Lower and Upper bounds for the magnetic lowest Dirichlet-to-Neumann eigenvalue in the strong magnetic limit

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in collaboration with
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May 2025.

Abstract

Inspired by some questions presented in a recent ArXiv preprint (version v1) by T. Chakradhar, K. Gittins, G. Habib and N. Peyerimhoff, we analyze their conjecture that the ground state energy of the magnetic Dirichlet-to-Neumann operator tends to $+\infty$ as the magnetic field tends to $+\infty$. More precisely, we explore refined conjectures for general domains in $\mathbb{R}^2$ or $\mathbb{R}^3$ based on the previous analysis in the case of the half-plane and the disk by Helffer-Nicoleau. This is a work in collaboration with Ayman Kachmar and François Nicoleau.

Presentation

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 2$, with smooth boundary. For any $u \in \mathcal{D}'(\Omega)$, the magnetic Schrödinger operator on Ω is defined as

$$H_A \; u = (D - A)^2 u = -\Delta u - 2i \; A \cdot \nabla u + (A^2 - i \; \mathrm{div} \; A)u, \quad (1)$$

where $D = -i\nabla$, $-\Delta$ is the usual positive Laplace operator on $\mathbb{R}^n$ and $A = \sum_{j=1}^{n} A_j dx_j$ is the 1-form magnetic potential. We often identify the 1-form magnetic potential A with the vector field $\overrightarrow{A} = (A_1, ..., A_n)$.

We assume that $A \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$. The magnetic field is given by the 2-form $B = dA$.

Since zero does not belong to the spectrum of the Dirichlet realization of H_A, the boundary value problem

$$\begin{cases} H_A\, u &= 0 \text{ in } \Omega, \\ u_{|\partial\Omega} &= f \in H^{1/2}(\partial\Omega), \end{cases} \tag{2}$$

has a unique solution $u \in H^1(\Omega)$.
The Dirichlet-to-Neumann map (D-to-N map) is defined by

$$\begin{aligned} \Lambda_A : H^{1/2}(\partial\Omega) &\longmapsto H^{-1/2}(\partial\Omega) \\ f &\longmapsto (\partial_{\vec{\nu}} u + i\langle A, \vec{\nu}\rangle\ u)_{|\partial\Omega}\,, \end{aligned} \tag{3}$$

where $\vec{\nu}$ is the outward normal unit vector field on $\partial\Omega$.

More precisely, we define the D-to-N map using the equivalent weak formulation :

$$\langle \Lambda_A f, g \rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} = \int_\Omega \langle (D-A)u, (D-A)v \rangle \, dx \,, \quad (4)$$

for any $g \in H^{1/2}(\partial\Omega)$ and $f \in H^{1/2}(\partial\Omega)$ such that u is the unique solution of (2) and v is any element of $H^1(\Omega)$ so that $v_{|\partial\Omega} = g$. Clearly, the D-to-N map is a positive operator.

We recall that when Ω is bounded and regular he spectrum of the D-to-N operator is discrete and is given by an increasing sequence of eigenvalues

$$0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \leq \ldots \to +\infty \,.$$

Due to the variational characterization, the ground state energy

$$\mu_1 := \lambda^{\mathrm{DN}}(A, \Omega)$$

can be expressed as

$$\lambda^{\mathrm{DN}}(A, \Omega) = \inf_{u \in C^\infty(\overline{\Omega}),\ \|u\|_{\partial\Omega}=1} \|(-i\nabla - A)u\|_\Omega^2 \,,$$

where $\|u\|_\Omega$ and $\|u\|_{\partial\Omega}$ denote the L^2-norms in $L^2(\Omega;\mathbb{C})$ and $L^2(\partial\Omega;\mathbb{C})$ resp.

Comparison with the Neumann magnetic problem

Our analysis is parallel with the analysis of the Neumann magnetic problem for which there is a huge literature in the last forty years including two books (Fournais-Helffer, N. Raymond).
In this case the Neumann magnetic ground state is given by

$$\lambda^{\mathrm{Ne}}(A,\Omega) = \inf_{u \in C^\infty(\overline{\Omega}),\ \|u\|_\Omega = 1} \|(-i\nabla - A)u\|_\Omega^2\,,$$

We were also inspired by the recent work of T. Chakradhar, K. Gittins, G. Habib and N. Peyerimhoff, (see [5], Example 2.8).

We consider the following magnetic 1-form defined in the unit disk $D(0,1) \subset \mathbb{R}^2$ by :

$$A_0(x,y) = (-y dx + x dy), \tag{5}$$

It has been proven by Helffer-Nicoleau the following

Theorem HN1

One has the asymptotic expansion as $b \to +\infty$,

$$\lambda^{DN}(bA_0) = \alpha b^{1/2} - \frac{\alpha^2+2}{6} + \mathcal{O}(b^{-1/2}), \tag{6}$$

where $-\alpha$ is the unique negative zero of the so-called parabolic cylinder function $D_{\frac{1}{2}}(z)$.

We recall that the parabolic cylinder functions $D_\nu(z)$ are the (normalized) solutions of the differential equation

$$w'' + (\nu + \frac{1}{2} - \frac{z^2}{4})\, w = 0\,,$$

which tend to 0 as $z \to +\infty$.

At last, the positive real α appearing in this theorem is approximately equal to

$$\alpha = 0.7649508673....$$

The main idea for treating the case of more general domains, following what has been done in Surface Superconductivity, is to use the previous result for disks, the radius being locally chosen as the inverse of the curvature when it is positive.

The starting point is in the case of the disk $\mathcal{B}_R$ of radius R and $\mathrm{curl}\, A = 1$,

$$\lambda^{\mathrm{DN}}(bA, \mathcal{B}_R) = \hat{\alpha}\, b^{1/2} - \frac{\hat{\alpha}^2 + 1}{3} R^{-1} + \mathcal{O}(b^{-1/2})\,,$$

with

$$\hat{\alpha} = \alpha/\sqrt{2}\,.$$

The analysis in [24] can be adapted to the case $\mathcal{B}_R^{\mathrm{ext}}$, the exterior of the disk $\mathcal{B}_R$ (see also another work in progress with F. Nicoleau [25]), and we get

$$\lambda^{\mathrm{DN}}(bA, \mathcal{B}_R^{\mathrm{ext}}) = \hat{\alpha}\, b^{1/2} + \frac{\hat{\alpha}^2 + 1}{3} R^{-1} + \mathcal{O}(b^{-1/2})\,.$$

Hence we can also consider boundary points with negative curvature.

To cover later all the cases with one notation we introduce for $R \in \mathbb{R}$

$$\lambda^{\mathrm{DN}}(b, R) = \begin{cases} \lambda^{\mathrm{DN}}(bA_0, \mathcal{B}_R) & \text{if } R > 0 \\ \hat{\alpha}\, b^{1/2} & \text{if } R = 0 \\ \lambda^{\mathrm{DN}}(bA_0, \mathcal{B}^{\mathrm{ext}}_{-R}) & \text{if } R < 0\,, \end{cases}$$

and observe that

$$\lambda^{\mathrm{DN}}(b, R) = \hat{\alpha}\, b^{1/2} - \frac{\hat{\alpha}^2 + 1}{3} R^{-1} + \mathcal{O}(b^{-1/2})\,.$$

Comparison with Neumann

The result for Neumann was (Baumann-Phillips-Tang [3] 1998, Bernoff-Sternberg, Lu-Pan, Helffer-Morame, Fournais-Helffer,....) , for some "spectral invariant" $C_1 > 0$,

$$\lambda^{\mathrm{Ne}}(b, R) = 2\Theta_0 b - C_1 b^{\frac{1}{2}} R^{-1} + \mathcal{O}(1)\,.$$

For the half-plane (formally $R = +\infty$)

$$\lambda^{\mathrm{Ne}}(b, +\infty) = 2\Theta_0 b\,.$$

This corresponds with the analysis of the spectrum of the Neumann realization in $\{t > 0\}$

$$D_t^2 + (D_x + 2bt)^2$$

After partial Fourier transform and dilation, we get the family

$$\mathfrak{h}(\xi) := D_t^2 + (\xi + t)^2$$

We get

$$\Theta_0 = \inf_{\xi} \mu(\xi)$$

where $\mu(\xi)$ is the Neumann ground state energy of $\mathfrak{h}^{Ne}(\xi)$.

For the D-to-N operator we have to solve in $\mathbb{R}^+$

$$\mathfrak{h}(\xi)u_\xi = 0\,,\, u_\xi(0) = u_0(\xi)\,,$$

and then compute $u'_\xi(0)$.

Here we see how the special function D_ν appears

Saint-James picture in the case of the disk

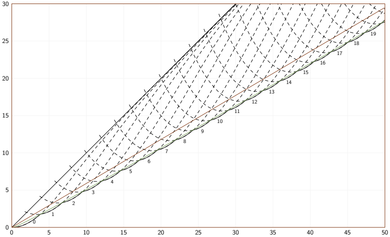

Figure: The magnetic Neumann eigenvalues

Our second result for the disk is concerned by diamagnetism. We recall that by diamagnetism we mean that $\lambda^{DN}(bA_0)$ is minimal for $b = 0$. This result has been proved in full generality in ter Elst-Ouhabaz [10] (see also Helffer-Nicoleau [23] for variants of this result).
We prove a strong diamagnetism result:

Theorem HN 2

The map $b \mapsto \lambda^{DN}(bA_0, \mathcal{B}_1)$ is increasing on $(0, +\infty)$.

Notice that the corresponding problem is open in the case of Neumann (see Fournais-Helffer (2007), Helffer-Léna (2024)) (only proven for $b \geq b_0$ or for $0 < b \leq b_1$).

Constant magnetic field in the disk

In polar coordinates (r, θ), the D-to-N map is defined by :

$$\begin{array}{rcl} \Lambda_{bA_0} : H^{\frac{1}{2}}(S^1) & \rightarrow & H^{-\frac{1}{2}}(S^1) \\ \Psi & \rightarrow & \partial_r v(r,\theta)|_{r=1}. \end{array} \tag{7}$$

Writing

$$v(r,\theta) = \sum_{n\in\mathbb{Z}} v_n(r) e^{in\theta} \quad , \quad \Psi(\theta) = \sum_{n\in\mathbb{Z}} \Psi_n e^{in\theta}, \tag{8}$$

we see that $v_n(r)$ solves:

$$\begin{cases} -v_n''(r) - \frac{v_n'(r)}{r} + (br - \frac{n}{r})^2 v_n(r) = 0 \quad \text{for } r \in (0,1), \\ v_n(1) = \Psi_n. \end{cases} \tag{9}$$

A bounded solution to the differential equation (9) is given by:

$$v_n(r) = e^{-\frac{br^2}{2}} r^n L^n_{-\frac{1}{2}}(br^2) \quad \text{for } n \geq 0, \tag{10}$$

where $L^\alpha_\nu(z)$ denotes the generalized Laguerre function. For $n \leq -1$, thanks to symmetries in (9), we get a similar expression for $v_n(r)$ changing the parameters (n,b) into $(-n,-b)$.

We recall that the generalized Laguerre functions $L_\nu^\alpha(z)$ satisfy the differential equation:

$$z\frac{d^2w}{dz^2} + (1+\alpha-z)\frac{dw}{dz} + \nu w = 0, \tag{11}$$

and are given by

$$L_\nu^\alpha(z) = \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)\Gamma(\nu+1)} M(-\nu, \alpha+1, z), \tag{12}$$

where $M(a, c, z)$ is the Kummer's confluent hypergeometric function, defined as

$$M(a, c, z) = \sum_{n=0}^{+\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}. \tag{13}$$

Here $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$.

For $0 < a < c$, we have the following formula:

$$M(a,c,z) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{c-a-1} dt \,. \qquad (14)$$

The derivative of the Kummer's function with respect to z is given by:

$$\partial_z M(a,c,z) = \frac{a}{c} \; M(a+1, c+1, z) \,. \qquad (15)$$

Now, let us return to the study of the Steklov eigenvalues. Obviously, they are given by

$$\lambda_n = \frac{v_n'(1)}{v_n(1)} \quad \text{for } n \in \mathbb{Z}. \tag{16}$$

Thus, using (10) and (12), we see that the Steklov spectrum is the set:

$$\sigma(\Lambda(b)) = \{\lambda_0(b)\} \cup \{\lambda_n(b), \lambda_n(-b)\ \}_{n\in\mathbb{N}^*}, \tag{17}$$

where for $n \geq 0$,

$$\lambda_n(b) = n - b + 2b\ \frac{\partial_z M(\frac{1}{2}, n+1, b)}{M(\frac{1}{2}, n+1, b)}. \tag{18}$$

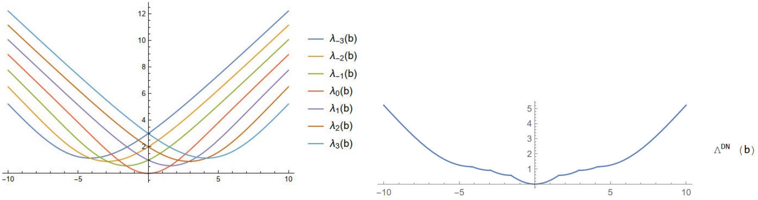

Figure: The magnetic Steklov eigenvalues $\lambda_n(b)$ (left) and the ground state energy $\lambda^{DN}(b)$ (right).

Constant magnetic field in general domains

We will prove the following theorem

Theorem HKN1

Let Ω be a regular domain in $\mathbb{R}^2$ and A be a vector potential with a magnetic field $B = \mathrm{curl\,A}$ such that B is C^1 on $\overline{\Omega}$ and $B = 1$ on a neighborhood of $\partial\Omega$. Then, the ground state energy of the D-to-N map Λ_{bA} satisfies

$$\lambda^{\mathrm{DN}}(bA, \Omega) = \hat{\alpha}\, b^{\frac{1}{2}} - \frac{\hat{\alpha}^2 + 1}{3} \max_{x \in \partial\Omega} k(x) + \mathcal{O}(b^{-1/6}) \quad , \quad b \to +\infty \, ,$$

where k is the curvature of $\partial\Omega$.

For comparison, the result in the case of the Neumann problem was:

Theorem LuPan–HeMo

Let Ω be a regular domain in $\mathbb{R}^2$ and A be a vector potential with a magnetic field $B = 1$. Then, the Neumann ground state energy satisfies

$$\lambda^{\mathrm{Ne}}(bA, \Omega) = \Theta_0 b - C_1 b^{1/2} \max_{x \in \partial\Omega} k(x) + \mathcal{O}(1) \quad , \quad b \to +\infty \, ,$$

where k is the curvature of $\partial\Omega$.

Coming back to ground state energy estimates

We will also consider the case of variable magnetic field in $2D$ and in $3D$ in the same spirit as for the analysis of the Neumann problem appearing in surface superconductivity [32, 20, 37, 38, 21].

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Theorem HKN2

Let Ω be a regular domain in $\mathbb{R}^2$, A be a magnetic potential with non vanishing magnetic field $B(x)$ in $\partial\Omega$, then the ground state energy of the D-to-N map Λ_{bA} satisfies

$$\lambda^{\mathrm{DN}}(bA,\Omega) = \hat{\alpha}(\inf_{x\in\partial\Omega} |B(x)|)^{\frac{1}{2}} b^{\frac{1}{2}} + o(b^{\frac{1}{2}})\,. \tag{19}$$

Neumann case

Theorem LuPan

Let Ω be a regular domain in $\mathbb{R}^2$, A be a magnetic potential with non vanishing magnetic field $B(x)$ in $\overline{\Omega}$, then the ground state energy of the Neumann realization of the magnetic Laplacian satisfies

$$\lambda^{\mathrm{Ne}}(bA,\Omega) = b\min\left(\inf_{x\in\Omega}|B(x)|, \Theta_0 \inf_{x\in\partial\Omega}|B(x)|\right) + o(b)\,. \tag{20}$$

Extension to $(3D)$

We have a similar theorem for variable magnetic fields in 3D which is in correspondence with known results obtained in the analysis of the ground state energy of the Neumann realization of the magnetic Laplacian (see [33, 21, 38]:

Theorem HKN3

Let Ω be a regular bounded domain in $\mathbb{R}^3$, A be a magnetic potential with non vanishing magnetic field $B(x)$ in $\partial\Omega$, then the ground state energy of the D-to-N map Λ_{bA}^{DN} satisfies

$$\lim_{b\to+\infty} b^{-1/2}\,\lambda^{\mathrm{DN}}(bA,\Omega) = \inf_{x\in\partial\Omega}\left(\lambda^{\mathrm{DN}}(\vartheta(x))|B(x)|^{\frac{1}{2}}\right), \qquad (21)$$

where, for $x \in \partial\Omega$,

- $\vartheta(x)$ is defined by $\langle \vec{H}(x)\,|\,\vec{\nu}\rangle = -|B(x)|\sin\vartheta(x)$.
- $\vec{H}(x)$ is the magnetic vector field associated with $B(x)$ considered as a 2-form by the Hodge-map
- $\vec{\nu}$ is the exterior normal at $x \in \partial\Omega$,
- $\lambda^{\mathrm{DN}}(\vartheta)$ is the ground state energy relative to the half space when the magnetic field is constant.

There are two important consequences

- When B is constant with magnitude 1, it follows that

$$\lim_{b\to+\infty} b^{-1/2}\,\lambda^{\mathrm{DN}}(bA,\Omega)=\hat{\alpha}.$$

- More generally, if we know only that $|B(x)|$ is constant, as for the helical magnetic field $B(x)=(\cos(\tau x_3),\sin(\tau x_3),0)$ encountered in liquid crystals [35, 17] (Pan, Helffer-Kachmar), then

$$\lim_{b\to+\infty} b^{-1/2}\,\lambda^{\mathrm{DN}}(bA,\Omega)=\inf_{x\in\partial\Omega}\lambda^{\mathrm{DN}}(\vartheta(x))\,.$$

  If $\partial\Omega$ is homeomorphic to S^2, then

$$\inf_{x\in\partial\Omega}\lambda^{\mathrm{DN}}(\vartheta(x))=\hat{\alpha}$$

Recent results by Z. Shen.

Using some parallel between Dirichlet, Neumann problem and the Dirichlet-to-Neumann problem, Z. Shen (2025) has considerably enlarged the number of results by considering possibly vanishing magnetic fields. Here we meet quantities which play an important role for other problems on magnetic bottles:

$$m_r(x) := \sum_{|\alpha|\leq r, i, j} |D_x^\alpha b_{ij}(x)|$$

and the results are obtained in any dimension. The asymptotics are nevertheless limited to the main term.

It is also interesting to look at the Robin problem [12], whose variational definition is

$$H^1(\Omega) \ni u \mapsto \Big(\|(-i\nabla - A)u\|_\Omega^2 + \gamma \int_{\partial\Omega} |u|^2 \Big) / \Big(\int_\Omega |u|^2 dx \Big) \,.$$

The problem is interesting in itself but also for technical reasons for the analysis of the D-to-N operator.

See Kachmar (2006) Robin condition is called De Gennes condition and more recently Fahs-Le Treust-Raymond-Vu Ngoc (2025).

Happy 75-76-77 birthdays Grigori !

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To be completed.