On Courant's nodal domain property for linear combinations of eigenfunctions (after P. Bérard and B. Helffer).

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Abstract

We revisit Courant's nodal domain property for linear combinations of eigenfunctions, and propose new, simple and explicit counterexamples for domains in $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{T}^2$, or $\mathbb{R}^3$.

This work has been done in collaboration with P. Bérard and has benefitted from the precious help of V. Bonnaillie-Noël.

Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain or, more generally, a compact Riemannian manifold with boundary.

Consider the eigenvalue problem

$$\begin{cases} -\Delta u & = \lambda\, u \quad \text{in } \Omega\,, \\ B(u) & = 0 \qquad \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where $B(u)$ is some boundary condition on $\partial\Omega$, so that we have a self-adjoint boundary value problem (including the empty condition if Ω is a closed manifold).
For example, $D(u) = u\big|_{\partial\Omega}$ for the Dirichlet boundary condition, or $N(u) = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$ for the Neumann boundary condition.

Call $H(\Omega, B)$ the associated self-adjoint extension of $-\Delta$, and list its eigenvalues in nondecreasing order, counting multiplicities,

$$0 \leq \lambda_1(\Omega, B) < \lambda_2(\Omega, B) \leq \lambda_3(\Omega, B) \leq \cdots \tag{2}$$

For any $n \geq 1$, define the number

$$\tau(\Omega, B, n) = \min\{k \mid \lambda_k(\Omega, B) = \lambda_n(\Omega, B)\}. \tag{3}$$

E_{λ_n} will denote the eigenspace associated with λ_n.

The Courant nodal theorem

For a real continuous function v on Ω, we define its *nodal set*

$$\mathfrak{Z}(v) = \overline{\{x \in \Omega \mid v(x) = 0\}}\,, \tag{4}$$

and call $\beta_0(v)$ the number of connected components of $\Omega \setminus \mathfrak{Z}(v)$ i.e., the number of *nodal domains* of v.

Courant's nodal Theorem (1923)

For any nonzero $u \in E_{\lambda_n(\Omega,B)}$,

$$\beta_0(u) \leq \tau(\Omega, B, n) \leq n\,. \tag{5}$$

Courant's noda l domain theorem can be found in Courant-Hilbert [7].

The extended Courant nodal property

Given $r > 0$, denote by $\mathfrak{L}(\Omega, B, r)$ the space

$$\mathfrak{L}(\Omega, B, r) = \left\{ \sum_{\lambda_j(\Omega,B)\leq r} c_j\, u_j \mid c_j \in \mathbb{R}, u_j \in E_{\lambda_j(\Omega,B)} \right\}. \quad (6)$$

Extended Courant Property:= (ECP)

We say that $v \in \mathfrak{L}\,(\Omega, B, \lambda_n(\Omega, B))$ satisfies (ECP) if

$$\beta_0(v) \leq \tau(\Omega, B, n)\,. \quad (7)$$

A footnote in Courant-Hilbert [7] indicates that this property also holds for any linear combination of the n first eigenfunctions, and refers to the PhD thesis of Horst Herrmann (Göttingen, 1932) [13].

Historical remarks : Sturm (1836), Pleijel (1956).

1. (ECP) is true for Sturm-Liouville equations. This was first announced by C. Sturm in 1833, [26] and proved in [27]. Other proofs were later on given by J. Liouville and Lord Rayleigh who both cite Sturm explicitly.

2. Å. Pleijel mentions (ECP) in his well-known paper [23] on the asymptotic behaviour of the number of nodal domains of a Dirichlet eigenfunction associated with the n-th eigenvalue in a plane domain. At the end of the paper, he writes:
"In order to treat, for instance the case of the free three-dimensional membrane $[0, \pi]^3$*, it would be necessary to use, in a special case, the theorem quoted in [6].... However, as far as I have been able to find there is no proof of this assertion in the literature."*

Historical remarks: V. Arnold (1973)

3. As pointed out by V. Arnold [1], when $\Omega = \mathbb{S}^d$, (ECP) is related to Hilbert's $16-$th problem. Arnold [2] mentions that he actually discussed the footnote with R. Courant, that (ECP) cannot be true, and that O. Viro produced in 1979 counter-examples for the 3-sphere $\mathbb{S}^3$, and any degree larger than or equal to 6, [28].

More precisely V. Arnold wrote:
"Having read all this, I wrote a letter to Courant: "Where can I find this proof now, 40 years after Courant announced the theorem?". Courant answered that one can never trust one's students: to any question they answer either that the problem is too easy to waste time on, or that it is beyond their weak powers."

And V. Arnold continues:

The point is that for the sphere $\mathbb{S}^2$ *(with the standard Riemannian metric) the eigenfunctions (spherical functions) are polynomials. Therefore, their linear combinations are also polynomials, and the zeros of these polynomials are algebraic curves (whose degree is bounded by the number n of the eigenvalue). Therefore, from the generalized Courant theorem one can, in particular, derive estimates for topological invariants of the complements of projective real algebraic curves (in terms of the degrees of these curves).*

Knowing this, I immediately deduced from the generalized Courant theorem new results in Hilbert's famous 16th-problem: "Study the topological properties of the arrangement of real algebraic curves of degree n on the real projective plane."

This Hilbert problem (for $n > 7$) is still unsolved, although many interesting estimates for different invariants have been obtained by Petrovskii, Oleinik, Gudkov, and others. About 1970, I associated this theory with the topology of four-dimensional manifolds, and my successors (Rokhlin, Viro, Kharlamov, Givental, Gromov, Witten, Floer, McDuff, and others) included all this into symplectic topology and quantum field theory. And then it turned out that the results of the topology of algebraic curves that I had derived from the generalized Courant theorem contradict the results of quantum field theory. Nevertheless, I knew that both my results and the results of quantum field theory were true. Hence, the statement of the generalized Courant theorem is not true (explicit counterexamples were soon produced by Viro).

Historical remarks: Gladwell-Zhu (2003)

4. In [9], Gladwell and Zhu refer to (ECP) as the *Courant-Herrmann conjecture*.
They claim that this extension of Courant's theorem is not stated, let alone proved, in Herrmann's thesis or subsequent publications.
They consider the case in which Ω is a rectangle in $\mathbb{R}^2$, stating that they were not able to find a counter-example to (ECP) in this case.
They also provide numerical evidence that there are counter-examples for more complicated (non convex) domains.

They suggest that may be the conjecture could be true in the convex case.

Historical remarks: looking for the PHD thesis of H. Herrmann

5. After a personal investigation, what we can add, after getting from the BNF, the manuscript of the PHD-thesis is that Herrmann's thesis has three parts. Only the second part was accepted by the evaluating committee for publication. This part does not contain any mention of (ECP). The first part, was published later in [14] in Math. Z. in 1936 in a different form. Finally, the third part was never published. The title of this chapter indicates that this part was devoted to the analysis of the Fourier-Robin problem and to analyze how the eigenvalues tend to the eigenvalues of the Dirichlet problem as the Robin parameter tends to $+\infty$. Nothing to do with (ECP).

The purpose in this talk is to provide simple counter-examples to the *Extended Courant Property* for domains in $\mathbb{R}^2$, $\mathbb{S}^2$ or $\mathbb{R}^3$, including convex domains. No algebraic topology will be involved.

Rectangular membrane, Dirichlet boundary condition

We summarize the ideas from Gladwell-Zhu [9] and consider $\mathfrak{Q}_\pi =]0,\pi[^2$, with Dirichlet condition. The eigenvalues are given by the numbers

$$q_2(m,n) = m^2 + n^2\,, \quad \text{for } m,n \in \mathbb{N}\,.$$

More precisely,

$$\begin{aligned} &\delta_1\ [2] < \delta_2 = \delta_3\ [5] < \delta_4\ [8] < \delta_5 = \delta_6\ [10] < \dots \\ &< \delta_7 = \delta_8\ [13] < \delta_9 = \delta_{10}\ [17] < \delta_{11}\ [18] < \cdots \end{aligned}$$

In this list the numbers in brackets are the actual values of the eigenvalues, for example, $\delta_2 = \delta_3 = 5$.

A corresponding orthonormal basis is given by the functions $\phi_{m,n}(x,y) = \sin(mx)\sin(ny)$ for $m, n \in \mathbb{N}$.
Using the classical Chebyshev polynomials, we have

$$\phi_{m,n}(x,y) = \phi_{1,1}(x,y)\, U_{m-1}(\cos x)\, U_{n-1}(\cos y)\,.$$

For some $r > 0$, we write $\mathfrak{L}_r := \mathfrak{L}(\mathfrak{Q}_\pi, D, r)$.

An element $\Phi \in \mathfrak{L}_r$ has the form

$$\Phi(x,y) := \sum_{q_2(m,n)\leq r} c_{m,n}\, \phi_{m,n}(x,y)\,.$$

We can factor out the non-vanishing $\phi_{1,1}$, and consider instead the nodal pattern of the function,

$$\Phi_1(x,y) = \sum_{q_2(m,n)\leq r} c_{m,n}\, U_{m-1}(\cos x)\, U_{n-1}(\cos y)\,.$$

On the other-hand, using the diffeomorphism

$$F :]0, \pi[\ni (x, y) \mapsto (X, Y) := (\cos x, \cos y) \in]-1, 1[\,,$$

we see that the nodal pattern of Φ is diffeomorphic to the nodal pattern of

$$\Psi(X, Y) := \sum_{q_2(m,n) \leq r} c_{m,n}\, U_{m-1}(X)\, U_{n-1}(Y)\,,$$

for $(X, Y) \in]-1, 1[^2$.

Choosing $r = \delta_6 = 10$, the linear combinations Ψ generate the subspace of $\mathbb{R}[X, Y]$ spanned by the family

$$\{1, X, Y, X^2, XY, Y^2\}.$$

Hence for any $d \in \{1, 2, 3, 4, 5\}$ is achieved as $\beta_0(\Phi)$ for some $\Phi \in \mathfrak{L}_{10}$. Notice that 5 is precisely Courant's bound $\tau(\mathfrak{Q}_\pi, D, 6)$, see Figure.

Choosing $r = \delta_{10} = 17$, it seems impossible to find an example of $\Phi \in \mathfrak{L}_{17}$ with nine or more nodal domains.

Table 1 CHC is true for the first 13 eigenfunctions on the square.

	m, n	New High Order Terms		Maximum No. of Nodal domains	
1	1, 1	1		1	
2, 3	1, 2 2, 1	X	Y	2	
4	2, 2	XY		4	
5, 6	1, 3 3, 1	X^2	Y^2	5	
7, 8	2, 3 3, 2	X^2Y	XY^2	7	
9, 10	1, 4 4, 1	X^3	Y^3	8	
11	3, 3	X^2Y^2		10	
12, 13	2, 4 4, 2	X^3Y	$X\ Y^3$	12	

Figure: The pictures and computations of Gladwell-Zhu [9]

Rectangle with a crack

Let $\mathfrak{R}_0$ be the rectangle $]0, 4\pi[\times]0, 2\pi[$. For $0 < a \leq 1$, let $C_a :=]0, a] \times \{\pi\}$ and $\mathfrak{R}_a := \mathfrak{R}_0 \setminus C_a$ and consider the Neumann condition. The setting is described in Dauge-Helffer [8].

We call

$$\begin{cases} 0 < \delta_1(0) < \delta_2(0) \leq \delta_3(0) \leq \cdots \\ \text{resp.} \\ 0 = \nu_1(0) < \nu_2(0) \leq \nu_3(0) \leq \cdots \end{cases} \tag{8}$$

the Neumann eigenvalues of $-\Delta$ in $\mathfrak{R}_0$.

They are given by the $\frac{m^2}{16} + \frac{n^2}{4}$ for pairs (m, n) of non-negative integers.

Corresponding eigenfunctions are products of cosines.

Similarly, the Neumann eigenvalues of $-\Delta$ in $\mathfrak{R}_a$ are denoted by

$$\begin{cases} 0 < \delta_1(a) < \delta_2(a) \leq \delta_3(a) \leq \cdots \\ \text{resp.} \\ 0 = \nu_1(a) < \nu_2(a) \leq \nu_3(a) \leq \cdots \end{cases} . \tag{9}$$

The first three Neumann eigenvalues for the rectangle $\mathfrak{R}_0$ are as follows.

$\nu_1(0)$	0	$(0,0)$	$\psi_1(x,y)=1$
$\nu_2(0)$	$\frac{1}{16}$	$(1,0)$	$\psi_2(x,y)=\cos(\frac{x}{4})$
$\nu_3(0)$		$(0,1)$	$\psi_3(x,y)=\cos(\frac{y}{2})$
$\nu_4(0)$	$\frac{1}{4}$	$(2,0)$	$\psi_4(x,y)=\cos(\frac{x}{2})$

(10)

Dauge-Helffer (1993) prove:

Theorem

For $i \geq 1$,

1. $[0,1] \ni a \mapsto \nu_i(a)$ is non-increasing.
2. $]0,1[\ni a \mapsto \nu_i(a)$, is continuous.
3. $\lim_{a\to 0+} \nu_i(a) = \nu_i(0)$.

It follows that for $0 < a$, small enough, we have

$$0 = \nu_1(a) = \nu_1(0) < \nu_2(a) \leq \nu_2(0) < \nu_3(a) \leq \nu_4(a) \leq \nu_3(0)\,. \tag{11}$$

Observe that for $i = 1$ and 2, $\frac{\partial \psi_i}{\partial y}(x, y) = 0$. Hence for a small enough, ψ_1 and ψ_2 are the first two eigenfunctions for $\mathfrak{R}_a$ with the Neumann condition with associated eigenvalues 0 and $\frac{1}{4}$. We have

$$\alpha \psi_1(x, y) + \beta \psi_2(x, y) = \alpha + \beta \cos(\frac{x}{4}).$$

We can choose the coefficients α, β in such a way that these linear combinations of the first two eigenfunctions have two or three nodal domains in $\mathfrak{R}_a$.

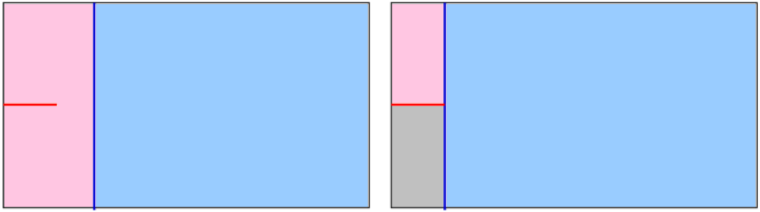


Figure: Rectangle with a crack (Neumann condition)

This proves that (ECP) is false in $\mathfrak{R}_a$ with Neumann condition.

Notice that we can introduce several cracks

$$\{(x, b_j) \mid 0 < x < a_j\}_{j=1}^{k}$$

so that for any $d \in \{2, 3, \dots k + 2\}$ there exists a linear combination of 1 and $\cos(\frac{x}{4})$ with d nodal domains.

Sphere $\mathbb{S}^2$ with cracks

On the round sphere $\mathbb{S}^2$, we consider the geodesic lines $(x, y, z) \mapsto (\sqrt{1-z^2}\cos\theta_i, \sqrt{1-z^2}\sin\theta_i, z)$ through the north pole $(0, 0, 1)$, with distinct $\theta_i \in [0, \pi[$.
Removing the geodesic segments $\theta_0 = 0$ and $\theta_2 = \frac{\pi}{2}$ with $1 - z \leq a \leq 1$, we obtain a sphere $\mathbb{S}^2_a$ with a crack in the form of a cross.
We consider the Neumann condition on the crack.
We then easily produce a function in the space generated by the two first eigenspaces of the sphere with a crack having five nodal domains.

The function z is also an eigenfunction of $\mathbb{S}^2_a$ with eigenvalue 2.
For a small enough, $\lambda_4(a) = 2$, with eigenfunction z.
For $0 < b < a$, the linear combination $z - b$ has five nodal domains in $\mathbb{S}^2_a$, see Figure below in spherical coordinates.

It follows that (ECP) does not hold on the sphere with cracks.

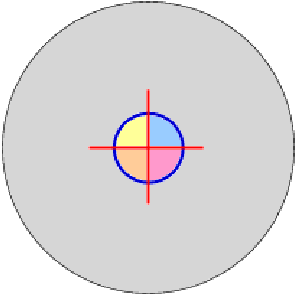


Figure: Sphere with crack, five nodal domains

Remark. Removing more geodesic segments around the north pole, we can obtain a linear combination $z - b$ with as many nodal domains as we want.

The cube with Dirichlet boundary condition

We can adapt the method of Gladwell-Zhu to the $3D$-case.

Consider the cube $\mathfrak{C}_\pi =]0,\pi[^3$. The eigenvalues are the numbers

$$q_3(k,m,n) = k^2 + m^2 + n^2\,, \quad k,m,n \in \mathbb{N}.$$

A corresponding complete set of eigenfunctions is given by

$$\phi_{k,m,n}(x,y,z) = \sin(kx)\,\sin(my)\,\sin(nz)\,, \quad k,m,n \in \mathbb{N}.$$

The first Dirichlet eigenvalues of the cube are given by

$$\delta_1\ [3] < \delta_2 = \delta_3 = \delta_4\ [6] < \delta_5 = \delta_6 = \delta_7\ [9] < \dots$$
$$\delta_8 = \delta_9 = \delta_{10}\ [11] < \delta_{11} \cdots .$$

Using Chebyshev polynomials, for $k, m, n \in \mathbb{N}$ we have

$$\phi_{k,m,n}(x,y,z) = \phi_{1,1,1}(x,y,z)\, U_{k-1}(\cos x)\, U_{m-1}(\cos y)\, U_{n-1}(\cos z)\,.$$

The factor $\phi_{1,1,1}$ does not vanish in the cube $\mathfrak{C}_\pi$. The map

$$\mathfrak{C}_\pi \ni (x,y,z) \mapsto (X,Y,Z) := (\cos(x), \cos(y), \cos(z)) \in]-1,1[^3$$

is a diffeomorphism from $\mathfrak{C}_\pi$ to the cube $]-1,1[^3$.

Let

$$\mathfrak{L}_r := \mathfrak{L}(\mathfrak{C}_\pi, D, r)\,.$$

As in the $2D$-case the counting of the nodal domains of a linear combination $\Phi \in \mathfrak{L}_r$,

$$\Phi = \sum_{q_3(k,m,n)\leq r} c_{k,m,n}\, \phi_{k,m,n}$$

in the cube $\mathfrak{C}_\pi$, is the same as the counting for,

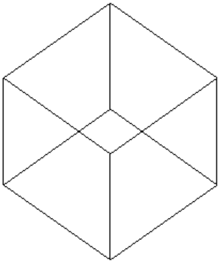
$$\Psi = \sum_{q_3(k,m,n)\leq r} c_{k,m,n}\, U_{k-1}(X) U_{m-1}(Y) U_{n-1}(Z)$$

in the cube $]-1,1[^3$.

Using the formulas for the Chebyshev polynomials, one gets that the linear combinations Ψ for $k^2 + m^2 + n^2 \leq 11 = \delta_{10}$ correspond to the polynomials of degree ≤ 2 in the variables X, Y and Z.

In particular, $f_a(X, Y, Z) := X^2 + Y^2 + Z^2 - a$ is a linear combination Ψ with $k^2 + m^2 + n^2 \leq 11$. Since $11 = \delta_8 = \delta_9 = \delta_{10}$, Courant's upper bound is $8 = \tau(\mathfrak{C}_\pi, D, 10)$.

It follows that when $\sqrt{2} < a < \sqrt{3}$, the function ϕ_a provides a counter-example to (ECP) for the $3D$-cube with Dirichlet boundary condition.



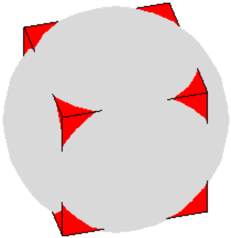
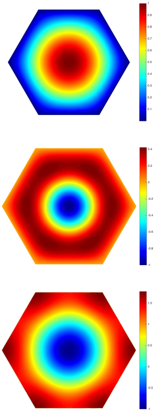


Figure: Cube with Dirichlet boundary condition

Numerical simulations for Regular polygons (Virginie Bonnaillie-Noël).

In (2D) Gladwell-Zhu were not successful for the square. One can be successfull for the hexagone for Neumann and for Dirichlet (Numerics).



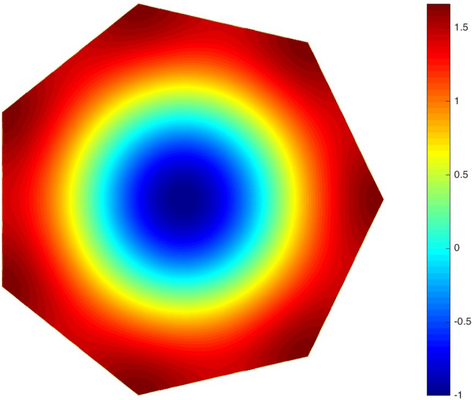


Figure: Level lines of $\frac{w_{6,D}}{w_{1,D}}$ for the Dirichlet problem in the regular heptagon

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