymond–Vu-Ngoc (2013)) show that no odd powers of $h^{\frac{1}{2}}$ actually occur.

Interpretation with some effective Hamiltonian

If we look at the bottom of the spectrum of

$$h\left(\hat{\beta}^w(x, hD_x) + h\gamma^w(x, hD_x, h^{\frac{1}{2}})\right) ,$$

this gives the result modulo $\mathcal{O}(h^2)$, hence it was natural to find a direct proof of this reduction (which is in the physical litterature is called the lowest Landau level approximation), permitting to give a spectral information in a larger window (excited states).
$\hat{\beta}$ is related to β by an explicit map:

$$\hat{\beta} = \beta \circ \phi \,.$$

See below for the definition of ϕ.

Sketch of the initial quasimode proof.

We try to do what was successfull for $-h^2\Delta + V(x)$ near the minimum of V: the so called harmonic (quadratic) approximation. The toy model is

$$h^2 D_x^2 + \left(hD_y - b(x + \frac{1}{3}x^3 + xy^2)\right)^2 .$$

We obtain this toy model by taking a Taylor expansion of the magnetic field centered at the minimum and chosing a suitable gauge.

The second point is to use a blowing up argument $x = h^{\frac{1}{2}} s$, $y = h^{\frac{1}{2}} t$.

Dividing by h this leads (taking $b = 1$) to

$$D_s^2 + (D_t - s + h(\frac{1}{3}s^3 + st^2))^2 .$$

Partial Fourier transform

$$D_s^2 + (\tau - s + h(\frac{1}{3}s^3 + s(D_\tau)^2)^2\,,$$

and translation

$$D_s^2 + \left((-s + h\left(\frac{1}{3}(s+\tau)^3 + (s+\tau)(D_\tau - D_s)^2\right)\right)^2\,.$$

Expand as $\sum_{j\in\mathbb{N}/2} L_j h^j$, with

- $L_0 = D_s^2 + s^2\,,$
- $L_{\frac{1}{2}} = 0$
- $L_1 = -\frac{2}{3}s(s+\tau)^3 - s(s+\tau)(D_\tau - D_s)^2 - (s+\tau)(D_\tau - D_s)^2 s\,.$

Look formally for an eigenpair

$$\sum_{j\in\mathbb{N}/2} u_j(s,\tau)h^j\,,\ \sum_{j\in\mathbb{N}/2} h^j\lambda_j$$

we get first

$$L_0 u_0 = \lambda_0 u_0\,.$$

This leads to $\lambda_0 = 1$ (if we look for the lowest eigenvalue) and to

$$u_0(s,\tau) = \phi_0(s)\psi_0(\tau)\,.$$

We take $\lambda_{\frac{1}{2}} = 0$ and $u_{\frac{1}{2}} = 0$.
Then we get

$$(L_0-\lambda_0)u_1 + L_1 u_0 = \lambda_1 u_0\,.$$

We now project on $\mathrm{span}(\phi_0) \otimes L^2(\mathbb{R}_\tau)$.
The second harmonic oscillator appears in the τ variable by considering

$$\psi \mapsto \langle \phi_0(s), L_1(\phi_0(s)\psi(\tau))\rangle_{L^2(\mathbb{R}_s)} .$$

Note that ϕ_0 is even or odd.
λ_1 has to be chosen as an eigenvalue of this harmonic oscillator in the τ variable.
This approach can be extended to any order.
The lower bound is more difficult ! (Helffer-Mohamed for the ground state).

Improvements

In 2013, Helffer-Kordyukov on one side, and Raymond–Vu-Ngoc on the other side reanalyze the problem with two close but different points of view.

The proof of Helffer-Kordyukov is based on

- A change of variable : $(x, y) \mapsto \phi(x, y)$
- Normal form near a point (the minimum of the magnetic field)
- Construction of a Grushin's problem.

The last point is reminiscent of the analysis of the hypoellipticity with loss of $3/2$ derivatives (Grusin (1970), Sjöstrand (1972), Helffer (1975)).

This approach is local near the point where the intensity of the magnetic field is assumed to be minimum.

The proof of Raymond–Vu Ngoc is based on the research of Birkhoff normal form. A former reference is the paper by S. Vu Ngoc: Quantum Birkhoff normal forms and semiclassical analysis. In Noncommutativity and singularities (2009), volume 55 of Adv. Stud. Pure Math., 99–116. Math. Soc. Japan, Tokyo.

A change of variable

After a gauge transform, we assume that $A_1 = 0$ and $A_2 = A$. We just take :

$$x_1 = A(x, y), \quad y_1 = y$$

This defines ϕ.
In these coordinates the magnetic field reads

$$B = dx_1 \wedge dy_1 \,.$$

Birkhoff normal form (after Raymond-Vu Ngoc)

The proof of Raymond–Vu-Ngoc is reminiscent of Ivrii's approach (see his book in different versions) and uses a Birkhoff normal form. This approach involves more general symplectomorphisms and their quantification in comparison with Helffer-Kordyukov's approach which only uses linear canonical transformations.
We consider the h-symbol of the Schrödinger operator with magnetic potential A:

$$H(x,y,\xi,\eta) = |\xi - A_1(x,y)|^2 + |\eta - A_2(x,y)|^2 \,.$$

Theorem (Ivrii—Raymond–Vu-Ngoc)

$\exists$ a symplectic diffeomorphism Φ defined in an open set $\widetilde{\Omega} \subset \mathbb{C}_{z_1} \times \mathbb{C}_{z_2}$ with value in $T^*\mathbb{R}^2$ which sends $z_1 = 0$ into the surface $H = 0$ and such that

$$H \circ \Phi(z_1, z_2) = |z_1|^2 f(z_2, |z_1|^2) + \mathcal{O}(|z_1|^\infty)\,,$$

where f is a smooth real function.
Moreover, the map

$$\Omega \ni (x, y) \mapsto \phi(x, y) := \Phi^{-1}(x, y, \mathbf{A}(x, y)) \in \{\{0\} \times \mathbb{C}_{z_2}\} \cap \widetilde{\Omega}$$

is a local diffeomorphism and

$$f(\phi(x, y), 0) = B(x, y)\,.$$

Quantum version (after Raymond–Vu-Ngoc)

Theorem: Quantum Normal Form

For h small enough, there exists a global Fourier-Integral operator $\mathcal{U}_h$ (essentially unitary modulo $\mathcal{O}(h^\infty)$) such that

$$\mathcal{U}_h^* \mathcal{H} \mathcal{U}_h = \mathcal{I}_h \, F_h + R_h \,,$$

where

$$I_h = -h^2 \frac{d^2}{dx_1^2} + x_1^2 \,,$$

F_h is a classical h-pseudodifferential operator which commutes with I_h, and R_h is a remainder (with $\mathcal{O}(h^\infty)$ property in the important region).

More precisely, the restriction to the invariant space $H_n \otimes L^2(\mathbb{R}_{x_2})$ (H_n is the n-th eigenfunction) can be seen as a h-pseudodifferential operator in the x_2 variable, whose principal symbol is B. In Ivrii, the relevant statement seems Theorem 13.2.8.

A short visit to the Pauli operator

In $2D$, another connected interesting question is to analyze the bottom of the spectrum of the Pauli operator:

$$P_{\pm}^{D}(\mathbf{A}, h, \Omega) := \mathcal{H}^{D}(\mathbf{A}, h, \Omega) \pm hB(x)\,. \tag{6}$$

These two operators being positive, it is interesting to analyze if the bottom of the spectrum is exponentially small as $h \to 0$ and to measure the decay rate.
This problem has been analyzed by Erdös (Disk), Helffer-Mohamed (2001) (Disk) in the constant magnetic case,
Ekholm-Kowarik-Portmann (2015) (lower bounds) and Helffer–M. Persson Sundqvist (2016).

This is the case when $B(x) > 0$ for

$$P_-^D(\mathbf{A}, h, \Omega) = \mathcal{H}^D(\mathbf{A}, h, \Omega) - hB(x)$$

Let Ω be simply connected and let ψ be the solution of

$$\Delta\psi = B(x) \text{ in } \Omega\,,\ \psi_{/\partial\Omega} = 0\,.$$

We have

$$\psi_{min} = \inf \psi < 0\,.$$

Then (theorem by Helffer–Persson Sundqvist)

$$\lim_{h \to +\infty} h \log \lambda_1(P_-^D) = 2\psi_{min}\,.$$

Witten Laplacians or Dirichlet forms

The problem we study is quite close with the question of analyzing the smallest eigenvalue of the Dirichlet realization of:

$$C_0^\infty(\Omega) \ni v \mapsto h^2 \int_\Omega |\nabla v(x)|^2 \; e^{-2f(x)/h} \, dx \, .$$

For this case, we can mention Theorem 7.4 in Freidlin-Wentcel (1998), which says (in particular) that, if f has a unique non-degenerate local minimum x_{min}, then the lowest eigenvalue $\lambda_1(h)$ of the Dirichlet realization $\Delta^{(0)}_{f,h}$ in Ω satisfies:

$$\lim_{h\to 0} -h \log \lambda_1(h) = \inf_{x\in\partial\Omega} (f(x) - f(x_{min})) \, . \tag{7}$$

More precise or general results (prefactors) are given in Bovier-Eckhoff-Gayrard-Klein (2004), Bovier-Gayrard-Klein (2004), Helffer-Nier (2006). This is connected with the semi-classical analysis of Witten Laplacians (see Witten (1982), Helffer-Sjöstrand (1985), Helffer-Klein-Nier (2005)).

Toward the 3D case (a toy model)

The problem was partially open (see however Helffer-Kordyukov (2013) for quasimodes) till 2015 in the $3D$ case. What the generic model should be is more delicate. The toy model is

$$h^2 D_x^2 + (hD_y - x)^2 + (hD_z + (\alpha zx - P_2(x,y)))^2$$

with $\alpha \neq 0$, P_2 homogeneous polynomial of degree 2 where we assume that the linear forms $(x,y,z) \mapsto \alpha z - \partial_x P_2$ and $(x,y,z) \mapsto \partial_y P_2$ are linearly independent. The hope is to prove :

$$\lambda_1^D(\mathbf{A}, h) = bh + \Theta_{\frac{1}{2}} h^{\frac{3}{2}} + \Theta_1 h^2 + o(h^2)\,. \tag{8}$$

A suitable blowing up permits indeed to find a quasi-mode state with this expansion.

More in dimension 3, magnetic geometry

Let us now describe the geometry of the problem. The configuration space is

$$\mathbb{R}^3 = \{q_1 e_1 + q_2 e_2 + q_3 e_3, \quad q_j \in \mathbb{R}, \quad j = 1, 2, 3\},$$

where $(e_j)_{j=1,2,3}$ is the canonical basis of $\mathbb{R}^3$. The phase space is

$$\mathbb{R}^6 = \{(q, p) \in \mathbb{R}^3 \times \mathbb{R}^3\}$$

and we endow it with the canonical 2-form

$$\omega_0 = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + dp_3 \wedge dq_3. \tag{9}$$

We use the Euclidean scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$ and $\| \cdot \|$ the associated norm. We can rewrite ω_0 as

$$\omega_0((u_1, u_2), (v_1, v_2)) = \langle v_1, u_2 \rangle - \langle v_2, u_1 \rangle, \quad \forall u_1, u_2, v_1, v_2 \in \mathbb{R}^3.$$

The main object is the magnetic Hamiltonian, defined for $(q,p) \in \mathbb{R}^6$ by

$$H(q,p) = \|p - \mathbf{A}(q)\|^2, \tag{10}$$

where $\mathbf{A} \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$.
The vector field $\mathbf{A} = (A_1, A_2, A_3)$ is associated (via the Euclidean structure) with the following 1-form

$$\alpha = A_1 dq_1 + A_2 dq_2 + A_3 dq_3$$

and its exterior derivative is a 2-form, called magnetic 2-form and expressed as $d\alpha$. The form $d\alpha$ may be identified with a vector field. If we let:

$$\mathbf{B} = \nabla \times \mathbf{A} = (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1) = (B_1, B_2, B_3),$$

then, we can write

$$d\alpha = B_3 dq_1 \wedge dq_2 - B_2 dq_1 \wedge dq_3 + B_1 dq_2 \wedge dq_3. \tag{11}$$

See also the talk of Y. Colin de Verdière in this conference for a more geometric pressentation (contact structure).

An important role will be played by the characteristic hypersurface

$$\Sigma = H^{-1}(0),$$

which is the submanifold defined by the parametrization:

$$\mathbb{R}^3 \ni q \mapsto j(q) := (q, \mathbf{A}(q)) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

The relation between Σ, the symplectic structure and the magnetic field is given by:

$$j^* \omega_0 = d\alpha \,. \tag{12}$$

Confinement assumptions and discrete spectrum

We would like to analyze the semiclassical analysis of the discrete spectrum of the magnetic Laplacian $\mathcal{L}_{\hbar,\mathbf{A}} := (-i\hbar\nabla_q - \mathbf{A}(q))^2$, which is the semiclassical Weyl quantization of H.
Let us recall the assumptions under which discrete spectrum actually exist. In two dimensions, with a non vanishing magnetic field, we have used

$$\hbar \int_{\mathbb{R}^2} |B(q)||u(q)|^2 \, dq \leq \langle \mathcal{L}_{\hbar,\mathbf{A}} u \mid u \rangle \, , \, \forall u \in \mathcal{C}_0^\infty(\mathbb{R}^2) \, .$$

In dimension ≥ 3, we should impose a control of the oscillations of $\mathbf{B}$ at infinity.

Let us now state the confining assumptions.
We set

$$b(q) := \|\mathbf{B}(q)\|.$$

Assumption A

$$b(q) \geq b_0 := \inf_{q \in \mathbb{R}^3} b(q) > 0\,, \tag{13}$$

and

$$\|\nabla \mathbf{B}(q)\| \leq C\,(1 + b(q))\,, \forall q \in \mathbb{R}^3\,. \tag{14}$$

Under these assumptions, it is proven that there exist $h_0 > 0$ and $C_0 > 0$ such that, for all $\hbar \in (0, h_0)$,

$$\hbar(1 - C_0 \hbar^{\frac{1}{4}}) \int_{\mathbb{R}^3} b(q)|u(q)|^2 \, dq \leq \langle \mathcal{L}_{\hbar,\mathbf{A}} u \mid u \rangle \,, \forall u \in \mathcal{C}_0^\infty(\mathbb{R}^3) \,. \quad (15)$$

As a corollary, using Persson's theorem, we obtain that the bottom of the essential spectrum is asymptotically above $\hbar b_1$, where $b_1 := \liminf_{|q| \to +\infty} b(q)$.

Assumption B (confining)

We assume that

$$0 < b_0 < b_1 \,. \tag{16}$$

Moreover we will assume that b has a unique minimum at q_0 and that this minimum is non degenerate

The second part in Asssumption B is only useful in the last step of the study.

Description of the results in Helffer-Kordyukov-Raymond-Vu Ngoc (2015)

Thanks to F. Faure for discussions.

Let us now walk through the main results of [HKRV]. We will assume that the magnetic field does not vanish and is confining.

Of course, for eigenvalues of order $\mathcal{O}(\hbar)$, the corresponding eigenfunctions are microlocalized in the semi-classical sense (there is a notion of frequency set replacing the notion of Wave front set) near the characteristic manifold Σ (see for instance the books of Guillemin-Sternberg, D. Robert (87) or M. Zworski (2013)).

Moreover the confinement assumption implies that the eigenfunctions of $\mathcal{L}_{\hbar,\mathbf{A}}$ associated with eigenvalues less that $\beta_0\hbar$ enjoy localization estimates *à la* Agmon.
Roughly speaking, $h|\beta(x)|$ plays here the role of an electric potential.

Therefore we only investigate the magnetic geometry locally in space near a point $q_0 = 0 \in \mathbb{R}^3$ belonging to the confinement region.
Then, in a neighborhood of $(0, \mathbf{A}(0)) \in \Sigma$, there exist symplectic coordinates $(x_1, \xi_1, x_2, \xi_2, x_3, \xi_3)$ such that

$$\Sigma = \{x_1 = \xi_1 = \xi_3 = 0\}$$

and $(0, \mathbf{A}(0))$ has coordinates $0 \in \mathbb{R}^6$.
Hence Σ is parametrized by (x_2, ξ_2, x_3).

Classical motion

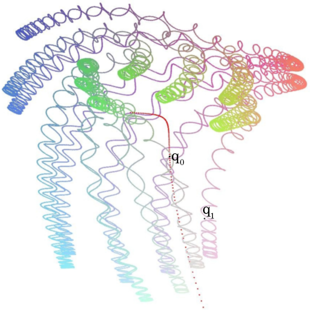

Figure: The dashed line represents the integral curve of the confining magnetic field through $q_0 = (0.5, 0.6, 0.7)$ for $B(x, y, z) = \left(\frac{y}{2}, \frac{z}{2}, \sqrt{1+x^2}\right)$ and the full line represents the projection in the q-space of the Hamiltonian trajectory with initial condition (q_0, p_0) (with $p_0 = (-0.6, 0.01, 0.2)$) ending at (q_1, p_1).

First Birkhoff form

In these coordinates, it is possible to perform a semiclassical Birkhoff normal form and to microlocally unitarily conjugate $\mathcal{L}_{\hbar,\mathbf{A}}$ to a first normal form $\mathcal{N}_\hbar = \mathrm{Op}_\hbar^w(N_\hbar)$ with an operator valued symbol $N_\hbar$ depending on (x_2, ξ_2, x_3, ξ_3) in the form

$$N_\hbar = \xi_3^2 + b(x_2, \xi_2, x_3)\mathcal{I}_\hbar + f^\star(\hbar, \mathcal{I}_\hbar, x_2, \xi_2, x_3, \xi_3) + \mathcal{O}(|\mathcal{I}_\hbar|^\infty, |\xi_3|^\infty).$$

where

- $\mathcal{I}_h = \hbar^2 D_{x_1}^2 + x_1^2$ is the first encountered harmonic oscillator
- $(\hbar, I, x_2, \xi_2, x_3, \xi_3) \mapsto f^\star(\hbar, I, x_2, \xi_2, x_3, \xi_3)$ satisfies, for $I \in (0, I_0)$,

$$|f^\star(\hbar, I, x_2, \xi_2, x_3, \xi_3)| \leq C \left(|I|^{\frac{3}{2}} + |\xi_3|^3 + \hbar^{\frac{3}{2}} \right).$$

Since we wish to describe the spectrum in a spectral window containing at least the lowest eigenvalues, we are led to replace $\mathcal{I}_\hbar$ by its lowest eigenvalue $\hbar$ and thus, we are reduced to the two-dimensional pseudo-differential operator

$$\mathcal{N}_\hbar^{[1]} = \mathrm{Op}_\hbar^w \left(N_\hbar^{[1]} \right)$$

where

$$N_\hbar^{[1]} = \xi_3^2 + b(x_2, \xi_2, x_3)\hbar + f^\star(\hbar, \hbar, x_2, \xi_2, x_3, \xi_3) + \mathcal{O}(\hbar^\infty, |\xi_3|^\infty).$$

Second Birkhoff form

If we want to continue the normalization, we use that $x_3 \mapsto b(x_2, \xi_2, x_3)$ admits a unique and non-degenerate minimum denoted by $s(x_2, \xi_2)$.
Then, by using a new symplectic transformation in order to center the analysis at the partial minimum $s(x_2, \xi_2)$, we get a new operator $\underline{\mathcal{N}}_\hbar^{[1]}$ with Weyl's symbol form

$$\underline{N}_\hbar^{[1]} = \nu^2(x_2, \xi_2)(\xi_3^2 + \hbar x_3^2) + \hbar b(x_2, \xi_2, s(x_2, \xi_2)) + \text{remainders},$$

with

$$\nu(x_2, \xi_2) = (\tfrac{1}{2}\partial_3^2 b(x_2, \xi_2, s(x_2, \xi_2)))^{1/4} \tag{17}$$

and where the remainders have been properly normalized to be at least formal perturbations of the second harmonic oscillator $\xi_3^2 + \hbar x_3^2$.

Since the frequency of this oscillator is $\hbar^{-\frac{1}{2}}$ in the classical picture, we are led to introduce the new semiclassical parameter

$$h = \hbar^{\frac{1}{2}}$$

and the new impulsion $\xi = \hbar^{\frac{1}{2}} \tilde{\xi}$ so that

$$\mathrm{Op}_{\hbar}^{w}\left(\xi_3^2 + \hbar x_3^2\right) = h^2 \mathrm{Op}_{h}^{w}\left(\tilde{\xi}_3^2 + x_3^2\right).$$

We therefore get the h-symbol of $\underline{\mathcal{N}}_{\hbar}^{[1]}$:

$$\underline{\mathsf{N}}_h^{[1]} = h^2 \nu^2(x_2, h\tilde{\xi}_2)(\tilde{\xi}_3^2 + x_3^2) + h^2 b(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2)) + \text{remainders.}$$

We can again perform a Birkhoff analysis in the space of formal series given by $\mathfrak{E} = \mathfrak{F}[\![x_3, \tilde{\xi}_3, h]\!]$ where $\mathfrak{F}$ is a space of symbols in the form $c(h, x_2, h\tilde{\xi}_2)$.
We get the new operator $\mathfrak{M}_h = \mathrm{Op}_h^w(\mathsf{M}_h)$, with

$$\begin{aligned}\mathsf{M}_h = h^2 b(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2)) + h^2 \mathcal{J}_h \nu^2(x_2, h\tilde{\xi}_2) \\ + h^2 g^\star(h, \mathcal{J}_h, x_2, h\tilde{\xi}_2) + \text{remainders},\end{aligned}$$

where

$$\mathcal{J}_h = \mathrm{Op}_h^w\left(\tilde{\xi}_3^2 + x_3^2\right)$$

and $g^\star(h, J, x_2, \xi_2)$ is of order three with respect to $(J^{\frac{1}{2}}, h^{\frac{1}{2}})$.

Motivated again by the perspective of describing the low lying eigenvalues, we replace $\mathcal{J}_h$ by h and rewrite the symbol with the old semiclassical parameter $\hbar$ to get the operator $\mathcal{M}_\hbar^{[1]} = \mathrm{Op}_h^w\left(\mathsf{M}_h^{[1]}\right) = \mathrm{Op}_\hbar^w\left(M_\hbar^{[1]}\right)$, with

$$\begin{aligned} M_\hbar^{[1]} &= \hbar b(x_2, \xi_2, s(x_2, \xi_2)) + \hbar^{\frac{3}{2}} \nu^2(x_2, \xi_2) \\ &\quad + \hbar g^\star(\hbar^{\frac{1}{2}}, \hbar^{\frac{1}{2}}, x_2, \xi_2) + \text{remainders.} \end{aligned} \tag{18}$$

Third Birkhoff form

The last step is to consider the minimum of the new *principal* symbol

$$(x_2, \xi_2) \mapsto b(x_2, \xi_2, s(x_2, \xi_2))$$

that occurs at $(0, 0)$. Up to an $\hbar^{\frac{1}{2}}$-dependent translation in the phase space and a rotation, we are essentially reduced to a standard Birkhoff normal form with respect to the third harmonic oscillator $\mathcal{K}_\hbar = \hbar^2 D_{x_2}^2 + x_2^2$.
Here one can also use more directly the spectral results for the 1D-hamiltonian.
Note that all our normal forms may be used to describe the classical dynamics of a charged particle in a confining magnetic field.

A semiclassical eigenvalue estimate

Let us state one of the consequences of our investigation. It will follow from the third normal form that we have a complete description of the spectrum below the threshold $b_0\hbar + 3\nu^2(0,0)\hbar^{\frac{3}{2}}$. This description is reminiscent of the results *à la* Bohr-Sommerfeld of Helffer-Robert (1984) [HR] and Helffer-Sjöstrand (1989) [HS2] obtained in the case of one dimensional semiclassical operators.

Main theorem

Assume that b admits a unique and non degenerate minimum at q_0. Let

$$\sigma = \frac{\mathrm{Hess}_{q_0} b\,(\mathbf{B},\mathbf{B})}{2b_0^2}\,, \quad \delta = \sqrt{\frac{\det \mathrm{Hess}_{q_0} b}{\mathrm{Hess}_{q_0} b\,(\mathbf{B},\mathbf{B})}}\,. \tag{19}$$

There exists a function $k^\star \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with arbitrarily small compact support, and $k^\star(\hbar^{\frac{1}{2}}, Z) = \mathcal{O}((\hbar + |Z|)^{\frac{3}{2}})$, such that, for all $c \in (0,3)$, the spectrum of $\mathcal{L}_{\hbar,\mathbf{A}}$ below $b_0\hbar + c\sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}}$ coincides modulo $\mathcal{O}(\hbar^\infty)$ with the spectrum of:

$$\mathcal{F}_\hbar = b_0\hbar + \sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}} - \frac{\zeta}{2\delta}\hbar^2 + \hbar\left(\frac{\delta}{2}\mathcal{K}_\hbar + k^\star(\hbar^{\frac{1}{2}}, \mathcal{K}_\hbar)\right),$$

with

$$\mathcal{K}_\hbar = \hbar^2 D_{x_2}^2 + x_2^2\,.$$

Corollary

Under the previous assumptions, for any $c \in (0,3)$,

$$\#\{m \in \mathbb{N}^*; \quad \lambda_m(\hbar) \leq \hbar b_0 + c\sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}}\} = \mathcal{O}(h^{-\frac{1}{2}}).$$

There exist $\upsilon_1, \upsilon_2 \in \mathbb{R}$ and $\hbar_0 > 0$ such that

$$\begin{aligned}\lambda_m(\hbar) = \hbar b_0 + \sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}} + \left[\delta(m - \tfrac{1}{2}) - \tfrac{\zeta}{2\delta}\right]\hbar^2 \\ +\upsilon_1(m - \tfrac{1}{2})\hbar^{\frac{5}{2}} + \upsilon_2(m - \tfrac{1}{2})^2\hbar^3 + \mathcal{O}(\hbar^{\frac{5}{2}}),\end{aligned}$$

uniformly for $\hbar \in (0, \hbar_0)$ and $m \in \mathsf{N}_{\hbar,c}$,

with

$$\mathsf{N}_{\hbar,c} := \{m \in \mathbb{N}^*; \quad \lambda_m(\hbar) \leq \hbar b_0 + c\sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}}\}.$$

An upper bound of $\lambda_m(\hbar)$ for fixed $\hbar$-independent m with remainder in $\mathcal{O}(\hbar^{\frac{9}{4}})$ was obtained in Helffer-Kordyukov (2013) through a quasimodes construction involving powers of $\hbar^{\frac{1}{4}}$. This corollary gives the most accurate description of magnetic eigenvalues in three dimensions, in a larger spectral window.

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