Spectral theory for the complex Airy operator: the case of a semipermeable barrier and applications to the Bloch-Torrey equation
Talk at NYU Shanghai.

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The transmission boundary condition which is considered appears in various exchange problems such as molecular diffusion across semi-permeable membranes [35, 32], heat transfer between two materials [10, 17, 7], or transverse magnetization evolution in nuclear magnetic resonance (NMR) experiments [19]. In the simplest setting of the latter case, one considers the local transverse magnetization $G(x, y\,; t)$ produced by the nuclei that started from a fixed initial point y and diffused in a constant magnetic field gradient g up to time t. This magnetization is also called the propagator or the Green function of the Bloch-Torrey equation [37] (1956):

$$\frac{\partial}{\partial t} G(x, y\,; t) = (D\Delta - i\gamma g x_1)\, G(x, y\,; t)\,, \tag{1}$$

with the initial condition

$$G(x, y\,; t = 0) = \delta(x - y), \tag{2}$$

where D is the intrinsic diffusion coefficient, $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_d^2$ the Laplace operator in $\mathbb{R}^d$, γ the gyromagnetic ratio, and x_1 the coordinate in a prescribed direction.

In this talk, we focus in the first part on the one-dimensional situation ($d = 1$), in which the operator

$$D_x^2 + ix = -\frac{d^2}{dx^2} + ix$$

is called the complex Airy operator and appears in many contexts: mathematical physics, fluid dynamics, time dependent Ginzburg-Landau problems and also as an interesting toy model in spectral theory (see [3]). We consider a suitable extension $\mathcal{A}_1^+$ of this differential operator and its associated evolution operator $e^{-t\mathcal{A}_1^+}$. The Green function $G(x, y\,; t)$ is the distribution kernel of $e^{-t\mathcal{A}_1^+}$.

For the problem on the line $\mathbb{R}$, an intriguing property is that this non self-adjoint operator, which has compact resolvent, has empty spectrum. However, the situation is completely different on the half-line $\mathbb{R}_+$. The eigenvalue problem

$$(D_x^2 + ix)u = \lambda u,$$

for a spectral pair (u, λ) with u in $H^2(\mathbb{R}_+)$, $xu \in L^2(\mathbb{R}_+)$ has been thoroughly analyzed for both Dirichlet ($u(0) = 0$) and Neumann ($u'(0) = 0$) boundary conditions.
The spectrum consists of an infinite sequence of eigenvalues of multiplicity one explicitly related to the zeroes of the Airy function (see [34, 25]).
The space generated by the eigenfunctions is dense in $L^2(\mathbb{R}_+)$ (completeness property) but there is no Riesz basis of eigenfunctions.
Finally, the decay of the associated semi-group has been analyzed in detail. The physical consequences of these spectral properties for NMR experiments have been first revealed by Stoller, Happer and Dyson [34] and then thoroughly discussed by De Sviet et al. and D. Grebenkov [14, 18, 21].

In this talk, we consider another problem for the complex Airy operator on the line but with a transmission property at 0 which reads (cf Grebenkov [21]),

$$\left\{ \begin{array}{lcl} u'(0_+) & = & u'(0_-)\,, \\ u'(0) & = & \kappa\left(u(0_+) - u(0_-)\right)\,, \end{array} \right. \tag{3}$$

where $\kappa \geq 0$ is a real parameter.
The case $\kappa = 0$ corresponds to two independent Neumann problems on $\mathbb{R}_-$ and $\mathbb{R}_+$ for the complex Airy operator.
When κ tends to $+\infty$, the second relation in (3) becomes the continuity condition, $u(0_+) = u(0_-)$, and the barrier disappears.
Hence, the problem tends (at least formally) to the standard problem for the complex Airy operator on the line.

We summarize our main (1D)-results in the following:

Theorem

The semigroup $\exp(-t\mathcal{A}_1^+)$ *is contracting. The operator* $\mathcal{A}_1^+$ *has a discrete spectrum* $\{\lambda_n(\kappa)\}$. *The eigenvalues* $\lambda_n(\kappa)$ *are determined as (complex-valued) solutions of the equation*

$$2\pi \mathrm{Ai}'(e^{2\pi i/3}\lambda)\mathrm{Ai}'(e^{-2\pi i/3}\lambda) + \kappa = 0, \tag{4}$$

where $\mathrm{Ai}'(z)$ *is the derivative of the Airy function.*
For all $\kappa \geq 0$, *there exists* N *such that, for all* $n \geq N$, *there exists a unique eigenvalue of* $\mathcal{A}_1^+$ *in the ball* $B(\lambda_n^\pm, 2\kappa|\lambda_n^\pm|^{-1})$, *where* $\lambda_n^\pm = e^{\pm 2\pi i/3} a_n'$, *and* a_n' *are the zeros of* $\mathrm{Ai}'(z)$.
Finally, for any $\kappa \geq 0$ *the space generated by the generalized eigenfunctions of the complex Airy operator with transmission is dense in* $L^2(\mathbb{R}_-) \times L^2(\mathbb{R}_+)$.

Note that due to the possible presence of eigenvalues with Jordan blocks, we do not prove in full generality that the eigenfunctions of $\mathcal{A}_1^+$ span a dense set in $L^2(\mathbb{R}_-) \times L^2(\mathbb{R}_+)$. Numerical computations suggest actually that all the spectral projections have rank one (no Jordan block) but we can only prove that there are at most a finite number of eigenvalues with nontrivial Jordan blocks.

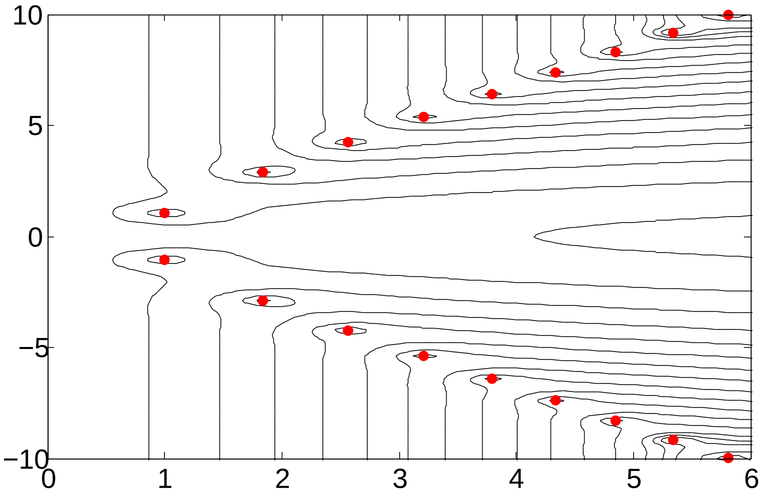

Figure: Numerically computed pseudospectrum in the complex plane of the complex Airy operator with the transmission boundary condition at the origin with $\kappa = 1$. The red points show the poles $\lambda_n^{\pm}(\kappa)$.

Basic properties of the Airy function

We recall that the Airy function is the unique solution of

$$(D_x^2 + x)u = 0\,,$$

on the line such that $u(x)$ tends to 0 as $x \to +\infty$ and $\mathrm{Ai}(0) = 1/\left(3^{\frac{2}{3}}\,\Gamma(\frac{2}{3})\right)$. This Airy function extends into an holomorphic function in $\mathbb{C}$.

Ai is positive decreasing on $\mathbb{R}_+$ but has an infinite number of zeros in $\mathbb{R}_-$. We denote by a_n ($n \in \mathbb{N}$) the decreasing sequence of zeros of Ai. Similarly we denote by a_n' the sequence of zeros of Ai'. Moreover

$$a_n \underset{n\to+\infty}{\sim} -\left(\frac{3\pi}{2}(n-1/4)\right)^{2/3}\,, \tag{5}$$

and

$$a_n' \underset{n\to+\infty}{\sim} -\left(\frac{3\pi}{2}(n-3/4)\right)^{2/3}\,. \tag{6}$$

$\mathrm{Ai}(e^{i\alpha}z)$ and $\mathrm{Ai}(e^{-i\alpha}z)$ (with $\alpha = 2\pi/3$) are two independent solutions of the differential equation

$$\left(-\frac{d^2}{dz^2} - iz\right) w(z) = 0\,.$$

Considering their Wronskian, one gets

$$e^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}z)\mathrm{Ai}(e^{i\alpha}z) - e^{i\alpha}\mathrm{Ai}'(e^{i\alpha}z)\mathrm{Ai}(e^{-i\alpha}z) = \frac{i}{2\pi} \quad \forall\, z \in \mathbb{C}\,. \tag{7}$$

Note the identity

$$\mathrm{Ai}(z) + e^{-i\alpha}\mathrm{Ai}(e^{-i\alpha}z) + e^{i\alpha}\mathrm{Ai}(e^{i\alpha}z) = 0 \quad \forall\, z \in \mathbb{C}\,. \tag{8}$$

The Airy function and its derivative satisfy different asymptotic:
(i) For $|\arg z| < \pi$,

$$\mathrm{Ai}(z) = \frac{1}{2}\pi^{-\frac{1}{2}} z^{-1/4} \exp\left(-\frac{2}{3}z^{3/2}\right)\left(1+\mathcal{O}(|z|^{-\frac{3}{2}})\right), \quad (9)$$

$$\mathrm{Ai}'(z) = -\frac{1}{2}\pi^{-\frac{1}{2}}\, z^{1/4} \exp\left(-\frac{2}{3}z^{3/2}\right)\left(1+\mathcal{O}(|z|^{-\frac{3}{2}})\right). \quad (10)$$

(ii) For $|\arg z| < \frac{2}{3}\pi$,

$$\begin{aligned}\mathrm{Ai}(-z) &= \pi^{-\frac{1}{2}} z^{-1/4} \left(\sin\left(\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right)(1+\mathcal{O}(|z|^{-\frac{3}{2}})\right. \quad (11)\\ &\left. -\frac{5}{72}\left(\frac{2}{3}z^{\frac{3}{2}}\right)^{-1}\cos\left(\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right)(1+\mathcal{O}(|z|^{-\frac{3}{2}})\right)\\ \mathrm{Ai}'(-z) &= -\pi^{-\frac{1}{2}} z^{1/4}\left(\cos(\frac{2}{3}z^{3/2}+\frac{\pi}{4})(1+\mathcal{O}(|z|^{-\frac{3}{2}}))\right. , \quad (12)\\ &\left. +\frac{7}{72}\left(\frac{2}{3}z^{3/2}\right)^{-1}\sin(\frac{2}{3}z^{3/2}+\frac{\pi}{4})(1+\mathcal{O}(|z|^{-\frac{3}{2}}))\right).\end{aligned}$$

Analysis of the resolvent of $\mathcal{A}^+$ on the line for $\lambda > 0$

On the line $\mathbb{R}$, $\mathcal{A}^+$ is the closure of the operator $\mathcal{A}_0^+$ defined on $C_0^\infty(\mathbb{R})$ by $\mathcal{A}_0^+ = D_x^2 + ix$. A detailed description of its properties can be found in my book in Cambridge [?] (2013). We give the asymptotic control of the resolvent $(\mathcal{A}^+ - \lambda)^{-1}$ as $\lambda \to +\infty$. We successively discuss the control in $\mathcal{L}(L^2(\mathbb{R}))$ and in the Hilbert-Schmidt space $\mathcal{C}^2(L^2(\mathbb{R}))$.

Control in $\mathcal{L}(L^2(\mathbb{R}))$.

Here we follow an idea present in an old paper of I. Herbst, the book of Davies [12] and used in Martinet's PHD [31] (see also [25]).

Proposition

For all $\lambda > \lambda_0$,

$$\|(\mathcal{A}^+ - \lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \sqrt{2\pi}\ \lambda^{-\frac{1}{4}} \exp\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right)(1 + o(1))\,. \qquad (13)$$

Proof

The proof is obtained by considering $\mathcal{A}^+$ in the Fourier space, i.e.

$$\widehat{\mathcal{A}}^+ = \xi^2 + \frac{d}{d\xi}\,. \tag{14}$$

The associated semi-group $T_t := \exp(-\widehat{\mathcal{A}}^+ t)$ is given by

$$T_t u(\xi) = \exp\left(-\xi^2 t - \xi t^2 - \frac{t^3}{3}\right) u(\xi - t)\,, \quad \forall\, u \in \mathcal{S}(\mathbb{R})\,. \tag{15}$$

T_t is the composition of a multiplication by $\exp(-\xi^2 t - \xi t^2 - \frac{t^3}{3})$ and of a translation by t.

Computing $\sup_\xi \exp(-\xi^2 t - \xi t^2 - \frac{t^3}{3})$ leads to

$$\| T_t\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \exp\left(-\frac{t^3}{12}\right) . \tag{16}$$

It is then easy to get an upper bound for the resolvent. For $\lambda > 0$, we have

$$\begin{aligned}
\|(\mathcal{A}^+ - \lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} &= \|(\widehat{\mathcal{A}}^+ - \lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} && (17)\\
&\leq \int_0^{+\infty} \exp(t\lambda)\| T_t\|_{\mathcal{L}(L^2(\mathbb{R}))} dt && (18)\\
&\leq \int_0^{+\infty} \exp\left(t\lambda - \frac{t^3}{12}\right) dt\,. && (19)
\end{aligned}$$

Control in Hilbert-Schmidt norm

As previously, we use the Fourier representation and analyze $\widehat{\mathcal{A}}^+$. Note that

$$\|(\widehat{\mathcal{A}}^+ - \lambda)^{-1}\|_{HS}^2 = \|(\mathcal{A}^+ - \lambda)^{-1}\|_{HS}^2 \tag{20}$$

We have then an explicit description of the resolvent by

$$(\widehat{\mathcal{A}}^+ - \lambda)^{-1} u(\xi) = \int_{-\infty}^{\xi} u(\eta) \exp\left(\frac{1}{3}(\eta^3 - \xi^3) + \lambda(\xi - \eta)\right) d\eta\,.$$

Hence, we have to compute

$$\|(\widehat{\mathcal{A}}^+ - \lambda)^{-1}\|_{HS}^2 = \int\int_{\eta<\xi} \exp\left(\frac{2}{3}(\eta^3 - \xi^3) + 2\lambda(\xi - \eta)\right) d\eta d\xi\,.$$

Again, this can be analyzed after a scaling in the spirit of the Laplace method.

Analysis of the resolvent for the Dirichlet realization in the half-line.

It is not difficult to define the Dirichlet realization $\mathcal{A}^{\pm,D}$ of $D_x^2 \pm ix$ on $\mathbb{R}_+$ (the analysis on the negative semi-axis is similar). One can use for example the Lax Milgram theorem and take as form domain

$$V^D := \{u \in H_0^1(\mathbb{R}_+)\,,\, x^{\frac{1}{2}} u \in L^2_+\}\,.$$

It can also be shown that the domain is

$$\mathcal{D}^D := \{u \in V^D\,,\, u \in H^2_+\}\,.$$

This implies

Proposition

The resolvent $\mathcal{G}^{\pm,D}(\lambda) := (\mathcal{A}^{\pm,D} - \lambda)^{-1}$ is in the Schatten class C^p for any $p > \frac{3}{2}$ (see [15] for definition), where $\mathcal{A}^{\pm,D} = D_x^2 \pm ix$ and the superscript D refers to the Dirichlet case.

More precisely we provide the distribution kernel $\mathcal{G}^{-,D}(x,y\,;\lambda)$ of the resolvent for the complex Airy operator $D_x^2 - ix$ on the positive semi-axis with Dirichlet boundary condition at the origin. Matching the boundary conditions, one gets

$$\mathcal{G}^{-,D}(x,y\,;\lambda) = \begin{cases} 2\pi \frac{\mathrm{Ai}(e^{-i\alpha}w_y)}{\mathrm{Ai}(e^{-i\alpha}w_0)} \big[\mathrm{Ai}(e^{i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_0) \\ \qquad\qquad -\mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{i\alpha}w_0)\big] \quad (0 < x < y)\,, \\ 2\pi \frac{\mathrm{Ai}(e^{-i\alpha}w_x)}{\mathrm{Ai}(e^{-i\alpha}w_0)} \big[\mathrm{Ai}(e^{i\alpha}w_y)\mathrm{Ai}(e^{-i\alpha}w_0) \\ \qquad\qquad -\mathrm{Ai}(e^{-i\alpha}w_y)\mathrm{Ai}(e^{i\alpha}w_0)\big] \quad (x > y)\,, \end{cases} \tag{21}$$

where $\mathrm{Ai}(z)$ is the Airy function, $w_x = ix + \lambda$, and $\alpha = 2\pi/3$.
We have the decomposition

$$\mathcal{G}^{-,D}(x,y\,;\lambda) = \mathcal{G}_0^{-}(x,y\,;\lambda) + \mathcal{G}_1^{-,D}(x,y\,;\lambda), \tag{22}$$

where $\mathcal{G}_0^{-}(x,y\,;\lambda)$ is the resolvent for the Airy operator $D_x^2 - ix$ on the whole line,

$$\mathcal{G}_0^-(x,y\,;\lambda) = \begin{cases} 2\pi \mathrm{Ai}(e^{i\alpha} w_x)\mathrm{Ai}(e^{-i\alpha} w_y) & (x<y), \\ 2\pi \mathrm{Ai}(e^{-i\alpha} w_x)\mathrm{Ai}(e^{i\alpha} w_y) & (x>y), \end{cases} \tag{23}$$

and

$$\mathcal{G}_1^{-,D}(x,y\,;\lambda) = -2\pi \frac{\mathrm{Ai}(e^{i\alpha}\lambda)}{\mathrm{Ai}(e^{-i\alpha}\lambda)} \mathrm{Ai}\big(e^{-i\alpha}(ix+\lambda)\big)\, \mathrm{Ai}\big(e^{-i\alpha}(iy+\lambda)\big). \tag{24}$$

The resolvent is compact. The poles of the resolvent are determined by the zeros of $\mathrm{Ai}(e^{-i\alpha}\lambda)$, i.e., $\lambda_n = e^{i\alpha} a_n$, where the a_n are zeros of the Airy function: $\mathrm{Ai}(a_n) = 0$. The eigenvalues have multiplicity 1 (no Jordan block).

As a consequence of the analysis of the numerical range of the operator, we have

Proposition

$$||\mathcal{G}^{\pm,D}(\lambda)|| \leq \frac{1}{|\mathrm{Re}\,\lambda|}\,, \quad \textit{if}\ \mathrm{Re}\,\lambda < 0\,; \tag{25}$$

and

$$||\mathcal{G}^{\pm,D}(\lambda)|| \leq \frac{1}{|\mathrm{Im}\,\lambda|}\,, \quad \textit{if}\ \mp \mathrm{Im}\,\lambda > 0\,. \tag{26}$$

This proposition together with the Phragmen-Lindelöf principle (see Agmon [2] or Dunford-Schwartz [15])

Proposition

The space generated by the eigenfunctions of the Dirichlet realization $\mathcal{A}^{\pm,D}$ of $D_x^2 \pm ix$ is dense in L^2_+.

It is proven by R. Henry in [27] that there is no Riesz basis of eigenfunctions.

The Hilbert-Schmidt norm of the resolvent for $\lambda > 0$

At the boundary of the numerical range of the operator, it is interesting to analyze the behavior of the resolvent. Numerical computations lead to the observation that

$$\lim_{\lambda \to +\infty} ||\mathcal{G}^{\pm,D}(\lambda)||_{\mathcal{L}(L^2_+)} = 0\,. \tag{27}$$

As a new result, we will prove

Proposition

When λ tends to $+\infty$, we have

$$||\mathcal{G}^{\pm,D}(\lambda)||_{HS} \approx \lambda^{-\frac{1}{4}} (\log \lambda)^{\frac{1}{2}}\,. \tag{28}$$

About the proof

The Hilbert-Schmidt norm of the resolvent can be written as

$$||\mathcal{G}^{-,D}||_{HS}^2 = \int_{\mathbb{R}_+^2} |\mathcal{G}^{-,D}(x,y\,;\lambda)|^2 dxdy = 8\pi^2 \int_0^{\infty} Q(x;\lambda)dx, \qquad (29)$$

where

$$Q(x;\lambda) = \frac{|\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2}{|\mathrm{Ai}(e^{-i\alpha}\lambda)|^2} \times \\ \times \int_0^x \left|\mathrm{Ai}(e^{i\alpha}(iy+\lambda))\mathrm{Ai}(e^{-i\alpha}\lambda) - \mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\mathrm{Ai}(e^{i\alpha}\lambda)\right|^2 dy. \qquad (30)$$

Using the identity (8), we observe that

$$\begin{aligned}&\mathrm{Ai}(e^{i\alpha}(iy+\lambda))\mathrm{Ai}(e^{-i\alpha}\lambda)-\mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\mathrm{Ai}(e^{i\alpha}\lambda)\\&\quad=e^{-i\alpha}\left(\mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\mathrm{Ai}(\lambda)-\mathrm{Ai}(iy+\lambda)\mathrm{Ai}(e^{-i\alpha}\lambda)\right)\,.\end{aligned}\tag{31}$$

Hence we get

$$Q(x;\lambda)=|\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2\int_0^x\left|\mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\frac{\mathrm{Ai}(\lambda)}{\mathrm{Ai}(e^{-i\alpha}\lambda)}-\mathrm{Ai}(iy+\lambda)\right|^2dy\,.\tag{32}$$

More on Airy expansions

As a consequence of (9), we can write

$$|\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))| = \frac{\exp\left(-\frac{2}{3}\lambda^{3/2}u(x/\lambda)\right)}{2\sqrt{\pi}(\lambda^2+x^2)^{1/8}}(1+\mathcal{O}(\lambda^{-\frac{3}{2}})), \tag{33}$$

where

$$\begin{aligned} u(s) &= -(1+s^2)^{3/4}\cos\left(\frac{3}{2}\tan^{-1}(s)\right) \\ &= \frac{\sqrt{\sqrt{1+s^2}+1}\,(\sqrt{1+s^2}-2)}{\sqrt{2}}. \end{aligned} \tag{34}$$

We note indeed that $|e^{-i\alpha}(ix+\lambda)| = \sqrt{x^2+\lambda^2} \geq \lambda \geq \lambda_0$ and that we have a control of the argument $\arg(e^{-i\alpha}(ix+\lambda)) \in [-\frac{2\pi}{3}, -\frac{\pi}{6}]$ which permits to apply (9).

Similarly, we obtain

$$|\mathrm{Ai}(ix+\lambda)| = \frac{\exp\left(\frac{2}{3}\lambda^{3/2}u(x/\lambda)\right)}{2\sqrt{\pi}(\lambda^2+x^2)^{1/8}}(1+\mathcal{O}(\lambda^{-\frac{3}{2}}))\,. \tag{35}$$

We note indeed that $|ix+\lambda| = \sqrt{x^2+\lambda^2}$ and that $\arg((ix+\lambda)) \in [0,+\frac{\pi}{2}]$ and one can then again apply (9). In particular the function $|\mathrm{Ai}(ix+\lambda)|$ grows super-exponentially as $x \to +\infty$.

Basic properties of u.

Note that

$$u'(s) = \frac{3}{2\sqrt{2}}\,\frac{s}{\sqrt{1+\sqrt{1+s^2}}} \geq 0 \qquad (s \geq 0), \tag{36}$$

and u has the following expansion at the origin

$$u(s) = -1 + \frac{3}{8}s^2 + \mathcal{O}(s^4)\,. \tag{37}$$

For large s, one has

$$u(s) \sim \frac{s^{3/2}}{\sqrt{2}}\,, \qquad u'(s) \sim \frac{3s^{1/2}}{2\sqrt{2}}\,. \tag{38}$$

One concludes that the function u is monotonously increasing from -1 to infinity.

Upper bound

We start from the simple upper bound (for any $\epsilon > 0$)

$$Q(x,\lambda) \leq (1+\frac{1}{\epsilon})Q_1(x,\lambda) + (1+\epsilon)Q_2(x,\lambda)\,, \tag{39}$$

with

$$Q_1(x,\lambda) := |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2 \,\frac{|\mathrm{Ai}(\lambda)|^2}{|\mathrm{Ai}(e^{-i\alpha}\lambda)|^2} \int_0^x |\mathrm{Ai}(e^{-i\alpha}(iy+\lambda))|^2\, dy$$

and

$$Q_2(x,\lambda) := |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2 \int_0^x |\mathrm{Ai}(iy+\lambda)|^2\, dy\,.$$

We then write

$$Q_1(x,\lambda) \leq |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2 \,\frac{|\mathrm{Ai}(\lambda)|^2}{|\mathrm{Ai}(e^{-i\alpha}\lambda)|^2} \int_0^{+\infty} |\mathrm{Ai}(e^{-i\alpha}(iy+\lambda))|^2\, dy$$

and integrating over x

$$\int^{+\infty} \ldots \quad |\mathrm{Ai}(\lambda)|^2$$

Using (9), we obtain

$$\int_0^{+\infty} Q_1(x,\lambda)dx \leq C\lambda^{-\frac{1}{2}}\,. \tag{40}$$

Hence at this stage, we have proven the existence of $C>0$, $\epsilon_0>0$ and λ_0 such that such that for any $\epsilon \in (0,\epsilon_0]$ and any $\lambda \geq \lambda_0$:

$$||\mathcal{G}^{-,D}||_{HS}^2 \leq (1+\epsilon)\left(8\pi^2 \int\limits_0^{\infty} Q_2(x;\lambda)dx\right) + C\lambda^{-1}\epsilon^{-1}\,. \tag{41}$$

It remains to estimate

$$\int_0^{+\infty} Q_2(x,\lambda)dx = \int_0^{+\infty} dx \int\limits_0^{x} |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))\mathrm{Ai}(iy+\lambda)|^2\, dy\,. \tag{42}$$

Using the estimates (33) and (35), we obtain

Lemma

There exist C *and* ϵ_0*, such that, for any* $\epsilon \in (0, \epsilon_0)$*, for* $\lambda > \epsilon^{-\frac{2}{3}}$*, the integral of* $Q_2(x;\lambda)$ *can be bounded as*

$$\frac{1}{2}(1 - C\epsilon)\, I(\lambda) \ \leq 8\pi^2 \int_0^{+\infty} Q_2(x,\lambda)dx \leq \frac{1}{2}(1 + C\epsilon)\, I(\lambda)\,, \qquad (43)$$

where

$$I(\lambda) = \int_0^{\infty} dx\, \frac{\exp\left(-\frac{4}{3}\lambda^{3/2}u(x/\lambda)\right)}{(\lambda^2+x^2)^{1/4}} \int_0^{x} dy\, \frac{\exp\left(\frac{4}{3}\lambda^{3/2}u(y/\lambda)\right)}{(\lambda^2+y^2)^{1/4}}. \qquad (44)$$

Control of $I(\lambda)$.

It remains to control $I(\lambda)$ as $\lambda \to +\infty$. Using a change of variables, we get

$$I(\lambda) = \lambda \int_0^\infty dx \, \frac{\exp\left(-\frac{4}{3}\lambda^{3/2} u(x)\right)}{(1+x^2)^{1/4}} \int_0^x dy \, \frac{\exp\left(\frac{4}{3}\lambda^{3/2} u(y)\right)}{(1+y^2)^{1/4}} \,. \tag{45}$$

Hence, introducing

$$t = \frac{4}{3}\lambda^{\frac{3}{2}} \,, \tag{46}$$

we reduce the analysis to $\hat{I}(t)$ defined for $t \geq t_0$ by

$$\hat{I}(t) := \int_0^\infty dx \, \frac{1}{(1+x^2)^{1/4}} \int_0^x dy \, \frac{\exp\left(t(u(y) - u(x))\right)}{(1+y^2)^{1/4}} \,, \tag{47}$$

with

$$I(\lambda) = \lambda \, \widehat{I}(t) \,. \tag{48}$$

The analysis is close to that of the asymptotic behavior of a Laplace integral.

Asymptotic upper bound of $\hat{I}(t)$.

Let us start by a heuristic discussion. The maximum of $u(y) - u(x)$ should be on $x = y$. For $x - y$ small, we have $u(y) - u(x) \sim (y - x)u'(x)$. This suggests a concentration near $x = y = 0$, whereas a contribution for large x is of smaller order

The main term is

$$\widehat{I}_1(t,\epsilon) = \int_0^\epsilon dx\, \frac{1}{(1+x^2)^{1/4}} \int\limits_0^x dy\, \frac{\exp\big(t(u(y)-u(x))\big)}{(1+y^2)^{1/4}}. \qquad (49)$$

Its asymptotics is obtained using the asymptotics of

$$J_\epsilon(\sigma) := \int_0^\epsilon dx \int_0^x \exp\big(\sigma(y^2-x^2)\big)\, dy\,,$$

which has now to be estimated for large σ.
Here appears the Dawson function (cf Abramowitz-Stegun [1], p. 295 and 319)

$$s \mapsto D(s) := \int_0^s \exp(y^2 - s^2)\, dy$$

and its asymptotics as $s \to +\infty$,

$$D(s) = \frac{1}{2s}(1 + \mathcal{O}(s^{-1})\,. \qquad (50)$$

Hence we have shown the existence of a constant $C > 0$ and of ϵ_0 such that if $t \geq C\epsilon^{-3}$ and $\epsilon \in (0, \epsilon_0)$

$$\widehat{l}_1(t,\epsilon) \leq \frac{2}{3}\frac{\log t}{t} + C(\epsilon\frac{\log t}{t} + \frac{1}{\epsilon}\frac{1}{t})\,. \tag{51}$$

Taking $\epsilon = (\log \lambda)^{-\frac{1}{2}}$, we obtain

Lemma

There exists $C > 0$ *and* λ_0 *such that for* $\lambda \geq \lambda_0$

$$||\mathcal{G}^{-,D}(\lambda)||_{HS}^2 \leq \frac{3}{8}\lambda^{-\frac{1}{2}} \log \lambda (1 + C\,(\log \lambda)^{-\frac{1}{2}})\,.$$

Lower bound

Once the upper bounds established, the proof of the lower bound is easy. We start from the simple lower bound (for any $\epsilon > 0$)

$$Q(x,\lambda) \geq -\frac{1}{\epsilon} Q_1(x,\lambda) + (1-\epsilon) Q_2(x,\lambda))\,, \qquad (52)$$

and consequently

$$\int_0^{+\infty} Q(x,\lambda)\,dx \geq (1-\epsilon) \int_0^{+\infty} Q_2(x,\lambda)\,dx - \frac{1}{\epsilon} \int_0^{+\infty} Q_1(x,\lambda)\,. \quad (53)$$

Similar estimates to the upper bound give the proof of

Lemma

There exists $C > 0$ and λ_0 such that for $\lambda \geq \lambda_0$

$$||\mathcal{G}^{-,D}(\lambda)||_{HS}^2 \geq \frac{3}{8} \lambda^{-\frac{1}{2}} \log \lambda \,(1 - C\,(\log \lambda)^{-\frac{1}{2}})\,.$$

The complex Airy operator with a semi-permeable barrier: definition and properties

We consider the sesquilinear form a_ν defined for $u = (u_-, u_+)$ and $v = (v_-, v_+)$ by

$$\begin{aligned} a_\nu(u,v) \;=\; & \int_{-\infty}^{0} \Big(u_-'(x)\bar{v}_-'(x) + i\, x u_-(x)\bar{v}_-(x) + \nu\, u_-(x)\bar{v}_-(x) \Big)\, dx \\ & + \int_0^{+\infty} \Big(u_+'(x)\bar{v}_+'(x) + i\, x u_+(x)\bar{v}_+(x) + \nu\, u_+(x)\bar{v}_+(x) \Big)\, dx \\ & + \kappa \big(u_+(0) - u_-(0)\big)\big(\overline{v_+(0) - v_-(0)}\big)\,, \end{aligned} \tag{54}$$

where the form domain $\mathcal{V}$ is

$$\mathcal{V} := \Big\{ u = (u_-, u_+) \in H^1_- \times H^1_+ : |x|^{\frac{1}{2}} u \in L^2_- \times L^2_+ \Big\}\,.$$

The space $\mathcal{V}$ is endowed with the Hilbertian norm

$$\|u\|_{\mathcal{V}} := \sqrt{ \|u_-\|^2_{H^1_-} + \|u_+\|^2_{H^1_+} + \| |x|^{1/2} u \|^2_{L^2} }\,.$$

We first observe

Lemma

For any $\nu \geq 0$, the sesquilinear form a_ν is continuous on $\mathcal{V}$.

As the imaginary part of the potential $V(x) = ix$ changes sign, it is not straightforward to determine whether the sesquilinear form a_ν is coercive. Due to the lack of coercivity, the standard version of the Lax-Milgram theorem does not apply. We shall instead use the following generalization introduced in Almog-Helffer [4].

Theorem

Let $\mathcal{V} \subset \mathcal{H}$ be two Hilbert spaces such that $\mathcal{V}$ is continuously embedded in $\mathcal{H}$ and $\mathcal{V}$ is dense in $\mathcal{H}$. Let a be a continuous sesquilinear form on $\mathcal{V} \times \mathcal{V}$, and assume that there exists $\alpha > 0$ and two bounded linear operators Φ_1 and Φ_2 on $\mathcal{V}$ such that, for all $u \in \mathcal{V}$,

$$\left\{ \begin{array}{lcl} |a(u,u)| + |a(u,\Phi_1 u)| & \geq & \alpha \, \|u\|_{\mathcal{V}}^2 \,, \\ |a(u,u)| + |a(\Phi_2 u, u)| & \geq & \alpha \, \|u\|_{\mathcal{V}}^2 \,. \end{array} \right. \tag{55}$$

Assume further that Φ_1 extends to a bounded linear operator on $\mathcal{H}$. Then there exists a closed, densely-defined operator S on $\mathcal{H}$ with domain

$$\mathcal{D}(S) = \left\{ u \in \mathcal{V} : v \mapsto a(u,v) \;\; \textit{can be extended continuously on} \;\; \mathcal{H} \right\},$$

such that, for all $u \in \mathcal{D}(S)$ and $v \in \mathcal{V}$,

$$a(u,v) = \langle Su, v \rangle_{\mathcal{H}} \,.$$

Moreover, from the characterization of the domain and its inclusion in $\widehat{D}$, we deduce the stronger

Proposition

There exists λ_0 *(*$\lambda_0 = 0$ *for* $\kappa > 0$*) such that* $(\mathcal{A}_1^+ - \lambda_0)^{-1}$ *belongs to the Schatten class* $\mathcal{C}^p$ *for any* $p > \frac{3}{2}$.

Note that if it is true for some λ_0 it is true for any λ in the resolvent set.

Remark

The adjoint of $\mathcal{A}_1^+$ *is the operator associated by the same construction with* $D_x^2 - ix$. $\mathcal{A}_1^- + \lambda$ *being injective, this implies by a general criterion [25] that* $\mathcal{A}_1^+ + \lambda$ *is maximal accretive, hence generates a contraction semigroup.*

The following statement summarizes the previous discussion.

Proposition

The operator $\mathcal{A}_1^+$ acting as

$$u \mapsto \mathcal{A}_1^+ u = \left(-\frac{d^2}{dx^2}u_- + ixu_-, -\frac{d^2}{dx^2}u_+ + ixu_+\right)$$

on the domain

$$\mathcal{D}(\mathcal{A}_1^+) = \{u \in H_-^2 \times H_+^2 : xu \in L_-^2 \times L_+^2 \text{ and } u \text{ satisfies conditions } (3)\} \quad (56)$$

is a closed operator with compact resolvent.
There exists some positive λ such that the operator $\mathcal{A}_1^+ + \lambda$ is maximal accretive.

Remark

We have

$$\Gamma \mathcal{A}_1^+ = \mathcal{A}_1^- \,, \tag{57}$$

where Γ *denotes the complex conjugation:*

$$\Gamma(u_- \,,\, u_+) = (\bar{u}_- \,,\, \bar{u}_+) \,.$$

Remark (PT-Symmetry)

If (λ, u) *is an eigenpair, then* $(\bar{\lambda}, \bar{u}(-x))$ *is also an eigenpair.*

Integral kernel of the resolvent

Lengthy but elementary computations give:

$$\mathcal{G}^-(x,y\,;\lambda,\kappa) = \mathcal{G}_0^-(x,y\,;\lambda) + \mathcal{G}_1(x,y\,;\lambda,\kappa)\,, \tag{58}$$

where $\mathcal{G}_0^-(x,y\,;\lambda)$ is the distribution kernel of the resolvent of the operator $\mathcal{A}_0^* := -\frac{d^2}{dx^2} - ix$ on the line, whereas $\mathcal{G}_1(x,y\,;\lambda,\kappa)$ is given by the following expressions

$$\mathcal{G}_1(x,y\,;\lambda,\kappa) = \begin{cases} -4\pi^2 \frac{e^{2i\alpha}[\mathrm{Ai}'(e^{i\alpha}\lambda)]^2}{f(\lambda)+\kappa} \mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_y)\,, & \text{for } x>0\,, \\ -2\pi \frac{f(\lambda)}{f(\lambda)+\kappa} \mathrm{Ai}(e^{i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_y)\,, & \text{for } x<0\,, \end{cases} \tag{59}$$

for $y>0$, and

$$\mathcal{G}_1(x,y\,;\lambda,\kappa) = \begin{cases} -2\pi \frac{f(\lambda)}{f(\lambda)+\kappa} \mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{i\alpha}w_y)\,, & x>0\,, \\ -4\pi^2 \frac{e^{-2i\alpha}[\mathrm{Ai}'(e^{-i\alpha}\lambda)]^2}{f(\lambda)+\kappa} \mathrm{Ai}(e^{i\alpha}w_x)\mathrm{Ai}(e^{i\alpha}w_y)\,, & x<0\,, \end{cases} \tag{60}$$

for $y<0$.

Hence the poles are determined by the equation

$$f(\lambda) = -\kappa\,, \tag{61}$$

with f defined by

$$f(\lambda) := 2\pi \mathrm{Ai}'(e^{-i\alpha}\lambda)\mathrm{Ai}'(e^{i\alpha}\lambda)\,. \tag{62}$$

Remark

For $\kappa = 0$, one recovers the conjugated pairs associated with the zeros a'_n of Ai'. We have indeed as poles

$$\lambda_n^+ = e^{i\alpha}a'_n\,, \quad \lambda_n^- = e^{-i\alpha}a'_n\,, \tag{63}$$

where a'_n is the n-th zero of Ai'.

We also know that the eigenvalues for the Neumann problem are simple. Hence by the local inversion theorem we get the existence of a solution close to each $\lambda_n^\pm$ for κ small enough (possibly depending on n) if we show that $f'(\lambda_n^\pm) \neq 0$. For λ_n^+, we have, using the Wronskian relation (7) and $\mathrm{Ai}'(e^{-i\alpha}\lambda_n^+) = 0$,

$$\begin{aligned} f'(\lambda_n^+) &= 2\pi\, e^{-i\alpha}\ \mathrm{Ai}''(e^{-i\alpha}\lambda_n^+)\mathrm{Ai}'(e^{i\alpha}\lambda_n^+) \\ &= 2\pi e^{-2i\alpha}\lambda_n^+ \mathrm{Ai}(e^{-i\alpha}\lambda_n^+)\mathrm{Ai}'(e^{i\alpha}\lambda_n^+) \\ &= -i\lambda_n^+\,. \end{aligned} \tag{64}$$

Similar computations hold for λ_n^-. We recall that

$$\lambda_n^+ = \overline{\lambda_n^-}\,.$$

Applications to (2D)-problems

In higher dimension, an extension of the complex Airy operator is the differential operator that we call the Bloch-Torrey operator or simply the BT-operator:

$$-D\Delta + igx_1 \,,$$

where $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2$ is the Laplace operator in $\mathbb{R}^n$, and D and g are real parameters. More generally, we will study the spectral properties of some realizations of the differential operator

$$\mathcal{A}_h^{\#} = -h^2\Delta + i\,V(x)\,, \tag{65}$$

in an open set Ω, where h is a real parameter and $V(x)$ a real-valued potential with controlled behavior at ∞, and the superscript $\#$ distinguishes Dirichlet (D), Neumann (N), Robin (R), or transmission (T) conditions.

More precisely we discuss

1. the case of a bounded open set Ω with Dirichlet, Neumann or Robin boundary condition;
2. the case of a complement $\Omega := \complement\overline{\Omega}_-$ of a bounded set Ω_- with Dirichlet, Neumann or Robin boundary condition;
3. the case of two components $\Omega_- \cup \Omega_+$, with $\Omega_- \subset \overline{\Omega}_- \subset \Omega$ and $\Omega_+ = \Omega \backslash \overline{\Omega}_-$, with Ω bounded and transmission conditions at the interface between Ω_- and Ω_+;
4. the case of two components $\Omega_- \cup \complement\,\overline{\Omega}_-$, with Ω_- bounded and transmission conditions at the boundary;
5. the case of two unbounded components Ω_- and Ω_+ separated by a hypersurface and transmission conditions at the boundary.

The state u (in the first two items) or the pair (u_-, u_+) in the last items should satisfy some boundary or transmission condition at the interface. We consider the following situations:

- the Dirichlet condition: $u_{|\partial\Omega} = 0$;
- the Neumann condition: $\partial_\nu u_{|\partial\Omega} = 0$, where $\partial_\nu = \nu \cdot \nabla$, with ν being the outwards pointing normal;
- the Robin condition: $h^2 \partial_\nu u_{|\partial\Omega} = -\mathcal{K} u_{|\partial\Omega}$, where ≥ 0 denotes the Robin parameter;
- the transmission condition:

$$h^2 \partial_\nu u_{+\,|\partial\Omega_-} = h^2 \partial_\nu u_{-\,|\partial\Omega_-} = \mathcal{K}(u_{+\,|\partial\Omega_-} - u_{-\,|\partial\Omega_-}) \,,$$

where $\mathcal{K} \geq 0$ denotes the transmission parameter.

$\Omega^\#$ denotes Ω if $\# \in \{D, N, R\}$ and Ω_- if $\# = T$.
$L^2_\#$ denotes $L^2(\Omega)$ if $\# \in \{D, N, R\}$ and $L^2(\Omega_-) \times L^2(\Omega_+)$ if $\# = T$.
In the first part of this talk, we have described various realizations of the complex Airy operator $A_0^\# := -\frac{d^2}{d\tau^2} + i\tau$ in the four cases.
The boundary conditions read respectively:

- $u(0) = 0$ (Dirichlet)
- $u'(0) = 0$ (Neumann)
- $u'(0) = \kappa\, u(0)$ (Robin)
- $u'_-(0) = u'_+(0) = \kappa\,(u_+(0) - u_-(0))$ (Transmission)

(with $\kappa \geq 0$ in the last items). For all these cases, we have proven the existence of a discrete spectrum and the completeness of the corresponding generalized eigenfunctions.

We have started the analysis of the spectral properties of the BT operator in dimension 2 or higher that are relevant for applications in superconductivity theory (Almog, Almog-Helffer-Pan, Almog-Helffer), in fluid dynamics (Martinet), in control theory (Beauchard-Helffer-Henry-Robbiano) and in diffusion magnetic resonance imaging (Grebenkov) . We mainly focus on

- definition of the operator,
- construction of approximate eigenvalues in some asymptotic regimes,
- localization of quasimode states near certain boundary points,
- numerical simulations.

In particular, it is interesting to discuss the semiclassical asymptotics $h \to 0$, the large domain limit, the asymptotics when $g \to 0$ or $+\infty$, the asymptotics when the transmission or Robin parameter tends to 0. Some other important questions remain unsolved like the existence of eigenvalues close to the approximate eigenvalues (a problem which is only solved in particular situations).

When $g = 0$, the BT-operator is reduced to the Laplace operator for which the answers are well known. In particular, the spectrum is discrete in the case of bounded domains and equals $[0, +\infty)$ when one or both components are unbounded. In the case $g \neq 0$, we show that if there is at least one boundary point at which the normal vector to the boundary is parallel to the coordinate x_1, then there exist approximate eigenvalues of the BT-operator *suggesting* the existence of eigenvalues while the associated eigenfunctions are localized near this point. This localization property has been already discussed in physics literature for bounded domains (deSwiet et al. 1994) for which the existence of eigenvalues is trivial. Since our asymptotic constructions are local and thus hold for unbounded domains, the localization behavior can be conjectured for exterior problems involving the BT-operator.

Some of these questions have been already analyzed by Y. Almog (see [3] (2008) and references therein for earlier contributions), R. Henry in his PHD (2013) (+ CPDE paper 2014) and Almog-Henry (2015) but they were mainly devoted to the case of a Dirichlet realization in bounded domains in $\mathbb{R}^2$ or particular unbounded domains like $\mathbb{R}^2$ and $\mathbb{R}^2_+$, these two last cases playing an important role in the local analysis of the global problem.

We consider $\mathcal{A}_h$ and the corresponding realizations in Ω are denoted by $\mathcal{A}_h^D$, $\mathcal{A}_h^N$, $\mathcal{A}_h^R$ and $\mathcal{A}_h^T$. These realizations will be properly under the condition that, when Ω is unbounded, there exists $C > 0$ such that

$$|\nabla V(x)| \leq C\sqrt{1 + V(x)^2}\,. \qquad (66)$$

Our main construction is local and summarized in the following

Main (2D)-theorem

Let $\Omega \subset \mathbb{R}^2$ as above, $V \in \mathcal{C}^\infty(\overline{\Omega};\mathbb{R})$ and $x^0 \in \partial\Omega^\#$ such that

$$\nabla V(x^0) \neq 0\,, \quad \nabla V(x^0) \wedge \nu(x^0) = 0\,, \tag{67}$$

where $\nu(x^0)$ denotes the outward normal on $\partial\Omega$ at x^0 .
Assume that, in the local curvilinear coordinates, the second derivative $2\,v_{20}$ of the restriction of V to the boundary at x^0 satisfies

$$v_{20} \neq 0\,.$$

For the Robin and transmission cases, we assume that for some $\kappa > 0$

$$\mathcal{K} = h^{\frac{4}{3}}\kappa\,. \tag{68}$$

Main theorem continued

If $\mu_0^\#$ is a simple eigenvalue of the realization "#" $-\frac{d^2}{dx^2} + ix$ in $L^2_\#$,
and μ_2 is an eigenvalue of Davies operator $-\frac{d^2}{dy^2} + iy^2$ on $L^2(\mathbb{R})$,
then there exists a pair $(\lambda_h^\#, u_h^\#)$ with $u_h^\#$ in the domain of $\mathcal{A}_h^\#$, such that

$$\lambda_h^\# = i\,V(x^0) + h^{\frac{2}{3}} \sum_{j\in\mathbb{N}} \lambda_{2j}^\#\, h^{\frac{j}{3}} + \mathcal{O}(h^\infty)\,, \tag{69}$$

$$(\mathcal{A}_h^\# - \lambda_h^\#)\, u_h^\# = \mathcal{O}(h^\infty) \text{ in } L^2_\#(\Omega)\,, \quad ||u_h^\#||_{L^2} \sim 1\,, \tag{70}$$

where

$$\lambda_0^\# = \mu_0^\#\, |\, v_{01}|^{\frac{2}{3}} \exp\left(i\frac{\pi}{3}\, v_{01}\right)\,, \quad \lambda_2 = \mu_2 |v_{20}|^{\frac{1}{2}} \exp\left(i\frac{\pi}{4} v_{20}\right)\,, \tag{71}$$

with $v_{01} := \nu\cdot\nabla V(x^0)$.

We can also discuss a physically interesting case when κ in (68) depends on h and tends to 0.

The proof of this theorem provides a general scheme for quasimode construction in an arbitrary planar domain with smooth boundary $\partial\Omega$. In particular, this construction allowed us to retrieve and further generalize the asymptotic expansion of eigenvalues obtained by de Swiet and Sen for the Bloch-Torrey operator in the case of a disk. The generalization is applicable for any smooth boundary, with Neumann, Dirichlet, Robin, or transmission boundary condition. Moreover, since the analysis is local, the construction is applicable to both bounded and unbounded components.

In a work in progress, Almog-Grebenkov-Helffer plan to prove the existence of the eigenvalues and the rate of the associated semi-group.

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