

# Asymptotics for the magnetic lowest Dirichlet-to-Neumann eigenvalue in the strong magnetic limit

Bernard Helffer (Nantes Université)

in collaboration with  
Ayman Kachmar and François Nicoleau

**March 2026 in Stuttgart.**

# Abstract

Inspired by some questions presented in a recent ArXiv preprint (version v1) by T. Chakradhar, K. Gittins, G. Habib and N. Peyerimhoff, we analyze their conjecture that the ground state energy of the magnetic Dirichlet-to-Neumann operator tends to  $+\infty$  as the magnetic field tends to  $+\infty$ . More precisely, we explore refined conjectures for general domains in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  based on the previous analysis in the case of the half-plane and the disk. This part is a work in collaboration with Ayman Kachmar and François Nicoleau. In connexion with old works on the magnetic Schrödinger operator, we will also discuss, if time permits, recent results by Zhongwei Shen and the case (after Helffer-Kachmar-Nicoleau) of the exterior problem.

# Presentation

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^n$ ,  $n \geq 2$ , with smooth boundary. For any  $u \in \mathcal{D}'(\Omega)$ , the magnetic Schrödinger operator on  $\Omega$  is defined as

$$H_A u = (D - A)^2 u = -\Delta u - 2i A \cdot \nabla u + (A^2 - i \operatorname{div} A)u, \quad (1)$$

where  $D = -i\nabla$ ,  $-\Delta$  is the usual positive Laplace operator on  $\mathbb{R}^n$

and  $A = \sum_{j=1}^n A_j dx_j$  is the 1-form magnetic potential. We often

identify the 1-form magnetic potential  $A$  with the vector field

$$\vec{A} = (A_1, \dots, A_n).$$

We assume that  $A \in C^\infty(\bar{\Omega}; \mathbb{R}^n)$ . The magnetic field is given by the 2-form  $B = dA$ .

Since zero does not belong to the spectrum of the Dirichlet realization of  $H_A$ , the boundary value problem

$$\begin{cases} H_A u = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = f & \in H^{1/2}(\partial\Omega), \end{cases} \quad (2)$$

has a unique solution  $u \in H^1(\Omega)$ .

The Dirichlet-to-Neumann map (D-to-N map) is defined by

$$\begin{aligned} \Lambda_A : H^{1/2}(\partial\Omega) &\longmapsto H^{-1/2}(\partial\Omega) \\ f &\longmapsto (\partial_{\vec{\nu}} u + i\langle A, \vec{\nu} \rangle u)|_{\partial\Omega}, \end{aligned} \quad (3)$$

where  $\vec{\nu}$  is the outward normal unit vector field on  $\partial\Omega$ .

More precisely, we define the D-to-N map using the equivalent weak formulation :

$$\langle \Lambda_A f, g \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} = \int_{\Omega} \langle (D - A)u, (D - A)v \rangle dx, \quad (4)$$

for any  $g \in H^{1/2}(\partial\Omega)$  and  $f \in H^{1/2}(\partial\Omega)$  such that  $u$  is the unique solution of (2) and  $v$  is any element of  $H^1(\Omega)$  so that  $v|_{\partial\Omega} = g$ . Clearly, the D-to-N map is a positive operator.

We recall that when  $\Omega$  is bounded and regular the spectrum of the D-to-N operator is discrete and is given by an increasing sequence of eigenvalues

$$0 \leq \mu_1 \leq \mu_2 \leq \dots \leq \mu_n \leq \dots \rightarrow +\infty.$$

Due to the variational characterization, the ground state energy

$$\mu_1 := \lambda^{\text{DN}}(A, \Omega)$$

can be expressed as

$$\lambda^{\text{DN}}(A, \Omega) = \inf_{u \in C^\infty(\bar{\Omega}), \|u\|_{\partial\Omega} = 1} \|(-i\nabla - A)u\|_\Omega^2,$$

where  $\|u\|_\Omega$  and  $\|u\|_{\partial\Omega}$  denote the  $L^2$ -norms in  $L^2(\Omega; \mathbb{C})$  and  $L^2(\partial\Omega; \mathbb{C})$  resp.

## Comparison with the Neumann magnetic problem

Our analysis is parallel with the analysis of the Neumann magnetic problem for which there is a huge literature in the last forty years including two books (Fournais-Helffer (2010) [21], N. Raymond (2017) [68]).

In this case the Neumann magnetic ground state is given by

$$\lambda^{\text{Ne}}(A, \Omega) = \inf_{u \in C^\infty(\bar{\Omega}), \|u\|_\Omega=1} \|(-i\nabla - A)u\|_\Omega^2,$$

We were also inspired by the recent work of T. Chakradhar, K. Gittins, G. Habib and N. Peyerimhoff, (see [9], Example 2.8).

We consider the following magnetic 1-form defined in the unit disk  $D(0, 1) \subset \mathbb{R}^2$  by :

$$A_0(x, y) = (-ydx + xdy), \quad (5)$$

Using special functions, it has been proven by Helffer-Nicoleau the following

### Theorem HNi1

One has the asymptotic expansion as  $b \rightarrow +\infty$ ,

$$\lambda^{DN}(bA_0) = \alpha b^{1/2} - \frac{\alpha^2 + 2}{6} + \mathcal{O}(b^{-1/2}), \quad (6)$$

where  $-\alpha$  is the unique negative zero of the so-called parabolic cylinder function  $D_{\frac{1}{2}}(z)$ .

We recall that the parabolic cylinder functions  $D_\nu(z)$  are the (normalized) solutions of the differential equation

$$w'' + \left(\nu + \frac{1}{2} - \frac{z^2}{4}\right) w = 0,$$

which tend to 0 as  $z \rightarrow +\infty$ .

At last, the positive real  $\alpha$  appearing in this theorem is approximately equal to

$$\alpha = 0.7649508673\dots$$

The main idea for treating the case of more general domains, following what has been done in Surface Superconductivity, is to use the previous result for disks, the radius being locally chosen as the inverse of the curvature when it is positive. We have also to analyze the half plane case and the case of the exterior of the disk (to cover the negative curvature case).

The starting point is in the case of the disk  $\mathcal{B}_R$  of radius  $R$  and  $\text{curl } A = 1$ ,

$$\lambda^{\text{DN}}(bA, \mathcal{B}_R) = \hat{\alpha} b^{1/2} - \frac{\hat{\alpha}^2 + 1}{3} R^{-1} + \mathcal{O}(b^{-1/2}),$$

with

$$\hat{\alpha} = \alpha / \sqrt{2}.$$

The analysis in Helffer-Nicoleau [38] can be adapted to the case  $\mathcal{B}_R^{\text{ext}}$ , the exterior of the disk  $\mathcal{B}_R$  (see another work with F. Nicoleau [39] or the end of the talk), and we get

$$\lambda^{\text{DN}}(bA, \mathcal{B}_R^{\text{ext}}) = \hat{\alpha} b^{1/2} + \frac{\hat{\alpha}^2 + 1}{3} R^{-1} + \mathcal{O}(b^{-1/2}).$$

Hence we can also consider boundary points with negative curvature.

To cover later all the cases with one notation we introduce for  $R \in \mathbb{R}$

$$\lambda^{\text{DN}}(b, R) = \begin{cases} \lambda^{\text{DN}}(bA_0, \mathcal{B}_R) & \text{if } R > 0 \\ \hat{\alpha}b^{1/2} & \text{if } R = 0, \\ \lambda^{\text{DN}}(bA_0, \mathcal{B}_{-R}^{\text{ext}}) & \text{if } R < 0 \end{cases},$$

and observe that

$$\lambda^{\text{DN}}(b, R) = \hat{\alpha}b^{1/2} - \frac{\hat{\alpha}^2 + 1}{3}R^{-1} + \mathcal{O}(b^{-1/2}).$$

## Comparison with Neumann

The result for Neumann was (Baumann-Phillips-Tang [5] 1998, Bernoff-Sternberg, Lu-Pan, Helffer-Morame, Fournais-Helffer,....) , for some "spectral invariants"  $\Theta_0$  and  $C_1 > 0$ ,

$$\lambda^{\text{Ne}}(b, R) = 2\Theta_0 b - C_1 b^{\frac{1}{2}} R^{-1} + \mathcal{O}(1).$$

For the half-plane (formally  $R = +\infty$ )

$$\lambda^{\text{Ne}}(b, +\infty) = 2\Theta_0 b.$$

This corresponds with the analysis of the spectrum of the Neumann realization in  $\{t > 0\}$

$$D_t^2 + (D_x + 2bt)^2.$$

After partial Fourier transform and dilation, we get the family

$$h(\xi) := D_t^2 + (\xi + t)^2.$$

We get

$$\Theta_0 = \inf_{\xi} \mu(\xi)$$

where  $\mu(\xi)$  is the Neumann ground state energy of  $h^{\text{Ne}}(\xi)$ .

For the D-to-N operator we have to solve in  $\mathbb{R}^+$

$$h(\xi)u_\xi = 0, u_\xi(0) = u_0(\xi),$$

and then compute  $u'_\xi(0)$ .

Here we see how the special function  $D_\nu$  and consequently  $\alpha$  appears.

# Saint-James picture in the case of the disk

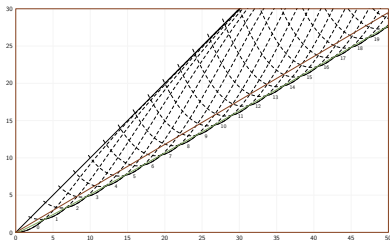


Figure: The magnetic Neumann eigenvalues

Our second result for the disk is concerned by diamagnetism. We recall that by diamagnetism we mean that  $\lambda^{DN}(bA_0)$  is minimal for  $b = 0$ . This result has been proved in full generality in ter Elst-Ouhabaz [14] (see also Helffer-Nicoleau [37] for variants of this result).

We prove a stronger diamagnetism result:

### Theorem HN 2

The map  $b \mapsto \lambda^{DN}(bA_0, B_1)$  is increasing on  $(0, +\infty)$ .

Notice that the corresponding problem is open in the case of Neumann (see Fournais-Helffer (2007), Helffer-Léna (2024)) (only proven for  $b \geq b_0$  or for  $0 < b \leq b_1$ ).

For the Dirichlet problem see Baur-Weidl (2024).

# Constant magnetic field in the disk

In polar coordinates  $(r, \theta)$ , the D-to-N map is defined by :

$$\begin{aligned} \Lambda_{bA_0} : H^{\frac{1}{2}}(S^1) &\rightarrow H^{-\frac{1}{2}}(S^1) \\ \Psi &\rightarrow \partial_r v(r, \theta)|_{r=1}. \end{aligned} \quad (7)$$

Writing

$$v(r, \theta) = \sum_{n \in \mathbb{Z}} v_n(r) e^{in\theta}, \quad \Psi(\theta) = \sum_{n \in \mathbb{Z}} \Psi_n e^{in\theta}, \quad (8)$$

we see that  $v_n(r)$  solves:

$$\begin{cases} -v_n''(r) - \frac{v_n'(r)}{r} + (br - \frac{n}{r})^2 v_n(r) = 0 & \text{for } r \in (0, 1), \\ v_n(1) = \Psi_n. \end{cases} \quad (9)$$

A bounded solution to the differential equation (9) is given by:

$$v_n(r) = e^{-\frac{br^2}{2}} r^n L_{-\frac{1}{2}}^n(br^2) \quad \text{for } n \geq 0, \quad (10)$$

where  $L_\nu^\alpha(z)$  denotes the generalized Laguerre function. For  $n \leq -1$ , thanks to symmetries in (9), we get a similar expression for  $v_n(r)$  changing the parameters  $(n, b)$  into  $(-n, -b)$ .

We recall that the generalized Laguerre functions  $L_\nu^\alpha(z)$  satisfy the differential equation:

$$z \frac{d^2 w}{dz^2} + (1 + \alpha - z) \frac{dw}{dz} + \nu w = 0, \quad (11)$$

and are given by

$$L_\nu^\alpha(z) = \frac{\Gamma(\alpha + \nu + 1)}{\Gamma(\alpha + 1)\Gamma(\nu + 1)} M(-\nu, \alpha + 1, z), \quad (12)$$

where  $M(a, c, z)$  is the Kummer's confluent hypergeometric function, defined as

$$M(a, c, z) = \sum_{n=0}^{+\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}. \quad (13)$$

Here  $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ .

For  $0 < a < c$ , we have the following formula:

$$M(a, c, z) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{c-a-1} dt. \quad (14)$$

The derivative of the Kummer's function with respect to  $z$  is given by:

$$\partial_z M(a, c, z) = \frac{a}{c} M(a+1, c+1, z). \quad (15)$$

Now, let us return to the study of the Steklov eigenvalues. Obviously, they are given by

$$\lambda_n = \frac{v_n'(1)}{v_n(1)} \quad \text{for } n \in \mathbb{Z}. \quad (16)$$

Thus, using (10) and (12), we see that the Steklov spectrum is the set:

$$\sigma(\Lambda(b)) = \{\lambda_0(b)\} \cup \{\lambda_n(b), \lambda_n(-b)\}_{n \in \mathbb{N}^*}, \quad (17)$$

where for  $n \geq 0$ ,

$$\lambda_n(b) = n - b + 2b \frac{\partial_z M(\frac{1}{2}, n+1, b)}{M(\frac{1}{2}, n+1, b)}. \quad (18)$$

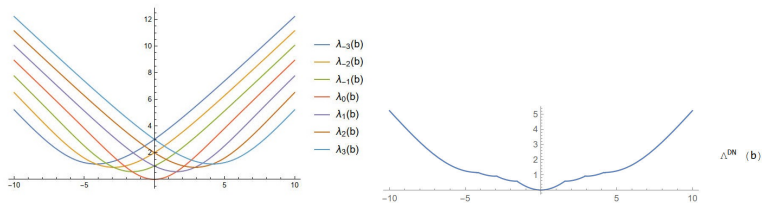


Figure: The magnetic Steklov eigenvalues  $\lambda_n(b)$  (left) and the ground state energy  $\lambda^{DN}(b)$  (right).

# Constant magnetic field in general domains

We will prove the following theorem

## Theorem HKN1

Let  $\Omega$  be a regular domain in  $\mathbb{R}^2$  and  $A$  be a vector potential with a magnetic field  $B = \text{curl} A$  such that  $B$  is  $C^1$  on  $\overline{\Omega}$  and  $B = 1$  on a neighborhood of  $\partial\Omega$ . Then, the ground state energy of the D-to-N map  $\Lambda_{bA}$  satisfies

$$\lambda^{\text{DN}}(bA, \Omega) = \hat{\alpha} b^{\frac{1}{2}} - \frac{\hat{\alpha}^2 + 1}{3} \max_{x \in \partial\Omega} k(x) + \mathcal{O}(b^{-1/6}) \quad , \quad b \rightarrow +\infty ,$$

where  $k$  is the curvature of  $\partial\Omega$ .

For comparison, the result in the case of the Neumann problem was:

### Theorem LuPan–HeMo 1999-2000

Let  $\Omega$  be a regular domain in  $\mathbb{R}^2$  and  $A$  be a vector potential with associated magnetic field  $B = 1$ . Then, the Neumann ground state energy satisfies

$$\lambda^{\text{Ne}}(bA, \Omega) = \Theta_0 b - C_1 b^{1/2} \max_{x \in \partial\Omega} k(x) + \mathcal{O}(1) \quad , \quad b \rightarrow +\infty ,$$

where  $k$  is the curvature of  $\partial\Omega$ .

## Coming back to ground state energy estimates

We will also consider the case of variable magnetic field in  $2D$  and in  $3D$  in the same spirit as for the analysis of the Neumann problem appearing in surface superconductivity [55, 34, 66, 67, 35].

### Theorem HKN2

Let  $\Omega$  be a regular domain in  $\mathbb{R}^2$ ,  $A$  be a magnetic potential with non vanishing magnetic field  $B(x)$  in  $\partial\Omega$ , then the ground state energy of the D-to-N map  $\Lambda_{bA}$  satisfies

$$\lambda^{\text{DN}}(bA, \Omega) = \hat{\alpha} \left( \inf_{x \in \partial\Omega} |B(x)| \right)^{\frac{1}{2}} b^{\frac{1}{2}} + o(b^{\frac{1}{2}}). \quad (19)$$

# Neumann case

## Theorem LuPan 1999

Let  $\Omega$  be a regular domain in  $\mathbb{R}^2$ ,  $A$  be a magnetic potential with non vanishing magnetic field  $B(x)$  in  $\bar{\Omega}$ , then the ground state energy of the Neumann realization of the magnetic Laplacian satisfies

$$\lambda^{\text{Ne}}(bA, \Omega) = b \min\left(\inf_{x \in \Omega} |B(x)|, \Theta_0 \inf_{x \in \partial\Omega} |B(x)|\right) + o(b). \quad (20)$$

## Extension to 3D

We have a similar theorem for variable magnetic fields in 3D which is in correspondence with known results obtained in the analysis of the ground state energy of the Neumann realization of the magnetic Laplacian (see Lu-Pan, Helffer-Morame, Raymond, Helffer-Kachmar, Hérau-Raymond, Alfa-Aafarani-Hérau-Raymond [56, 35, 67, 26, 42, 1]):

## Theorem HKN3

Let  $\Omega$  be a regular bounded domain in  $\mathbb{R}^3$ ,  $A$  be a magnetic potential with non vanishing magnetic field  $B(x)$  in  $\partial\Omega$ , then the ground state energy of the D-to-N map  $\Lambda_{bA}^{DN}$  satisfies

$$\lim_{b \rightarrow +\infty} b^{-1/2} \lambda^{DN}(bA, \Omega) = \inf_{x \in \partial\Omega} \left( \lambda^{DN}(\vartheta(x)) |B(x)|^{\frac{1}{2}} \right), \quad (21)$$

where, for  $x \in \partial\Omega$ ,

- ▶  $\vartheta(x)$  is defined by  $\langle \vec{H}(x) | \vec{\nu} \rangle = -|B(x)| \sin \vartheta(x)$  . ,
- ▶  $\vec{H}(x)$  is the magnetic vector field associated with  $B(x)$  considered as a 2-form by the Hodge-map,
- ▶  $\vec{\nu}$  is the exterior normal at  $x \in \partial\Omega$ ,
- ▶  $\lambda^{DN}(\vartheta)$  is the ground state energy relative to the half space when the magnetic field is constant.

There are two important consequences

- ▶ When  $B$  is constant with magnitude 1, it follows that

$$\lim_{b \rightarrow +\infty} b^{-1/2} \lambda^{\text{DN}}(bA, \Omega) = \hat{\alpha}.$$

- ▶ More generally, if we know only that  $|B(x)|$  is constant, as for the helical magnetic field  $B(x) = (\cos(\tau x_3), \sin(\tau x_3), 0)$  encountered in liquid crystals [64, 26] (Pan, Helffer-Kachmar), then

$$\lim_{b \rightarrow +\infty} b^{-1/2} \lambda^{\text{DN}}(bA, \Omega) = \inf_{x \in \partial\Omega} \lambda^{\text{DN}}(\vartheta(x)).$$

If  $\partial\Omega$  is homeomorphic to  $S^2$ , then

$$\inf_{x \in \partial\Omega} \lambda^{\text{DN}}(\vartheta(x)) = \hat{\alpha}$$

## Recent results by Z. Shen.

Using some parallel between Dirichlet, Neumann problem and the Dirichlet-to-Neumann problem, Z. Shen (2025) has considerably enlarged the number of results by considering possibly vanishing magnetic fields. Here we meet quantities which play an important role for other problems on magnetic bottles:

$$m_r(x) := \sum_{|\alpha| \leq r, i, j} |D_x^\alpha b_{ij}(x)|$$

and the results are obtained in any dimension.  
The asymptotics are nevertheless limited to the main term.

## Shen's first theorem

Suppose that  $m_r(x) \geq c_0 > 0$ ,  $m_{r-1}(x_0) = 0$  for some  $x_0 \in \overline{\Omega}$ , and  $m_{r+1}(x) \leq C_0$ , then, with positive  $c, C$

$$cb^{\frac{2}{r+2}} \leq \lambda^N(bA, \Omega) \leq Cb^{\frac{2}{r+2}}$$

## Shen's second theorem

Suppose that  $m_r(x) \geq c_0 > 0$ ,  $m_{r-1}(x_0) = 0$  for some  $x_0 \in \partial\Omega$ , and  $m_{r+1}(x) \leq C_0$ , then, with positive  $c, C$

$$cb^{\frac{1}{r+2}} \leq \lambda^{DN}(bA, \Omega) \leq Cb^{\frac{1}{r+2}}$$

One element in the proof is to use estimates of the type (with  $c > 0$ )

$$c \int_{\Omega} m(x, B)^2 |\psi|^2 \leq \int_{\Omega} |\nabla_A \psi|^2 dx$$

or

$$c \int_{\partial\Omega} m(x, B)^2 |\psi|^2 \leq \int_{\Omega} |\nabla_A \psi|^2 dx$$

where

$$\frac{1}{m(x, B)} = \sup\{r > 0 : \max_{Q(x,r)} |B(x)| \leq \frac{1}{r^2}\}.$$

More accurate statements (with equivalents) are obtained by Z. Shen under additional assumptions.

## The exterior case

This was already considered in the strong magnetic limit for treating the points of the boundary with negative curvature.

While the weak magnetic field limit reduces to a regular perturbation for bounded domains, exterior domains exhibit singular behavior. This was recently studied for the magnetic Laplacian in [49] and for the magnetic Steklov problem in [38]. An alternative Steklov-specific regularization via a weak scalar potential has been studied in [10] (Chaigneau-Grebenkov) and [8] (Lukas Bundrock, Alexandre Girouard, Denis S. Grebenkov, Michael Levitin, Iosif Polterovich).

We (=Helffer-Nicoleau-Kachmar) study the influence of the magnetic flux on the lowest eigenvalue of the magnetic Laplace and Steklov operators in the exterior of the unit disk,

$$\Omega = \{x \in \mathbb{R}^2 : |x| > 1\},$$

addressing both strong and weak magnetic field regimes.

For the strong field limit, we derive three-term asymptotic expansions for both the magnetic Laplacian and the magnetic Steklov problem. In the weak field regime, we prove accurate asymptotics for the Neumann magnetic Laplacian. The case of a general domain is open.

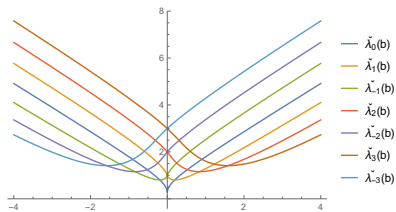


Figure: The exterior of the disk

We consider a vector potential  $\mathbf{F}: \Omega \rightarrow \mathbb{R}^2$  of class  $C^1$  with constant curl,

$$\operatorname{curl} \mathbf{F} = b > 0,$$

and consider the eigenvalues

$$\mu(\mathbf{F}, \beta) = \inf_{u \in H^1(\Omega)} \frac{\|(-i\nabla - \mathbf{F})u\|_{\Omega}^2 + \beta \|u\|_{\Gamma}^2}{\|u\|_{\Omega}^2},$$
$$\lambda(\mathbf{F}) = \inf_{u \in H^1(\Omega)} \frac{\|(-i\nabla - \mathbf{F})u\|_{\Omega}^2}{\|u\|_{\Gamma}^2},$$

where  $\|\cdot\|_{\Omega}$  and  $\|\cdot\|_{\Gamma}$  denote the  $L^2$ -norms on  $\Omega$  and  $\Gamma := \partial\Omega$  respectively, and  $\beta \in \mathbb{R}$  is a given parameter.

The quantity  $\mu(\mathbf{F}, \beta)$  corresponds to the lowest eigenvalue of the magnetic Laplacian,

$$\mathcal{L} = (-i\nabla - \mathbf{F})^2 = (-i\nabla - \mathbf{F}) \cdot (-i\nabla - \mathbf{F}) \text{ in } \Omega, \quad (22a)$$

subject to the Robin boundary condition

$$\mathbf{n} \cdot (\nabla - i\mathbf{F})u = \beta u \quad \text{on } \Gamma, \quad (22b)$$

where  $\mathbf{n}$  is the unit normal vector to  $\Gamma$  pointing inward  $\Omega$ .

A slight modification of the argument in [23, Theorem 1.1] (M. Goffeng, A. Kachmar, M. Persson Sundqvist) yields that the essential spectrum of the Robin magnetic Laplacian is the same as the essential spectrum of the Landau Hamiltonian with Aharonov-Bohm solenoid<sup>1</sup>. Thanks to [15] (P. Exner, P. Štoviček, P. Vytřas), this consists of the Landau levels  $b, 3b, 5b, \dots$ .

Similarly,  $\lambda(\mathbf{F})$  is the lowest eigenvalue of the magnetic Steklov problem, characterized by the existence of a non-zero function  $u$  satisfying

$$\begin{cases} (-i\nabla - \mathbf{F})^2 u = 0 & \text{on } \Omega, \\ \mathbf{n} \cdot (\nabla - i\mathbf{F})u = -\lambda u & \text{on } \Gamma. \end{cases} \quad (23)$$

---

<sup>1</sup>That is the operator  $\mathcal{H} = (-i\nabla - \mathbf{F})^2$  in  $\mathbb{R}^2$  with  $\mathbf{F}$  as in (25); when  $\nu = 0$ , we recover the Landau Hamiltonian.

The magnetic Laplacian with Robin boundary condition provides key insights into the magnetic Steklov eigenvalue  $\lambda(\mathbf{F})$ . This connection arises from the relation

$$\mu(\mathbf{F}, \beta) = 0 \quad \text{if and only if} \quad \beta = -\lambda(\mathbf{F}). \quad (24)$$

$\mu(\mathbf{F}, b)$  and  $\lambda(\mathbf{F})$  are uniquely determined by the magnetic field  $\text{curl } \mathbf{F} = \mathbf{b}$  and since  $\Omega$  is not simply connected, by the renormalized magnetic flux

$$\Phi := \frac{1}{2\pi} \int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{x}.$$

We fix the choice of the vector potential as

$$\mathbf{F}(x) = \frac{b}{2}(-x_2, x_1) + \frac{\nu}{|x|^2}(-x_2, x_1), \quad \nu = \Phi - \frac{b}{2}, \quad (25)$$

and the Robin parameter  $\beta$  as

$$\beta = b^{1/2}\gamma, \quad \gamma \in \mathbb{R}.$$

We denote the lowest magnetic Laplace and Steklov eigenvalues by

$$\mu(b, \nu, \gamma) := \mu(\mathbf{F}, \beta), \quad \lambda(b, \nu) := \lambda(\mathbf{F}). \quad (26)$$

Furthermore, these eigenvalues are periodic with respect to  $\nu$  with period **1**. Thus, there is no loss of generality in restricting the parameter  $\nu$  to  $(-1/2, 1/2]$ .

## Strong magnetic field

Recently, for the Steklov problem in the exterior of the disk, [39] (Helffer-Nicoleau) established as already mentioned the asymptotics,

$$\lambda(b, \nu) = \hat{\alpha} b^{1/2} + \frac{\hat{\alpha}^2 + 1}{3} + \mathcal{O}(b^{-1/2}) \quad \text{as } b \rightarrow +\infty. \quad (27)$$

As this formula does not display the flux effects dependent on  $\nu$ , our aim is to capture those hidden terms.

To that end, we establish an expansion with three terms for the lowest eigenvalue for the magnetic Laplacian, which involves spectral quantities dependent on  $\gamma$ , namely:

1.  $\Theta(\gamma)$ , an increasing and smooth function of  $\gamma$ , is the infimum of the lowest eigenvalues for the family of harmonic oscillators on the positive semi-axis

$$h_0[\xi, \gamma] = -\frac{d^2}{dt^2} + (t - \xi)^2,$$

with Robin boundary condition  $u'(0) = \gamma u(0)$ .

2.  $\xi(\gamma) = \sqrt{\Theta(\gamma) + \gamma^2}$ .
3.  $\varphi_\gamma$  is a normalized ground state to  $h_0[\xi(\gamma), \gamma]$ .
4.  $\mathcal{C}(\gamma) = \frac{1}{3}(1 - \gamma\xi(\gamma))\Theta'(\gamma)$ .

At this stage, we can describe the constant  $\hat{\alpha}$  appearing in (27) by the equation  $\Theta(-\hat{\alpha}) = 0$ .

The three-term asymptotics below for the lowest eigenvalue  $\mu(b, \nu, \gamma)$  generalizes<sup>2</sup> (Fournais-Helffer) [18, Theorem 2.5] and [21, Theorem 5.3.1], which addressed the case  $\nu = 0$  and  $\gamma = 0$ .

## Theorem HKN1

Let  $\gamma \in \mathbb{R}$  be fixed. There are constants

$$C_0(\gamma) \in \mathbb{R} \quad \text{and} \quad C_1(\gamma) \in \mathbb{R}$$

such that, for  $\nu \in (-1/2, 1/2]$ , we have as  $b \rightarrow +\infty$ ,

$$\mu(b, \nu, \gamma) = \Theta(\gamma)b + C(\gamma)b^{1/2} + \xi(\gamma)\Theta'(\gamma) \inf_{m \in \mathbb{Z}} \Delta_m(b, \nu, \gamma) + \mathcal{O}(b^{-1/2}), \quad (28a)$$

where

$$\Delta_m(b, \nu, \gamma) = \left( m - \nu - \frac{b}{2} - b^{1/2}\xi(\gamma) - C_0(\gamma) \right)^2 + C_1(\gamma). \quad (28b)$$

<sup>2</sup>See also Helffer-Léna [32] for related recent results in the disk 

## Remarks

- i) Under Neumann boundary conditions ( $\gamma = 0$ ), we recover:
  - ▶ Fournais-Helffer [21] for  $\nu = 0$ ;
  - ▶ a particular case of Fraser-Schoen [27] for  $\nu \neq 0$ .
- ii) When  $\gamma \neq 0$  and  $\nu = 0$ , the first two terms in (28) are obtained from either Kachmar [47] or from the general spectral reduction to an effective operator given in R. Fehs, L. Le Treust, N. Raymond, and S. Vũ Ngọc [22].
- iii) Thanks to (25), the oscillatory term in (28) depends on the flux and the intensity of the magnetic field:

$$\widehat{\Delta}(\Phi, b, \gamma) := \inf_{m \in \mathbb{Z}} \left( m - \Phi - b^{1/2} \xi(\gamma) - \mathcal{C}_0(\gamma) \right)^2.$$

- iv)  $\mathbb{R} \ni \gamma \mapsto \mathcal{C}_0(\gamma)$  and  $\mathbb{R} \ni \gamma \mapsto \mathcal{C}_1(\gamma)$  can be expressed explicitly in terms of the spectral parameters  $\Theta(\gamma)$ ,  $\xi(\gamma)$  and the ground state  $\varphi_\gamma$ .

## Weak magnetic field

We investigate the limit  $b \rightarrow 0^+$  in the Neumann case ( $\gamma = 0$ ).

We prove

### Theorem HKN

Let  $\nu \in (-1/2, 1/2]$ . There exists  $b_0 > 0$  such that, for  $b \in (0, b_0)$ , we have the following. The ground state energy of the Neumann magnetic Laplacian is a simple eigenvalue and it satisfies as  $b \rightarrow 0^+$ ,

$$\mu(b, \nu, 0) = \begin{cases} b - \frac{2^\nu}{\Gamma(1-\nu)} b^{2-\nu} + o(b^{2-\nu}) & \text{if } \nu \geq 0, \\ b - \frac{2^{1+\nu}}{\Gamma(-\nu)} b^{1-\nu} + o(b^{1-\nu}) & \text{if } \nu < 0. \end{cases}$$

## Remarks

The study of the non-Neumann case is analyzed in Helffer-Nicoleau [38]:

- ▶ for  $b$  sufficiently small,  $\mu(b, \nu, \gamma(b, \nu))$  is a simple eigenvalue with a radially symmetric ground state;
- ▶ the Robin parameter  $\beta = b^{1/2}\gamma(b, \nu)$  satisfies as  $b \rightarrow 0^+$ :

$$\beta = \begin{cases} \frac{2}{\log b} + \mathcal{O}\left(\frac{1}{(\log b)^2}\right) & \text{if } \nu = 0, \\ -|\nu| - \frac{2 \Gamma(1 - |\nu|) \Gamma(|\nu| + \frac{1}{2})}{\sqrt{\pi} \Gamma(|\nu|)} b^{|\nu|} + \mathcal{O}(b^{2|\nu|}) & \text{if } \nu \neq 0. \end{cases}$$

# Open questions

What can we say for general exterior domains as  $b \rightarrow 0$ .

What can we say (with in mind what has been done for the Neumann problem) for polygonal domains in the strong magnetic field limit for the D-to-N map ? What is the role of the angles in the localization of the ground state?

## [Bibliography]



Khaled Abou Alfa, Maha Aafarani, Frédéric Hérau, Nicolas Raymond

Spectral asymptotics of the Neumann Laplacian with variable magnetic field on a smooth bounded domain in three dimensions.

[arXiv:2306.17446](#) (2023). *Anal. PDE* 18, No. 8, 1897–1920 (2025)



Y. Almog and B. Helffer.

On the spectrum of non-selfadjoint Schrödinger operators with compact resolvent.

*Comm. in PDE* 40 (8) (2015), pp. 1441–1466.



W. Assaad.

The breakdown of superconductivity in the presence of magnetic steps.

*Commun. Contemp. Math.* 23 (2021).



Y. Avron, I. Herbst, and B. Simon.

Schrödinger operators with magnetic fields I. General interactions.

[Duke Math. J. 45 \(4\), p. 847-883 \(1978\).](#)



[P. Bauman, D. Phillips, Q. Tang.](#)

Stable nucleation for the Ginzburg-Landau system with an applied magnetic field,

[Arch. Rational Mech. Anal., Volume 142 \(1998\), 1-43.](#)



[M. Baur and T. Weidl. Eigenvalues of the magnetic Dirichlet-Laplacian with constant magnetic field on discs in the strong field limit.](#)

[arXiv:2402.01474 \(2024\).](#)



[A. Bernoff and P. Sternberg.](#)

Onset of superconductivity in decreasing fields for general domains.

[J. Math. Phys. 39 \(1998\), 1272–1284.](#)



[Lukas Bundrock, Alexandre Girouard, Denis S. Grebenkov, Michael Levitin, Iosif Polterovich](#)

[The exterior Steklov problem for Euclidean domains](#)

ArXiv.



T. Chakradhar, K. Gittins, G. Habib, and N. Peyrerimhoff.  
A note on the magnetic Steklov operator on functions.  
[arXiv:2410.07462](#), (2024).



A. Chaigneau, D. S. Grebenkov.  
The Steklov problem for exterior domains: asymptotic  
behavior and applications.  
[J. Math. Phys. 66 \(2025\)](#), art. no. 061502.



D. Daners and J.B. Kennedy.  
On the asymptotic behaviour of the eigenvalues of a Robin  
problem.  
[Differential Integral Equations 23 \(2010\)](#), 659–669.



P.G. de Gennes.  
Boundary effects in superconductors.  
[Rev. Mod. Phys. January 1964](#).



A. F. M. ter Elst and El Maati Ouhabaz.

Convergence of the Dirichlet-to-Neumann operator on varying domains.

[Oper. Theory Adv. Appl.](#), 250 Birkhäuser/Springer, Cham, 2015, 147–154.



[A. F. M. ter Elst and El Maati Ouhabaz.](#)

The diamagnetic inequality for the Dirichlet-to-Neumann operator.

[Bull. Lond. Math. Soc.](#) 54 (2022), no. 5, 1978–1997.



[P. Exner, P. Št'ovíček, P. Vytřas.](#)

Generalized boundary conditions for the Aharonov-Bohm effect combined with a homogeneous magnetic field.

[J. Math. Phys.](#) 43 (2002), 2151–2168.



[C. Fefferman, D. Phong.](#)

On positivity of pseudo-differential operators.

[Proc. Nat. Acad. Sci. U.S.A.](#), 75(10):4673–4674, 1978



[S. Fournais and B. Helffer.](#)

Accurate eigenvalue asymptotics for the magnetic Neumann Laplacian.

Ann. Inst. Fourier 56(1), 1-67 (2006).



S. Fournais, B. Helffer.

On the third critical field in Ginzburg-Landau theory.  
Comm. Math. Phys. 266 (2006), 153–196.



S. Fournais and B. Helffer.

Strong diamagnetism for general domains and applications.  
Annales de l'Institut Fourier, Tome 57 (2007) no. 7,  
2389-2400.



S. Fournais and B. Helffer.

Inequalities for the lowest magnetic Neumann eigenvalue.  
Letters in Mathematical Physics, Volume 109, 1683–1700,  
(2019).



S. Fournais and B. Helffer.

*Spectral Methods in Surface Superconductivity.*  
Progress in Nonlinear Differential Equations and Their  
Applications, Vol. 77, Birkhäuser, (2010).



R. Fahs, L. Le Treust, N. Raymond, and S. Vũ Ngọc.

Boundary states of the Robin magnetic Laplacian.

Doc. Math. 29 (2024), 1157–1200.



M. Goffeng, A. Kachmar, M.P. Sundqvist.

Clusters of eigenvalues for the magnetic Laplacian with Robin condition.

J. Math. Phys. 57 (2016), art. no. 063510.



D. Guibourg.

Inégalités maximales pour l'opérateur de Schrödinger,  
PhD Thesis, Université de Rennes 1, 1992.



D. Guibourg.

Inégalités maximales pour l'opérateur de Schrödinger.  
CRAS 316 (1993), 249–252.



B. Helffer and A. Kachmar.

Helical magnetic fields and semi-classical asymptotics of the lowest eigenvalue.

Doc. Math. 28 (2023), 857–901.



A. Fraser and R. Schoen.

The first Steklov eigenvalue, conformal geometry, and minimal surfaces.

[Adv. Math. 203 \(3\): 823 –890 \(2016\).](#)



[A. Girouard and I. Polterovich.](#)

Spectral geometry of the Steklov problem (Survey).

[In book \(Henrot editor\).](#)



[B. Helffer, A. Kachmar, and F. Nicoleau.](#)

Lower and Upper bounds for the magnetic lowest Dirichlet-to-Neumann eigenvalue for general domains.

[ArXiv 2501.00947 \(2025\).](#)



[B. Helffer, A. Kachmar, F. Nicoleau.](#)

Asymptotics for the magnetic Dirichlet-to-Neumann eigenvalues in general domains.

[arXiv:2501.00947 \(2025\).](#)



[B. Helffer, A. Kachmar, and F. Nicoleau.](#)

Flux effects on magnetic Laplace and Steklov eigenvalues in the exterior of the disk.

[ArXiv 2508.18119 \(2025\).](#)



B. Helffer and C. Léna.

Eigenvalues of the Neumann magnetic Laplacian in the unit disk.

[arXiv:2411.11721](#), (2024). To appear in JMP (2025).



B. Helffer and A. Mohamed.

*Sur le spectre essentiel des opérateurs de Schrödinger avec champ magnétique.*

[Ann. Inst. Fourier 38\(2\)](#), pp. 95–113 (1988).



B. Helffer and A. Morame.

Magnetic bottles in connection with superconductivity.

[J. Funct. Anal. 185 \(2\)](#), 604–680, (2001).



B. Helffer and A. Morame.

Magnetic bottles for the Neumann problem: the case of dimension 3.

[Proc. Indian Acad. Sci. Vol. 112 \(1\)](#), February 2002, 71–84.



B. Helffer and A. Morame.

Magnetic bottles for the Neumann problem: curvature effects in the case of dimension 3 (general case)

[Ann. Scient. Éc. Norm. Sup., 4e série, t. 37, 105–170 \(2004\).](#)



[B. Helffer and F. Nicoleau.](#)

Trace formulas for the magnetic Dirichlet to Neumann operator.

[ArXiv and to appear in Asymptotic analysis \(2025\).](#)



[B. Helffer and F. Nicoleau.](#)

On the magnetic Dirichlet to Neumann operator on the disk—strong diamagnetism and strong magnetic field limit—

[ArXiv:2411.15522 v2, \(2024\).](#)



[B. Helffer and F. Nicoleau.](#)

On the magnetic Dirichlet to Neumann operator on the exterior of the disk.

[To appear in JMPA \(2026\).](#)



[B. Helffer and J. Nourrigat.](#)

*Hypoellipticité Maximale pour des Opérateurs Polynômes de Champs de Vecteurs.*

Progress in Mathematics, Birkhäuser, Vol. 58 (1985).



B. Helffer and J. Nourrigat.

On the domain of a magnetic Schrödinger operator with complex electric potential.

Analysis and operator theory, 149–165, Springer Optim. Appl., 146, Springer, Cham, 2019.



F. Hérau and N. Raymond.

Semiclassical spectral gaps of the 3D Neumann Laplacian with constant magnetic field.

Ann. Inst. Fourier 74, No. 3, 915–972 (2024)).



L. Hörmander.

Hypoelliptic second order differential equations.

Acta Math. 119 (1967), 147-171.



L. Hörmander.

The Analysis of Linear Partial Differential Operators.

Volume III, Springer, 1985.



T. Ikebe and T. Kato.

Uniqueness of the self-adjoint extension of singular elliptic differential operators.

[Arch. for Rat. Mech. and Anal. 9 \(1962\), 77–92.](#)



A. Iwatsuka.

Magnetic Schrödinger operators with compact resolvent.

[J. Math. Kyoto Univ. 26, 357-374 \(1986\).](#)



A. Kachmar. On the ground state energy for a magnetic Schrödinger operator and the effect of the de Gennes boundary condition. [J. Math. Phys. 47 \(2006\), art. no. 072106.](#)




A. Kachmar. Diamagnetism versus Robin condition and concentration of ground states. [Asymptot. Anal. 98 \(2016\), no. 4, 341–375.](#)





A. Kachmar, V. Lotoreichik, and M. Persson-Sundqvist.


On the Laplace operator with a weak magnetic field in exterior domains.

[ArXiv:2405.18154v1 \(2024\).](#)

 A. Kachmar and G. Miranda. The magnetic Laplacian on the disc for strong magnetic fields.  
Arxiv: 2407.11241v1 (2024).

 J. Kohn.  
Lectures on degenerate elliptic problems.  
Pseudodifferential operators with applications, C.I.M.E.,  
Bressanone 1977, 89-151 (1978).

 V. Kondratiev, V. Maz'ya, and M. Shubin.  
Gauge optimization and spectral properties of magnetic  
Schrödinger operators.  
Comm. Partial Differential Equations 34 (2009), no. 10-12,  
1127–1146.

 V. Kondratiev and M. Shubin.  
Discreteness of spectrum for the magnetic Schrödinger  
operators.  
Comm. Partial Differential Equations 27 (3-4), 477-526  
(2002).



K. Lu and X. Pan.

Gauge invariant eigenvalue problems in  $\mathbb{R}^2$  and in  $\mathbb{R}_+^2$ .  
Trans. Amer. Math. Soc. 352 (2000): 1247–1276.



K. Lu and X. Pan.

Eigenvalue problems of Ginzburg-Landau operator in bounded domains.

Journal of Mathematical Physics 4 (1999): 2647–2667.



K. Lu and X. Pan.

Surface nucleation of superconductivity in 3-dimensions.  
J. of Diff. Equations 168 (2000) 386–452.



W. Magnus, F. Oberhettinger and R.P. Soni.

Formulas and theorems for the special functions of  
mathematical physics, 3rd enlarged ed,  
Grundlehren der Mathematischen Wissenschaften,  
Volume 52, Springer, (1965).



J.T. Mba Yébé.

Réalisation et spectre d'opérateurs de Schrödinger et de Klein-Gordon avec des potentiels irréguliers.

Thèse de Doctorat (Université de Reims) (1995).



M. Meftah.

Conditions suffisantes pour la compacité de la résolvante d'un opérateur de Schrödinger avec un champ magnétique.

J. Math. Kyoto Univ. 31 (3), p. 875-880 (1991).



A. Mohamed and J. Nourrigat.

Encadrement du  $N(\lambda)$  pour un opérateur de Schrödinger avec un champ magnétique et un potentiel électrique.

Journal de Mathématiques Pures et Appliquées (9) 70, no 1, 87-99 (1991).



J. Nourrigat.

Inégalités  $L^2$  et représentations de groupes nilpotents.

Journal of Functional Analysis 74. October 1987.



J. Nourrigat.

Une inégalité  $L^2$ , unpublished manuscript ( $\sim$  1990).



J. Nourrigat.

$L^2$  inequalities and representations of nilpotent groups.  
CIMPA School of Harmonic Analysis. Wuhan (China),  
April-May 1991.



X.B. Pan.

An eigenvalue variation problem of magnetic Schrödinger  
operator in three dimensions.  
Discrete Contin. Dyn. Syst. 24 (2009), 933–978.



X.B. Pan and K.H. Kwek.

Schrödinger operators with non-degenerately vanishing  
magnetic fields in bounded domains.  
Trans. Amer. Math. Soc. 354 (2002) 4201–4227.



N. Raymond.

Sharp asymptotics for the Neumann Laplacian with variable  
magnetic field: case of dimension 2.  
Ann. Henri Poincaré 10(1), 95-122 (2009).



N. Raymond.

On the semiclassical (3D)- Neumann Laplacian with variable magnetic field.

*Asymptotic Analysis* 68 (1-2), 1–40 (2010).



N. Raymond.

Bound States of the Magnetic Schrödinger Operator  
*EMS Tracts in Mathematics* (2017).



D. Robert.

Comportement asymptotique des valeurs propres d'opérateurs du type Schrödinger à potentiel "dégénéré".  
*J. Math. Pures Appl.* (9) 61 (1982), no. 3, 275–300 (1983).



L.P. Rothschild and E. Stein.

Hypoelliptic operators and nilpotent groups.  
*Acta Mathematica* 137 (1977), 248-315.



D. Saint-James.

Etude du champ critique  $HC3$  dans une géométrie cylindrique.  
*Physics Letters*, (15)(1), 13-15, (1965).



Z. Shen.

$L^p$  estimates for Schrödinger with certain potentials.  
[Ann. Inst. Fourier 45 \(1995\), 513–546.](#)



[Z. Shen.](#)

Estimates in  $L^p$  for magnetic Schrödinger operators.  
[Indiana Univ. Math. J., 45 \(1996\), 817-841.](#)



[Z. Shen.](#)

Eigenvalue asymptotics and exponential decay of eigenfunctions for Schrödinger operators with magnetic fields.  
[TAMS 348, 4465–4488 \(1996\).](#)



[Z. Shen.](#)

The magnetic Schrödinger operator and reverse Hölder class  
[Journées équations aux dérivées partielles \(1996\), 1-10.](#)



[B. Simon.](#)

Some quantum operators with discrete spectrum but classically continuous spectrum.  
[Ann. Physics 146, p. 209-220 \(1983\).](#)



[Zhongwei Shen.](#)

The magnetic Laplacian with a higher-order vanishing magnetic field in a bounded domain.

[arXiv:2505.03690](https://arxiv.org/abs/2505.03690) (2025).