On the semi-classical analysis of the groundstate energy of the Dirichlet Pauli operator (after Helffer-P. Sundqvist and Helffer-Kowarik-P. Sundqvist) .

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Abstract

Motivated by a recent paper by Ekholm–Kowarik–Portmann, we analyze the semi-classical analysis of the ground state energy of the Dirichlet-Pauli operator. Tunneling effect can be measured with some analogy with the semi-classical analysis of the small eigenvalues of a Witten Laplacian, as analyzed in papers by Helffer-Sjöstrand, Helffer-Klein-Nier, Helffer-Nier,.... The presented works are in collaboration with Mikael Persson Sundqvist (University of Lund) (two papers [HPS1] and [HPS2]) and Hynek Kowarik and M. P. Sundqvist for one paper [HKPS].

Pauli operator

Let Ω be a connected, regular domain in $\mathbb{R}^2$, $B = B(x)$ be a magnetic field in $C^\infty(\bar{\Omega})$, and $h > 0$ a semiclassical parameter. We are interested in the analysis of the ground state energy $\lambda^D_{P_-}(h, \mathbf{A}, B, \Omega)$ of the Dirichlet realization of the Pauli operator

$$P_\pm(h, \mathbf{A}, B, \Omega) := (hD_{x_1} - A_1)^2 + (hD_{x_2} - A_2)^2 \pm hB(x)\,.$$

Here $D_{x_j} = -i\partial_{x_j}$ for $j = 1, 2$.
The vector potential $\mathbf{A} = (A_1, A_2)$ satisfies

$$B(x) = \partial_{x_1} A_2(x) - \partial_{x_2} A_1(x)\,. \qquad (1)$$

For the spectral analysis of $P_\pm(h, \mathbf{A}, B, \Omega)$, the reference to $\mathbf{A}$ is not necessary when Ω is simply connected (gauge invariance) but plays a role in the non simply connected case.

The Pauli operator is non-negative on $C_0^\infty(\Omega)$.
This follows by an integration by parts or think also of the square of the Dirac operator

$$D_A := \sum_j \sigma_j (hD_{x_j} - \mathbf{A}_j)\,,$$

where the σ_j ($j = 1, 2$) are the 2×2-Pauli matrices.
We have, on $C_0^\infty(\Omega; \mathbb{C}^2)$

$$D_A^2 := (P_-(h, \mathbf{A}, B, \Omega), P_+(h, \mathbf{A}, B, \Omega))\,.$$

This implies that

$$\lambda^D_{P_\pm}(h, \mathbf{A}, B, \Omega) \geq 0\,.$$

When $\Omega = \mathbb{R}^2$ and $B > 0$ constant, we have

$$\lambda_{P_-}(h, \mathbf{A}, B, \mathbb{R}^2) = 0\,,$$

and, under weak assumptions on $B(x)$ (see Helffer-Nourrigat-Wang (1989), Thaller (book 1992)),

$$\sigma_{ess}(P_-(h, \mathbf{A}, B, \mathbb{R}^2)) \cup \sigma_{ess}(P_+(h, \mathbf{A}, B, \mathbb{R}^2)) \neq \emptyset\,.$$

So a natural question is:

Question

What is going on when Ω is bounded ?

Three years ago, T. Ekholm, H. Kowarik and F. Portmann [2] give a lower bound which has a universal character

Theorem EKP

Let Ω be regular, bounded, simply connected in $\mathbb{R}^2$. If B does not vanish identically in Ω , $\exists\ \epsilon > 0$ s.t. $\forall h > 0$, $\forall \mathbf{A}$ s.t. $\mathrm{curl} \mathbf{A} = B$,

$$\lambda^D_{P\_}(h, \mathbf{A}, B, \Omega) \geq \lambda^D(\Omega)\, h^2 \exp\big(-\epsilon/h\big)\,. \qquad (2)$$

where $\lambda^D(\Omega)$ denotes the ground state energy of the Laplacian on Ω.

Main goals

It is clear that if $B < 0$, this estimate cannot be optimal.

Our goal is to determine, the optimal ϵ, to compare with exponentially small upper bounds [5] and to analyze the non simply connected case [6]. We will consider three cases:

- Ω simply connected, $B > 0$
- Ω non simply connected, $B > 0$
- Ω , B changing sign.

This will be done in the semi-classical limit: $h \to 0$.

The main theorem in HPS1 is

Theorem HPS1

If $B(x) > 0$, Ω is simply connected and if ψ_0 is the solution of

$$\Delta\psi_0 = B(x) \text{ in } \Omega\,,\ \psi_{0/\partial\Omega} = 0\,,$$

then, for any $h > 0$,

$$\lambda^D_{P\_}(h, B, \Omega) \geq \lambda^D(\Omega)\, h^2 \exp\big(2 \inf \psi_0 / h\big)\,. \tag{3}$$

and, in the semi-classical limit

$$\lim_{h\to 0} h \log \lambda^D_{P\_}(h, B, \Omega) \leq 2 \inf \psi_0\,.$$

In the non simply connected case, one is expecting that the result depends on the circulations of the magnetic potential along the components of the boundary.
Nevertheless the next theorem states that circulation effects disappear in the semi-classical limit for the rate of the exponential decay.

Theorem HPS2A

If $B(x) > 0$, Ω is connected, and if ψ_0 is the solution of

$$\Delta \psi_0 = B(x) \text{ in } \Omega\,,\; \psi_{0/\partial\Omega} = 0\,, \tag{4}$$

then, for any $\mathbf{A}$ such that $\mathrm{curl}\,\mathbf{A} = B$,

$$\lim_{h\to 0} h \log \lambda^D_{P\_}(h, \mathbf{A}, B, \Omega) = 2 \inf \psi_0\,. \tag{5}$$

Note that the lower bound is only true in the semi-classical limit. The proof will use strongly the gauge invariance of the problem.

What remains when B changes sign.

First we rediscuss [EKP]-theorem. The proof says in particular, without sign-condition on B,

Theorem (EKP-HPS)

Assume that Ω is a simply connected domain in $\mathbb{R}^2$ and that ψ_0 satisfies (4). Then

$$\lambda^D_{P_-}(h, \mathbf{A}, B, \Omega) \geq \lambda^D(\Omega)\, h^2 \exp\bigl(-2\mathrm{Osc}_\Omega \psi_0 / h\bigr)\,. \tag{6}$$

Note that this lower bound is universal.

In the non-simply connected ψ_0 cannot be used for a universal lower bound.

Remark

If $B > 0$ then, by the maximum principle $\psi_0 < 0$ in Ω, hence

$$\mathrm{Osc}_\Omega \psi_0 = -\inf_\Omega \psi_0 \,.$$

For B with varying sign, it might still be the case that ψ_0 is of constant sign in Ω, but that will depend on B, and the situation is delicate.
We can propose simple examples for the various situations.

Remarks (continued)

With in mind the identity

$$\lambda^D_{P_-}(h, B, \Omega) = \lambda^D_{P_+}(h, -B, \Omega)$$

we focus on the eigenvalue $\lambda^D_{P_-}(h, B, \Omega)$.

Actually, the natural question for Pauli is at the end to analyze

$$\min(\lambda^D_{P_-}(h, B, \Omega), \lambda^D_{P_+}(h, B, \Omega)) \,.$$

Other boundary conditions ?

The Dirichlet condition is natural as a mathematical problem but one could think of more physical boundary conditions.

They should be connected to boundary conditions for the magnetic Dirac Operator D_B, the pair of Pauli operators appearing as the square D_B^2, with the resulting boundary conditions (see Letreust-Raymond-..., Barbaroux-Stockmeyer-...).

Dirichlet conditions have NOT this property.

Further results

The first result concerns the case when ψ_0 attains negative values in Ω. Without to assume a sign condition on B we have:

Theorem HPS2B

Assume that Ω is a simply connected domain in $\mathbb{R}^2$, and that $\inf_\Omega \psi_0 < 0$, where ψ_0 is satisfying (4). Then

$$\limsup_{h\to 0} h \log \lambda^D_{P\_}(h, B, \Omega) \leq 2 \inf_\Omega \psi_0 .$$

We define

$$\Omega_B^{\pm} = \{\pm B > 0\}$$

and will now turn to the case when both of them are non-void. We assume in addition that $\Gamma := B^{-1}(0) \subset \overline{\Omega}$ is of class $C^{2,+}$ and that $\Gamma \cap \partial\Omega$ is either empty or, if non empty, that the intersection is a finite set, avoiding the corner points, with transversal intersection.

Under this assumption $\Omega_B^{\pm}$ satisfies the same condition as Ω and we will denote by $\hat{\psi}_0$ the solution of

$$\Delta\hat{\psi}_0 = B(x) \text{ in } \Omega_B^+, \quad \hat{\psi}_0 = 0 \text{ on } \partial\Omega_B^+. \tag{7}$$

By domain monotonicity, with $\Omega_B^+ \subset \Omega$, we can apply Theorem HPS2 with Ω replaced by Ω_B^+ and get

Corollary HKPS

Assume that Ω is a connected domain in $\mathbb{R}^2$. Assume further that $\hat{\psi}_0$ satisfies (7). Then

$$\limsup_{h\to 0} h \log \lambda_-^D(h, B, \Omega) \leq 2 \inf_{\Omega_B^+} \hat{\psi}_0 \,. \qquad (8)$$

Now, the main problem is to determine if one of the bounds above, i.e. (6) and (8), is optimal.

Without to have a complete answer, we propose rather trivial but enlightening statements on this question. The first one gives a simple criterion under which the upper bound given above is not optimal.

Theorem HKPS1

Assume that Ω is a simply connected domain in $\mathbb{R}^2$, $\Omega_B^+ \neq \varnothing$, and that $\hat{\psi}_0$ satisfies (7). If $B^{-1}(0)$ is

- either a compact $C^{2,+}$ closed curve in Ω
- or a $C^{2,+}$ line crossing $\partial\Omega$ transversally away from the corners,

then

$$\limsup_{h\to 0} h \log \lambda_{P_-}^D(h, B, \Omega) < 2 \inf_{\Omega_B^+} \hat{\psi}_0 \,.$$

Note that this is a STRICT inequality. Two examples where this condition is satisfied are when Ω is a disk, and the magnetic field is either radial, vanishing on a circle, or affine, vanishing on a line. We will come back to the second example at the end of the talk.

We mentioned earlier that even though B changes sign, it might happen that the scalar potential ψ_0 does not. Our second statement says that in this case we actually have the optimal result.

Theorem HKPS 2

Assume Ω is a simply connected domain in $\mathbb{R}^2$.
If $\psi_0 < 0$ in Ω, where ψ_0 is the solution of (4), then

$$\lim_{h \to 0} h \log \lambda^D_{P_-}(h, B, \Omega) = 2 \inf_{\Omega} \psi_0 \,.$$

Our final result concerns a case when ψ_0 changes sign.

Theorem HKPS 3

Assume that $\psi_{\min} = \inf_\Omega \psi_0 < 0 < \sup_\Omega \psi_0 = \psi_{\max}$.
Assume further that $\psi_0^{-1}(\psi_{\max})$ contains a closed curve enclosing a non-empty part of $\psi_0^{-1}(\psi_{\min})$ or the same with exchange of min and max.
Then

$$\lim_{h\to 0} h \log \Lambda^D(h, B, \Omega) = -2\mathrm{Osc}\, \psi_0 \,, \tag{9}$$

where

$$\Lambda^D(h, B, \Omega) = \min \left(\lambda^D_{P_-}(h, B, \Omega), \lambda^D_{P_+}(h, B, \Omega) \right) \,.$$

The main application concerns the disk with a radial magnetic field.

Canonical choice of the magnetic potential

Following what is done for example in superconductivity, given some magnetic potential $\mathbf{A}$ in Ω satisfying curl $\mathbf{A} = B$, we can after a gauge transformation assume that $\mathbf{A}$ satisfies in addition

$$\operatorname{div} \mathbf{A} = 0 \text{ in } \Omega\,;\ \mathbf{A} \cdot \nu = 0 \text{ on } \partial\Omega\,. \tag{10}$$

If this is not satisfied for say $\mathbf{A}_0$, we can construct

$$\mathbf{A} = \mathbf{A}_0 + \nabla\phi$$

satisfying in addition (10), by choosing ϕ as a solution of

$$-\Delta\phi = \operatorname{div} \mathbf{A}_0 \text{ in } \Omega\,;\ \nabla\phi \cdot \nu = -\mathbf{A}_0 \cdot \nu \text{ on } \partial\Omega\,.$$

From now on, we suppose that $\mathbf{A}$ satisfies (10). In this case and when Ω is simply connected

$$\mathbf{A} = \nabla^\perp \psi_0\,,$$

where ψ_0 is a solution of (4).

The role of the circulations in the case with holes

The second point to observe is a standard proposition (also present in Hodge-De Rham theory)

Proposition HdR

Let Ω be an open connected set with k holes Ω_j. Given B in $C^\infty(\bar{\Omega})$ and k real numbers Φ_j, then there exists a unique $\mathbf{A}(\Phi)$ satisfying (1), (10) and

$$\int_{\partial\Omega_j} \mathbf{A}(\Phi) = \Phi_j \tag{11}$$

Hence, the only relevant parameters in the analysis of spectral problems for the Pauli operator are the magnetic field B and the circulations Φ_j along the boundaries of the holes (if any).

Generating function

Associated with $\mathbf{A} := \mathbf{A}(\Phi)$ satisfying (10), $\exists$ a unique ψ^{Φ} s.t.

$$\mathbf{A}(\Phi) = \nabla^{\perp}\psi^{\Phi}$$

and $\psi^{\Phi}_{/\partial\widetilde{\Omega}} = 0\,,$ where $\widetilde{\Omega}$ is the simply connected envelope of Ω.
This ψ^{Φ} is constant on each connected component of the boundary $\partial\Omega$.
If we denote by $p_k(\Phi)$ the value of the trace of ψ^{Φ} on ∂D_k, one can show

Lemma

The map $\mathbb{R}^k \ni \Phi \mapsto p(\Phi) \in \mathbb{R}^k$ is a bijection from $\mathbb{R}^k$ onto $\mathbb{R}^k$.

We note that Ψ^{Φ} is the unique solution of

$$\Delta\Psi^{\Phi} = B\,,\ \psi^{\Phi}_{/\partial D_k} = p(\Phi)\,,\ \psi^{\Phi}_{/\partial\widetilde{\Omega}} = 0\,. \tag{12}$$

Remarks and notation

This is a rather standard fact in Hodge-De Rham theory in the case with boundary.
We write

$$\psi_p = \psi^{\Phi}\,, \text{ when } p = p(\Phi)\,,$$

where

$$\Delta \Psi_p = B\,,\ (\psi_p)_{/\partial D_k} = p\,,\ (\psi_p)/\partial\widetilde{\Omega} = 0\,. \tag{13}$$

In the case, with no hole ($k = 0$), there is no Φ and we recover ψ_0 as given previously.

Variation of the oscillation of ψ_p

The oscillation is defined by

$$\mathrm{Osc}(\psi_p) = \sup \psi_p - \inf \psi_p \,.$$

When $p = 0$ and $B > 0$

$$\mathrm{Osc}(\psi_0) = - \inf \psi_0 \,. \tag{14}$$

Using that $p \mapsto \psi_p - \psi_0$ is a linear map, more precisely:

$$\psi_p - \psi_0 = \sum_j p_j \, \theta_j \,,$$

with

$$0 \leq \theta_j \leq 1 \,,\, (\theta_j)_{/\partial D_i} = \delta_{ij} \,,\;\; (\theta_j)_{/\partial\Omega} = 0 \,,\; \text{ and } \Delta\theta_j = 0$$

we get:

$$|\mathrm{Osc}(\psi_p) - \mathrm{Osc}(\psi_0)| \leq \sum_j |p_j| \,. \tag{15}$$

Variation of the oscillation of ψ^Φ

Similarly, observing that

$$(\Phi - \Phi^0)_j = \sum_{\ell=1}^{k} M_{j\ell}\, p_\ell$$

with M invertible and Φ^0 the family of circulations such that $p(\Phi^0) = 0$, we get, for some constant $C > 0$,

$$|\mathrm{Osc}(\psi^\Phi) - \mathrm{Osc}(\psi_0)| \leq C\,|\Phi - \Phi^0|\,. \tag{16}$$

Isospectrality

In the non simply connected case, it is important to have in mind the following standard proposition:

Proposition Gauge equivalence

Let $\Omega \subset \mathbb{R}^2$ be bounded and connected, $\mathbf{A} \in C^1(\overline{\Omega})$, $\widetilde{\mathbf{A}} \in C^1(\overline{\Omega})$ satisfying

$$\mathrm{curl}\,\mathbf{A} = \mathrm{curl}\,\widetilde{\mathbf{A}}\ , \tag{17}$$

and

$$\frac{1}{2\pi h}\int_\gamma (\mathbf{A} - \widetilde{\mathbf{A}}) \in \mathbb{Z}\,, \text{on any closed path } \gamma \text{ in } \Omega \tag{18}$$

then the associated Dirichlet realizations of $(hD - \mathbf{A})^2 + V$ and $(hD - \widetilde{\mathbf{A}})^2 + V$ are unitary equivalent.

This can be applied to the Pauli operator with $V = \pm hB$.

Towards lower bounds: EKP-approach.

To get lower bounds, we consider, following EKP, the expression $\|(hD - \mathbf{A})u\|^2$ by writing

$$u = \exp\left(-\frac{\psi^\Phi}{h}\right) v\,.$$

We have if $\mathbf{A}$ satisfies (1) and (10):

$$\begin{aligned} &\|(hD - \mathbf{A})u\|^2 - h\int_\Omega B(x)|u(x)|^2\,dx \\ &\quad = h^2 \int_\Omega \exp\left(-2\frac{\psi^\Phi}{h}\right)|(\partial_{x_1} + i\partial_{x_2})v|^2\,dx\,. \end{aligned} \tag{19}$$

With u (and consequently v) in $H_0^1(\Omega)$, this implies

$$\begin{aligned} &\|(hD - \mathbf{A})u\|^2 - h\int_\Omega B(x)|u(x)|^2\,dx \\ &\quad \geq h^2 \exp\left(\frac{-2(\sup\psi^\Phi)}{h}\right)\int_\Omega |(\partial_{x_1} + i\partial_{x_2})v|^2\,dx \\ &\quad \geq h^2 \exp\left(\frac{-2(\sup\psi^\Phi)}{h}\right)\int_\Omega |\nabla v|^2\,dx \\ &\quad \geq \lambda^D(\Omega)h^2 \exp\left(\frac{-2\mathrm{Osc}(\psi^\Phi)}{h}\right)\int_\Omega |u|^2 dx\,. \end{aligned}$$

So we have obtained

Theorem EKP Lowerbound

Assume that Ω is a bounded connected domain. Then

$$\lambda_{P\_}^D(h, \mathbf{A}, B, \Omega) \geq h^2 \lambda^D(\Omega) \exp\left(\frac{-2\mathrm{Osc}(\psi^\Phi)}{h}\right). \tag{20}$$

Note that the statement is just the statement given in [EKP]-paper, but with a special choice of ψ^Φ. This gives the existence of some $\epsilon > 0$ but we can be far from optimality. When Ω has no hole, we recover when $B > 0$, having in mind (15), the lower bound statement in "Theorem-HPS1" as established in [HPS1].

Implementing gauge invariance

Implementing the gauge invariance and the control of the oscillation established in (16), we obtain

Theorem HPS2B

Assume that Ω is a bounded connected domain. Then there exists $C > 0$ such that, for any $\Phi \in \mathbb{Z}^k$, any $h > 0$

$$\lambda^D_{P\_}(h, \mathbf{A}, B, \Omega) \geq \lambda^D(\Omega) h^2 \exp -\frac{C}{h}\, d(\Phi_0, \Phi + 2\pi\mathbb{Z}^k h) \exp\left(\frac{-2\mathrm{Osc}(\psi_0)}{h}\right). \tag{21}$$

When $B > 0$, we get

$$\lambda^D_{P\_}(h, \mathbf{A}, B, \Omega) \geq h^2 \lambda^D(\Omega) \exp -\frac{C}{h}\, d(\Phi_0, \Phi + 2\pi\mathbb{Z}^k h) \exp\left(\frac{2(\inf \psi_0)}{h}\right). \tag{22}$$

Upper bounds in the non-simply connected case

In the simply connected case, with the explicit choice of ψ_0, it is easy to get:

Proposition HPS1

Under assumption $B > 0$ and assuming that Ω is simply connected, we have, for any $\eta > 0$, there exists $C_\eta > 0$

$$\lambda^D_{P\_}(h, B, \Omega) \leq C \exp \frac{2 \inf \psi_0}{h} \exp \frac{2\eta}{h} \,. \tag{23}$$

The proof is obtained by taking as trial state $u = \exp -\frac{\psi_0}{h} v_\eta$, with v_η with compact support in Ω and $v_\eta = 1$ outside a sufficiently small neighborhood of the boundary and implementing this quasimode in (29). One concludes by the max-min principle.

It has been shown in [HPS1] how to have a (probably) optimal upper bound by using as trial state

$$u = \exp(-\frac{\psi_0}{h}) \ - \ \exp(\frac{\psi_0}{h})\,.$$

This is at least the case for the disk. For the disk, we refer to Erdös, Helffer-Mohamed, Fournais-Helffer (for these two references, note the same mystake), Ekholm-Kowarik-Portman, and the optimal statement in HPS1.

In the non simply connected case the situation is much more complicate.

We can get general results by considering an $\check{\Omega}$ simply connected in Ω. This proves that $\lambda^D_{P_-}(h, \mathbf{A}, B, \Omega)$ is indeed exponentially small, independently of the circulations along each component of the boundary.

When, for some ψ, ψ_{min} is attained at a point in Ω i.e. if

$$p_j := \psi/\partial\Omega_j > \psi_{min}\,, \tag{24}$$

for $j = 0, \cdots, k$, where $\partial\Omega_0 = \partial\tilde{\Omega}$.

In this case, one gets

Proposition

If $\psi/\partial\Omega_j > \psi_{min}$, then, for any $\eta > 0$, there exists $C_\eta > 0$ and h_η such that, for $h \in (0, h_\eta)$,

$$\lambda^D_{P_-}(h, \mathbf{A}, B, \Omega) \leq C \exp \frac{2(\inf \psi - \inf_j \psi/\partial\Omega_j)}{h} \exp \frac{2\eta}{h}\,. \tag{25}$$

We observe that the assumption $\psi/\partial\Omega_j > \inf \psi$ is stable when a small variation of the Φ is performed. This assumption is evidently satisfied for $B > 0$ and $\Phi = \Phi_0$.
By gauge transformation, we can find $\alpha \in \mathbb{Z}^k$ such that $\Phi_h := \Phi + 2\pi\alpha h$ is $O(h)$-close to Φ_0 and similarly $p(\Phi_h)$ is $\mathcal{O}(h)$. We can then apply the previous proposition to ψ^{Φ_h} which is close to ψ_0.

The case when $B(x)$ changes sign

The proof of Theorem HKPS1 is based on an argument of "pushing the boundary".
We have seen that we can have $\psi < 0$ in Ω without to assume $B > 0$ and that once this property is satisfied we can obtain an upper bound by restricting to the subset of Ω where ψ is negative instead.
Hence a natural idea is to consider the family of subdomains of Ω defined by

$$\mathcal{F} = \{\omega \subset \Omega,\, \partial\omega \in C^{2,+}_{\mathrm{pw}} :\ \Delta\psi = B \text{ in } \omega \ \text{ and } \ \psi|_{\partial\omega} = 0 \ \Rightarrow\ \psi < 0 \text{ in } \ \omega\}$$

$\Omega_B^+ \in \mathcal{F}$.
The idea behind the proof of Theorem HKPS1 is to show that there exists $\omega \in \mathcal{F}$ s.t. $\Omega_B^+ \subset \omega$ with strict inclusion.

Pushing the boundary

More precisely, we have

Proposition HKPS

Let $\omega \in \mathcal{F}$ and let ψ_ω be the solution of $\Delta\psi = B$ in ω such that $\psi_\omega = 0$ on $\partial\omega$. If $\partial_\nu \psi_\omega > 0$ at some point M_ω of $\partial\omega \cap \Omega$, there exists $\tilde{\omega} \in \mathcal{F}$, with $\omega \subset \tilde{\omega}$ (strict inclusion) such that for the corresponding $\psi_{\tilde{\omega}}$

$$\inf_{\tilde{\omega}} \psi_{\tilde{\omega}} < \inf_{\omega} \psi_\omega \,. \tag{26}$$

Question

Analyze, when B changes sign, the optimal domains.

Dirichlet forms and Witten Laplacians

The analysis of the Pauli operator problem is quite close to the question of analyzing the smallest eigenvalue of the Dirichlet realization of the operator associated with the quadratic form:

$$C_0^\infty(\Omega) \ni v \mapsto h^2 \int_\Omega |\nabla v(x)|^2 \; e^{-2f(x)/h} \, dx \, . \tag{27}$$

For this case, we can mention Theorem 7.4 in Freidlin-Wentcell, which says (in particular) that, if f has a unique non-degenerate local minimum x_{min} and no other critical points, then the lowest eigenvalue $\lambda_1(h)$ of the Dirichlet realization $\Delta^{(0)}_{f,h}$ in Ω satisfies:

$$\lim_{h \to 0} -h \log \lambda_1(h) = 2 \inf_{x \in \partial\Omega} (f(x) - f(x_{min})) \, . \tag{28}$$

Link between Pauli and Witten

Starting from

$$\langle u, P_- u\rangle = h^2 \int_\Omega \exp\Big(-2\frac{\psi}{h}\Big)\left|(\partial_{x_1} + i\partial_{x_2})v\right|^2 dx, \qquad (29)$$

with $u = \exp\big(-\frac{\psi}{h}\big)v$, we observe that if v is real, then

$$|(\partial_{x_1} + i\partial_{x_2})v|^2 = |\nabla v|^2 .$$

Using the min-max caracterization, this implies that the ground state energy of the Pauli operator P_- is lower than the the ground state energy of the Dirichlet realization of the semi-classical Witten Laplacian on 0-forms:

$$-h^2\Delta + |\nabla\psi|^2 - h\Delta\psi .$$

References on former results

This problem has been analyzed in detail in Bovier-Eckoff-Gayrard-Klein (2004), Helffer-Klein-Nier (2004) and Helffer-Nier (2006) with computation of prefactors *but* under generic conditions on ψ which are not satisfied in our case. The restriction of ψ at the boundary is indeed not a Morse function. Hence it is difficult to define the points at the boundary which should be considered as saddle points.

Another remark is that, analyzing the proof in Helffer-Nier, the Morse assumption at the boundary appears only at the points where the normal exterior derivative of ψ at the boundary is strictly positive.

Finally, non generic situations are treated recently by L. Michel (2016), Di Gesu-Lelièvre-Lepeutrec-Nectoux (2017), Nectoux (2017).

One can illustrate this discussion with various instructive examples

First example: affine magnetic field

One consider in the unit disk:

$$\psi_\beta(x_1,x_2) = \frac{1}{8}(x_1 - 2\beta)\big(1 - x_1^2 - x_2^2\big)\,,$$

The critical points of ψ_β are consequently either given by $x_2 = 0$ and $1 - 3x_1^2 + 4\beta x_1 = 0$, or by $x_1 = 2\beta$ and $x_2^2 = 1 - 4\beta^2$. If $\beta \in (-\frac{1}{2}, \frac{1}{2})$, we have on $x_2 = 0$ two critical points corresponding to a maximum and a minimum of ψ_β, and, on $x_1 = 2\beta$, two symmetric critical points corresponding if $\beta \neq 0$ to two saddle points.

If we apply (generalization of) the results of Helffer–Nier, the rate of decay will correspond to the difference between the minimum (which is unique) and the value at a saddle point (which is in any case 0).
Actually, we can apply more directly Freidlin–Wentzell theorem and get a decay corresponding to the difference between the minimum and the value at the boundary which is zero. We can consequently not get in this way the oscillation of ψ_β.

Hence for this example, the upper bound given by the Witten Laplacian does not lead to any improvment.

Second example

We consider in the unit disk the function

$$\psi_0(x_1, x_2) = (1 - x_1^2 - x_2^2)(6x_1^2 + 3x_2^2 - x_2^4 - 1)\,.$$

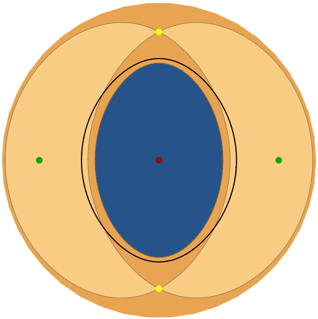

Figure: Blue means $\psi_0 < 0$. The (common) level set of the yellow saddle points are represented by continuous lines. The two maxima are indicated with green dots, the saddle points with yellow, and the minima in red. The bold black line shows the set $\{\Delta\psi_0 = 0\}$.

Critical points.

It is easy to compute the critical points living inside the unit disk on $\{x_1 = 0\}$ and $\{x_2 = 0\}$.
On $\{x_1 = 0\}$, we get $x_{2,\mathrm{sp}}^{\pm} = \pm\sqrt{2/3}$ and one can verify that this corresponds to saddle points.
On $\{x_2 = 0\}$, we get $x_{1,\min} = 0$ corresponding to a non degenerate minimum. We also get two non degenerate maxima with $x_{1,\max}^{\pm} = \pm\sqrt{7/12}$.
At each of these non degenerate critical points, the computation of the Hessian and the fact in the case of the extrema that the two eigenvalues are distinct determine the local picture of the integral curves of $\nabla\psi_0$.
Finally, there are no critical point outside the coordinate axes

Conclusions for this example.

According to the previous remark, we can apply the interior results (see Helffer-Klein-Nier) for getting the main asymptotics. Here we note that the more recent paper by
Di Gesu–Lelièvre–Lepeutrec–Nectoux treats the case of saddle points of same value (this case was excluded in Helffer-Nier).
The magnetic field in the unit disk is given by

$$B(x_1, x_2) = 22 - 90x_1^2 - 66x_2^2 + 12x_1^2x_2^2 + 32x_2^4\,.$$

and vanishes at order 1 along a closed regular curve (see Figure 1). We can now apply either

1. Theorems HPS and HKPS1 in $\Omega_B^+ := B^{-1}((-\infty, 0))$,
   or
2. the following generalization of Theorem HKPS3.

Theorem HKPS 4

Let Ω be a simply connected domain in $\mathbb{R}^2$, and let ψ_0 be given by (4). Assume that there exists a critical point $\mathbf{x}_{\rm sp}$

$$\psi_{\rm min} = \inf_{\Omega} \psi_0 < 0 < \psi_0(\mathbf{x}_{\rm sp}) := \psi_{\rm sp} \,.$$

Assume further that $\psi_0^{-1}(\psi_{\rm sp})$ contains a closed curve γ enclosing ω_{min} such that

$$\omega_{min} \cap \psi_0^{-1}(\psi_{\rm min}) \neq \emptyset \,.$$

Then

$$\lim_{h \to 0} h \log \Lambda^D(h, B, \Omega) \leq -2(\psi_{\rm sp} - \psi_{\rm min}) \,. \tag{30}$$

Application

In our case, the curve

$$\gamma := \psi_0^{-1}(\psi_{\rm sp})$$

consists of two symmetric curves joining the two saddle points. But this is not the optimal result. Along γ the exterior normal derivative to ω_m is strictly positive (except at the saddle points). Hence Proposition HKPS (with $\psi_\omega = \psi - \psi_{\rm sp}$) shows that we can improve the upper bound:

$$\lim_{h\to 0} h \log \Lambda^D(h, B, \Omega) < -2(\psi_{\rm sp} - \psi_{\rm min})\,. \qquad (31)$$

This example explicitely shows that the Witten Laplacian upper bound is *not* optimal for our example.

Remark

More generic situations can be considered by killing the symmetry by addition of a small perturbation of our symmetric example, by introducing for $\epsilon > 0$.

$$\psi_1(x_1, x_2; \epsilon) = (1 - x_1^2 - x_2^2)\big(6x_1^2 + 3x_2^2 - x_2^4 - 1 + \epsilon x_1(x_2 - \sqrt{3/2})\big).$$

In this case, this is the saddle point with lowest energy which has to be used for applying Theorem HKPS4. See Figure 2.

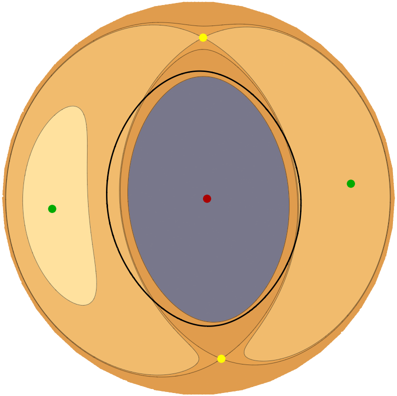

Figure: An asymmetric example with $\epsilon = 1/2$: The two saddle points (yellow) and the two maximas (green) have no more the same energy. The perturbation has been chosen in order to keep one saddle point fixed. The bolder black curve is the zero set of $\Delta\psi_0$.

Remark

A candidate to be the optimal set with respect to the "pushing the boundary" procedure could be the basin of attraction of ψ_0 relative to the minimum (see Figure 3). The boundary of this basin consists of four integral curves joining the saddle points to the maxima. Unfortunately, we have no theoretical proof for this optimality. Numerical analysis could be with this respect interesting.

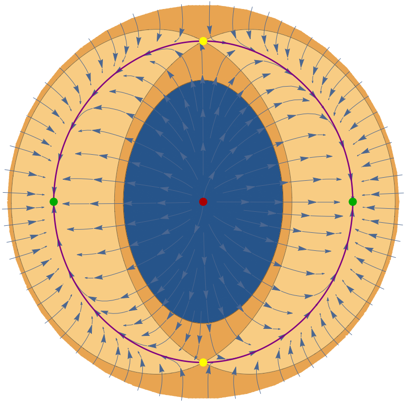

Figure: The symmetric picture with integral curves of $\nabla \psi_0$. The boundary of the basin of attraction is plotted in purple and is C^∞ except at the maxima.

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