Around Helffer-Nourrigat Conjecture (history, proof, open questions, applications)

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Introduction

The conjecture of Helffer-Nourrigat, which was formulated in its local version (1979), has been solved recently (2022) by Ianivos Androulidakis, Omar Mohsen and Robert Yuncken.
After some reminder on the history of the conjecture, I would like to discuss some remaining problems around the microlocal version and if time permits some applications obtained along the years to the spectral properties of Schrödinger operators with or without magnetic fields, Witten Laplacians, Fokker-Planck operators.

This talk has benefited in its preparation from discussions in 2022-2023 with O. Mohsen and J. Nourrigat. I of course should mention the recent talk by Claire Debord in Bourbaki and the course in Collège de France by Omar Mohsen, this year.

A few definitions

Lie Algebra.

A Lie algebra $\mathcal{G}$ on $\mathbb{R}$ is a vector space on $\mathbb{R}$ together with a bilinear map (Lie-Bracket)

$$\mathcal{G} \times \mathcal{G} \ni (x, y) \mapsto [x, y] \in \mathcal{G}\,,$$

such that

- $\forall x \in \mathcal{G}$, $[x, x] = 0\,$,
- Jacobi Identity holds:

$$\forall x, y, z \in \mathcal{G}, [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0\,.$$

Graded Lie Algebras

We only consider **graded Lie Algebras**, i.e. which can be written, for some $r \in \mathbb{N} \setminus \{0\}$, as a direct sum

$$\mathcal{G} = \oplus_{j=1}^{r} \mathcal{G}_j \,,$$

with

$$[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j} \text{ if } i + j \leq r$$

and

$$[\mathcal{G}_i, \mathcal{G}_j] = 0 \text{ if } i + j > r \,.$$

In addition, we assume that $\mathcal{G}$ is stratified, i.e. generated by $\mathcal{G}_1$.

We denote by $\mathcal{G}^{r,p}$ the maximal stratified algebra of rank r with p generators.

G and $\mathcal{G}$

One can put on $\mathcal{G}$ a group structure by using the Campbell-Hausdorff formula

$$a \circ b := a + b + \frac{1}{2}[a, b] + \frac{1}{12}([a, [a, b]] - [b, [a, b]]) + \cdots$$

In the nilpotent case, this Campbell-Hausdorff formula becomes finite. We can also write

$$G = \exp \mathcal{G} \,,$$

and we have with $g_1 = \exp a_1$ and $g_2 = \exp a_2$

$$g_1 \cdot g_2 = \exp(a_2 \circ b_2) \,.$$

The elements of $\mathcal{G}$ can be identified with the left invariant vector fields on the group G. (We can first identify $\mathcal{G}$ and the tangent space at the neutral element e $T_e G$.)

Enveloping Algebra

The enveloping algebra $\mathcal{U}(\mathcal{G})$ can be defined in the stratified case as the space of the non commutative polynomials in the form

$$P = \sum_{\alpha} a_\alpha Y^\alpha$$

where $Y^\alpha = Y_{\alpha_1} Y_{\alpha_2} \cdots Y_{\alpha_k}$, Y_i ($i = 1, \ldots, p$) is a basis of $\mathcal{G}_1$, $\alpha_\ell \in \{1, \ldots, p\}$ and $a_\alpha \in \mathbb{C}$.

We have a natural family of dilations defined by

$$\delta_t(\sum_{j=1}^{r} a_j) = \sum_{j=1}^{r} t^j a_j \,,\; a_j \in \mathcal{G}_j \,.$$

Using this dilation, we can introduce the subspace $\mathcal{U}_m(\mathcal{G})$ of the homogeneous elements

$$\delta_t P = t^m P \,.$$

For example, the operator $\sum_{i=1}^{p} Y_i^2$ belongs to $\mathcal{U}_2(\mathcal{G})$. Notice that it can also be considered as a left invariant operator on G, which is a particular case of the Hörmander operator.

Examples

We focus on two particular Lie Algebra.

Heisenberg group.
A basis of its Lie Algebra is given by Y_1, Y_2, Z, $[Y_1, Y_2] = Z$.
In exponential coordinates

$$Y_1 = \partial_{u_1} - \frac{1}{2} u_2 \partial_{u_3}\,,\; Y_2 = \partial_{u_2} + \frac{1}{2} u_1 \partial_{u_3}\,,\; Z = \partial_{u_3}\,.$$

Here $p = 2$, $r = 2$.

Engel group.
Y_1, Y_2, Z, W, $[Y_1, Y_2] = Z$, $[Y_1, [Y_1, Y_2] = W$, with
$[Y_2, [Y_1, Y_2]] = 0\,.$
In exponential coordinates

$$Y_1 = \partial_{u_1}\,,\; Y_2 = \partial_{u_2} + u_1 \partial_{u_3} + \frac{1}{2} u_1^2 \partial_{u_4}\,,\; Z = \partial_{u_3} + u_1 \partial_{u_4}\,,\; W = \partial_{u_4}\,.$$

Here $p = 2$ and $r = 3$.

Induced representations

Let $\mathcal{H}$ a subalgebra in $\mathcal{G}$ (respecting the stratification) of codimension k and $\ell : \mathcal{H} \mapsto \mathbb{R}$ a linear form such that

$$\ell([X, Y]) = 0\,,\ \forall X, Y \in \mathcal{H}\,.$$

One can show that one can find a basis e_j ($j = 1, \dots, k$) of a supplementary space to $\mathcal{H}$ (each e_j being homogeneous with respect to δ_t) such that for any $a \in \mathcal{G}$, we can write

$$\exp a = \exp h \cdot \exp t_k e_k \cdots \exp t_1 e_1 := \exp h \cdot \exp \gamma(t)\,,$$

where the map $a \mapsto (h, t)$ is a global diffeomorphism of $\mathcal{G}$ onto $\mathcal{H} \times \mathbb{R}^k$.

We then introduce $h(t,a)$ and $\sigma(t,a)$ by the relation

$$\gamma(t) \circ a = h(t,a) \circ \sigma(t,a)\,.$$

We can now define the induced representation $\pi_{\ell,\mathcal{H}}$ of the group G in $L^2(\mathbb{R}^k)$ by

$$(\pi_{\ell,\mathcal{H}}(\exp a) f)(t) = e^{i<\ell, h(t,a)>} f(\sigma(t,a))\,,\ \forall t \in \mathbb{R}^k, \forall a \in \mathcal{G}\,.$$

Note that for $k=0$, $L^2(\mathbb{R}^k) = \mathbb{C}$.
When $\ell = 0$, $\pi_{0,\mathcal{H}}$ is the natural representation (on the right) of G in $L^2(H\backslash G)$.

Induced representation of the Lie algebra

For $f \in \mathcal{S}(\mathbb{R}^k)$ and $a \in \mathcal{G}$ we define

$$\pi_{\ell,\mathcal{H}}(a)f = \frac{d}{ds}(\pi_{\ell,\mathcal{H}}(\exp sa)f)_{/s=0}$$

which after computation gives

$$\pi_{\ell,\mathcal{H}}(a) = i < \ell, h'(t,a) > + \sum_{j=1}^{k} \sigma_j'(t,a)\frac{\partial}{\partial t_j}\,,$$

where

$$h'(t,a) := \frac{d}{ds}h(t,sa)_{/s=0}\,,\, \sigma'(t,a) := \frac{d}{ds}\sigma(t,sa)_{/s=0}\,.$$

We can then naturally extend $\pi_{\ell,\mathcal{H}}$ to $\mathcal{U}(\mathcal{G})$.

Examples

- For $G =$ Heisenberg, $\mathcal{H} = \mathbb{R} Y_2$, $\ell = 0$, we get with $k = 2$,

$$X_1 := \pi_{0,\mathcal{H}}(Y_1) = \partial_{t_1}\,,\; X_2 := \pi_{0,\mathcal{H}}(Y_2) = t_1 \partial_{t_2}\,.$$

  $X_1^2 + X_2^2$ is a Baouendi-Grushin operator. The analysis of the hypoellipticity of $X_1^2 + X_2^2 + \lambda[X_1, X_2]$ is due to V. Grushin.

- For $G =$ Engel, $\mathcal{H} = \mathbb{R}\, Y_2$, $\ell = 0$, we get with $k = 3$,

$$\begin{aligned} X_1 &:= \pi_{0,\mathcal{H}}(Y_1) = \partial_{t_1} \\ X_2 &:= \pi_{0,\mathcal{H}}(Y_2) = \tfrac{1}{2} t_1^2 \partial_{t_3} + t_1 \partial_{t_2} \\ [X_1, X_2] &= \pi_{0,\mathcal{H}}(Z) = t_1 \partial_{t_3} + \partial_{t_2}\,. \\ [X_1[X_1, X_2]] &= \pi_{0,\mathcal{H}}(W) = \partial_{t_3}\,. \end{aligned}$$

- For $G =$ Engel, $\mathcal{H} = \mathbb{R}\, Y_2 + \mathbb{R} Z$, $\ell = 0$, we get with $k = 2$,

$$\begin{array}{rl} X_1 & := \pi_{0,\mathcal{H}}(Y_1) = \partial_{t_1} \\ X_2 & := \pi_{0,\mathcal{H}}(Y_2) = \frac{1}{2} t_1^2 \partial_{t_2} \\ [X_1, X_2] & = \pi_{0,\mathcal{H}}(Z) = t_1 \partial_{t_2} \,. \\ [X_1 [X_1, X_2]] & = \pi_{0,\mathcal{H}}(W) = \partial_{t_2} \,. \end{array}$$

$X_1^2 + X_2^2$ is a (more degenerate) Baouendi-Grushin operator:

$$X_1^2 + X_2^2 = \partial_{t_1^2} + \frac{1}{4} t_1^4 \partial_{t_2}^2 \,.$$

Kirillov's theory

For $\ell \in \mathcal{G}^*$, we consider the bilinear form on $\mathcal{G} \times \mathcal{G}$

$$B_\ell(x, y) = \langle \ell\,,\, [x, y] \rangle\,.$$

We now consider a subalgebra $\mathcal{H}$ which is isotropic for B_ℓ and look at the induced representation $\pi_{\ell,\mathcal{H}}$. One can show that $\pi_{\ell,\mathcal{H}}$ is irreducible iff $\mathrm{Codim}\mathcal{H} = \frac{1}{2}\mathrm{rank} B_\ell$.
Moreover for any $\ell \in \mathcal{G}^*$, there exists a (non unique) maximal $\mathcal{H}$.
Hence we can associate to each ℓ an irreducible unitary representation of G $\pi_{\ell,\mathcal{H}}$ which is unique up to unitary conjugation, hence defining a map κ of $\mathcal{G}^*$ to $\hat{G}$ the set of the irreducible representations of G.

This map is not injective. To understand this non injectivity we have to explain how G naturally acts on $\mathcal{G}^*$.
If $g = \exp a$ and $\ell \in \mathcal{G}^*$, we define (coadjoint action)

$$g \cdot \ell = \sum_{k=0}^{r} \frac{1}{k!} \mathrm{ad}^*(-a)^k \ell \,,$$

where

$$((\mathrm{ad}^*(b))\ell))(c) = \ell([b, c])\,.$$

Kirillov's theory says that the (equivalent class of the) representation π_ℓ depends only on the orbit of ℓ and that in this way we recover all the irreducible unitary representations of G.

Exercise 1. Irreducible representation of Heisenberg (by hand)

This presentation follows the way we use for the proof of Rockland's conjecture.
We will look at the representation of the corresponding Lie algebra: Y_1, Y_2, Z, $[Y_1, Y_2] = Z$ starting of the regular representation (in exponential coordinates)

$$Y_1 = \partial_{u_1} - \frac{1}{2} u_2 \partial_{u_3}\,,\; Y_2 = \partial_{u_2} + \frac{1}{2} u_1 \partial_{u_3}\,,\; Z = \partial_{u_3}\,.$$

A partial Fourier transform with respect to u_3 gives the family (parametrized by ℓ_3)

$$\pi_{\ell_3}(Y_1) = \partial_{u_1} - \frac{i}{2}\ell_3 u_2\,,\; Y_2 = \partial_{u_2} + \frac{i}{2}\ell_3 u_1\,,\; \pi_{\ell_3}(Z) = i\ell_3\,.$$

After a gauge transformation, we get the family

$$\tilde{\pi}_{\ell_3}(Y_1) = \partial_{u_1}\,,\; \tilde{\pi}_{\ell_3}(Y_2) = \partial_{u_2} + i\ell_3 u_1\,,\; \tilde{\pi}_{\ell_3}(Z) = i\ell_3\,.$$

This is clearly not irreducible. A partial Fourier transform in u_2 gives the family (parametrized by ℓ_2, ℓ_3)

$$\tilde{\pi}_{\ell_2,\ell_3}(Y_1) = \partial_{u_1}\,,\; \tilde{\pi}_{\ell_2,\ell_3}(Y_2) = i(\ell_2 + \ell_3 u_1)\,,\; \tilde{\pi}_{\ell_2,\ell_3}(Z) = i\ell_3\,.$$

This is irreducible if $\ell_3 \neq 0$. In this case, a translation in u_1 shows that it is enough to consider $\ell_2 = 0$. The orbit of $(0,0,\ell_3)$ is

$$\mathcal{O}((0,0,\ell_3)) = \{(\ell_1,\ell_2,\ell_3)\,,\, (\ell_1,\ell_2) \in \mathbb{R}^2\}\,.$$

If $\ell_3 = 0$, $\tilde{\pi}_{\ell_2,0}$ is not irreducible. A partial Fourier transform in u_1 gives

$$\pi_{\ell_1,\ell_2,0}(Y_1) = i\ell_1\,,\; \pi_{\ell_1,\ell_2,0}(Y_2) = i\ell_2\,,\; \pi_{\ell_1,\ell_2,0}(Z) = 0\,.$$

Rockland calls these representations the "degenerate" representations (corresponding with the ℓ vanishing on $\mathcal{G}_2 = \mathbb{R}Z$). The orbits are reduced to points.

Exercise 2. Engel.

We start from

$$Y_1 = \partial_{u_1}\,,\ Y_2 = \partial_{u_2} + u_1 \partial_{u_3} + \frac{1}{2} u_1^2 \partial_{u_4}\,,\ Z = \partial_{u_3} + u_1 \partial_{u_4}\,, W = \partial_{u_4}\,.$$

A Fourier transform in (u_2, u_3, u_4) leads to

$$\pi_{\ell_2,\ell_3,\ell_4}(Y_1) = \partial_{u_1}\,,\ \pi_{\ell_2,\ell_3,\ell_4}(Y_2) = i(\ell_2 + u_1 \ell_3 + \frac{\ell_4}{2} u_1^2)\,, \dots\,.$$

This is irreducible if $\ell_4 \neq 0$.

The orbit of $(0, \ell_2, \ell_3, \ell_4)$ is parametrized by (ℓ_1, β)

$$\mathcal{O}((0, \ell_2, \ell_3, \ell_4)) = \{(\ell_1, \ell_2 + \beta \ell_3 + \frac{1}{2} \beta^2 \ell_4, \ell_3 + \beta \ell_4, \ell_4), (\ell_1, \beta) \in \mathbb{R}^2\}$$

If $\ell_4 = 0$,

$$\begin{aligned} &\pi_{\ell_2,\ell_3,0}(Y_1) = \partial_{u_1}\,, \quad \pi_{\ell_2,\ell_3,0}(Y_2) = i(\ell_2 + u_1 \ell_3)\,,\\ &\pi_{\ell_2,\ell_3,0}(Z) = i\ell_3\,, \quad \pi_{\ell_2,\ell_3,0}(W) = 0\,. \end{aligned}$$

We can continue like for Heisenberg.

Rockland's conjecture

Theorem of Helffer-Nourrigat (1979)

Let $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_r$ a graded stratified Lie algebra and let $P \in \mathcal{U}_m(\mathcal{G})$, then the three following conditions are equivalent

1. P is hypoelliptic in G ($G = \exp \mathcal{G}$ is the associated Lie group and P is identified with a left invariant operator on G).
2. For any non trivial irreducible representation in $\widehat{G}$, $\pi(P)$ is injective in $\mathcal{S}_\pi$, the space of C^∞ vectors of the representation.
3. For any $Q \in \mathcal{U}_m(\mathcal{G})$, there exists C_Q s.t. for any $\pi \in \widehat{G}$, any $u \in \mathcal{S}_\pi$ we have

$$||\pi(Q)u||^2 \leq C_Q ||\pi(P)u||^2$$

Historics

- The formulation of the conjecture is due to Charles Rockland (1976) (published in (1978) [40]) who proves the conjecture in the case of the Heisenberg group.
- B. Helffer and R. Beals observed independently that when $r = 2$ the theorem, modulo the establishment of a dictionary, was a consequence of general theorems about the hypoellipticity of operators with multiple characteristics (J. Sjöstrand (1974), L. Boutet de Monvel (1974), Boutet de Monvel-Grigis-Helffer (1976)). The proof (for the two last papers) was based on a very nice class of pseudo-differential operators introduced by L. Boutet de Monvel and adapted to operators with multiple characteristics.
- Extension to nondifferential convolution operators is considered by P. Glowacki in [10].

- R. Beals (1977) also proves in full generality "(1) implies (2)". Helffer and Nourrigat prove that "(2) implies (3)" in two steps: first $r = 3$ (1978) and one year later the general case. Kirillov's theory [32] plays an important role but cannot be used as a black box.
  The feeling at this time was that one cannot use a standard class of pseudo-differential operators and that $r = 2$ was in some sense the limit for this kind of approach.
- Since this proof, only A. Melin (1981) gives a partially alternative proof using a group adapted pseudo-differential calculus but he can not avoid to use an important step of Helffer-Nourrigat's proof to complete his proof. See also later [5], P. Glowacki [11] and references therein.
- More properties of the so-called positive Rockland's operators are presented in the book of V. Fischer and M. Ruzhansky [8].

Towards Helffer-Nourrigat's conjecture

At about the same time appears the fundamental paper of Rothschild-Stein (1976) (C. Rockland is citing the paper which was submitted to Acta Mathematica in June 1975) which gives a new light on the paper of Lars Hörmander (1967) on the operator $\sum X_j^2 + X_0$, where the X_j's are vector fields satisfying the celebrated

Hörmander condition $(CH)_r$

The X_j and all their brackets up to rank r generate at each point the whole tangent space.

We write $(CH)_r(x)$ if the condition is satisfied at x.

One important step was that this condition implies

$$||u||_{1/r}^2 \leq C\Big(\sum_j ||X_j u||^2 + ||u||_2^2\Big)\,.$$

Except Kohn's paper (1973) giving an alternative easier proof of the hypoellipticity (but with weaker estimates), no progress was done except in the case $r = 2$ (see above).

From the PDE point of view, the interest of the paper by Rotschild-Stein was that they get maximal estimates for an operator in the form

$$P := \sum_{|\alpha|\leq m} a_\alpha(x) X^\alpha$$

i.e. it holds

$$\sum_{|\alpha|\leq m} ||X^\alpha u||^2 \leq C\left(||Pu||^2 + ||u||^2\right), \forall u \in C_0^\infty,$$

as a consequence of construction of a nice calculus modelled on nilpotent groups.
Note that the two inequalities imply hypoellipticity but maximal hypoellipticity is much stronger.

Without to enter in the details, I would like to mention the following points

- The lifting theorem (see also Folland, Hörmander–Melin, Helffer-Nourrigat). This lifting (addition of variable) permits to associate with a polynomial of vector fields $\sum_{|\alpha|\leq m} a_\alpha(x) X^\alpha$ an operator $\sum_{|\alpha|\leq m} a_\alpha(\lambda(x))\tilde{X}^\alpha$ where the $\tilde{X}_j$ are this time well approximated by corresponding Y_j generating a free nilpotent, stratified, Lie Algebra of rank r with p generators.
- Assuming that

$$\mathcal{P}_{x_0} := \sum_{|\alpha|=m} a_\alpha(x_0) Y^\alpha$$

is hypoelliptic for any x_0, a singular integral calculus for hypoelliptic operators which are polynomial of these vector fields.

If this approach worked perfectly well for $\sum_j X_j^2$ or more generally for $\sum_j X_j^{2k}$ (the lifted operator is hypoelliptic), this does not work in general. Hence the assumption that $\mathcal{P}_{x_0} := \sum_{|\alpha|=m} a_\alpha(x_0) Y^\alpha$ is hypoelliptic is too strong.
The first idea was to consider the case when the lifting can be done with a smaller Lie Algebra. This case was for example considered by L.P. Rothschild (1979) (see also G. Métivier for the corresponding theory) and combined with the proved Rockland's Conjecture.
Thinking of Rockland's conjecture and many particular cases (Grushin's like results) one is led to the formulation of our conjecture.

Conjecture of Helffer-Nourrigat (1979)

Conjecture

We assume that at some point x_0 the vector fields X_i satisfy $(CH)_r(x_0)$. Then there exists a closed subset $\widehat{\Gamma}_{x_0}$ in $\widehat{G}$ such that the following conditions are equivalent

1. P is maximally hypoelliptic in x_0
2. For any non trivial representation π in $\widehat{\Gamma}_{x_0}$, $\pi(\mathcal{P}_{x_0})$ is injective $\mathcal{S}_\pi$.

The conjecture gives in addition the candidate !

If λ is the lifting map, i.e. the unique linear application of $\mathcal{G}$ into the algebra of the vector fiels defined on Ω such that

$$\lambda(Y_i) = X_i$$

which is a partial homomorphism of rank r, we define λ_x by $\lambda_x(a) = \lambda(a)(x)$ and denote by λ_x^* the transposed map.

Definition of Γ_x

Assuming $(CH)_r(x_0)$ we introduce $\Gamma_{x_0} \subset \mathcal{G}^*$ as the set of the ℓ such that there exists a sequence (t_n, x_n, ξ_n) in $\mathbb{R}^+ \times T^*\Omega \setminus \{0\}$ such that

$$\left\{ \begin{array}{l} t_n \to 0\,,\ x_n \to x_0, |\xi_n| \to +\infty \\ t_n^r|\xi_n| \text{ is bounded} \\ \ell = \lim_{n\to+\infty} \delta_{t_n}^* \lambda_{x_n}^* \xi_n\,. \end{array} \right.$$

One can prove that Γ_{x_0} is a closed set in $\mathcal{G}^*$ which is invariant by dilation and by the coadjoint action of G on $\mathcal{G}^*$. By definition $\widehat{\Gamma}_{x_0}$ is the corresponding set (via Kirillov's theory) in $\widehat{G}$.

The book of 1985 by B. Helffer and J. Nourrigat [24]

The book is the result of five years of investigations around this conjecture by the two authors separately or together. It presents the proof of Rockland's conjecture in a self contained way.

Then it explores particular cases where the conjecture of Helffer-Nourrigat can be proved.

The book is also exploring cases where one can make Rockland's conditions more explicit, in particular for the analysis of problems connected with complex analysis $\bar{\partial}_b$, $\Box_b$.

The following result obtained by J. Nourrigat in 1987 ([36]) is enlightning for some of the techniques appearing in the proof of Rockland's Conjecture and other results in the book

Nourrigat's Theorem

Let F be a closed subset of $\mathcal{G}^*$ stable by dilation and the coadjoint action of G. Let $P \in \mathcal{U}_m(\mathcal{G})$.
Then if $\pi_\ell(P)$ is injective for any $\ell \in F \setminus \{0\}$, then for any $Q \in \mathcal{U}_m(\mathcal{G})$, there exists C_Q s.t. for any $\pi \in \widehat{F}$, any $u \in \mathcal{S}_\pi$ we have

$$||\pi(Q)u|| \leq C_Q ||\pi(P)u||$$

Note that this result can have many other applications than for Hypoellipticity. The case when $F = \mathcal{G}^*$ corresponds to Rockland's conjecture. The case when $F = \overline{G.\mathcal{H}^\perp}$ where $\mathcal{H}$ is a graded subalgebra of $\mathcal{G}$ appears also naturally and was analyzed in the book.

In 1998, W. Hebisch [14] (Theorem 2) gives a nice simple proof of this theorem, modulo the extension of Rockland's conjecture and some adapted pseudo-differential calculus due to M. Christ, D. Geller, P. Glowacki, and L. Polin [5].

Proof of the local conjecture in full generality

Around 40 years later the conjecture of Helffer-Nourrigat is proven (2022) by Iakovos Androulidakis, Omar Mohsen and Robert Yuncken [1] by mixing the proved Rockland's conjecture
These techniques can be completely avoided as shown in more recent proofs by Omar Mohsen
with techniques coming from the groupoids theory (see for the complete references in their paper and the Peccot course of Omar Mohsen at the Collège de France or this week in Lausanne).

As a consequence, one can recovers in a more general way results by H. Maire and many results mentioned in the two books [24] and [19] for semi-classical aspects.
Perhaps it is not the end of the story since a "microlocal" version of the conjecture was left open in 2022 (but see recent announcements of Omar Mohsen) and this is what we want to discuss now following mainly J. Nourrigat.

Microlocal questions

In order to present the problem "microlocally", one has

- first to mention a microlocalized version of Hörmander-Kohn inequality (due to Bolley-Camus-Nourrigat (1982) [3]),
- then to give a microlocalized definition of maximal estimates,
- finally to define the microlocal analog of Γ_x, i.e define the cone $\Gamma_{x,\xi}$.

The vector field is considered as a pseudo-differential operator of degree 1

To the vector field $X_j = \sum_k a_{jk}(x)\partial_{x_k}$, we can attach its symbol

$$U_j(x,\xi) = i\sum_k a_{jk}(x)\xi_k\,.$$

The symbol of $[X_j, X_k]$ is the Poisson bracket $-i\{U_j, U_k\}$.
In other words, X_j can be considered as a pseudo-differential operator of symbol U_j and of degree 1

Microlocalized Hörmander condition $(CH)_r(x_0,\xi_0)$

Let $(x_0,\xi_0) \in T^*\Omega \setminus \{0\}$. We say that $(CH)_r(x_0,\xi_0)$ holds if the system of the U_j and all their Poisson brackets up to rank r is elliptic at (x_0,ξ_0).

This definition immediately extends to pseudo-differential operators $U_j(x,D_x)$ of degree one with purely imaginary symbols.

Bolley-Camus-Nourrigat (1982) have shown in [3]

BoCaNo theorem

If $(CH)_r(x_0,\xi_0)$ holds, then there exists a pseudo-differential operator of degree 0 $\psi(x,D_x)$, elliptic at (x_0,ξ_0) such that

$$||\psi(x,D_x)u||^2_{1/r} \leq C\Big(\sum_j ||U_j(x,D_x)u||^2 + ||u||_2^2\Big)\,.$$

Here we recall the definition of a pseudo-differential operator:

$$u \mapsto (\psi(x,D_x)u)(x) = (2\pi)^{-n}\int e^{ix\xi}\psi(x,\xi)\hat{u}(\xi)d\xi\,.$$

Definition of Γ_{x_0,ξ_0}

Let $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$ and assume $(CH)_r(x_0, \xi_0)$.
$\Gamma_{x_0,\xi_0} \subset \mathcal{G}^*$ is the set of the ℓ such that there exists a sequence (t_n, x_n, ξ_n) in $\mathbb{R}^+ \times T^*\Omega \setminus \{0\}$ such that

$$\begin{cases} t_n \to 0\,,\ x_n \to x_0, |\xi_n| \to +\infty, \xi_n/|\xi_n| \to \xi_0/|\xi_0| \\ t_n^r|\xi_n| \text{ is bounded} \\ \ell = \lim_{n\to+\infty} \delta_{t_n}^* \lambda_{x_n}^* \xi_n\,. \end{cases}$$

The last condition can also be written in the following way:
For any bracket Y_I of length $|I|$ of the generators of $\mathcal{G}$ we have

$$\ell(Y_I) = (-i)^{|I|} \lim_{n\to+\infty} t_n^{|I|} U_I(x_n, \xi_n)\,,$$

where U_I denotes the iterated Poisson bracket of the symbols of the pseudo-differential operators U_i. In this way we can define Γ_{x_0,ξ_0} for a family of pseudo-differential operators of degree one U_i satisfying $(CH)_r(x_0, \xi_0)$. Note that Γ_{x_0,ξ_0} is a closed G-invariant cone.

Wave front set of a distribution

If Ω is an open set in $\mathbb{R}^n$ and u is a distribution in Ω the Wave front set is a cone in $T^*\Omega \setminus \{0\} := \Omega \times (\mathbb{R}^n \setminus \{0\}$. It is easier to define the "complementary".

(x_0, ξ_0) does not belong to the wave front set of the distribution of u, if there exists a neighborhood ω of x_0 and a conic neighborhood Γ of ξ_0 such that $\forall \phi \in C_0^\infty(\omega)$ the Fourier transform $\widehat{\phi u}(\xi)$ is rapidly decreasing in Γ.

Alternately there is a pseudo-differential operator of degree 0 elliptic at (x_0, ξ_0) such that $\psi(x, D)u$ is C^∞.

Maximal Microhypoellipticity

We consider an operator in the form

$$P := \sum_{|\alpha|\leq m} a_\alpha(x) U^\alpha$$

More generally one can replace $a_\alpha(x)$ by pseudo-differential operators of order 0: $a_\alpha(x, D_x)$.

We say that P is maximally microhypoelliptic at (x_0, ξ_0) if there exists a pseudo-differential operator of degree 0 $\psi(x, D_x)$, elliptic at (x_0, ξ_0) such that

$$\sum_{|\alpha|\leq m} ||\psi(x, D_x) U^\alpha u||^2 \leq C\left(||Pu||^2 + ||u||^2\right), \forall u \in C_0^\infty,$$

Together with $(CH)_r(x_0, \xi_0)$ this implies micro-hypoellipticity at (x_0, ξ_0).

Microlocal conjecture

Conjecture

We assume that at some point (x_0, ξ_0) the operators U_i satisfy $(CH)_r(x_0, \xi_0)$. Then the following conditions are equivalent

1. P is maximally microhypoelliptic at (x_0, ξ_0)
2. For any non trivial representation π in $\widehat{\Gamma}_{x_0,\xi_0}$, $\pi(\mathcal{P}_{x_0,\xi_0})$ is injective $\mathcal{S}_\pi$, where

$$\mathcal{P}_{x_0,\xi_0} := \sum_{|\alpha|=m} a_\alpha(x_0, \xi_0) Y^\alpha .$$

J. Nourrigat has shown the necessary part. The sufficient part is rather well understood when $r = 2$ since the end of the seventies. J. Nourrigat has shown in [37] the sufficiency part for a class of systems of order 1 (see in the next slides). The proof is extremely technical, and inspired by Fefferman-Phong techniques.

Applications by J. Nourrigat 1986-1991

J. Nourrigat has focused on a system of p pseudo-differential operators in the form

$$L_j = U_j + i\, U_{p+j}\,,$$

where the U_j's are pseudo-differential operators of order 1 with purely imaginary principal symbol.
Given $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n \setminus \{0\}$ and assuming to simplify $(CH)_r(x_0, \xi_0)$ we say that the system $(L_1, \dots, L_p)$ is microlocally maximally hypoelliptic at (x_0, ξ_0) if there exists a neighborhood V of x_0 and a pseudo-differential operator of order 0 elliptic at (x_0, ξ_0) such that

$$\sum_{j=1}^{2p} ||\psi(x, D) U_j u|| \leq C \Big(\sum_{j=1}^{p} ||L_j u|| + ||u|| \Big)\,,$$

for all $u \in C_0^\infty(V)$.

As a consequence, J. Nourrigat gives a new proof af the Egorov theorem for subelliptic operators, and an extension to overdetermined systems. In some cases, the abstract condition could be made more explicit: some functions should have no local minimum. The most interesting is perhaps the case on an overdetermined system of complex vector fields.

Following Nourrigat 1990-1991 [38] and a recent discussion with J. Nourrigat in Athenes (2023)

The case $p = 1$ is already quite interesting.

Egorov-Hörmander Theorem

We assume that $(CH)_r(x_0, \xi_0)$, $(\Psi)(x_0, \xi_0)$-condition and $\text{grad}_{x,\xi} U_1(x_0, \xi_0) \neq 0$. Then $U_1 + iU_2$ is maximally microhypoelliptic at (x_0, ξ_0).

Here the (Ψ) condition at (x_0, ξ_0) is defined by

$(\Psi)(x_0, \xi_0)$-condition

There exists a conic neighborhood W of (x_0, ξ_0) and there is no complex valued, non zero, positively homogeneous q in W such that $\mathrm{Im}\,(q(U_1 + iU_2)$ changes of sign from $+$ to $-$ when moving along a bicharacteristic curve of $\mathrm{Re}\,(q(U_1 + iU_2)$.

One can then apply the microlocal Rockland like criterion (proved in this case by J. Nourrigat) and determine rather explicitly the cone Γ_{x_0,ξ_0} when $U_1(x_0,\xi_0) = U_2(x_0,\xi_0) = 0$. Condition (Ψ) implies that the only irreducible representations involved in Γ_{x_0,ξ_0} can be defined on $L^2(\mathbb{R}^s)$ with $0 \leq s \leq 2$.

For $s = 2$ the condition (Ψ) implies that there exists a unitary operator U in $L^2(\mathbb{R}^2)$ such that

$$U^{-1}\pi(Y_1)U = \frac{\partial}{\partial x_1}$$

$$U^{-1}\pi(Y_2)U = A_2(x_1)\frac{\partial}{\partial x_2} + iB(x)$$

where B is a polynomial and

$$x_1 \mapsto F(x_1, x_2, \xi_2) := A_2(x_1)\xi_2 + B(x_1, x_2)$$

does not change of sign from $+$ to $-$ as x_1 is increasing.

It remains then to show that

$$\frac{\partial}{\partial x_1} + iA_2(x_1)\frac{\partial}{\partial x_2} - B(x_1, x_2)$$

is injective in $\mathcal{S}(\mathbb{R}^2)$. This can be shown by using the maximum principle.

When $s = 1$, (Ψ) implies that there exists a unitary operator U in $L^2(\mathbb{R})$ such that

$$U^{-1}\pi(Y_1)U = \frac{d}{dx}$$

$$U^{-1}\pi(Y_2)U = iB(x)$$

where B is a polynomial and

$$x \mapsto B(x)$$

does not change of sign from $+$ to $-$ as x is increasing.
It remains then to show that

$$\frac{d}{dx} - B(x)$$

is injective in $\mathcal{S}(\mathbb{R})$.

Maximal inequalities independently of hypoellipticity

Maximal estimates are not only important for proving hypoellipticity but can also be useful for the analysis of the domains of operators occuring in mathematical Physics. Many of these operators appear in the form $\pi(P)$ where π is not necessarily an irreducible representation, but simply an induced representation.

We will focus on the Schrödinger operator $-\Delta + V$ where V is not necessarily real and on the Fokker-Planck operator in order to present more recent results.

Applications to the Schrödinger operator

The aim is to review and compare the spectral properties of (the closed extension of) $-\Delta + U$ ($U \geq 0$) and $-\Delta + iV$ in $L^2(\mathbb{R}^d)$ for C^∞ potentials U or V with polynomial behavior.
The most recent results have been in collaboration with Y. Almog or with J. Nourrigat.
By maximal inequalities, we mean the existence of $C > 0$ s. t.

$$||u||^2_{H^2} + ||Uu||^2_{L^2} \leq C \left(||(-\Delta + U)u||^2_{L^2} + ||u||^2_{L^2} \right) , \forall u \in C_0^\infty(\mathbb{R}^d) , \tag{1}$$

or

$$||u||^2_{H^2} + ||Vu||^2 \leq C \left(||(-\Delta + iV)u||^2 + ||u||^2 \right) , \forall u \in C_0^\infty(\mathbb{R}^d) . \tag{2}$$

We can also discuss the magnetic case:

$$P_{\mathbf{A},\mathbf{V}} = -\Delta_A + W := \sum_{j=1}^{d} (D_{x_j} - A_j(x))^2 + W(x)\,,$$

(with $W = U + iV$) and the notion of maximal regularity is expressed in terms of the magnetic Sobolev spaces.

The question of analyzing $-\Delta + iV$ or more generally $P_{\mathbf{A},\mathbf{iV}} := -\Delta_A + iV$ appears in many situations:

- Time dependent Ginzburg-Landau theory:

$$(D_x^2 + (D_y - \frac{x^2}{2})^2 + iy$$

- Bloch-Torrey (complex Airy) equation

$$-\Delta + ix$$

Moreover, V does not satisfy necessarily a sign condition $V \leq 0$ as for dissipative systems.

Maximal regularity

We only mention L^2 estimates. In the case, when $U = \sum_j U_j(x)^2$, the maximal L^2 estimate is obtained as a byproduct of the analysis of the hypoellipticity.
This was then generalized to the case when U is a positive polynomial by J. Nourrigat in 1990 (unpublished) and then used by his PHD D. Guibourg [13] (1992), which considers the case when the electric potential is real $W = U \geq 0$ and the magnetic potential $\mathbf{A}$ are polynomials.
See also Zhong (1993), Z. Shen (1995) [44] and Mba-Yébé (1995).

The analysis of $-\Delta + iV$ seems more recent.

Maximal estimates for $-\Delta + iV$

For $V \in C^\infty$, we introduce:

- $(H1)$ $\exists C_2 \geq 1$ and $\exists r \in \mathbb{N}$ s.t. , $\forall x \in \mathbb{R}^d$, $\forall R > 0$,

$$\frac{1}{C_2} \sup_{|y-x| \leq R} |V(y)| \leq \sum_{|\alpha| \leq r} R^{|\alpha|} |\partial^\alpha V(x)| \leq C_2 \sup_{|y-x| \leq R} |V(y)| .$$

- $(H2(r))$ $\exists C_0 > 0$ and $\exists r \in \mathbb{N}$ s.t.

$$\max_{|\beta|=r+1} |D_x^\beta V(x)| \leq C_0 \, m(x) ,$$

where

$$m(x) := m_V^{(r)}(x) = \sqrt{\sum_{|\alpha| \leq r} |D_x^\alpha V(x)|^2 + 1} .$$

We note that any polynomial of degree r satisfies these conditions.

Main theorem

Theorem (Helffer-Nourrigat 2017)

If V satisfies for some $r \in \mathbb{N}$ assumptions (H1) and (H2), there exists $C > 0$ s.t. $\forall u \in C_0^\infty$

$$\|Vu\|^2 + \||V|^{1/2}\nabla u\|^2 \leq C\left(\|P_{iV}\, u\|^2 + ||u||^2\right) . \qquad (3)$$

Applications to the magnetic Fokker-Planck operator (2019-2022)

For $d = 2$ or 3, we consider the Kramers-Fokker-Planck operator K with an external electromagnetic field B_e defined on $\mathbb{R}^d$ with value in $\mathbb{R}^{d(d-1)/2}$ and an electric real valued potentiel V defined on $\mathbb{R}^d$:

$$K = v \cdot \nabla_x - \nabla_x V \cdot \nabla_v - (v \wedge B_e) \cdot \nabla_v - \Delta_v + v^2/4 - d/2, \tag{4}$$

where $v \in \mathbb{R}^d$ represents the velocity, $x \in \mathbb{R}^d$ represents the space variable. In the previous definition of our operator, $(v \wedge B_e) \cdot \nabla_v$ means:

$$(v \wedge B_e) \cdot \nabla_v = \begin{cases} b(x)\,(v_1 \partial_{v_2} - v_2 \partial_{v_1}) & \text{if } d = 2 \\ b_1(x)(v_2 \partial_{v_3} - v_3 \partial_{v_2}) + b_2(x)(v_3 \partial_{v_1} - v_1 \partial_{v_3}) & \\ + b_3(x)(v_1 \partial_{v_2} - v_2 \partial_{v_1}) & \text{if } d = 3. \end{cases}$$

The operator K is considered as an unbounded operator on $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ with domain

$$D(K) = C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)\,.$$

We denote by $\mathbf{K}$ the minimal extension of K where $D(\mathbf{K})$ is the closure of $D(K)$ with respect to the graph norm.

Maximal accretivity

The existence of a strongly continuous semi-group associated with the operator **K** is shown in [ZK1] (Karaki 2019) when the magnetic field is regular and [ZK2] (Karaki 2020) with weaker regularity, by combining with the results of Rothschild-Stein [43] for the operators introduced by Hörmander [26] in 1967:

$$\sum_j X_j^2 + X_0 \,.$$

Characterization of the domain

We are now interested in specifying the domain of the operator **K**. For this goal, we prove a maximal estimate for **K**, using nilpotent techniques

Theorem K (Helffer-Karaki 2022)

Let $d = 2$ or 3. Assume that $B_e \in C^1(\mathbb{R}^d, \mathbb{R}^{d(d-1)/2}) \cap L^\infty$ and $\exists C >$, $\exists \rho_0 > \frac{1}{3}$ and $\exists \gamma_0 < \frac{1}{3}$ s. t.

$$|\nabla_x B_e(x)| \leq C < \nabla V(x) >^{\gamma_0}\,, \tag{5}$$

$$|D_x^\alpha V(x)| \leq C < \nabla V(x) >^{1-\rho_0}\,, \forall \alpha \text{ s.t. } |\alpha| = 2\,, \tag{6}$$

Then $\exists C_1 > 0$ s. t. $\forall u \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\begin{aligned} &\|\,|\nabla V(x)|^{\frac{2}{3}}\, u\,\| + \|(v\cdot\nabla_x - \nabla_x V\cdot\nabla_v - (v\wedge B_e)\cdot\nabla_v)u\| + \|u\|_{\tilde{B}^2} \\ &\quad \leq C_1(\,\|Ku\| + \|u\|\,). \end{aligned} \tag{7}$$

Here

- $B^2_v(\mathbb{R}^d) := \{u \,|\, \forall(\alpha,\beta) \in \mathbb{N}^{2d},\, |\alpha| + |\beta| \leq 2\,,\; v^\alpha\, \partial_v^\beta\, u \in L^2\}$
- $\tilde{B}^2(\mathbb{R}^d \times \mathbb{R}^d) := L^2_x \widehat{\otimes} B^2_v$

Using the density of $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ in $D(\mathbf{K})$, we obtain:

Corollary B1

$$D(\mathbf{K}) = \{u \in \tilde{B}^2 \,| \\ (v \cdot \nabla_x - \nabla V(x) \cdot \nabla_v - (v \wedge B_e) \cdot \nabla_v)\, u \in L^2 \\ \text{and } |\nabla V|^{\frac{2}{3}}\, u \in L^2\}.$$

Note that if in addition $|\nabla V(x)|$ tends to $+\infty$ as $|x| \to +\infty$, then $\mathbf{K}$ has compact resolvent.

Notice also that if $\mathbf{K}$ has compact resolvent then the Witten-Laplacian $-\Delta + \frac{1}{4}|\nabla V|^2 - \frac{1}{2}\Delta V$ has compact resolvent (see Helffer-Nier for the case $B = 0$).

Proof of Theorem K: General strategy

The proof consists in constructing a graded and stratified algebra $\mathcal{G}$ of type 3, and, at any point $x \in \mathbb{R}^d$, an homogeneous element $\mathcal{F}_x$ in the enveloping algebra $\mathcal{U}_2(\mathcal{G})$ which satisfies the Rockland condition. As we have seen Helffer-Nourrigat's proof is based on maximal estimates which not only hold for the operator but also (and uniformly) for $\pi(\mathcal{F}_x)$ where π is any induced representation of the Lie Algebra.
It remains to find π_x s. t. $\pi_x(\mathcal{F}_x) = \mathcal{K}_x + 1$ is a good approximation of **K** in suitable balls and to patch together the estimates through a partition of unity as used by L. Hörmander in his Weyl calculus. Actually, we first define $\mathcal{K}_x$ and then look for the Lie Algebra, the operator and the induced representation.

Concretely, we construct a graded Lie algebra $\mathcal{G}$ verifying

- $\mathcal{G}_1$ is generated by $Y_{1,1}^{'}$, $Y_{2,1}^{'}$, $Y_{1,1}^{''}$ and $Y_{2,1}^{''}$,
- $\mathcal{G}_2$ is generated by $Y_{1,2}$ and $Y_{2,2}$
- $\mathcal{G}_3$ is generated by $Y_{1,3}$ and $Y_{2,3}$.

$$Y_{2,2} := [Y_{1,1}^{'}, Y_{1,1}^{''}] = [Y_{2,1}^{'}, Y_{2,1}^{''}],$$
$$Y_{1,3} := [Y_{1,2}, Y_{1,1}^{'}]\,,\; Y_{2,3} := [Y_{1,2}, Y_{2,1}^{'}]\,,$$

and

$$[Y_{1,1}^{'}, Y_{2,1}^{'}] = [Y_{1,1}^{''}, Y_{2,1}^{''}] = 0\,,$$
$$[Y_{j,1}^{'}, Y_{k,3}] = [Y_{j,1}^{''}, Y_{k,3}] = [Y_{k,3}, Y_{2,2}] = ... = 0 \quad , \forall j, k = 1, 2\,.$$

We then introduce

$$\mathcal{F}_{b'} = Y_{1,2} - \sum_{k=1}^{2} \left((Y'_{k,1})^2 + \frac{1}{4}(Y''_{k,1})^2 \right) - ib'_1 \left(Y'_{1,1} Y''_{2,1} - Y'_{2,1} Y''_{1,1} \right) - ib'_2 \left(Y''_{1,1} Y''_{2,1} + Y'_{1,1} Y'_{2,1} \right) \tag{8}$$

Conclusion

We have tried to explain the history of this Rockland conjecture and its proof, initially developed for the analysis of hypoellipticity but then more generally as a powerful tool for proving maximal estimates. We have also tried to explain that the microlocal aspects of the Helffer-Nourrigat conjecture are still to be understood.

Thanks for your attention.

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