A walk in Georgi Raikov's mathematical world
Promenade dans l'univers mathématique de Georgi Raikov.

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Introduction

It is of course impossible to present the whole contribution of G. Raikov to spectral theory and mathematical physics.
Hence I just propose[1] a walk through Georgi's world which sometimes has intersected my own world but most of the time explores other domains.
May be a good picture says more than a long talk, so let us start by reconsidering the picture realized by Constanza Rojas-Molina.

[1] Thanks to Vincent Bruneau for his help

When Constanza meets Georgi's world

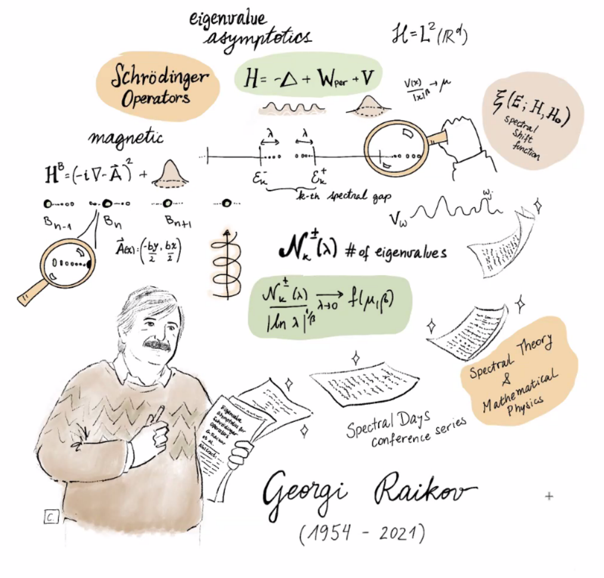


Figure: Picture realized by Constanza Roja-Molina

About his (academic) life.

Gueorgui (Georgi) Dimitrov Raikov is born in 1954 in Bulgaria.

He becomes student in Russia starting from 1975. He defended in 1986 his PHD (Maths and Physics) in Saint-Petersbourg (Leningrad at the time) with M.S. Birman as advisor.
The title of the thesis was:
The spectrum of some linear problems of magnetic hydrodynamics.

After coming back in Bulgaria, he defended a "bulgarian" Doctoral Thesis (Mathematics) in 1992 with the title:
Non-classical eigenvalue asymptotics for the Schrödinger operator with electromagnetic potential.

Academic Appointments.

- Research Fellow: Department of Mathematical Physics, Institute of Mathematics, Bulgarian Academy of Sciences: January 1987 - December 1992;
- Senior Research Fellow: Department of Analysis, Geometry and Topology; Department of Real and Functional Analysis; Department of Mathematical Physics, Institute of Mathematics, Bulgarian Academy of Sciences: since December 1992;
- Associate Professor: Department of Mathematics, University of Chile, Santiago, Chile: August 2002 - July 2006;
- Associate Professor: Faculty of Mathematics, Catholic University of Chile, Santiago, Chile: August 2006 - May 2012;
- Professor: Faculty of Mathematics, Catholic University of Chile, Santiago, Chile: since June 2012;

Visiting Positions.

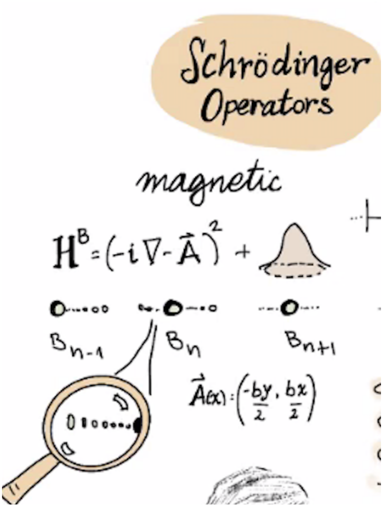
- University of Bordeaux 1, France: 1 semester, 1994, and numerous one month visits.
- University of Nantes, France: 1989, then 3 semesters, 1995, 1999, and 2000;
- University of Reims, France: 2 semesters, 1996 - 1997;
- Purdue University, USA: 1 semester, 2001;
- University of Chile, 2 semesters, 2001 - 2002;

Numerous contacts with France where he was hoping to get a position at the end of the nineties.

Many Collaborators around the world

M.S. Birman, J. Behrndt, M. Holzman, V. Lotoreichik, F. Klopp, S. Warzel, V. Bruneau (13), A. Pushnitski (3), J.-F. Bony (5), A. Khochman, M.A. Astaburuaga, P. Briet (8), C. Fernandez, T. Lungenstrass, C. Villegas-Blas, P. Miranda (4), D. Parra, H. Kovarik, P.D. Hislop, E. Soccorsi (3), E. Cardenas, I. Tejeda, W. Kirsch (3), D. Krejcirik, N. Dombrowski, F. Germinet, J.M. Combes, M. Dimassi, A. Mohamed(Morame).

Zoom inside the picture



At the end of the eighties, G. Raikov starts to move from the study of the magnetohydrodynamic to the analysis of Schrödinger with magnetic field. The common point remains spectral theory.
It seems that a visit of Georgi in Nantes in 1989 plays some role in the change of interest as is mentioned in the introduction of the survey he wrote with my former student A. Mohamed (Morame), a survey finally achieved in 1994 !
A. Morame wrote to me three days ago the following:
C'est une bonne chose de rendre hommage à G. Raikov! C'est toi qui trouvais que ses travaux et son premier passage à Nantes représentaient une opportunité de collaboration avec moi. ...Nous nous sommes mis d'accord sur un "survey", ... de rendre "lisible" pour les écoles de l'Est et de l'Ouest certains résultats fondamentaux sur Schrödinger avec Champ Magnétique. A ma grande surprise, il connaissait mieux que moi les travaux des grenoblois[2]*.*
I also visited Bulgaria during this period (invited by V. Petkov) and met G. Raikov at this occasion.

[2] Y. Colin de Verdière, J.P. Demailly.

On Schrödinger with magnetic field

Our basic object is the Schrödinger operator with constant magnetic field b and possibly perturbed by an electric potential V. The physical cases correspond to the dimension 2 and the dimension 3. For lack of time, we will mainly consider the $(2D)$-case.

$$H^B := H(A,V) = (-i\frac{\partial}{\partial x} + \frac{by}{2})^2 + (-i\frac{\partial}{\partial y} - \frac{bx}{2})^2 + V$$

The problem could be in $\mathbb{R}^2$, but also in an open set (possibly an exterior domain). One can also consider perturbations of the constant magnetic fields.

The case $V = 0$.

Let Λ_q $(q \geq 0)$ the Landau-Levels[3]

$$\Lambda_q = (2q + 1)b$$

which describe the spectrum when $V = 0$. Each of these eigenvalues has infinite multiplicity.
We can introduce the spectral projector Π_q relative to the Landau level Λ_q relative to H^0

[3] B_n in the picture

Perturbing by a potential V

When adding a potential V say with compact support it is standard that the essential spectrum is conserved. But eigenvalues can appear outside the Landau levels. It is then natural to ask if there is only a finite number of such eigenvalues or in the case it is infinite the question is how to count them. For this we need a magnifying glass in the neighborhood of one Landau level.
In this context, and to consider the operator

$$\Pi_q V \Pi_q \,,$$

The questions are the following:

- What is the link between the spectrum of $\Pi_q V \Pi_q$ and the eigenvalues close to Λ_q as $q \to +\infty$.
- What is the nature of the operator $\Pi_q V \Pi_q$ and how can we analyze its spectrum.

For notational convenience, we define

$$\Lambda_{-1} = \tilde{\Lambda}_{-1} = -\infty$$

For $q \in \mathbb{Z}_+$, we consider $\tilde{\Lambda}_q \in (\Lambda_q, \Lambda_{q+1})$, which is not an eigenvalue of $H(V, A)$. We can then introduce

$$\mathcal{N}_q^+(\lambda) = N_{(\Lambda_q+\lambda, \tilde{\Lambda}_q)}\,,\ \lambda \in (0, \tilde{\Lambda}_q - \Lambda_q)$$

and

$$\mathcal{N}_q^-(\lambda) = N_{(\tilde{\Lambda}_{q-1}, \Lambda_q-\lambda)}\,,\ \Lambda_q - \lambda \in (\tilde{\Lambda}_{q-1}, \Lambda_q)\,.$$

Further let $\lambda_{k,q}^+$ (resp. $\lambda_{k,q}^-$) be the non increasing (resp. non decreasing) sequence set of the eigenvalues of $H(V, A)$ lying in $(\Lambda_q, \tilde{\Lambda}_q)$ (resp. in $(\tilde{\Lambda}_{q-1}, \Lambda_q)$) and counted with multiplicity.

Set

$$m_q(V) = \#(\sigma(H(V,A)) \cap (\Lambda_{q-1}, \Lambda_q))\,.$$

Assume that $V \geq 0$. Then the discrete eigenvalues of $H(V,A)$ may only accumulate to a Landau Level from above (from below for $H(-V,A)$). Accordingly, we can choose $\tilde{\Lambda}_q$ so that $\sigma(H(V,A)) \cap (\tilde{\Lambda}_q, \Lambda_{q+1}) = \emptyset$.

Raikov-Warzel–Fixed Landau level.

Theorem of Raikov-Warzel (2002) and extensions Filonov-Pushnitski (2006) [1]

Assume that $V \in C^0(\mathbb{R}^2)$, $\mathrm{supp}\, V = \overline{\Omega}$ where $\Omega \subset \mathbb{R}^2$ is a bounded domain and $V > 0$ on Ω. Then for any $q \in \mathbb{Z}_+$ we have

$$m_q(\pm V) = +\infty\,,$$

and, as $q \to +\infty$,

$$\log\Big(\pm(\lambda^{\pm}_{k,q}(\pm V) - \Lambda_q)\Big) = -k \log k + \big(1 + \log(b\,\mathrm{Cap}(\Omega)^2/2)\big)k + o(k)\,.$$

This is actually an improvement of Raikov and Warzel [RaWa] which were only counting the eigenvalues and giving the main term of the asymptotic. At about the same time, one should mention a contribution by Melgaard and Rozenblum in any dimension.
See also Filonov-Pushnitski for this version of the statement.

Main steps of the initial result

We just consider the particular case. The first step is to introduce convenient orthonormal bases of the subspaces $\Pi_q L^2(\mathbb{R}^2)$.
For $\mathbf{x} = (x, y) \in \mathbb{R}^2$, $q \in \mathbb{Z}_+$ and $k \in \mathbb{Z}_+ - q$, we set

$$\varphi_{q,k}(\mathbf{x}) := c_{k,q,b}(x + iy)^k L_q^{(k)}\left(\frac{b(x^2 + y^2)}{2}\right) \exp(-b(x^2 + y^2)/4)\,,$$

where

$$L_q^{(\alpha)}(\xi) = \sum_{m=0}^{q} C_{q+\alpha}^{q-m} \frac{(-\xi)^m}{m!}\,,\ \xi \geq 0.\,,$$

are the generalized Laguerre polynomials.

We can introduce the integral kernel of Π_q

$$K_q(\mathbf{x},\mathbf{x}') = \sum_{k=-q}^{+\infty} \varphi_{q,k}(\mathbf{x})\overline{\varphi_{q,k}(\mathbf{x}')}\,,$$

and we have

$$K_q(\mathbf{x},\mathbf{x}') = \frac{b}{2\pi} L_q^{(0)}\left(\frac{b|\mathbf{x}-\mathbf{x}'|^2}{2}\right) \exp\left(-\frac{b}{4}((|\mathbf{x}-\mathbf{x}'|^2+2i(x'y-xy'))\right).$$

The next step consists in investigating the eigenvalue asymptotics of the so-called Toeplitz operator $\Pi_q V \Pi_q$.
We have seen another example in the talk of A. Pushnitski.
Here we observe [Ra0] (1990), assuming of V is radial, that the eigenvalues of $\Pi_q V \Pi_q$ are given by

$$\langle V\varphi_{q,k}, \varphi_{q,k}\rangle = \frac{q!}{(k+q)!}\int_0^{+\infty} V\big((\sqrt{(2\xi/b}, 0)\big)e^{-\xi}\xi^k L_q^{(k)}(\xi)^2\, d\xi\,.$$

This permits to obtain asymptotics of the corresponding eigenvalues.
We cannot hope such an explicit formula in more general cases.
The question is then to insert these explicit expressions in a more general calculus. The Berezin-Toeplitz calculus will play this role.

The last step consists in comparing the spectrum of $\Pi_q V \Pi_q$ with the spectrum of $H(V, A)$.

Comparison proposition

Let $E' \in (\Lambda_q, \Lambda_{q+1})$, $q \in \mathbb{Z}_+$. Assume that V satisfies "suitable assumptions" including $V \geq 0$. We can start with V with compact support ! Then for any $\epsilon \in (0,1)$ we have as $E \to O^+$

$$\begin{aligned} n_+(E; (1-\epsilon)\Pi_q V \Pi_q) + \mathcal{O}(1) \\ \leq N(\Lambda_q + E, E'; H(V)) \\ \leq n_+(E; (1+\epsilon)\Pi_q V \Pi_q) + \mathcal{O}(1) \,. \end{aligned}$$

The proof is based on the Birman Schwinger principle which gives

$$\begin{aligned} & N(\Lambda_q + E, E'; H(V)) \\ & = n_+(1; V^{1/2}(\Lambda_q + E - H(0))^{-1} V^{1/2}) \\ & \quad - n_+(1; V^{1/2}(E' - H(0))^{-1} V^{1/2}) \\ & \quad - \dim \operatorname{Ker}\big(H(V) - E'\big) \,. \end{aligned}$$

[PuRaVi] (2013) and [LuRa] (2014)

Theorem of G.R. with A. Pushnitski and T. Lungenstrass

Assume that V satisfies

$$|V(x)| \leq C < x >^{-\gamma}$$

with $\gamma > 0$.
Then there exists $\hat{C} > 0$ such that

$$\sigma(H(A,V)) \subset \cup_{q \in \mathbb{Z}_+} \left(\Lambda_q - \hat{C} \Lambda_q^{-\inf(1,\gamma)/2}, \Lambda_q + \hat{C} \Lambda_q^{-\inf(1,\gamma)/2} \right)$$

Then under suitable assumptions one can obtain more precise results by considering for example

$$\lim_{q \to +\infty} \Lambda_q^{-1/2} \int \rho(\lambda) d\mu_q(\lambda) \, ,$$

where

$$\mu_q([\alpha, \beta]) = \mathrm{rank} \mathbf{1}_{[\Lambda_q + \Lambda_q^{-1/2}\alpha, \Lambda_q + \Lambda_q^{-1/2}\beta]}(H) \, .$$

The limit is proved to be

$$\int \rho(\lambda) d\mu(\lambda) \, ,$$

where the measure μ is associated with b and the Radon transform of V

$$\tilde{V}(\omega, r) := \frac{1}{2\pi} \int_0^{+\infty} V(r\omega + t\omega^{\perp}) dt \, , \, \omega \in S^1, r > 0$$

by

$$\mu(\mathcal{O}) = \frac{1}{2\pi} |\tilde{V}^{-1}(b^{-1}\mathcal{O})| \, ,$$

where $| \cdot |$ denotes the Lebesgue measure.

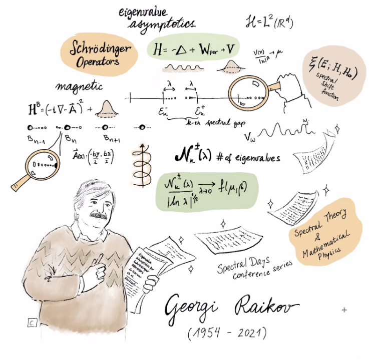
About other problems considered by Georgi.

I have not discussed

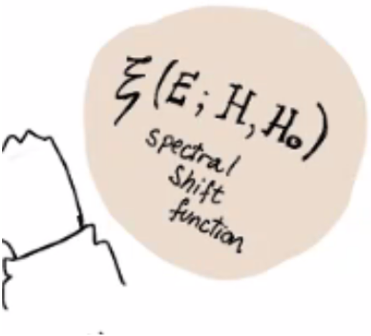
- various contributions for the strong magnetic field limit
- His analysis of the spectral shift function, the scattering phase and the analysis of resonances (contributions with V. Bruneau, P. Briet, J-F. Bony,....) (or scattering poles)

An important element in the analysis of these problems is the crucial role of the Berezin-Toeplitz operators as effective Hamiltonians in different asymptotic regimes.

The spectral shift function (SSF) is present at the top right corner of the picture.



May be you see it better with a zoom



or with a mathematical definition:

$$\mathrm{Tr}\left(f(H_V) - f(H_0)\right) = \int_{\mathbb{R}} \xi(H_V, H_0; \lambda)\, f'(\lambda)\, d\lambda\,,$$

for each $f \in C_0^\infty(\mathbb{R})$ with the convention that $\xi(H_V, H_0; \lambda) = 0$ for $\lambda < \inf \sigma(H_V)$.

A book !

The Berezin–Toeplitz Calculus is the main subject of a book that Georgi was preparing in collaboration with Vincent Bruneau who was his main collaborator since 2004 (13 common papers !). We have seen that a primitive version of this calculus already plays an important role in [RaWa]. Setsuro Fujie has also recalled in his talk that a preliminary version of this book was taught in Japan.

V. Bruneau was writing in the introduction of the last version of the book:

The aim of the book was to provide an overview of these results to which the authors have contributed. Unfortunately the Covid-19 did not let Georgi Raikov bring this book project to fruition. In March 2021, Georgi Raikov, instigator of this book, succumbed to Covid-19 when we had written most of the three Chapters 2, 3 and 4.

Dear Vincent, the best way to honour Georgi's memory is surely to achieve the book. I know that you are almost done !

Figure: Picture of Oberwolfach in 2018

Thanks to Georgi..

and

thanks for your attention.

[Selecta]

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