Recent results on Quantum tunneling in deep potential wells and strong magnetic field

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11 February 2026. Talk in Roscoff.

Conference in honour of Christian Gérard.

Abstract

I would like to present a survey on the recent contributions on this subject and its application to the splitting. many new results have been obtained in the last five years has been very active due to the efforts of Charles Fefferman, Jakob Shapiro, Michael Weinstein, Helffer, Ayman Kachmar, Soeren Fournais, Guedes Bonthonneau, Léo Morin, Nicolas Raymond, M. Persson Sundqvist, F. Hérau and others.

Presentation

We briefly present what we are looking for, starting from the case considered by Fefferman-Shapiro-Weinstein (2022).

The Hamiltonian

We start from $v_0 \in C_c^\infty(\mathbb{R}^2)$ such that

$$\begin{cases} \mathfrak{v}_0(x) = v_0(|x|) \text{ is radial } \& \ v_0^{\min} := \min_{r \geq 0} v_0(r) < 0\,, \\ \operatorname{supp} \mathfrak{v}_0 \subset \overline{D(0,a)} := \{x \in \mathbb{R}^2 \ : \ |x| \leq a\}\,, \\ U_0 := \{\mathfrak{v}_0(x) = v_0^{\min}\} = \{0\} \quad \& \quad v_0''(0) > 0\,. \end{cases} \tag{1}$$

We suppose that $\overline{D(0,a)}$ is the smallest disc containing $\operatorname{supp} \mathfrak{v}_0$, i.e.

$$a = a(v_0) := \inf\{r > 0 \ : \ \operatorname{supp} \mathfrak{v}_0 \subset D(0,r)\}\,. \tag{2}$$

We introduce the *double well* potential

$$V(x) = \mathfrak{v}_0(x - z^\ell) + \mathfrak{v}_0(x - z^r)\,, \tag{3}$$

where

$$z^\ell = \Big(-\frac{L}{2}, 0\Big), \quad z_r = \Big(\frac{L}{2}, 0\Big)\,. \tag{4}$$

and

$$L > 2a(v_0)\,.$$

The potential wells of V associated with the energy v_0^{min} are z_ℓ and z_r.

Consider a constant magnetic field $b > 0$, so

$$b = \mathrm{curl}\,(\mathrm{b}\mathbf{A})$$

where $\mathbf{A}$ is defined in polar coordinates (r, θ) as follows,

$$\mathbf{A}(r, \theta) = \frac{r}{2}\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}. \tag{5}$$

Deep symmetric wells in a strong magnetic field

This is a particular case of a more general situation. We consider the Hamiltonian

$$\mathcal{H}_{b,\lambda} := (D - b\mathbf{A})^2 + \lambda^2 V\,, \quad D := \frac{1}{i}\nabla\,, \tag{6}$$

with a double well electric potential $\lambda^2 V$ and a magnetic potential $b\mathbf{A}$. Here, we suppose that $b = \lambda$ and $\lambda \gg 1$ is large.

Regimes where b does not scale like the coupling parameter λ have been considered a long time ago.
For instance, when $b \ll \lambda$, accurate estimates of the tunnel effect where obtained in Helffer-Sjöstrand [HelSj1987], while when $b \gg \lambda$, the effect of the potential well becomes weak and the magnetic effect is dominant (see Bellissard [Bel1988] and Helffer-Sjöstrand [HelSjSond1989]).

The potential function considered in (6) is not analytic, thereby making our setting significantly different from the one of [HelSj1987]. This will induce difficulties in deriving accurate bounds on the magnitude of the tunnel effect and highlights another interesting new phenomenon related to *tunneling* under a magnetic field compared to recent results:

- by Bonnaillie-Hérau-Raymond [BonHerRay2022] (tunneling inside the boundary Γ for the Neumann realization of the Schrödinger operator with constant magnetic field in an open set Ω)
- by Fournais-Helffer-Kachmar [FoHelKa2022] (tunneling along the discontinuity Γ of a magnetic step).
- by Khaled Abou Alfa [AbAl2022] who is considering a case where the magnetic field vanishes along a curve Γ.

Of course, in these questions an assumption of symmetry (or more generally the action of a finite group) should be done leading to the existence of symmetric (mini)-wells in Γ. See Helffer-Kachmar–Persson Sundqvist [HelKaPSun2023] for a general discussion.

To be short, we can say that by a controlled reduction to the boundary one can reduce the question to the spectral analysis of a $1D$-pseudodifferential operator on the boundary.

Similar problems were appearing in papers of V. Bonnaillie-Noël and collaborators (M. Dauge, S. Fournais) (tunneling in regular polygons) (see the book of Fournais-Helffer [FoHel2010] and additional references therein). But a complete proof in this case remains OPEN.

In order to exploit the connection with semi-classical analysis we consider instead

$$\mathcal{L}_h := (hD - \mathbf{A})^2 + V\,, \tag{7}$$

where $h = \lambda^{-1} \ll 1$.
With $(e_j^{v_0}(h))_{j\geq 1}$ the sequence of eigenvalues of $\mathcal{L}_h$, we will investigate the semi-classical asymptotics of

$$e_2^{v_0}(h) - e_1^{v_0}(h)\,, \tag{8}$$

and prove that, if $\mathfrak{v}_0$ does not vanish in $D\big(0, a\big)$, an asymptotics of the form

$$e_2^{v_0}(h) - e_1^{v_0}(h) \underset{h\to 0}{=} \exp\left(-\frac{S(v_0) + o(1)}{h}\right)$$

Our proof will be based on a mixing between what we get from the semi-classical analysis initiated in Helffer-Sjöstrand and Simon in the eighties with the approach of Fefferman-Shapiro-Weinstein. Notice that the optimal version was finally obtained by L. Morin [Mor2023].

Analysis of the Single well operator

Our investigation relies first on expanding the ground state $e^{\mathrm{sw}}(h)$ of the single well Hamiltonian

$$\mathcal{L}_h^{\mathrm{sw}} := (hD - \mathbf{A})^2 + \mathfrak{v}_0\,, \tag{9}$$

under the additional assumption that $\mathfrak{v}_0$ is radial.

We show that:

OneWell Theorem: Existence of radial ground states and precise expansions

1. The ground state energy $e^{\mathrm{sw}}(h)$ of $\mathcal{L}_h^{\mathrm{sw}}$, is a simple eigenvalue and
$$e^{\mathrm{sw}}(h) = v_0^{\min} + h\sqrt{1 + 2v_0''(0)} + \mathcal{O}(h^{3/2})\,. \tag{10}$$

2. There exists a unique positive ground state $\mathfrak{u}_h$, with the properties
   - $\mathfrak{u}_h(x) = u_h(|x|)$ is a radial function ;
   - $\int_{\mathbb{R}^2} |\mathfrak{u}_h(x)|^2 dx = 1\,.$

Theorem continued

3. There exists a positive radial function $\mathfrak{a}_0$ on $\mathbb{R}^2$ satisfying

$$\mathfrak{a}_0(0) = \frac{1}{2}\frac{\sqrt{1+2v_0''(0)}}{\pi}\,, \tag{11}$$

and s. t. $\forall R > 0$, the ground state $\mathfrak{u}_h$ satisfies, unif. in $B(0,R)$,

$$\left| e^{\mathfrak{d}(x)/h}\mathfrak{u}_h(x) - h^{-1/2}\mathfrak{a}_0(x)\right| = \mathcal{O}(h^{1/2})\,, \tag{12}$$

where

$$\mathfrak{d}(x) = d(|x|) = \int_0^{|x|}\sqrt{\frac{\rho^2}{4} + v_0(\rho) - v_0^{\min}}\,d\rho\,. \tag{13}$$

Except the "radial" statement, this is rather standard in semi-classical analysis since the works of [HelSj1984] and [Sim1983a].

The magnetic harmonic approximation

Here we can refer to Matsumoto [Mat], Matsumoto-Ueki which treat a more general case.
Consider the case where $v_0(x) = \mu|x|^2$, where μ is a positive constant. This means that we have replaced v_0 by its quadratic approximation at 0. The single well operator $\mathcal{L}_h^{sw}$ becomes approximated by

$$\mathcal{L}_h^{\mathrm{swap}} = (hD - \mathbf{A})^2 + \mu|x|^2 \,.$$

After rescaling[1] we get

$$\sigma(\mathcal{L}_h^{\mathrm{swap}}) = h\sigma(L_\mu^{\mathrm{mag}})$$

where

$$L_\mu^{\mathrm{mag}} = (D - \mathbf{A})^2 + \mu|x|^2 \,.$$

[1] We do the change of variable $y = h^{-1/2}x$.

We decompose the operator L_μ^{mag} via the orthogonal projections on the Fourier modes as follows

$$L_\mu^{\mathrm{mag}} \simeq \bigoplus_{m\in\mathbb{Z}} H_{m,\mu}$$

where

$$H_{m,\mu} := \pi_m L_\mu^{\mathrm{mag}} \pi_m^* = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \left(\frac{1}{4} + \mu\right) r^2 + \frac{m^2}{r^2} - m\,.$$

The min-max principle yields for $m < 0$

$$\lambda_1(H_{m,\mu}) > \inf_{u\neq 0} \frac{\left\langle (-\Delta + \left(\frac{1}{4}+\mu\right)|x|^2)u, u \right\rangle_{L^2(\mathbb{R}^2)}}{\|u\|_{L^2(\mathbb{R}^2)}} = 2\sqrt{\frac{1}{4}+\mu}\,.$$

Moreover, the rescaling $r \mapsto (1+4\mu)^{1/4} r$ yields the reduction to the unitary equivalent Landau Hamiltonian,

$$\widehat{H}_{m,\mu} = \sqrt{1+4\mu}\, H_{m,0} + \left(\sqrt{1+4\mu} - 1\right) m\,.$$

Consequently, we get

$$\inf_{m\in\mathbb{Z}} \lambda_1(H_m) = \lambda_1(H_0) = \sqrt{1+4\mu}\,, \quad \inf_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \lambda_1(H_m) > \sqrt{1+4\mu}\,.$$

This implies that

$$\lambda_1(L_\mu^{\mathrm{mag}}) = \sqrt{1+4\mu}$$

is a simple eigenvalue and that its (normalized) associated eigenfunction is radial:

$$\phi_\mu^{\mathrm{mag}}(x) = \pi^{-1/2}(1+4\mu)^{1/4} \exp\left(-\frac{\sqrt{1+4\mu}}{2}|x|^2\right) .$$

Eigenvalue asymptotics and radial ground states

We now have an accurate description of the spectrum of the operator $\mathcal{L}_h^{\mathrm{sw}}$

Proposition

For every fixed $j \in \mathbb{N}$, the j'th eigenvalue of $\mathcal{L}_h^{\mathrm{sw}}$ satisfies,

$$\lambda_j(\mathcal{L}_h^{\mathrm{sw}}) = v_0^{\mathsf{min}} + h\,\lambda_j(L_\mu^{\mathrm{mag}}) + \mathcal{O}(h^{3/2}) \quad (h \to 0_+)\,,$$

with $\mu = \frac{v_0''(0)}{2}$.
Moreover, the lowest eigenvalue of $\mathcal{L}_h^{\mathrm{sw}}$ is simple with a radial ground state.

Agmon estimates

If f is a radial function, then

$$\mathcal{L}_h^{\mathrm{sw}} f = -h^2 \Delta f + \mathfrak{w} f \tag{14}$$

with

$$\mathfrak{w}(x) = w(|x|)\,,$$

and

$$w(\rho) = v_0(\rho) + \frac{1}{4}\rho^2\,.$$

Therefore, when restricting the action of $\mathcal{L}_h^{\mathrm{sw}}$ to radial functions, we consider $\mathfrak{w}$ as the effective potential.
Hence, we can apply the semi-classical analysis relative to the Schrödinger operator without magnetic potential as considered in [HelSj1984] or [Sim1983a] (see [Hel1988] or [DimSj1999] for a more pedagogical presentation).

Decay

Proposition D

For all $\delta \in (0,1)$, there exist $a(\delta), C_\delta, h_0 > 0$ such that $\lim_{\delta\to 0_+} a(\delta) = 0$ and, if $\mathfrak{u}_h$ is a ground state of $\mathcal{L}_h^{\mathrm{sw}}$ and $h \in (0, h_0]$, then we have,

$$\left\| \nabla \left(e^{(1-\delta)\mathfrak{d}(x)/h} \mathfrak{u}_h \right) \right\|^2 + \left\| e^{(1-\delta)\mathfrak{d}(x)/h} \mathfrak{u}_h \right\|^2 \leq C_\delta \, e^{a(\delta)/h} \, \|\mathfrak{u}_h\|^2 \,,$$

where $\mathfrak{d}$ is the Agmon distance associated with $\mathfrak{w} - v_0^{min}$.

We note that $\mathfrak{d}$ is radial:

$$\mathfrak{d}(x) = d(|x|) \,.$$

WKB approximation

For all $S > 0$, we introduce the set $B_{\mathfrak{d}}(S) = \{x \in \mathbb{R}^2 \ : \ \mathfrak{d}(x) < S\}$, where $\mathfrak{d}$ is the Agmon distance to 0. We can then perform the WKB construction:

Proposition WKB1

There exist $N_0 \geq 1$ and two sequences $(E_k)_{k\geq 0} \subset \mathbb{R}$ and $(\mathfrak{a}_k)_{k\geq 0} \subset C^\infty(\mathbb{R}^2)$ s. t. , for all $N \geq 1$ and $S > 0$,

$$e^{\mathfrak{d}(x)/h}\Big(\mathcal{L}_h^{\mathrm{sw}} - E^N(h)\Big)\vartheta^N = \mathcal{O}(h^{N-N_0}) \quad \text{on } B_{\mathfrak{d}}(S)\,,$$

where

$$E^N(h) = \sum_{k=0}^{N} E_k h^k\,, \quad E_0 = v_0^{\min}, \quad E_1 = \sqrt{1 + 2v_0''(0)}$$

$$\vartheta^N(x) = h^{-1/2}\left(\sum_{k=0}^{N} \mathfrak{a}_k(x) h^k\right) e^{-\mathfrak{d}(x)/h}, \quad \mathfrak{a}_0(0) = \frac{1}{2}\sqrt{\frac{1+2v_0''(0)}{\pi}}\,.$$

Proposition WKB2

There exists $N_0 \geq 1$, and for all $h \in (0, h_0]$, there exists a normalized ground state $\mathfrak{u}_h$ of $\mathcal{L}_h^{\mathrm{sw}}$ s. t. for any N and any $R > 0$ the following holds

$$\left\| e^{\mathfrak{d}(x)/h}(\mathfrak{u}_h - \vartheta^N) \right\|_{H^2(D(0,R))} = \mathcal{O}\left(h^{N-N_0}\right).$$

This ends the sketch of the proof of Theorem OW.

Coming back to the main theorem

Theorem OW in particular clarifies the hypotheses imposed in Fefferman-Shapiro-Weinstein [FeShWe2022] which states then that when

$$v_0 \leq 0 \text{ and } L > 4\left(\sqrt{|v_0^{\min}|} + a(v_0)\right)\,, \tag{15}$$

then

$$\exp\left(-\frac{L^2 + 4\sqrt{|v_0^{\min}|}L + \gamma(v_0)}{4h}\right) \leq e_2^{v_0}(h) - e_1^{v_0}(h) \tag{16}$$

where $\gamma(v_0)$ is a positive constant, and

$$e_2^{v_0}(h) - e_1^{v_0}(h) \leq Ch^{-5/2}\exp\left(-\frac{(L - a(v_0))^2 - a(v_0)^2}{4h}\right)\,. \tag{17}$$

The most important was here that they get a lower bound but we will see that these estimates are far from optimal.

Interaction matrix or hopping coefficient

The bounds above follow from the asymptotics [FeShWe2022]

$$e_2^{v_0}(h) - e_1^{v_0}(h) \underset{h \to 0}{\sim} \left| 2 \int_{D(0,a)} v_0(x) \mathfrak{u}_h(x) \mathfrak{u}_h(x_1 + L, x_2) e^{\frac{iLx_2}{2h}} \, dx \right| \tag{18}$$

where $\mathfrak{u}_h$ is the radial ground state of $\mathcal{L}_h^{\mathrm{sw}}$.
The integral in the right hand side is called in Solid State Physics the *hopping coefficient*.
Under different conditions, it can be derived through a reduction to the restriction of $\mathcal{L}_h$ on a two dimensional space, yielding an *interaction matrix* like in [Hel1988] or [DimSj1999].
The hopping parameter corresponds with the off diagonal term in the 2×2 interaction matrix.

Using the improved expansion of the ground state $\mathfrak{u}_h$, we improve the bounds on the hopping coefficient and thereby on $e_2^{v_0}(h) - e_1^{v_0}(h)$ provided v_0 satisfies the conditions in (1).

Besides its role in capturing the tunneling asymptotics, precise estimates of the hopping coefficient (or the so-called interaction matrix) are key ingredients in the understanding of tight binding reductions in Solid State Physics (see [ShWe2022] and earlier [Out1987, Dau1994, DimSj1999, Frank2006] and Fournais-Morin 2023 for mathematical contributions).

Interaction matrix or hopping coefficient

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$$e_2^{\mathfrak{v}_0}(h) - e_1^{\mathfrak{v}_0}(h) \underset{h\to 0}{\sim} \left|2\int_{D(0,a)} \mathfrak{v}_0(x)\mathfrak{u}_h(x)\mathfrak{u}_h(x_1+L,x_2)e^{\frac{iLx_2}{2h}}\,dx\right| \tag{19}$$

where $\mathfrak{u}_h$ is the radial ground state of $\mathcal{L}_h^{\mathrm{sw}}$.
The integral in the right hand side is called in Solid State Physics the *hopping coefficient* which can be written in a more symmetric way as

$$\mathrm{Hop}(\mathfrak{v}_0,h,L) := \int \mathfrak{v}_0(x-z_\ell)\mathfrak{u}_h(x-z_\ell)\mathfrak{u}_h(x-z_r)e^{\frac{iLx_2}{2h}}\,dx$$

Heuristics leading to the hopping coefficient

We explain first the case without magnetic field. We have two approximate states u_ℓ and u_r constructed by using u_0 the eigenfunction of the one well operator with eigenvalue λ_0

$$u_\ell(x) = u_0(x - z_\ell)\,,\; u_r(x) = u_0(x - z_r)\,,$$

and we admit that "essentially" the true eigenspace is very close to the span of u_ℓ and u_r and that u_ℓ and u_r are "essentially" orthogonal. Then to compute the matrix of

$$\mathcal{L}_h - \lambda_0 = -h^2\Delta + V - \lambda_0$$

relative to this eigenspace in this "essentially orthogonal" basis, we just consider the off diagonal term

$$\langle (\mathcal{L}_h - \lambda_0) u_\ell, u_r \rangle\,.$$

But by construction

$$(\mathcal{L}_h - \lambda_0) u_\ell = v_r u_\ell\,, \text{with } v_r(x) = v_0(x - z_r)\,.$$

Note here that $u_r(x) = u_\ell(-x) = u_0(x - z_r)$.

This is no more the case in the magnetic case, where we have to use the magnetic translation for defining u_r and u_ℓ

$$u_r(x) = e^{iLx_2/2h} u_0(x - z_r)$$

$$\frac{L}{2} x_2 = \frac{L}{2} e_1 \wedge (x - z_r)\,.$$

$$u_\ell(x) = e^{-iLx_2/2h} u_0(x - z_\ell)$$

Our main result, on the eigenvalue splitting, is

[HK]-Theorem: Sharp asymptotics of the eigenvalue splitting

Under the previous assumptions, if $v_0 < 0$ in $D(0,a))$, then we have

$$h \ln \left(e_2^{v_0}(h) - e_1^{v_0}(h) \right) \underset{h \to 0}{\sim} -S(v_0),,$$

where $S(v_0)$ is a positive explicit constant.

Recently [HelKaPSun2023], we have proven. the same result under the much weaker and more natural condition

$$(1+\sqrt{3})a(v_0) < L\,.$$

Note that the most natural condition would be to have

$$2a(v_0) < L\,.$$

The formula for $S(v_0)$

$$S(v_0) = -F(v_0) + \inf_{\substack{r\in[0,a]\\ t\in(0,+\infty)}} \Psi(r,t)\,,$$

where

$$\Psi(r,t) := d(r) + \frac{r^2+L^2}{4}(2t+1) + \frac{|v_0^{\min}|}{2}\ln\left(1+\frac{1}{t}\right) - Lr\sqrt{t(t+1)} \tag{20}$$

and

$$F(v_0) = \frac{a}{4}\sqrt{a^2+4|v_0^{\min}|} + |v_0^{\min}|\ln\frac{\left(\sqrt{a^2+4|v_0^{\min}|}+a\right)^2}{4|v_0^{\min}|} - d(a) \tag{21}$$

Analyzing the infimum of Ψ

If $L > 2a$, then

$$\min_{(r,t)\in[0,a]\times\mathbb{R}_+} \Psi(r,t) = \Psi(a, t_a)\,,$$

where

$$t_a = \sqrt{\frac{1}{4} + s_+(a, L, v_0^{\min})} - \frac{1}{2}$$

and

$$s_+(a, L, v_0^{\min}) := \frac{2|v_0^{\min}|(L^2+a^2) + L^2a^2}{2(L^2-a^2)^2} + \frac{1}{L^2-a^2}\sqrt{\frac{(2|v_0^{\min}|(L^2+a^2)+L^2a^2)^2}{4(L^2-a^2)^2} - |v_0^{\min}|^2}\,.$$

Moreover, (a, t_a) is the unique minimum of Ψ.

An important representation formula

Representation formula

The radial ground state $\mathfrak{u}_h$ has the following representation for $\rho \geq a$,

$$\mathfrak{u}_h(\rho) = C_h \exp\left(-\frac{\rho^2}{4h}\right) \int_0^{+\infty} \exp\left(-\frac{\rho^2 t}{2h}\right) t^{\alpha-1}(1+t)^{-\alpha} dt\,,$$

where

$$\alpha = \frac{1}{2h}|v_0^{\min}| - \frac{1}{2}\left(\sqrt{1+2v_0''(0)} - 1\right) + \mathcal{O}(h^{1/2}) \underset{h\to 0}{\sim} \frac{1}{2h}|v_0^{\min}|\,,$$

and

$$C_h \underset{h\to 0}{\sim} C_h^{\mathrm{asy}} := \mathfrak{m}(v_0) h^{-1} \exp\left(\frac{F(v_0)}{h}\right)\,.$$

Here $a = a(v_0)$ and

$$F(v_0) = \frac{a}{4}\sqrt{a^2 + 4|v_0^{\min}|} + |v_0^{\min}| \ln \frac{\left(\sqrt{a^2 + 4|v_0^{\min}|} + a\right)^2}{4|v_0^{\min}|} - d(a)$$

$$m(v_0) = \frac{\mathfrak{a}_0(0)}{4|v_0^{\min}|\sqrt{2\pi a}} \left(a^2 + 4|v_0^{\min}|\right)^{1/4} \left(\sqrt{a^2 + 4|v_0^{\min}|} + a\right)^2.$$

$$\alpha = \frac{1}{2} - \frac{1}{2h} e^{\mathrm{sw}}(h).$$

Second representation formula

We start by expressing the hopping coefficient in polar coordinates

$$w_{\ell,r} = \int_0^a r\, v_0(r) u_h(r) \left(\int_0^{2\pi} K_h(r,\theta) d\theta \right) dr \,, \tag{22}$$

where

$$K_h(r,\theta) := u_h(r^2 + L^2 + 2Lr\cos\theta) e^{\frac{iLr\sin\theta}{h}} \,.$$

The integral of K_h with respect to θ is computed in [FeShWe2022, Prop. 5.1] as follows

$$\int_0^{2\pi} K_h(r,\theta)d\theta = C_h \exp\left(-\frac{r^2+L^2}{4h}\right)\int_0^{+\infty} G_h(r,t)dt\,, \qquad (23)$$

where

$$G_h(r,t) = \exp\left(-\frac{(r^2+L^2)t}{2h}\right) t^{\alpha-1}(1+t)^{-\alpha} I_0\left(\frac{Lr\sqrt{t(t+1)}}{h}\right) \qquad (24)$$

and

$$z \mapsto I_0(z) = \frac{1}{2\pi}\int_0^{\pi} e^{z\cos\theta}d\theta\,.$$

The advantage of the second representation formula is the absence of the oscillatory complex term and moreover, the integrand G_h is a positive function. The complex term disappears by using the following formula

$$\int_0^{2\pi} \exp(i\xi \sin\theta + \beta\cos\theta)\, d\theta = \int_0^{2\pi} \exp(-\sqrt{\beta^2 - \xi^2}\, \sin\theta)\, d\theta\,,$$

which is obtained by translation in the complex.

We can also implement a recent improvement by Leo Morin [Mor2023] in order to get, for $c \neq 0$,

$$e_2^{\mathfrak{v}_0}(h) - e_1^{\mathfrak{v}_0}(h) \underset{h \to 0}{=} c\, h^{\nu} \exp\left(-\frac{S(\mathfrak{v}_0)}{h}\right)$$

like in the case without magnetic field [HelSj1984, Sim1983b].

Third representation formula

As observed by [FoMoRa2023] and exploited in [Mor2023], it is better to come back to the trick appearing in [HelSj1984] and in the magnetic case in [HelSj1987].
Here we have the simplification that we can avoid to introduce a cut-off since our approximate eigenfunctions are defined in $\mathbb{R}^2$.

If we consider an open set Ω containing $B(z_\ell, a)$ and with empty intersection with $B(z_r, a)$ and denote its boundary by $\Sigma = \partial\Omega$, we can always write [HM2]

$$Hop(L, v_0) = ih \int_\Sigma \Big(u_\ell \cdot \overline{(-ih\nabla - A)u_r \cdot \vec{n}} + \overline{u_r} \cdot (-ih\nabla - A)u_\ell \cdot \vec{n} \Big) d\sigma \,,$$

where $\vec{n}$ is the outward normal to Σ.
Notice that the left hand side is independent of Σ, hence we have the freedom for the choice of Σ.

At this stage, we are very happy to have an example where we can have a very accurate estimate for the splitting but it remains difficult to interpret what could be the "magnetic" Agmon estimate between the two wells. It seems also difficult to determine if the assumption of radial one well-problems is simply a technical assumption or if this explains a smaller splitting as expected.
In the last two years, Fefferman-Shapiro-Weinstein explore this question in three papers [FeShWe2025-1, FeShWe2025-2, FeShWe2025-3] with partial answers.

Absence of tunneling– after Fefferman-Shapiro-Weinstein[FeShWe2025-2]

Since the results in [FeShWe2022] are highly non generic, it is interesting to see if the splitting is in some sense stable by perturbation.
Notice that the fact that the splitting can vanish when a magnetic field is present has been exhibited in various contexts (see for example [Hel1988] for the discussion of some Aharonov-Bohm effect (but there the magnetic field is compactly supported), [FoHel2010] in the context of superconductivity (partly heuristic) and [HelKaPSun2023], where the phenomenon appears when three wells are present).

Here the authors describe a family of examples where the splitting can vanish by exponentially small perturbation of the radial case..
A symmetry is kept permitting to analyze separately an odd spectrum and an even spectrum and the authors show that by destroying "continuously" the radial assumptions of the one well potential the difference between the odd ground state energy and the even ground state energy can change sign.
We do not give the detailed statement which at the end remains rather mysterious.

Generic lower bounds [FeShWe2025-1, FeShWe2025-3]

The authors present lower bounds on tunneling rates in magnetic double well systems for generic values of the coupling constant. These lower bounds are consistent with the possible vanishing of the splitting observed in their previous preprint and also with the other $1D$ results on the splitting.

Conclusion of the approach

Although we have obtained much more examples than five years ago, we are still very far from a complete understanding of the splitting.

Towards purely magnetic tunneling

Till now, we have mainly discussed cases where either the potential or the geometry of the boundary or the geometry relative to the discontinuity or the vanishing of the magnetic field was playing a role. We have not spoken about Aharonov-Bohm effects (see [Hel1988] or [HelKa2024]. We will now focus on what is called: purely magnetic tunneling.

In continuation of FSW choices to refer to a radial one well model, we can mention the case with constant magnetic field in the complementary of two (or more) symmetric discs [FoMo2024] or the case [FoMoRa2023] when the magnetic field B creates two symmetric wells

$$B(x_1, -x_2) = B(-x_1, x_2) = B(x_1, x_2)\,,$$

B is constant outside of two disks and radial in each disk. In each case, the authors give the main asymptotics of the splitting between the two first eigenvalues in the semi-classical limit.

A purely magnetic and generic tunneling result

The recent result [FGMR] by Soeren Fournais, Yannick Guedes Bonthonnneau, Léo Morin, Nicolas Raymond is an important breakthrough.

Here is their abstract:

The two-dimensional magnetic Laplacian is considered. We calculate the leading term of the splitting between the first two eigenvalues of the operator in the semiclassical limit under the assumption that the magnetic field does not vanish and has two symmetric magnetic wells with respect to the coordinate axes. This is the first result of quantum tunneling between purely magnetic wells under generic assumptions. The proof, which strongly relies on microlocal analysis, reveals a purely magnetic Agmon distance between the wells. Surprisingly, it is discovered that the exponential decay of the eigenfunctions away from the magnetic wells is not crucial to derive the tunneling formula. The key is a microlocal exponential decay inside the characteristic manifold, with respect to the variable quantizing the classical center guide motion.

The assumptions

- symmetry

$$B(x_1,-x_2)=B(-x_1,x_2)=B(x_1,x_2)\,.$$

- two local non degenerate minima. at $(0,c_\ell)$ and $(0,c_r)$, with $\inf B := b_0 > 0$
- Partial analyticity in x_1 and mild variation.
- Existence in $B^{-1}(b_0)\cap\mathbb{C}\times[c_\ell,c_r]$ of a complex path $i\gamma$ connecting the minima:

$$B(i\gamma(x_2),x_2)=b_0\,,\ \gamma(c_\ell)=\gamma(c_r)=0\,,\ \gamma'(c_\ell)>0\,,\ \gamma'(c_r)<0\,.$$

- $B(x_1,x_2)-b_0$ vanishes only on the curves $x_1=\pm i\gamma(x_2)$ and in a non degenerate way.
- $+$ technical assumptions

Theorem FBMR

There exists $c_0 > 0$ such that

$$\lambda_2 - \lambda_1 = c_0 h^{3/2} e^{-S/h}(1 + o(1)),$$

where

$$S = \int_{c_\ell}^{c_r} \int_0^{\gamma(x_2)} B(it, x_2) dt\, dx_2 > 0.$$

A wrong intuition (that hB should replace the electric potential V) leads me together with A. Mohamed-Morame to conjecture in [HelMo1996] that the decay should have the form $\exp -S/\sqrt{h}$.

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