Spectral theory for magnetic Schrödinger operators and applications

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Denoting by $\lambda_1^N(B\mathbf{F})$ the first eigenvalue of the magnetic Neumann Laplacian on the domain, we analyze its asymptotic behavior as $B \to +\infty$ and the corresponding location of ground states.

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- to superconductivity (when the magnetic field is constant)
- and (in the Serrambi conference) to the theory of liquid crystals (when the magnetic field is of constant strength).

We refer to a book written in collaboration with Soeren Fournais for more details.

The quadratic form

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) be a bounded open set with smooth boundary, let β be a magnetic field (satisfying $\text{div}\ \beta = 0$ in the $3D$ case), and we can specify, (by adddition of $\nabla\Phi$), $\mathbf{F}$ to be a vector field such that

$$\text{div}\ \mathbf{F} = 0\,, \tag{1}$$

$$\text{curl}\ \mathbf{F} = \beta, \qquad \text{in}\ \Omega\,, \tag{2}$$

$$\mathbf{F} \cdot N = 0 \qquad \text{on}\ \partial\Omega\,, \tag{3}$$

where $N(x)$ is the unit interior normal vector to $\partial\Omega$.

Define $Q_{B\mathbf{F}}$ to be the closed quadratic form

$$W^{1,2}(\Omega) \ni u \mapsto Q_{B\mathbf{F}}(u) := \int_\Omega |(-i\nabla + B\mathbf{F})u(x)|^2 \, dx \,, \qquad (4)$$

where

$$W^{1,2}(\Omega) = H^1(\Omega; \mathbb{C}) \,,$$

is the standard Sobolev space. This leads to the Neumann problem.

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If we replace $W^{1,2}(\Omega)$ by $W_0^{1,2}(\Omega)$ we will get the Dirichlet problem which is of less interest for our applications.
If we want to discuss the two problems for comparison, we will write $Q^N_{B\mathbf{F}}$ and $Q^D_{B\mathbf{F}}$. With no indication this will be Neumann.

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and is called the magnetic Neumann condition.
When Ω is regular and bounded, the operator $\mathcal{H}^N(B\mathbf{F})$ has compact resolvent. The bottom of the spectrum

$$\lambda_1^N(B\mathbf{F}) := \inf \; \mathrm{Spec} \; \mathcal{H}^N(B\mathbf{F}) \; . \tag{5}$$

is an eigenvalue.

Dirichlet problem

One can also consider the Dirichlet realization.
The form domain is $W_0^{1,2}(\Omega)$ and the operator is $\mathcal{H}(B\mathbf{F})$ with domain $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$).
In this case, we denote the operator by $\mathcal{H}^D(B\mathbf{F})$.

Diamagnetism and application.

The diamagnetic inequality says that, for $u \in H^1_{loc}$,

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Using the Max-Min principle, this gives that

$$\lambda_1^N(B\mathbf{F}) \geq \lambda_1^N(0) = 0 \ , \ \lambda_1^D(B\mathbf{F}) \geq \lambda_1^D(0) > 0 \ . \tag{6}$$

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This is more generally true in the case when we consider the lowest eigenvalue of the Neuman or the Dirichlet realization of the Schrödinger operator

$$-\mathcal{H}(B\mathbf{F}) + V ,$$

where V is an L^∞ function called the electric potential.

We will

- describe in various cases the asymptotic behavior of the mapping $B \mapsto \lambda_1^N(B\mathbf{F})$
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The asymptotic spectral analysis starts with the analysis of various models corresponding first to constant magnetic field (or affine magnetic fields) and to particular domains : plane, half-plane, disks ....

The Model in $\mathbb{R}^2$.

$$\mathcal{H}(B\mathbf{F}) = D_{x_1}^2 + (D_{x_2} + Bx_1)^2 \ . \tag{7}$$

Here $\mathbf{F} = (0, x_1)$ and $D_{x_j} = \frac{1}{i}\partial_{x_j}$.

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$$H(\xi) = D_t^2 + (t-\xi)^2 \,, \text{ on the line } \,, \tag{8}$$

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$$\sigma(\mathcal{H}(B\mathbf{F})) = \sigma_{ess}(\mathcal{H}(B\mathbf{F})) = |B| \left(\cup_{n\in\mathbb{N}}(2n+1)\right) \ . \tag{10}$$

These are the so called Landau-levels.

The Model in $\mathbb{R}^3$.

Here we consider (with $b_{ij} = -b_{ji}$ and $\sum_{i<j} b_{ij}^2 = 1$)

$$\begin{aligned} \mathcal{H}(B\mathbf{F}) := & (D_{x_1} + B(\tfrac{1}{2} b_{12} x_2 + \tfrac{1}{2} b_{13} x_3))^2 \\ & + (D_{x_2} + B(\tfrac{1}{2} b_{23} x_3 + \tfrac{1}{2} b_{21} x_1))^2 \\ & + (D_{x_3} + B(\tfrac{1}{2} b_{31} x_1 + \tfrac{1}{2} b_{32} x_2))^2 \, . \end{aligned} \tag{11}$$

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After a rotation and a gauge transform, we arrive at

$$\mathcal{H}(B\mathbf{F}) := D_{x_1}^2 + (D_{x_2} + Bx_1)^2 + D_{x_3}^2 \tag{12}$$

whose spectrum is

$$\sigma(\mathcal{H}(B\mathbf{F})) = B[1, +\infty[\,. \tag{13}$$

The result is independent of the direction of the magnetic vector field :

$$\beta_1 = b_{23} \; , \; \beta_2 = b_{31} \; , \; \beta_3 = b_{12} \; .$$

The De Gennes Model in $\mathbb{R}^{2,+}$.

$$\mathcal{H}^N(B\mathbf{F}) = D_{x_1}^2 + (D_{x_2} + Bx_1)^2 \ , \tag{14}$$

on $\mathbb{R}^{2,+} = \{x_1 > 0\}$, with Neumann condition on $\{x_1 = 0\}$.

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on $\mathbb{R}^{2,+} = \{x_1 > 0\}$, with Neumann condition on $\{x_1 = 0\}$.
After a dilation and a partial Fourier transform, we are let to analyze the family

$$H(\xi) = D_t^2 + (t - \xi)^2 \ , \tag{15}$$

on the half-line (Neumann at 0) whose lowest eigenvalue

$$\xi \mapsto \mu(\xi)$$

admits a unique minimum at $\xi_0 > 0$.

On the variation of μ

It is useful to combine two formulas

- Feynman-Hellmann formula :

$$\mu'(\xi) = -2\int_0^{+\infty}(t-\xi)u_\xi(t)^2\,dt\ ,$$

where u_ξ is the normalized groundstate of $H(\xi)$.

- Bolley-Dauge-Helffer formula :

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- Bolley-Dauge-Helffer formula :

$$\mu'(\xi) = u_\xi(0)^2 (\xi^2 - \mu(\xi)) \,.$$

This permits to show that μ has a unique minimum, which is attained at $\xi_0 > 0$. Morever

$$\lim_{\xi \to +\infty} \mu(\xi) = 1 \,, \lim_{\xi \to -\infty} \mu(\xi) = +\infty \,.$$

Graph of μ and comparison with Dirichlet

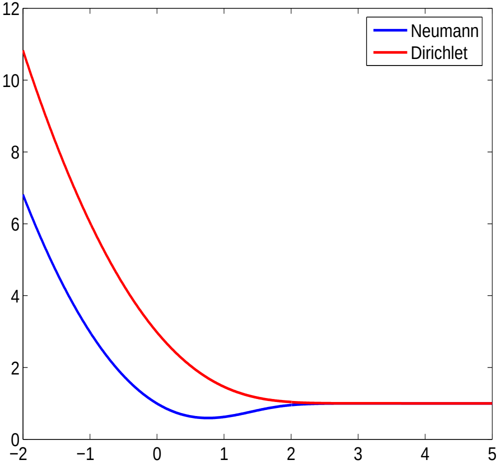

Figure: De Gennes model, computed by V. Bonnaillie-Noël

Hence two constants will play a role.
The first one is :

$$0 < \Theta_0 = \mu(\xi_0) = \inf_{\xi \in \mathbb{R}} \mu(\xi) < 1 \,. \tag{16}$$

We have

$$\xi_0^2 = \Theta_0 \sim 0,59\,. \tag{17}$$

$$\sigma(\mathcal{H}^N(B\mathbf{F})) = \Theta_0 B[1, +\infty[\,. \tag{18}$$

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The second one is :

$$\delta_0 = \frac{1}{2}\mu''(\xi_0) \,, \tag{19}$$

The Lu-Pan Models in $\mathbb{R}^{3,+}$.

The second model is quite specific of the problem in dimension 3. After a rotation respecting the half-space and a gauge transform, we look in $\{x_1 > 0\}$ at

$$\mathfrak{L}(\vartheta, -i\partial_{x_3}) = -\partial_{x_1}^2 - \partial_{x_2}^2 + (-i\partial_{x_3} + \cos\vartheta\, x_1 + \sin\vartheta\, x_2)^2 \,.$$

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By partial Fourier transform, we arrive to :

$$\mathfrak{L}(\vartheta, \tau) = -\partial_{x_1}^2 - \partial_{x_2}^2 + (\tau + \cos\vartheta\, x_1 + \sin\vartheta\, x_2)^2 \ ,$$

in $x_1 > 0$ and with Neumann condition on $x_1 = 0$.

It is enough to consider the variation with respect to $\vartheta \in [0, \frac{\pi}{2}]$.
The bottom of the spectrum is given by :

$$\varsigma(\vartheta) := \inf\ \mathrm{Spec}\ (\mathfrak{L}(\vartheta, -i\partial_{x_3})) = \inf_{\tau}(\inf\ \mathrm{Spec}\ (\mathfrak{L}(\vartheta, \tau)))\ .$$

We first observe the following lemma :

Lemma A

If $\vartheta \in]0, \frac{\pi}{2}]$, then $\text{Spec}\,(\mathfrak{L}(\vartheta, \tau))$ is independent of τ.

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This is trivial by translation in the x_2 variable.
One can then show that the function $\vartheta \mapsto \varsigma(\vartheta)$ is continuous on $]0, \frac{\pi}{2}[$.
This is based on the analysis of the essential spectrum of

$$\mathfrak{L}(\vartheta) := D_{x_1}^2 + D_{x_2}^2 + (x_1 \cos\vartheta + x_2 \sin\vartheta)^2 \,.$$

and on the fact that the bottom of the spectrum of this operator corresponds to an eigenvalue.

We then show easily that

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Finally, one shows that $\vartheta \mapsto \varsigma(\vartheta)$ is monotonically increasing and that

$$\varsigma(\vartheta) = \Theta_0 + \alpha_1 |\vartheta| + \mathcal{O}(\vartheta^2) \; , \tag{20}$$

with

$$\alpha_1 = \sqrt{\frac{\mu''(\xi_0)}{2}} \; . \tag{21}$$

Montgomery's model.

When the assumptions are not satisfied, and that the magnetic field β vanishes. Other models should be consider. An interesting example is when β vanishes along a line :

$$\mathcal{H}(B) := D_t^2 + (D_s - Bt^2)^2 \,. \tag{22}$$

This model was proposed by Montgomery in connection with subriemannian geometry but it appears also in the analysis of the dimension 3 case.

More precisely, we meet the following family (depending on ρ) of quartic oscillators :

$$D_t^2 + (t^2 - \rho)^2 \,. \tag{23}$$

Denoting by $\nu(\rho)$ the lowest eigenvalue, Pan-Kwek have shown that there exists a unique minimum of $\nu(\rho)$ leading to a new universal constant

$$\hat{\nu}_0 = \inf_{\rho \in \mathbb{R}} \nu(\rho) \,. \tag{24}$$

Hence, we get

$$\inf \sigma(\mathcal{H}(B)) = B^{\frac{2}{3}} \hat{\nu}_0 \,.$$

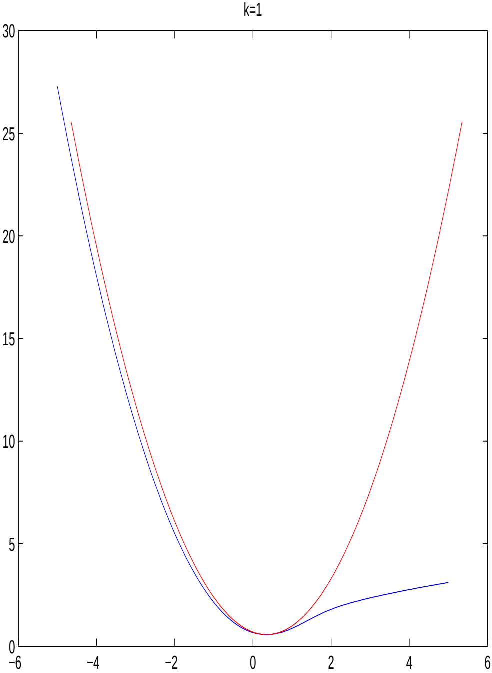

Figure: Montgomery's model

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Numerical computations confirm that the minimum is attained for a positive value of ρ :

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Numerical computations also suggest that the minimum is non degenerate.

The case of the disk : Bauman-Phillips-Tang model

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We consider the disc $D(0,R)$ in $\mathbb{R}^2$ and the case with constant magnetic field.
First we state a result on the case of Dirichlet boundary conditions (Erdös, Bolley-Helffer, Helffer-Morame).

Proposition A

As $R\sqrt{B}$ becomes large, the following asymptotics holds :

$$\lambda_1^D(B, D(0,R)) - B \sim 2^{\frac{3}{2}}\pi^{-\frac{1}{2}}B^{\frac{3}{2}}R\exp(-\frac{BR^2}{2})\,. \qquad (25)$$

As observed by Erdös there is a trick to reduce at a one dimensional problem.

For the Neumann problem, we will use the invariance by rotation and reduce the problem to the spectral analysis of a family (parametrized by $m \in \mathbb{Z}$.

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We will get (Bauman-Phillips-Tang with improvment of Fournais-Helffer) a three terms expansion showing the role of the curvature (coefficient of $B^{\frac{1}{2}}$).

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More precisely, the proof can be sketched as follows ...

We can first compare (modulo an exponentially small error) (we take a disc of radius 1) with a Dirichlet-Neumann problem in $D(0,1) \setminus D(0,\frac{1}{2})$. The Dirichlet condition is on the interior disk, the Neumann condition on the exterior disk.

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This is due to the property that groundstates are as $B \to +\infty$ localized at the boundary of the disk (this will be explained later through Agmon estimates).

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We take suitable coordinates adapted to the invariance by rotation :

$$t = 1 - r \ , \ s = \theta \ .$$

After some scaling and using Fourier series in the tangential variable, we see that the groundstate energy of the new problem $\lambda_1^{DN}(B)$ is given by

$$\lambda_1^{DN}(B) = B \inf_{m\in\mathbb{Z}} e_{\delta(m,B),B} \tag{26}$$

where

$$\delta(m,B) := m - \tfrac{B}{2} - \xi_0\sqrt{B}. \tag{27}$$

and $e_{\delta,B}$ is the lowest eigenvalue of the self-adjoint operator associated to the quadratic form

$$\tilde{q}_{\delta,B}[\phi] = \int_0^{\sqrt{B}/2} (1-\tfrac{\tau}{\sqrt{B}})^{-1}\big((\tau+\xi_0)+B^{-\frac{1}{2}}(\delta-\tfrac{\tau^2}{2})\big)^2|\phi(\tau)|^2 + (1-\tfrac{\tau}{\sqrt{B}})|\phi'(\tau)|^2\, d\tau\,. \tag{28}$$

This quadratic form is considered as a form defined on the H^1-Sobolev space associated to the space

$$L^2\big((0,\sqrt{B}/2);(1-\frac{\tau}{\sqrt{B}})d\tau\big)\,,$$

with Dirichlet condition on the interior circle.

The analysis of $e_{\delta,B}$ goes through a formal expansion in powers of $B^{-\frac{1}{2}}$ of the Hamiltonian :

$$\sum_{j\geq 0} B^{-\frac{j}{2}} \mathfrak{k}_j\,, \tag{29}$$

with

$$\begin{aligned}
\mathfrak{k}_0 &:= -\frac{d^2}{d\tau^2} + (\tau+\xi_0)^2 \quad , \\
\mathfrak{k}_1 &:= \frac{d}{d\tau} + 2(\tau+\xi_0)(\delta-\tfrac{\tau^2}{2}) + 2\tau(\tau+\xi_0)^2\,, \\
\mathfrak{k}_2 &:= \tau\frac{d}{d\tau} + (\delta-\tfrac{\tau^2}{2})^2 + 4\tau(\tau+\xi_0)(\delta-\tfrac{\tau^2}{2}) + 3\tau^2(\tau+\xi_0)^2\,.
\end{aligned} \tag{30}$$

These operators will actually be considered on $\mathbb{R}^+$ with Neumann condition at 0.

We get a corresponding expansion of $e_{\delta,B}$ in powers of $B^{-\frac{1}{2}}$. Keeping the first three terms, we obtain

$$\begin{aligned} e_{\delta,B} &= \Theta_0 - \mathcal{C}_1 B^{-\frac{1}{2}} + B^{-1} 3\mathcal{C}_1 \sqrt{\Theta_0} \left((\delta - \widehat{\delta}_0)^2 + \mathcal{C}_0 \right) \\ &+ \mathcal{O}(\delta^3 B^{-\frac{3}{2}}) + \mathcal{O}(B^{-\frac{3}{2}}) \,, \end{aligned}$$

with

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We then implement $\delta = \delta(m, B)$ and minimize over $m \in \mathbb{Z}$.

Proposition B : Asymptotics of λ_1^N for the disc

Define $\delta(m,B)$, for $m \in \mathbb{Z}$, $B > 0$, by

$$\delta(m,B) := m - \tfrac{B}{2} - \xi_0 \sqrt{B}. \tag{31}$$

Then there exist (computable) constants $\mathcal{C}_0$ and $\check{\delta}_0 \in \mathbb{R}$ such that, if

$$\widehat{\Delta}_B := \inf_{m\in\mathbb{Z}} \left|\delta(m,B) - \check{\delta}_0\right|, \tag{32}$$

then, for all $\eta > 0$,

$$\lambda_1^N(B\mathbf{F}) = \Theta_0 B - \mathcal{C}_1\sqrt{B} + 3\mathcal{C}_1\sqrt{\Theta_0}\big(\widehat{\Delta}_B^2 + \mathcal{C}_0\big) + \mathcal{O}(B^{\eta-\frac{1}{2}})\,. \tag{33}$$

Note that the third term in the expansion (which is due to Fournais-Helffer) is bounded and oscillatory.

Rough estimates for general magnetic Laplacians (Lu-Pan)

We introduce

$$b = \inf_{x \in \overline{\Omega}} |\beta(x)| \,, \tag{34}$$

$$b' = \inf_{x \in \partial\Omega} |\beta(x)| \,, \tag{35}$$

and, for $d = 2$,

$$b'_2 = \Theta_0 \inf_{x \in \partial\Omega} |\beta(x)| \,, \tag{36}$$

and, for $d = 3$,

$$b'_3 = \inf_{x \in \partial\Omega} |\beta(x)| \varsigma(\theta(x)) \tag{37}$$

Theorem 1 : Rough asymptotics

$$\lambda_1^N(B\mathbf{F},\Omega) = B\min(b, b'_d) + o(B)\,, \tag{38}$$

$$\lambda_1^D(B\mathbf{F},\Omega) = Bb + o(B) \tag{39}$$

Particular case, if $|\beta(x)| = 1$, then

$$\min(b, b'_d) = b'_d = \Theta_0\ . \tag{40}$$

Lower bounds in the 2D case

The first (trivial by integration by parts) estimate is that, when $\beta(x) \geq 0$, then we have, for any $\phi \in C_0^\infty(\Omega)$,

$$Q_B(\phi) \geq B \int \beta(x)|\phi(x)|^2 \, dx. \tag{41}$$

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Gaussians centered at a point where $|\beta|$ is minimum can be used for getting the upper bound.

For Neumann, we should introduce a partition of unity.

A partition of unity

Let $0 \leq \rho \leq 1$. Then there exists C s.t. $\forall R_0 > 0$, we can find a partition of unity χ_j^B satisfying in Ω,

$$\sum_j |\chi_j^B|^2 = 1\,, \tag{42}$$

$$\sum_j |\nabla \chi_j^B|^2 \leq C\, R_0^{-2}\, B^{2\rho}\,, \tag{43}$$

and

$$\text{supp}\,(\chi_j^B) \subset Q_j = D(z_j, R_0\, B^{-\rho})\,, \tag{44}$$

where $D(c, r)$ denotes the open disc of center c and radius r.
Moreover, we can add the property that :

$$\text{either}\ \ \text{supp}\ \chi_j \cap \partial\Omega = \emptyset\,,\ \text{either}\ z_j \in \partial\Omega\,. \tag{45}$$

According to the two alternatives in (45), we can decompose the sum in (42) in the form :

$$\sum = \sum_{int} + \sum_{bnd},$$

where 'int' is in reference to the j's such that $z_j \in \Omega$ and 'bnd' is in reference to the j's such that $z_j \in \partial\Omega$.

We now implement this partition of unity in the following way :

$$Q(u) = \sum_j Q(\chi_j^B u) - \sum_j \| \, |\nabla \chi_j^B| \, u \, \|^2 \, , \; \forall u \in H^1(\Omega) \, . \qquad (46)$$

Here $Q = Q^N_{B\mathbf{F},\Omega}$ denotes the magnetic quadratic form. This decomposition is some time called (IMS)-formula.

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Here $Q = Q^N_{B\mathbf{F},\Omega}$ denotes the magnetic quadratic form. This decomposition is some time called (IMS)-formula.
We can rewrite the right hand side of (46) as the sum of three (types of) terms.

$$Q(u) = \sum_{int} Q(\chi_j^B u) + \sum_{bnd} Q(\chi_j^B u) - \sum_j \| \, |\nabla \chi_j^B| \, u \, \|^2 \,, \; \forall u \in H^1(\Omega) \,. \tag{47}$$

For the last term on the right side of (47), we get using (43) :

$$\sum_j \| \, |\nabla \chi_j^B| \, u \, \|^2 \leq C \, B^{2\rho} \, R_0^{-2} \, \|u\|^2 \, . \tag{48}$$

This measures the price to pay when using a fine partition of unity :
If ρ is large, which seems the best for controlling the comparison with the models, the error due to this localization will be bad and of order $\mathcal{O}(B^{2\rho})$.

We shall see later what could be the best compromise for an optimal choice of ρ or of R_0 for our various problems (note that the play with R_0 large will be only interesting when $\rho = \frac{1}{2}$).

The first term to the right in (47) can be estimated from below using the basic estimate. The support of $\chi_j^B u$ is indeed contained in Ω. So we have :

$$\sum_{int} Q(\chi_j u) \geq B \sum_{int} \int \beta(x) |\chi_j^B u|^2 \, dx \,. \tag{49}$$

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The second term in the right hand side of (47) is the more delicate and corresponds to the specificity of the Neumann problem. We have to find a lower bound for $Q(\chi_j^B u)$ for some j such that $z_j \in \partial\Omega$. We emphasize that z_j depends on B, so we have to be careful in the control of the uniformity.

We use the standard boundary coordinates (s,t). Let $z \in \partial\Omega$ and consider functions u supported in the small disc $D(z, B^{-\rho})$. We now choose a convenient gauge. Define

$$\tilde{A}_1 := -\int_0^t (1-t'k(s))\tilde{\beta}(s,t')\,dt', \qquad \tilde{A}_2 := 0.$$

With a suitable gauge change, i.e. with the substitution $\tilde{v} := e^{iB\phi}v$ for some function ϕ, we have for $\text{supp}\, u \subset D(z, R_0 B^{-\rho})$,

$$\int |(-i\nabla + B\mathbf{F})u|^2\,dx = \int (1-tk(s))^{-1}\left|(-i\partial_s + B\tilde{A}_1)\tilde{v}\right|^2 + (1-tk(s))|\partial_t \tilde{v}|^2\,dsdt\,. \tag{50}$$

Define

$$
\begin{aligned}
k_0 &:= k(0), \\
\overline{A}(s,t) &:= -\tilde{\beta}(0,0)\big(t - \frac{1}{2}t^2 k(0)\big), \\
\Delta k(s) &:= k(s) - k(0), \\
\tilde{b}(s,t) &:= \big(1 - tk(s)\big)\tilde{\beta}(s,t) - \big(1 - tk(0)\big)\tilde{\beta}(0,0), \\
\tilde{a}_1(s,t) &:= -\int_0^t \tilde{b}(s,t')\,dt'.
\end{aligned}
$$

Then we have the estimates in the support of $\tilde{v}$,

$$
\begin{aligned}
|\Delta k| &\le CR_0 B^{-\rho}\,, \\
|b(s,t)| &\le CR_0 B^{-\rho}\,, \\
|\tilde{a}_1(s,t)| &\le CR_0 B^{-\rho} t\,.
\end{aligned}
$$

Of course, since $t = \mathcal{O}(B^{-\rho})$, one can be more specific about this last estimate, but we keep the t dependence for later use.
Let B be so large that $2^{-1} \leq (1 - tk(s)) \leq 2$ on $\operatorname{supp} \tilde{v}$. Then we can make the following comparison between (50) and the similar constant field, constant curvature formula :

$$\begin{aligned} &\int |(-i\nabla + B\mathbf{F})u|^2 \, dx \\ &\quad \geq (1-\eta) \int (1 - tk_0)^{-1} |(-i\partial_s + B\overline{A})\tilde{v}|^2 + (1 - tk_0)|\partial_t \tilde{v}|^2 \, dsdt \\ &\qquad - C \int t\Delta k \left\{ \left|(-i\partial_s + B\tilde{A}_1)\tilde{v}\right|^2 + |\partial_t \tilde{v}|^2 \right\} dsdt \\ &\qquad - \eta^{-1} \int (1 - tk_0)^{-1} B^2 \tilde{a}_1^2 |\tilde{v}|^2 \, dsdt, \end{aligned} \tag{51}$$

for any $0 < \eta < 2^{-1}$ and any u with $\operatorname{supp} u \subset B(z, R_0 B^{-\rho})$.

The first term on the right is the quadratic form corresponding to constant curvature and constant magnetic field, so we can estimate

$$\int (1 - tk_0)^{-1} |(-i\partial_s + B\overline{A})\tilde{v}|^2 + (1 - tk_0)|\partial_t \tilde{v}|^2 \, dsdt \geq \left(\Theta_0 B\beta(z) - \mathcal{C}_1 k\sqrt{B\beta(z)} - C\right)\|\tilde{v}\|_2^2 \,. \quad (52)$$

using the result for the disk. Notice that this estimate is uniform, since the boundary curvature is uniformly bounded.

The second term on the right is estimated by

$$C \int t\Delta k\{\big|(-i\partial_s + B\tilde{A}_1)\tilde{v}\big|^2 + |\partial_t\tilde{v}|^2\}\, dsdt$$
$$\leq C\widehat{C}B^{-2\rho} \int |(-i\nabla + B\mathbf{F})u|^2\, dx\,, \quad (53)$$

and consequently involves the left hand side. Here we use the property that $0 \leq t \leq CB^{-\rho}$ on $\text{supp }\tilde{v}$.

The third term is estimated by

$$\eta^{-1}\int(1-tk_0)^{-1}B^2\tilde{a}_1^2|\tilde{v}|^2\,dsdt \leq \widetilde{C}\eta^{-1}B^{2-4\rho}\,\|\widetilde{v}\|^2\,. \qquad (54)$$

To get a first not optimal estimate, we choose $R_0 = 1$, $\eta = B^{\frac{1}{2}-2\rho}$, $\rho = \frac{3}{8}$, and conclude from (51) and (52)-(54), that

$$\int|(-i\nabla + B\mathbf{F})u|^2\,dx \geq \left(\Theta_0 B\beta(z) - CB^{\frac{3}{4}}\right)\|u\|_2^2, \qquad (55)$$

for all u such that $\operatorname{supp} u \subset B(z, B^{-\rho})$.

Combining this with (46), (48) and (49), we find the lower bound. More precisely, we find constants C and B_0 such that, $\forall u \in H^1(\Omega)$ and $\forall B \geq B_0$,

$$\begin{aligned} Q(u) \;&\geq B \textstyle\sum_{int} \int \beta(x) |\chi_j^B u|^2 \, dx \\ &+ \Theta_0 B \textstyle\sum_{bnd} \int \beta(z_j) |\chi_j^B u|^2 \, dx - C B^{\frac{3}{4}} \textstyle\sum_j \int |\chi_j^B u|^2 \, dx \,. \end{aligned}$$

Upon replacing $\beta(z_j)$ by $\beta(x)$ in each of the terms in the boundary sum, we have actually proved the following.

Proposition a

There exist positive constants C and B_0 such that, with

$$U_\beta(x) := \begin{cases} B\beta(x), & d(x,\partial\Omega) \geq B^{-\frac{3}{8}}, \\ \Theta_0 B\beta(x), & d(x,\partial\Omega) < B^{-\frac{3}{8}}, \end{cases} \tag{56}$$

we have

$$\int_\Omega |(-i\nabla + B\mathbf{F})u|^2\,dx \geq \int_\Omega (U_\beta(x) - CB^{\frac{3}{4}})|u(x)|^2\,dx\,, \tag{57}$$

for all $u \in H^1(\Omega)$ and all $B \geq B_0$.

In particular, we get the following version of the lower bound.

Proposition b

There exist positive constants C and B_0 such that, for all $B \geq B_0$:

$$\lambda_1^N(B\mathbf{F}) \geq \big(\min(b, \Theta_0 b')\big)\, B - C\, B^{\frac{3}{4}}\,. \tag{58}$$

We can also make the choice $\rho = \frac{1}{2}$, $\eta = B^{-\frac{1}{8}}$ and R_0 large in (51). This gives an estimate which may look weaker than Proposition a, but which will be more efficient in the study of decay. The reason is that the boundary zone now has the right length scale, namely $B^{-\frac{1}{2}}$. The result analogous to Proposition a is :

Proposition c

There exist $C, B_0 > 0$ and, for all $R_0 > 0$, there exists $C(R_0)$ such that with

$$U_\beta^{(2)}(x) := \left\{ \begin{cases} B\beta(x), & d(x, \partial\Omega) \geq R_0 B^{-\frac{1}{2}}, \\ BC(R_0)\beta(x), & d(x, \partial\Omega) \leq R_0 B^{-\frac{1}{2}}, \end{cases} \right. \tag{59}$$

we have,

$$\int_\Omega |(-i\nabla + B\mathbf{F})u|^2 \, dx \geq \int_\Omega \left(U_\beta^{(2)}(x) - C\frac{B}{R_0^2} \right) |u(x)|^2 \, dx, \tag{60}$$

for all $u \in H^1(\Omega)$ and all $B \geq B_0$.

Consequences.

The consequences are that a ground state is localized as $B \to +\infty$,

- for Dirichlet, at the points of $\overline{\Omega}$ where $|\beta(x)|$ is minimum,
- for Neumann,
  - if $b < b'_d$, at the points of Ω where $|\beta(x)|$ is minimum (no difference with Dirichlet),
  - if $b > b'_d$ at the points of $\partial\Omega$ where $|\beta(x)|\varsigma(\theta(x)$ is minimum.

In particular, if $|\beta(x)| = 1$, we are, for Neumann, in the second case, hence the groundstate is localized at the boundary.

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Moreover, when $d = 3$, the groundstate is localized at the point where $\beta(x)$ is tangent to the boundary.

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Moreover, when $d = 3$, the groundstate is localized at the point where $\beta(x)$ is tangent to the boundary.

All the results of localization are obtained through Agmon estimates (as Helffer-Sjöstrand, Simon have done in the eighties for $-h^2\Delta + V$) .

Two-terms asymptotics in the case of a variable magnetic field

The interior case

If

$$b < \inf_{x \in \partial\Omega} |\beta(x)| \;\text{ for Dirichlet}$$

or if

$$b < b' \;\text{ for Neumann,}$$

the asymptotics of $\lambda_1^N(B\mathbf{F})$ and $\lambda_1^D(\mathbf{F}))$ are the same (modulo an exponentially small error).

If we assume in addition

Assumption A

- There exists a unique point $x_{min} \in \Omega$ such that $b = |\beta(x_{min})|$.
- This minimum is non degenerate.

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- This minimum is non degenerate.

we get in $2D$ (Helffer-Morame)

Theorem 2

$$\lambda_1^{D \text{ or } N}(B\mathbf{F}) = bB + \Theta_{\frac{1}{2}} B^{\frac{1}{2}} + o(B^{\frac{1}{2}}) \,. \tag{61}$$

where $\Theta_{\frac{1}{2}}$ is computed from the Hessian of β at the minimum.

The problem is still open (Helffer-Kordyukov, work in progress) in the $3D$ case.

There are also many results for the case when $b = 0$ (Montgomery, Helffer-Mohamed, Pan-Kwek, Helffer-Kordyukov ....).
The ground state is localized near the minimum.
When more than a minimum, tunneling can occur (Helffer-Sjöstrand, Helffer-Kordyukov).

Main results for Neumann with constant magnetic fields

The 2D case

We recall from the previous result that in a disk of radius R, we have

$$\lambda_1^N(B\mathbf{F}) = \Theta_0 B - \frac{1}{R}\mathcal{C}_1\sqrt{B} + \mathcal{O}(1)\,. \tag{62}$$

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In the two dimensional case, it was proved by DelPino-Felmer-Sternberg–Lu-Pan–Helffer-Morame the following

Theorem 3

$$\lambda_1(B) = \Theta_0 B - \mathcal{C}_1 k_0 B^{\frac{1}{2}} + o(B^{\frac{1}{2}})\,, \tag{63}$$

where k_0 is the maximal curvature of the boundary.

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where k_0 is the maximal curvature of the boundary.

Moreover (Fournais-Helffer) a complete expansion of λ_1^N exists if the points of maximal curvature are non degenerate.

The $3D$ case

We will work under the following geometric assumption

G-Assumptions

1. On the set of boundary points where β is tangent to $\partial\Omega$, i.e. on

$$\Gamma_\beta := \{x \in \partial\Omega \,\big|\, \beta \cdot N(x) = 0\}, \tag{64}$$

$$d^T(\beta \cdot N)(x) \neq 0 \, , \; \forall x \in \Gamma_\beta \, . \tag{65}$$

2. The set of points where β is tangent to Γ_β is finite.

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2. The set of points where β is tangent to Γ_β is finite.

These assumptions are rather generic, they imply that Γ_β is is a regular submanifold of $\partial\Omega$ and are for instance satisfied for ellipsoids.

Two terms asymptotics

We will need a two-term asymptotics of $\lambda_1^N(B\mathbf{F})$ (due to Helffer-Morame- Pan).

Theorem 4

If Ω and β satisfy **G-Assumptions**, then as $B \to +\infty$

$$\lambda_1^N(B\mathbf{F}) = \Theta_0 B + \widehat{\gamma}_0 B^{\frac{2}{3}} + \mathcal{O}(B^{\frac{2}{3}-\eta}),$$

for some $\eta > 0$.

In previous formula $\widehat{\gamma}_0$ is defined by

$$\widehat{\gamma}_0 := \inf_{x\in\Gamma_\beta} \widetilde{\gamma}_0(x), \tag{66}$$

where

$$\widetilde{\gamma}_0(x) := 2^{-2/3}\widehat{\nu}_0\delta_0^{1/3}|k_n(x)|^{2/3}\left(\delta_0+(1-\delta_0)|T(x)\cdot\beta|^2\right)^{1/3}. \tag{67}$$

Here $T(x)$ is the oriented, unit tangent vector to Γ_β at the point x and

$$k_n(x) = |d^T(\beta\cdot N)(x)| .$$

Monotonicity

$2D$ case

Two recent results obtained in collaboration with S. Fournais are the

Theorem 5

Let $\Omega \subset \mathbb{R}^2$ and $\beta = 1$. Then there exists B_0 such that $B \mapsto \lambda_1^N(B\mathbf{F})$ is monotonically increasing.

The proof results of the separate analysis of two cases :

- The case of the disk (results of the previous analysis),
- The case when the curvature is non constant (see below). When the curvature is not constant a groundstate is localized at the boundary but away from some interval of the boundary.

$3D$ case

Theorem 6

Let $\Omega \subset \mathbb{R}^3$ and β satisfying G- Assumptions
Let $\{\Gamma_1, \ldots, \Gamma_n\}$ be the collection of disjoint smooth curves making up Γ_β. We assume that, for all j there exists $x_j \in \Gamma_j$ such that $\widetilde{\gamma}_0(x_j) > \widehat{\gamma}_0$.
Then the directional derivatives
$(\lambda_{1,\pm}^N)' := \lim_{t\to 0\pm} \frac{\lambda_1^N(B+t)-\lambda_1^N(B)}{t}$,
exist.
Moreover

$$\lim_{B\to\infty} (\lambda_{1,+}^N)'(B) = \lim_{B\to\infty} (\lambda_{1,-}^N)'(B) = \Theta_0\,. \qquad (68)$$

We now sketch how one can derive the monotonicity result from the known asymptotics of the groundstate energy and localization estimates for the groundstate itself.

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Proof of Theorem 6

For simplicity, we assume that Γ_β is connected. Applying Kato's analytic perturbation theory to $\mathcal{H}(B)$ gives the first part.

Let $s_0 \in \Gamma$ be a point with $\widetilde{\gamma}(s_0) > \widehat{\gamma}_0$. Let $\widehat{\mathbf{A}}$ be the vector potential which is gauge equivalent to $\mathbf{A}$ to be chosen later.

Let $\widehat{Q}_B$ the quadratic form

$$W^{1,2}(\Omega) \ni u \mapsto \widehat{Q}_B(u) = \int_\Omega |-i\nabla u + B\widehat{\mathbf{A}}u|^2 dx\,,$$

and $\widehat{\mathcal{H}}(B)$ be the associated operator.
Then $\widehat{\mathcal{H}}(B)$ and $\mathcal{H}(B)$ are unitarily equivalent:
$\widehat{\mathcal{H}}(B) = e^{iB\phi}\mathcal{H}(B)e^{-iB\phi}$, for some ϕ independent of B.
With $\psi_1^+(\,\cdot\,;\beta)$ being a suitable choice of normalized groundstate of $\widehat{\mathcal{H}}(B)$, we get (by analytic perturbation theory applied to $\mathcal{H}(B)$ and the explicit relation between $\widehat{\mathcal{H}}(B)$ and $\mathcal{H}(B)$,

$$\begin{aligned} &\lambda'_{1,+}(B) \\ &= \langle \widehat{\mathbf{A}}\psi_1^+(\,\cdot\,;B)\,,\, p_{B\widehat{\mathbf{A}}}\psi_1^+(\,\cdot\,;B)\rangle \\ &\quad + \langle p_{B\widehat{\mathbf{A}}}\psi_1^+(\,\cdot\,;B)\,,\, \widehat{\mathbf{A}}\psi_1^+(\,\cdot\,;B)\rangle\,. \end{aligned} \tag{69}$$

We now obtain for any $\epsilon > 0$,

$$\lambda'_{1,+}(B) = \frac{\widehat{Q}_{B+\epsilon}(\psi_1^+(\,\cdot\,;B)) - \widehat{Q}_B(\psi_1^+(\,\cdot\,;B))}{\epsilon} - \epsilon \int_\Omega |\widehat{\mathbf{A}}|^2 \, |\psi_1^+(x;B)|^2 \, dx \tag{70}$$

$$\geq \frac{\lambda_1(B+\epsilon) - \lambda_1(B)}{\epsilon} - \epsilon \int_\Omega |\widehat{\mathbf{A}}|^2 \, |\psi_1^+(x;B)|^2 \, dx \,. \tag{71}$$

We choose $\epsilon := MB^{\frac{2}{3}-\eta}$, with η as before and $M > 0$ (to be taken arbitrarily large in the end). Then, (70) becomes

$$\begin{aligned} \lambda'_{1,+}(B) \quad & \geq \Theta_0 + \widehat{\gamma}_0 B^{-1/3} \tfrac{(1+\epsilon/B)^{2/3}-1}{\epsilon/B} \\ & - CM^{-1} - \epsilon \textstyle\int_\Omega |\widehat{\mathbf{A}}|^2 \, |\psi_1^+(x;B)|^2 \, dx \,, \end{aligned} \tag{72}$$

for some constant C independent of M, B.

If we can prove that we can find $\widehat{A}$ such that

$$B^{\frac{2}{3}} \int_{\Omega} |\widehat{\mathbf{A}}(x)|^2 \, |\psi_1^+(x;B)|^2 \, dx \leq C, \tag{73}$$

for some constant C independent of B, then we can take the limit $B \to \infty$ in (72) and obtain

$$\liminf_{B \to \infty} \lambda'_{1,+}(B) \geq \Theta_0 - CM^{-1}. \tag{74}$$

Since M was arbitrary this implies the lower bound for $\lambda'_{1,+}(B)$.
Applying the same argument to the derivative from the left, $\lambda'_{1,-}(B)$, we get (the inequality gets turned since $b < 0$)

$$\limsup_{B\to\infty} \lambda'_{1,-}(B) \leq \Theta_0. \tag{75}$$

Since, by perturbation theory, $\lambda'_{1,+}(B) \leq \lambda'_{1,-}(B)$ for all B, we get (68).
Thus it remains to prove (73).

We have

$$\begin{aligned}&\int_\Omega |\widehat{\mathbf{A}}|^2\, |\psi_1^+(x;B)|^2\, dx\\ &\leq C \int_{\Omega\setminus\widehat{\Omega}\left(\epsilon,s_0\right)} (t^2+r^4)|\psi_1^+(x;B)|^2\, dx\\ &\quad +\|\widehat{\mathbf{A}}\|_\infty^2 \int_{\widehat{\Omega}\left(\epsilon_0,s_0\right)} |\psi_1^+(x;B)|^2\, dx\,,\end{aligned}$$

where $\widehat{\Omega}\big(\epsilon_0, s_0\big)$ is a small neighborhood of s_0, $t=0$ defines $\partial\Omega$ and $r=0$ defines Γ_β.

We can indeed choose $\widehat{A}$ with $\text{curl}\,\widehat{A} = \text{curl}\,\mathbf{F}$ such that

$$|\widehat{\mathbf{A}}|^2 \leq C(t^2+r^4)\,,$$

in a neighborhood of Γ_β, but outside of $\Omega\big(\epsilon_0, s_0\big)$.

So it remains to find the existence of a constant $C > 0$ and, for any $N > 0$ $C_N > 0$, such that :

$$\int_{\Omega \setminus \widehat{\Omega}\left(\epsilon, s_0\right)} (t^2 + r^4) |\psi_1^+(x; B)|^2 \, dx \leq C \, B^{-1} \,, \tag{76}$$

and

$$\int_{\widehat{\Omega}\left(\epsilon, s_0\right)} |\psi_1^+(x; B)|^2 \, dx \leq C_N \, B^{-N} \,, \tag{77}$$

which will imply the needed estimate (73).

The proof involves various estimates on the localization of a ground state. They are all based on the following Agmon's identity for the Schrödinger operator $P_{BA,B^{2\sigma}V} = -\nabla_{BA} + B^{2\sigma}V$.

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Proposition Ag

Let Ω be a bounded regular open domain, $V \in C^0(\overline{\Omega};\mathbb{R})$, $A \in C^0(\overline{\Omega};\mathbb{R}^m)$ and ϕ a real valued lipschitzian function on $\overline{\Omega}$. Then, $\forall u \in C^2(\overline{\Omega};\mathbb{C})$ satisfying

- either the Dirichlet condition $u\big|_{\partial\Omega} = 0$,
- or the magnetic Neumann condition $N \cdot (\nabla u + iBAu)\big|_{\partial\Omega} = 0$,

we have

$$\int_\Omega |\nabla_{BA}(e^{B^\sigma\phi}u)|^2\,dx + B^{2\sigma}\int_\Omega (V - |\nabla\phi|^2)e^{2B^\sigma\phi}|u|^2\,dx = \Re\left(\int_\Omega e^{2B^\sigma\phi}\,\overline{(P_{BA,B^{2\sigma}V}u)(x)}\cdot u(x)\,dx\right). \quad (78)$$

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In our case $V = 0$, but some effective electric potential will reappear through a lower bound of the term

$$\int_\Omega \left|\nabla_{BA}\left(e^{B^\sigma \phi} u\right)\right|^2 dx$$

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of the type we have proven before.

It remains then to make a clever choice of ϕ which could be a multiple of the distance of the boundary, or some tangential Agmon distance inside the boundary.

Localization estimates at the boundary

We start by recalling the decay of a groundstate in the direction normal to the boundary. We use the notation

$$t(x) := \text{ dist } (x, \partial\Omega). \tag{79}$$

Now, if $\varphi \in C_0^\infty(\Omega)$, i.e. has support away from the boundary, we have already observe that

$$Q_B(\varphi) \geq B\|\varphi\|_2^2. \tag{80}$$

It is a consequence of this elementary inequality (and the fact that $\Theta_0 < 1$) that groundstates are exponentially localized near the boundary.

Agmon estimates.

Theorem 7

There exist positive constants C, a_1, B_0 such that

$$\int_\Omega e^{2a_1 B^{1/2} t(x)} \Big(|\psi_B(x)|^2 + B^{-1} |(-i\nabla + B\mathbf{F})\psi_B(x)|^2 \Big)\, dx \leq C\, \|\psi_B\|_2^2, \tag{81}$$

for all $B \geq B_0$, and all groundstates ψ_B of the operator $\mathcal{H}(B)$.

We will mainly use this localization result in the following form.

Corollary 8

For all $n \in \mathbb{N}$, there exists $C_n > 0$ and $B_n \geq 0$ such that, $\forall B \geq B_n$,

$$\int t(x)^n |\psi_B(x)|^2 \, dx \leq C_n \, B^{-n/2} \|\psi_B\|_2^2 \, .$$

Localization inside the boundary

We work in tubular neighborhoods of the boundary as follows. For $\epsilon > 0$, define

$$B(\partial\Omega, \epsilon) = \{x \in \Omega \,:\, t(x) \leq \epsilon\}. \tag{82}$$

For sufficiently small ϵ_0 we have that, for all $x \in B(\partial\Omega, 2\epsilon_0)$, there exists a unique point $y(x) \in \partial\Omega$ such that $t(x) = \text{ dist }(x, y(x))$. Define, for $y \in \partial\Omega$, the function $\vartheta(y) \in [-\pi/2, \pi/2]$ by

$$\sin\vartheta(y) := -\beta \cdot N(y). \tag{83}$$

We extend ϑ to the tubular neighborhood $B(\partial\Omega, 2\epsilon_0)$ by $\vartheta(x) := \vartheta(y(x))$.

In order to obtain localization estimates in the variable normal to Γ, we use the following operator inequality (due to Helffer-Morame).

Theorem 9

Let B_0 be chosen such that $B_0^{-3/8} = \epsilon_0$ and define, for $B \geq B_0, C > 0$ and $x \in \Omega$,

$$W_B(x) := \begin{cases} B - CB^{1/4}, & t(x) \geq 2B^{-3/8}, \\ B\varsigma(\vartheta(x)) - CB^{1/4}, & t(x) < 2B^{-3/8}. \end{cases} \tag{84}$$

Then, for C large enough

$$\mathcal{H}(B) \geq W_B, \tag{85}$$

(in the sense of quadratic forms) for all $B \geq B_0$.

We use this energy estimate to prove Agmon type estimates on the boundary.

Theorem 10

Suppose that $\Omega \subset \mathbb{R}^3$ and β satisfy G-Assumptions. Define for $x \in \partial\Omega$,

$$d_\Gamma(x) := \text{ dist } (x, \Gamma),$$

and extend d_Γ to a tubular neighborhood of the boundary by $d_\Gamma(x) := d_\Gamma(y(x))$.

Then there exist constants $C, a_2 > 0, B_0 \geq 0$, such that

$$\int_{B(\partial\Omega,\epsilon_0)} e^{2a_2 B^{1/2} d_\Gamma(x)^{3/2}} |\psi_B(x)|^2 \, dx \leq C \|\psi_B\|_2^2, \tag{86}$$

for all $B \geq B_0$ and all groundstates ψ_B of $\mathcal{H}(B)$.

We have the following easy consequence.

Corollary 11

Suppose that $\Omega \subset \mathbb{R}^3$ satisfies G-Assumptions relatively to β.
Then for all $n \in \mathbb{N}$ there exists $C_n > 0$ such that

$$\int_{B(\partial\Omega,\epsilon_0)} d_\Gamma(x)^n |\psi_B(x)|^2 \, dx \leq C_n B^{-n/3} \|\psi_B\|_2^2, \qquad (87)$$

for all $B > 0$ and all groundstates ψ_B of $\mathcal{H}(B)$.

Consider now the set $\mathcal{M}_\Gamma \subset \Gamma$ where the function $\widetilde{\gamma}_0$ is minimal,

$$\mathcal{M}_\Gamma := \{x \in \Gamma \, : \, \widetilde{\gamma}_0 = \widehat{\gamma}_0\}. \tag{88}$$

For simplicity, we asume that Γ is connected.

Theorem 12

Suppose that $\Omega \subset \mathbb{R}^3$ satisfies G-Assumptions relatively to β and let $\delta > 0$. Then for all $N > 0$ there exists C_N such that if ψ_B is a groundstate of $\mathcal{H}(B)$, then

$$\int_{\{x\in\Omega : \ \mathrm{dist}\ (x,\mathcal{M}_\Gamma)\geq\delta\}} |\psi_B(x)|^2 \, dx \leq C_N B^{-N}, \tag{89}$$

for all $B > 0$.

Ginzburg-Landau functional 2D

The Ginzburg-Landau functional is given, with

$$\beta = \text{ curl } \mathbf{F} = 1 \ ,$$

by

$$\mathcal{E}_{\kappa,\sigma}[\psi, \mathbf{A}] =$$
$$\int_\Omega \left\{ |\nabla_{\kappa\sigma\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \right\} dx$$
$$+\kappa^2\sigma^2 \int_\Omega |\text{ curl } \mathbf{A} - 1|^2 \, dx \ ,$$

with

- Ω simply connected,
- $(\psi, A) \in W^{1,2}(\Omega; \mathbb{C}) \times W^{1,2}_{\text{div}}(\Omega; \mathbb{R}^2)$,
- $\nabla_A = (\nabla + iA)$,
- $W^{1,2}_{\text{div}}(\Omega; \mathbb{R}^2) = \{A \in W^{1,2}(\Omega, \mathbb{R}^2) \,|\, \text{ div } A = 0 \,\}$.

Ginzburg-Landau functional 3D

The Ginzburg-Landau functional is given, with

$$\beta = \text{ curl } \mathbf{F} \ ,$$

by

$$\mathcal{E}_{\kappa,\sigma}[\psi, \mathbf{A}] = \int_\Omega \left\{ |\nabla_{\kappa\sigma\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \right\} dx + \kappa^2\sigma^2 \int_{\mathbb{R}^3} |\text{ curl } \mathbf{A} - \beta|^2 \, dx \ ,$$

with

- Ω simply connected,
- $(\psi, \mathbf{A}) \in W^{1,2}(\Omega; \mathbb{C}) \times \dot{W}^{1,2}_{\text{div },\mathbf{F}}(\mathbb{R}^3; \mathbb{R}^3)$,
- $\beta = (0, 0, 1)$,
- $\nabla_{\mathbf{A}} = (\nabla + i\mathbf{A})$,
- $\dot{W}^{1,2}_{\text{div },\mathbf{F}}(\mathbb{R}^3; \mathbb{R}^3) = \{A \mid \text{ div } A = 0 \ , \ A - \mathbf{F} \in \dot{W}^{1,2}(\mathbb{R}^3, \mathbb{R}^3)\}$.

Claim : Minimizers exist.

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As Ω is bounded, the existence of a minimizer is rather standard, so the infimum is actually a minimum. However, in general one does not expect uniqueness of minimizers. A minimizer should satisfy the Euler-Lagrange equation, which is called in this context the Ginzburg-Landau system.

This equation reads

$$\left.\begin{aligned} -\nabla_{\kappa\sigma A}^2\psi = \kappa^2(1-|\psi|^2)\psi\,, \\ \text{curl}\ \big(\ \text{curl}\ A - 1\big) = -\tfrac{1}{\kappa\sigma}\Im\Big(\overline{\psi}\,\nabla_{\kappa\sigma A}\psi\Big) \end{aligned}\right\} \quad \text{in} \quad \Omega\,, \qquad (90a)$$

$$\left.\begin{aligned} \nu\cdot\nabla_{\kappa\sigma A}\psi = 0\,, \\ \text{curl}\ A\ -\ 1 = 0 \end{aligned}\right\} \quad \text{on} \quad \partial\Omega\,. \qquad (90b)$$

Notice that the weak formulation of (90) is

$$\Re \int_\Omega \left(\overline{\nabla_{\kappa\sigma A}\phi} \cdot \nabla_{\kappa\sigma A}\psi - \kappa^2(1-|\psi|^2)\overline{\phi}\psi\right) \, dx = 0 \,, \tag{91a}$$

$$\int_\Omega (\operatorname{curl} \alpha)(\operatorname{curl} A - 1)\, dx = -\frac{1}{\kappa\sigma}\int_\Omega \Im\left(\overline{\psi}\, \nabla_{\kappa\sigma A}\psi\right) \alpha \, dx \,, \tag{91b}$$

for all $(\phi, \alpha) \in H^1(\Omega) \times H^1(\Omega, \mathbb{R}^2)$.

The analysis of the system (90) can be performed by PDE techniques. We note that this system is non-linear, that $H^1(\Omega)$ is, when Ω is bounded and regular in $\mathbb{R}^2$, compactly imbedded in $L^p(\Omega)$ for all $p \in [1, +\infty[$, and that, if $\operatorname{div} A = 0$, $\operatorname{curl}{}^2 A = (-\Delta A_1, -\Delta A_2)$.
Actually, the non-linearity is weak in the sense that the principal part is a linear elliptic system. One can show in particular that the solution in $H^1(\Omega, \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ of the elliptic system (90) is actually, when Ω is regular, in $C^\infty(\overline{\Omega})$.

Terminology for the minimizers

- The pair $(0, \mathbf{F})$ is called the Normal State.
- A minimizer (ψ, A) for which ψ never vanishes will be called SuperConducting State.
- In the other cases, one will speak about Mixed State.

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The general question is to determine the topology of the subset in $\mathbb{R}^+ \times \mathbb{R}^+$ of the (κ, σ) corresponding to minimizers belonging to each of these three situations.
Giorgi-Phillips' Theorem says that, for given $\kappa > 0$, there exists $\sigma_0(\kappa) > 0$, such that, for $\sigma \geq \sigma_0(\kappa)$ the global minimizer is $(0, \mathbf{F})$.

Local critical field = Global critical field

Looking at the Hessian of the Ginzburg-Landau functional computed at the point $(0, \mathbf{F})$ leads us to conjecture that a bifurcation between normal solutions and mixed solutions occurs when

$$\lambda_1^N(\kappa\sigma\mathbf{F}) = \kappa^2 \; .$$

This corresponds at a local critical field $H_{C3}^{loc}(\kappa)$ or at least at a local critical zone $[\underline{H}_{C3}^{loc}(\kappa)\,,\overline{H}_{C3}^{loc}(\kappa)]$.

More precisely, let us define the following subsets of the positive real axis :

$$\mathcal{N}(\kappa) := \{\sigma > 0 \,\big|\, \mathcal{E}_{\kappa,\sigma} \text{ has a non-trivial minimizer}\}\,, \tag{92}$$

$$\mathcal{N}^{\mathrm{loc}}(\kappa) := \{\sigma > 0 \,\big|\, \lambda_1^N(\kappa\sigma\,\mathbf{F}) < \kappa^2\}\,, \tag{93}$$

$$\mathcal{N}^{\mathrm{sc}}(\kappa) := \{\sigma > 0 \,\big|\, \text{ The Ginzburg-Landau equations have non-trivial solutions}\}\,. \tag{94}$$

$$\overline{H}_{C_3}(\kappa) := \sup\ \mathcal{N}(\kappa)\,, \qquad \underline{H}_{C_3}(\kappa) := \inf\ \mathbb{R}^+ \setminus \mathcal{N}(\kappa)\,. \tag{95}$$

Similarly, we define local fields and generalized fields by

$$\begin{aligned}\overline{H}_{C_3}^{\mathrm{loc}}(\kappa) &:= \sup\ \mathcal{N}^{\mathrm{loc}}(\kappa)\,, &\qquad \underline{H}_{C_3}^{\mathrm{loc}}(\kappa) &:= \inf\ \mathbb{R}^+ \setminus \mathcal{N}^{\mathrm{loc}}(\kappa)\,,\\ \overline{H}_{C_3}^{\mathrm{sc}}(\kappa) &:= \sup\ \mathcal{N}^{\mathrm{sc}}(\kappa)\,, &\qquad \underline{H}_{C_3}^{\mathrm{sc}}(\kappa) &:= \inf\ \mathbb{R}^+ \setminus \mathcal{N}^{\mathrm{sc}}(\kappa)\,.\end{aligned} \tag{96}$$

Main results

Our main result below is that all the critical fields above are contained in the interval $[\underline{H}^{\rm loc}_{C_3}(\kappa), \overline{H}^{\rm loc}_{C_3}(\kappa)]$, when κ is large. More precisely, the different sets $\mathcal{N}(\kappa)$, $\mathcal{N}^{\rm loc}(\kappa)$ and $\mathcal{N}^{\rm sc}(\kappa)$ coincide for large values of κ. The proof we give is identical for the 2- and 3-dimensional situations.

We first observe the following general inequalities.

Theorem CFa

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$ be a bounded, simply connected domain with smooth boundary. The following general relations hold between the different definitions of H_{C_3},:

$$\underline{H}_{C_3}^{\mathrm{loc}}(\kappa) \leq \underline{H}_{C_3}(\kappa)\,, \tag{97}$$

$$\overline{H}_{C_3}^{\mathrm{loc}}(\kappa) \leq \overline{H}_{C_3}(\kappa)\,. \tag{98}$$

For large values of κ, we have a converse to (98).

Theorem CFb

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, be a bounded, simply connected domain with smooth boundary. If $d = 2$, suppose that the external magnetic field β satisfies

$$0 < \Theta_0 b' < b\,. \tag{99}$$

If $d = 3$, we suppose that $\beta \in \mathbb{S}^2$ is constant.
Then there exists $\kappa_0 > 0$ such that for $\kappa \geq \kappa_0$,

$$\overline{H}^{\mathrm{loc}}_{C_3}(\kappa) = \overline{H}_{C_3}(\kappa)\,. \tag{100}$$

Theorem CFc

Furthermore, if the function $B \mapsto \lambda_1^N(B\mathbf{F})$ is strictly increasing for large B, then all the critical fields coincide for large κ and are given by the unique solution σ to the equation

$$\lambda_1^N(\kappa\sigma\,\mathbf{F}) = \kappa^2\,. \tag{101}$$

Remark.

This explains why we have analyzed the monotonicity of $B \mapsto \lambda_1^N(B\mathbf{F})$ for B large.

Around the proof of Theorem CFb

The crucial point leads in the following argument.
If, for some σ, there is a non trivial minimizer (ψ, A) so

$$\mathcal{E}_{\kappa,\sigma}(\psi, A) \leq 0 \,.$$

Then

$$0 < \Delta := \kappa^2 ||\psi||_2^2 - Q_{\kappa\sigma A}[\psi] = \kappa^2 ||\psi||_4^4 \,,$$

where $Q_{\kappa\sigma A}[\psi]$ is the energy of ψ.

The last equality is a consequence of the first G-L equation.

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Note that the localization of the minimizer leads to the proof of :

$$||\psi||_{L^2(\Omega)} \leq C\, \kappa^{-\frac{1}{4}} ||\psi||_{L^4(\Omega)}\ , \tag{102}$$

which is true for κ large enough.this gives

$$||\psi||_2 \leq C\, \kappa^{-\frac{3}{4}} \Delta^{\frac{1}{4}}\ .$$

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which is true for κ large enough.this gives

$$||\psi||_2 \leq C\,\kappa^{-\frac{3}{4}}\Delta^{\frac{1}{4}}\ .$$

By comparison of the quadratic forms Q respectively associated with A et $\mathbf{F}$, we get, with $\mathbf{a} = A - \mathbf{F}$:

$$\Delta \leq \left[\kappa^2 - (1-\rho)\lambda_1^N(\kappa\sigma\,\mathbf{F})\right]\|\psi\|_2^2 + \rho^{-1}(\kappa\sigma)^2\int_\Omega |\mathbf{a}\psi|^2\,dx\ , \tag{103}$$

for all $0 < \rho < 1$.

Note that by the regularity of the system Curl-Div, combined with the Sobolev's injection theorem, we get

$$\|\mathbf{a}\|_4 \leq C_1 \|\mathbf{a}\|_{W^{1,2}} \leq C_2 \| \text{ curl } \mathbf{a}\|_2 \ .$$

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Now Δ is also controlling $\| \text{ curl } \mathbf{a}\|_2^2$, so we get :

$$(\kappa\sigma)^2 \|\mathbf{a}\|_4^2 \leq C\, \Delta \ .$$

Combining all these inequalities leads to :

$$
\begin{aligned}
0 < \Delta \leq & \\
& \leq \left[\kappa^2 - (1-\rho)\lambda_1^N(\kappa\sigma\mathbf{F})\right]\|\psi\|_2^2 + \rho^{-1}(\kappa\sigma)^2\|\mathbf{a}\|_4^2\|\psi\|_4^2 \\
& \leq \left[\kappa^2 - \lambda_1^N(\kappa\sigma\mathbf{F})\right]\|\psi\|_2^2 \\
& \qquad + C\rho\lambda_1^N(\kappa\sigma\mathbf{F})\Delta^{\frac{1}{2}}\kappa^{-\frac{3}{2}} + C\rho^{-1}\Delta^{\frac{3}{2}}\kappa^{-1} \ .
\end{aligned}
$$

Chosing $\rho = \sqrt{\Delta}\kappa^{-\frac{3}{4}}$, and using the rough upper bound $\lambda_1^N(\kappa\sigma\,\mathbf{F}) < C\kappa^2$, we find

$$0 < \Delta \leq \left[\kappa^2 - \lambda_1^N(\kappa\sigma\,\mathbf{F})\right]\|\psi\|_2^2 + C\Delta\kappa^{-\frac{1}{4}}\ .$$

This shows finally, for κ large enough independently of σ sufficiently close to "any" third critical field (they have the same asymptoics)

$$0 < \Delta \leq \tilde{C}\left[\kappa^2 - \lambda_1^N(\kappa\sigma\,\mathbf{F})\right]\|\psi\|_2^2\ ,$$

so in particular

$$\kappa^2 - \lambda_1^N(\kappa\sigma\,\mathbf{F}) > 0\ .$$

Coming back to the definitions this leads to the statement.

The model

The energy for the model in Liquid Crystals can be written[1] as

$$\mathcal{E}[\psi, \mathbf{n}] = \int_\Omega \Big\{ |\nabla_{q\mathbf{n}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + K_1 \, |\operatorname{div} \mathbf{n}|^2 + \mathbf{K_2} \, |\mathbf{n} \cdot \operatorname{curl} \mathbf{n} + \tau|^2 + \mathbf{K_3} \, |\mathbf{n} \times \operatorname{curl} \mathbf{n}|^2 \Big\} \, \mathbf{dx}$$

where :

- $\Omega \subset \mathbb{R}^3$ is the region occupied by the liquid crystal,
- ψ is a complex-valued function called the *order parameter*,
- $\mathbf{n}$ is a real vector field of unit length called *director field*,
- q is a real number called *wave number*,
- τ is a real number measuring the chiral pitch,
- $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$ are called the *elastic coefficients*,
- $\kappa > 0$ depends on the material and on temperature.

[1] This is an already simplified model where boundary terms have been eliminated.

The two questions are then :

- What is the minimum of the energy ?
- What is the nature of the minimizers ?

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- What is the minimum of the energy ?
- What is the nature of the minimizers ?

Of course the answer depends heavily on the various parameters !!

As in the theory of superconductivity, a special role will be played by the following critical points of the functional, i.e. the pairs

$$(0, \mathbf{n}) \ ,$$

where $\mathbf{n}$ should minimize the second part of the functional called the Oseen-Frank functional :

$$\int_\Omega \Big\{ K_1 \,|\text{ div } \mathbf{n}|^2 + \mathbf{K_2}\, |\mathbf{n} \cdot \text{ curl } \mathbf{n} + \tau|^2 + \mathbf{K_3}\, |\mathbf{n} \times \text{ curl } \mathbf{n}|^2 \Big\}\, \mathbf{dx} \ .$$

These special solutions are called "nematic phases" and one is naturally asking if they are minimizers or local minimizers of the functional.

A first upper bound

For $\tau > 0$, let us consider $\mathcal{C}(\tau)$ the set of the $\mathbb{S}^2$-valued vectors satisfying :

$$\text{curl } \mathbf{n} = -\tau \mathbf{n} \,, \quad \text{div } \mathbf{n} = \mathbf{0} \,.$$

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It can be shown that $\mathcal{C}(\tau)$ consists of the vector fields $\mathbb{N}_\tau^Q$ such that, for some $Q \in \mathrm{SO}(3)$,

$$\mathbb{N}_\tau^Q(x) \equiv Q\mathbb{N}_\tau(Q^t x) , \quad \forall x \in \Omega, \tag{104}$$

where

$$\mathbb{N}_\tau(y_1, y_2, y_3) = (\cos(\tau y_3), \sin(\tau y_3), 0) , \ \forall y \in \mathbb{R}^3 . \tag{105}$$

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$$\mathbb{N}_\tau(y_1, y_2, y_3) = (\cos(\tau y_3), \sin(\tau y_3), 0) \,, \ \forall y \in \mathbb{R}^3 \,. \tag{105}$$

Note that is also equivalent, when $|\mathbf{n}|^2 = \mathbf{1}$ to

$$\text{div } \mathbf{n} = \mathbf{0}, \quad \mathbf{n} \cdot \text{ curl } \mathbf{n} + \tau = \mathbf{0}, \quad \mathbf{n} \times \text{ curl } \mathbf{n} = \mathbf{0} \,. \tag{106}$$

So the last three terms in the functional vanish iff $\mathbf{n} \in \mathcal{C}(\tau)$.

As a consequence, if we denote by

$$C(K_1, K_2, K_3, \kappa, q, \tau) = \inf_{(\psi, \mathbf{n}) \in \mathbb{V}(\mathbf{\Omega})} \mathcal{E}[\psi, \mathbf{n}] \, ,$$

the infimum of the energy over the natural maximal form domain of the functional, then

$$C(K_1, K_2, K_3, \kappa, q, \tau) \leq c(\kappa, q, \tau) \, , \tag{107}$$

where

$$c(\kappa, q, \tau) = \inf_{\mathbf{n} \in \mathcal{C}(\tau)} \inf_{\psi} \mathcal{G}_{q\mathbf{n}}(\psi) \tag{108}$$

and $\mathcal{G}_{q\mathbf{n}}(\psi)$ is the so called the reduced Ginzburg-Landau functional.

Reduced Ginzburg-Landau functional

Given a vector field $\mathbf{A}$, this functional is defined on $H^1(\Omega, \mathbb{C})$ by

$$\psi \mapsto \mathcal{G}_{\mathbf{A}}[\psi] = \int_\Omega \{|\nabla_{\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\}\, dx\,. \tag{109}$$

For convenience, we also write $\mathcal{G}_{\mathbf{A}}[\psi]$ as $\mathcal{G}[\psi, \mathbf{A}]$.
So we have

$$c(\kappa, q, \tau) = \inf_{\mathbf{n}\in\mathcal{C}(\tau), \psi\in\mathbf{H}^1(\boldsymbol{\Omega},\mathbb{C})} \mathcal{G}[\psi, q\mathbf{n}]\,. \tag{110}$$

and

$$\mathcal{E}(\psi, \mathbf{n}) = \mathcal{G}[\psi, \mathbf{qn}]\ , \tag{111}$$

if

$$\mathbf{n} \in \mathcal{C}(\tau)\ .$$

A limiting case

We have seen that in full generality that

$$C(K_1, K_2, K_3, \kappa, q, \tau) \leq c(\kappa, q, \tau)\ . \tag{112}$$

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Conversely, it can be shown (Bauman-Calderer-Liu-Phillips, Pan, Helffer-Pan), that when the elastic parameters tend to $+\infty$, the converse is asymptotically true.

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Proposition LCa

$$\lim_{K_1, K_2, K_3 \to +\infty} C(K_1, K_2, K_3, \kappa, q, \tau) = c(\kappa, q, \tau)\ . \tag{113}$$

So $c(\kappa, q, \tau)$ is a good approximation for the minimal value of $\mathcal{E}$ for large K_j's.
Note that an interesting open problem is to control the rate of convergence in (113).

Minimizers of the reduced G-L functional

We now examine the non-triviality of the minimizers realizing $c(\kappa, q, \tau)$.

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in Ω, with

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$$-\nabla^2_{q\mathbf{n}}$$

in Ω, with

$$\nabla_{q\mathbf{n}} = \nabla - iq\mathbf{n}\ ,$$

namely $\lambda = \lambda_1^N(q\mathbf{n})$ (in short $\lambda_1^N(q\mathbf{n})$) is the lowest eigenvalue of the following problem

$$\begin{cases} -\nabla^2_{q\mathbf{n}}\phi = \lambda\phi & \text{in } \Omega, \\ N \cdot \nabla_{q\mathbf{n}}\phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{114}$$

where N is the unit outer normal of $\partial\Omega$.

But the new point is that we will minimize over $\mathbf{n} \in \mathcal{C}(\tau)$. So we shall actually meet

$$\mu_*(q,\tau) = \inf_{\mathbf{n}\in\mathcal{C}(\tau)} \lambda_1^N(q\mathbf{n})\,. \tag{115}$$

Our main comparison statement (analogous to some statement in Fournais-Helffer for surface superconductivity) is :

Proposition LCb

$$-\frac{\kappa^2|\Omega|}{2}[1-\kappa^{-2}\mu_*(q,\tau)]_+^2 \leq c(\kappa,q,\tau) \tag{116}$$

and

$$c(\kappa,q,\tau) \leq -\frac{\kappa^2}{2}[1-\kappa^{-2}\mu_*(q,\tau)]_+^2 \sup_{\mathbf{n}\in\mathcal{C}(\tau)} \sup_{\phi\in\mathcal{S}p(q\mathbf{n})} \frac{(\int_\Omega |\phi|^2\,dx)^2}{\int_\Omega |\phi|^4\,dx}, \tag{117}$$

where $\mathcal{S}p(q\mathbf{n})$ is the eigenspace associated to $\mu(q\mathbf{n})$.

This shows also that $c(\kappa, q, \tau)$ is strictly negative if and only

$$\mu_*(\kappa, \tau) < \kappa^2 .$$

The link with spectral theory for Schrödinger with magnetic field

As a consequence of Proposition LCb, we obtain that the transition from nematic phases to non-nematic phases (the so called smectic phases) is strongly related to the analysis of the solution of

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This will permit indeed to find a unique solution of (118) permitting a natural definition of the critical value $Q_{C3}(\kappa,\tau)$.

We have proved with Pan that if τ stays in a bounded interval, then this quantity and $\mu_*(q,\tau)$ can be controlled in two regimes

- $\sigma \to +\infty$,
- $\sigma \to 0$,

where

$$\sigma = q\tau$$

which is in some sense the leading parameter in the theory.

Semi-classical case : $q\tau$ large

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The magnetic field $-q\tau\mathbf{n}$ (corresponding when $\mathbf{n} \in \mathcal{C}(\tau)$ to the magnetic potential $q\mathbf{n}$) is no more constant, so one should extend the analysis of Helffer-Morame ($d = 3$) to this case.

A first analysis (semi-classical in spirit) gives :

Theorem LCc

As $\sigma = q\tau \to +\infty$,

$$\mu_*(q,\tau) = \Theta_0(q\tau) + \mathcal{O}((q\tau)^{\frac{2}{3}}) \tag{119}$$

where the remainder is controlled uniformly for $\tau \in]0, \tau_0]$.

This condition can be relaxed (N. Raymond 2008) at the price of a worse remainder.

This leads (assuming the uniqueness of Q_{C3}), to

$$\tau\, Q_{C3}(\kappa,\tau) = \frac{\kappa^2}{\Theta_0} + \mathcal{O}(\kappa^{\frac{4}{3}})\,. \tag{120}$$

Coming back to the limit $\sigma \to +\infty$, an open question (but see Pan and work in progress by Helffer-Pan) is to find uniform two terms asymptotic for $\mu(qn_\tau)$ and for $\mu_*(q, \tau)$.

A simpler question

A simpler question which is partially solved in Pan (2007) (with the help of Helffer-Morame (d=3)) and corresponds to the case $\tau = 0$ is the following :

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A simpler question which is partially solved in Pan (2007) (with the help of Helffer-Morame (d=3)) and corresponds to the case $\tau = 0$ is the following :

Given a strictly convex open set, find the direction $\mathbf{h}$ of the constant magnetic field giving asymptotically as $\sigma \to +\infty$ the lowest energy for the Neumann realization in Ω of the Schrödinger operator with magnetic field $\sigma\, \mathbf{h}$.

Let us present shortly the answer to this question. We assume that

Assumption G'

At each point of $\partial\Omega$ the curvature tensor has two strictly positive eigenvalues $\kappa_1(x)$ and $\kappa_2(x)$, so

$$0 < \kappa_1(x) \leq \kappa_2(x) \, .$$

This assumption implies that for any $\mathbf{h}$, the corresponding set $\Gamma_{\mathbf{h}}$ of boundary points where $\mathbf{h}$ is tangent to $\partial\Omega$, i.e.

$$\Gamma_{\mathbf{h}} := \{x \in \partial\Omega \,|\, \mathbf{h} \cdot N(x) = 0\}, \tag{121}$$

is a regular submanifold of $\partial\Omega$.
For any given $\mathbf{h}$, let $F_{\mathbf{h}}$ be the magnetic potential such that

$$\text{curl } \mathbf{F_h} = \mathbf{h} \, , \quad \text{div } \mathbf{F_h} = 0 \, , \; \mathbf{F_h} \cdot N(x) = 0 \text{ on } \partial\Omega \, .$$

We have the following two-term asymptotics of $\lambda_1^N(\sigma \mathbf{F_h})$ of the Neuman Laplacian $\Delta_{\sigma \mathbf{F_h}}$, (due to Helffer-Morame-Pan).

Theorem 20

If Ω and $\mathbf{h}$ as above, then, as $\sigma \to +\infty$,

$$\lambda_1^N(\sigma \mathbf{F_h}) = \Theta_0 \sigma + \widehat{\gamma}_{\mathbf{h}} \sigma^{\frac{2}{3}} + \mathcal{O}(\sigma)^{\frac{2}{3}-\eta}, \tag{122}$$

for some $\eta > 0$.

Moreover η is independent of $\mathbf{h}$ and the control of the remainder is uniform with respect to $\mathbf{h}$.

and $\widehat{\gamma}_{\mathbf{h}}$ is defined by

$$\widehat{\gamma}_{\mathbf{h}} := \inf_{x\in\Gamma_{\mathbf{h}}} \widetilde{\gamma}_{\mathbf{h}}(x), \tag{123}$$

where

$$\widetilde{\gamma}_{\mathbf{h}}(x) := 2^{-2/3}\widehat{\nu}_0\delta_0^{1/3}|k_n(x)|^{2/3}\left(\delta_0+(1-\delta_0)|T_{\mathbf{h}}(x)\cdot\mathbf{h}|^2\right)^{1/3}. \tag{124}$$

Here $T_{\mathbf{h}}(x)$ is the oriented, unit tangent vector to $\Gamma_{\mathbf{h}}$ at the point x and

$$k_n(x) = |d^T(\beta\cdot N)(x)|\,.$$

Here is now the answer to the "simpler" question. We have just to determine $\inf_{\mathbf{h}}\widehat{\gamma}_{\mathbf{h}}$ or equivalently

$$\inf_{\mathbf{h},x\in\Gamma_{\mathbf{h}}} |k_n(x)|^{2/3}\left(\delta_0+(1-\delta_0)|T_{\mathbf{h}}(x)\cdot\mathbf{h}|^2\right)^{\frac{1}{3}}\,.$$

So everything is reduced to the analysis of the map

$$\Gamma_{\mathbf{h}}\ni x\mapsto k_n(x)^2\left(\delta_0+(1-\delta_0)|T_{\mathbf{h}}(x)\cdot\mathbf{h}|^2\right)\,.$$

This last expression can be written in the form

$$\Gamma_{\mathbf{h}} \ni x \mapsto \kappa_1(x)^2 \cos^2\phi(x) + \kappa_2(x)\sin^2\phi(x) - (1-\delta_0)(\kappa_1(x)-\kappa_2(x))^2 \sin^2\phi(x)\cos^2(\phi(x)),$$

where, for $x \in \partial\Omega$, $\phi(x)$ is defined by writing

$$\mathbf{h} = \cos\phi(x) u_1(x) + \sin\phi(x) u_2(x) ,$$

with $(u_1(x), u_2(x))$ being the orhonormal basis of the curvature tensor at x, associated to the eigenvalues $\kappa_1(x)$ and $\kappa_2(x)$. When minimizing over $\mathbf{h}$ and $x \in \Gamma_{\mathbf{h}}$, it is rather easy to show that the infimum is obtained by first choosing a point x_0 of $\partial\Omega$ where $\kappa_1(x)$ is minimum and then taking $\mathbf{h} = u_1(x_0)$.

This leads to the proposition

Proposition 21

Under Assumption (116), we have

$$\inf_{\mathbf{h}} \widehat{\gamma}_{\mathbf{h}} = \inf_{x \in \partial\Omega} (\kappa_1(x))^{\frac{2}{3}} \,. \tag{125}$$

This answers to our question.

Let Ω be a smooth, simply-connected domain in $\mathbb{R}^2$. Let

$$\gamma : \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \to \partial\Omega$$

be a parametrization of the boundary with $|\gamma'(s)| = 1$ for all s. Let $\nu(s)$ be the unit vector, normal to the boundary, pointing inward at the point $\gamma(s)$. We choose the orientation of the parametrization γ to be counter-clockwise, so

$$\det\left(\gamma'(s), \nu(s)\right) = 1\,.$$

The curvature $k(s)$ of $\partial\Omega$ at the point $\gamma(s)$ is now given by

$$\gamma''(s) = k(s)\nu(s)\,.$$

The map Φ defined by,

$$\begin{aligned}&\Phi : \mathbb{R}/(|\partial\Omega|\mathbb{Z})\times]0,t_0[\to \Omega\,,\\&(s,t)\mapsto \gamma(s)+t\nu(s)\,,\end{aligned}\tag{126}$$

is clearly a diffeomorphism, when t_0 is sufficiently small, with image

$$\Phi\big(\mathbb{R}/(|\partial\Omega|\mathbb{Z})\times]0,t_0[\big) = \{x\in\Omega \big|\ \mathrm{dist}\,(x,\partial\Omega) < t_0\} =: \Omega_{t_0}\,.$$

Furthermore, with the distance to the boundary $t(x)$,
$t(\Phi(s,t)) = t$.
The inverse Φ^{-1} defines a system of coordinates for a tubular neighborhood of $\partial\Omega$ in $\overline{\Omega}$ that we can use locally or semi-globally.

If A is a vector field on Ω_{t_0} with $\beta = \text{ curl } A$ we define the associated fields in (s,t)-coordinates by

$$\tilde{A}_1(s,t) = (1-tk(s))A(\Phi(s,t))\cdot\gamma'(s)\,, \quad \tilde{A}_2(s,t) = A(\Phi(s,t))\cdot\nu(s)\,, \tag{127}$$

$$\tilde{\beta}(s,t) = \beta(\Phi(s,t))\,. \tag{128}$$

Then

$$\partial_s\tilde{A}_2 - \partial_t\tilde{A}_1 = (1-tk(s))\tilde{\beta}\,. \tag{129}$$

Furthermore, for all $u \in H^1(\Omega_{t_0})$, we have, with $v = u\circ\Phi$,

$$\int_{\Omega_{t_0}} |(-i\nabla + A)u|^2\,dx \tag{130}$$

$$= \int \Big\{(1-tk)^{-2}\Big|(-i\partial_s + \tilde{A}_1)v\Big|^2 + \Big|(-i\partial_t + \tilde{A}_2)v\Big|^2\Big\}(1-tk)\,dsdt\,,$$

$$\int_{\Omega_{t_0}} |u(x)|^2\,dx = \int |v(s,t)|^2(1-tk(s))\,dsdt\,.$$

The next lemma is quite useful for the fine analysis in a tubular neighborhood of the boundary and gives a kind of normal form.

Lemma (Semi-global version)

Let θ on Ω_{t_0} s.t the corresponding $\tilde{\theta}$ is t-independent. Then $\exists C > 0$ s.t. , if A satisfies $\text{curl } A = \theta \quad \text{on } \partial\Omega\,,$ and with $\tilde{A}$ defined as in (127), then $\exists \varphi(s,t)$ on $\mathbb{R}/(|\partial\Omega|\mathbb{Z})\times]0,t_0[$ s.t.

$$\bar{A}(s,t) = \begin{pmatrix} \bar{A}_1(s,t) \\ \bar{A}_2(s,t) \end{pmatrix}$$

$$:= \tilde{A} - \nabla_{(s,t)}\varphi = \begin{pmatrix} \gamma_0 - \tilde{\theta}(s,0)t + \frac{t^2 k(s)}{2} + t^2 b(s,t) \\ 0 \end{pmatrix}, \quad (131)$$

where

$$\gamma_0 = \frac{1}{|\partial\Omega|} \int_\Omega \text{curl } A\, dx\,. \quad (132)$$

Local version

Furthermore, if $[s_0, s_1]$ is a subset of $\mathbb{R}/(|\partial\Omega|\mathbb{Z})$ with $s_1 - s_0 < |\partial\Omega|$, then we may choose φ on $]s_0, s_1[\times]0, t_0[$ s.t.

$$\bar{A}(s,t) = \begin{pmatrix} \bar{A}_1(s,t) \\ \bar{A}_2(s,t) \end{pmatrix} := \tilde{A} - \nabla_{(s,t)}\varphi = \begin{pmatrix} -\tilde{\theta}(s,0)t + \frac{t^2 k(s)}{2} + t^2 b(s,t) \\ 0 \end{pmatrix} . \quad (133)$$

No need to have some γ_0 !

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