Courant or Pleijel theorems for Hörmander's operators

Bernard Helffer (Nantes Université)

(after Eswarathasan–Letrouit (2023) [17], R. Frank and B. Helffer (2024) [20]–To appear in Journal de l'Ecole Polytechnique–)

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Motivated by some of the results by Eswarathasan–Letrouit in [17] and connected open problems initially discussed with C. Letrouit, we consider in this work with Rupert Frank the Dirichlet realization in an open set $\Omega \subset \mathbb{R}^n$ of a subLaplacian (also called Hörmander's operator)

$$-\Delta_{\mathbf{X}}^{\Omega} := \sum_{j=1}^{p} X_j^\star X_j \,,$$

where the X_j $(j = 1, \cdots, p)$ are C^∞ real vector fields satisfying the so-called Hörmander condition [36] which reads:

Assumption $(CH(r))$

For some $r \geq 1$ the X_j and the brackets up to order r generate at each point $x \in \overline{\Omega}$ the tangent space $T_x\Omega$.

Note that the terminology "subRiemannian Laplacian" or in short "subLaplacian" is posterior to the work of L. Hörmander and corresponds to the development of the subRiemannian geometry starting of the middle of the eighties .

More generally, we can consider a connected C^∞ Riemannian manifold M of dimension n with a given measure μ (with a C^∞-density with respect to the Lebesgue measure in a local system of coordinates) and a system of p C^∞ ($p \leq n$) vector fields satisfying Assumption $(CH(r))$.

These operators are known to be hypoelliptic (Hörmander 1968) in Ω.

Under additional conditions at the boundary which is supposed in this case to be C^∞ and satisfying the

Non-characteristic assumption

A system $\mathbf{X}$ is said non-characteristic for an open set Ω, if for each point $x \in \partial\Omega$ there exists a vector field X_i that is transverse to the boundary at x ,

we have regularity up to the boundary (see Derridj [15] 1971 and the talk by Brian Street in this conference). Note that we will not need for our results this condition which is rather strong due to topological considerations.

Maximal hypoellipticity.

These operators are also known to be maximally hypoelliptic (Rotschild-Stein 1976).
There is a characterization of the polynomials of vector fields which are maximally hypoelliptic using a Rockland's like criterion initially introduced by Helffer-Nourrigat in 1979. The proof in full generality of this criterion was recently proven by Androulidakis–Mohsen-Yuncken (2022).
We do not need this characterization here, but the pseudo-differential calculus introduced by Rothschild-Stein (1976), in the version given by L. Rothschild in the equiregular case (1979) will be important in the proof of Faber-Krahn's inequality.

Main questions

The operator has compact resolvent provided Ω is bounded and we can consider for its discrete spectrum all the questions that have been solved along the years concerning the Dirichlet realization $-\Delta^\Omega$ of the Euclidean Laplacian. We focus in this talk on

- Courant's theorem: comparison between the minimal labelling k of an eigenvalue λ_k and the number ν_k of the nodal domains of the eigenfunction in the eigenspace corresponding to λ_k.
- Pleijel's theorem

Åke Pleijel (1913-1989)

Appointed professor of mathematics at Lund University on 4 August 1952, he took up the appointment on 1 December that year. He served as dean of the Faculty of Mathematical and Natural Sciences from 1 July 1957 to 30 November 1961 and served again in the academic year 1964-1965. The 1959 International Handbook of Universities lists Lund University with Åke Pleijel as Dean of the Faculty of Mathematics and Natural Sciences with a staff of 102.

Courant's Theorem

As well known, Courant's theorem (1920)[1] states that in the case of the Dirichlet Laplacian in $\Omega \subset\subset \mathbb{R}^n$, an eigenfunction associated with the k-th eigenvalue has at most k nodal domains:

$$\nu_k \leq k \,.$$

If one looks at the standard proof of Courant's theorem, this mainly appears as a consequence of

- a restriction statement (the restriction of an eigenfunction to its nodal domain is the ground state of the Dirichlet realization of the Laplacian in this domain),
- the minimax characterization of the eigenvalue,
- the Unique Continuation theorem (UCT).

Hence the difficulty is to determine under which conditions, we can extend these results to the sub-Riemannian Laplacians.

[1] Richard Courant 1888-1972

Restriction

Having rather few informations about the nodal sets (i.e. the boundary of the nodal domains) we adapt to the subRiemannian case a proof of the restriction Lemma proposed by E. Müller-Pfeiffer (1985) which permits to avoid regularity assumptions at $\partial\Omega$.

Restriction Lemma

If u is an eigenfunction, the restriction of u to one of its nodal domains is the first eigenfunction of the associated Dirichlet (Sub)-Laplacian.

Unique Continuation

In the C^∞ category K. Watanabe [60] proves UCT in dimension 2, but H. Bahouri [2] gives a discouraging counter-example with two vector fields in $\mathbb{R}^3$.
Nevertheless, J. M. Bony at the end of the sixties [3] proved that (UCT) holds when the vector fields are analytic.
Hence Courant's theorem holds in the analytic category as proved by Eswarathasan–Letrouit in [17].
Actually, using our restriction lemma, we can extend statements given in [17] to the case when the boundary is not necessarily non-characteristic.

Weak version of Courant's theorem (Mangoubi)

We have:

$$\nu_k \leq k + \mathrm{Mult}(\lambda_k) - 1\,.$$

This version is only using the variational characterization, the interior regularity and the restriction Lemma.

Pleijel's Theorem

In the same spirit, one can hope for an asymptotic control of $\nu(k)/k$. In the case of the Dirichlet Laplacian in $\Omega \subset \mathbb{R}^n$, Pleijel's theorem (Pleijel 1956) says that, if $n \geq 2$, there exists an Ω-independent constant $\gamma(n) < 1$ such that

$$\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \gamma(n)\,.$$

In the case of the Euclidean Laplacian, the proof of Pleijel's theorem is a nice combination of Weyl's formula which gives the asymptotic behavior as $\lambda \to +\infty$ of the counting function

$$N(\lambda, -\Delta^{\Omega}_{eucl}) \sim W_n |\Omega| \, \lambda^{n/2} \,,$$

and Faber-Krahn inequality establishing that

$$\lambda_1(-\Delta^{\Omega}_{eucl}) \geq |\Omega|^{-2/n} \, \lambda_1(-\Delta^{B_1}) \,,$$

where B_1 is the ball of unit volume.
One can then establish that

$$\gamma(n) = W_n^{-1} \lambda_1(-\Delta^{B_1})^{-n/2} < 1 \,, \tag{1}$$

for $n \geq 2$.

If we consider the generalization to the subLaplacians, we can be optimistic on the side of Weyl's formula. Since the pioneering work of Guy Métivier (1976) [48, 49], we are rich in results, at least if we add to Assumption (CH(r)) some equiregularity condition, which permits at each point to approximate the X_j by the generators of a nilpotent Lie Algebra $\mathcal{G}_x$. More precisely the vector fields satisfy the following condition

Equiregularity assumption (CEq)

For each $j \leq r$ the dimension of the space spanned by the commutators of the X_k of length $\leq j$ at each point is constant.

In the modern language of sub-Riemannian geometry, this is called an equiregular distribution.

Métivier's Weyl formula

We denote by $\mathcal{D}_j$ the span of all vector fields obtained as brackets of length $\leq j$ of the X_k's.
We set $n_j := \dim(\mathcal{D}_j)$, which, by assumption (3) above does not depend on the point $x \in M$.
A family $(Y_1, \ldots, Y_n)$ of n vector fields is said to be adapted at $x \in M$ if for any $j \leq r$, $\text{Span}(Y_1(x), \ldots, Y_{n_j}(x)) = \mathcal{D}_j(x)$.

We can then introduce the homogeneous dimension

$$Q := \sum_{j=1}^{r} j\,(n_j - n_{j-1})\,, \tag{2}$$

with the convention that $n_0 = 0$.
Under Assumptions ($CH(r)$) and (CEq) , G. Métivier (1976) shows (using in particular the techniques of Rothschild-Stein and Rothschild [57, 56]) that there exists a constant $c(M,\mu)$ such that as $\lambda \to +\infty$

$$N_{-\Delta_{\mathbf{X}}^{M,\mu}}(\lambda) := \#\{j:\ \lambda_j(-\Delta) \leq \lambda\} \sim c(M,\mu)\lambda^{\frac{Q}{2}}\,. \tag{3}$$

Note that in the case $r = 2$, connected results are obtained in the eighties Menikoff–Sjöstrand [46, 47], Métivier [48], Abderemane Mohamed [50] and Métivier's theorem (together with many other results) has been revisited at the light of subRiemannian geometry in Colin de Verdière–Hillairet–Trélat [10, 11, 12] (2018-2022).

Faber-Krahn's inequality.

On the other side, our knowledge is rather poor concerning Faber–Krahn's inequality. In the case of the Heisenberg group, one can think of a result by P. Pansu [53] concerning the isoperimetric inequality. We follow another way by revisiting the nilpotenzation procedure permitting to deduce Faber–Krahn inequalities for sub-Laplacians from Faber–Krahn inequalities for sub-Laplacians on nilpotent groups. Combining with Weyl's formulas we get a sufficient condition for the validity of the theorem which relies on the corresponding Faber–Krahn constants to be established for Dirichlet realizations of subLaplacians in open set of nilpotent groups.

The second part will be devoted to the analysis of this question for the nilpotent groups $\mathbb{H}_n \times \mathbb{R}^k$ where $\mathbb{H}_n$ is the Heisenberg group. Unfortunately, we fail in the case $(n, k) = (1, 0)$, except if we admit a celebrated Pansu conjecture but fortunately we succeed assuming roughly that $k + n$ is large enough.

Combining our result about Faber–Krahn inequalities with Métivier's Weyl-type formula, we obtain a sufficient condition for the validity of a Pleijel-type bound; The upper bound on $\limsup_{k\to\infty} \nu_k/k$ is of the form

$$\left(\int_M (c_x^{\mathrm{FK}})^{-\frac{Q}{2}}\, d\mu(x)\right)\left(\int_M c_x^{\mathrm{Weyl}}\, d\mu(x)\right)^{-1}, \qquad (4)$$

where

- c_x^{FK} is a certain local Faber–Krahn constant, defined in terms of the nilpotentization of $-\Delta_{\mathbf{X}}^{M,\mu}$ at $x \in M$,
- c_x^{Weyl} is a certain local Weyl constant, defined in terms of the same nilpotentization.

This strengthening of our original result (2023) is due to Y. Colin de Verdière.

The role of the Borel measure

$$D \mapsto \int_D c_x^{\mathrm{Weyl}} \, d\mu(x)$$

on M is emphasized in the work of Colin de Verdière-Hillairet-Trélat [12], where it is called the *Weyl measure*.

Similarly, here we introduce what may be called the *Faber–Krahn measure*

$$D \mapsto \int_D (c_x^{\mathrm{FK}})^{-\frac{Q}{2}} \, d\mu(x) \, .$$

It is interesting to compare (4) with the Pleijel formula (1), to which it reduces in the case of open subsets of $\mathbb{R}^n$. More generally, in the Riemannian case (where $p = n$ and where μ is the Riemannian volume measure) the expression (4) reduces to (1) and we recover the result of Bérard and Meyer [5].
According to (4), a sufficient condition for the validity of Pleijel's theorem is the following bound on the "local Pleijel constants":

$$\left(c_x^{\mathrm{FK}}\right)^{-\frac{Q}{2}} \left(c_x^{\mathrm{Weyl}}\right)^{-1} < 1 \qquad \text{for all } x \in M\,;$$

We emphasize that the latter condition involves the corresponding Faber–Krahn constants for Dirichlet realizations of sub-Laplacians in open set of nilpotent groups.

Hence in the second part of the talk we will describe what we have obtained in this particular case.

On nilpotent approximation

Here we refer to Métivier [48], Rothschild–Stein [57]). and the presentation of Rothschild [56] (based on former assumptions and definitions given by Folland [18]). Since this period in the seventies, a huge literature has developed the so-called sub-Riemannian geometry analyzing in particular this nilpotent approximation.

We consider the situation presented in the introduction and assume that Assumptions (CH(r)) and (CEq) are satisfied. We impose (to simplify in this talk) in addition that

$$Y_j = X_j \text{ for } j = 1, \cdots, p\,.$$

Given a flag (special basis of $T_x M$ adapted to the $\mathcal{D}_j$ and completing the basis of $\mathcal{D}_1$) at $x \in M$ we can define canonical privileged coordinates at x by the mapping θ_x given by

$$\theta_x(y) := u = (u_i) \qquad \text{if } y = \exp(\sum u_i Y_i) \cdot x\,, \tag{5}$$

where exp denotes the exponential map defined in some small nhbd of x.

It is known that under assumption (2) above, around any $x \in M$ it is possible to choose a flag that is adapted at any point in a nhbd of x. Thus we identify a nhbd of $x \in M$ with a nhbd of 0 in $\mathbb{R}^n$. It has been shown by G. Métivier that everything depends smoothly on x.

We now introduce the notion of nilpotentized measure $d\widehat{\mu}_x$ at $x \in M$. There is a definition in the formalism of sub-Riemannian geometry but we prefer to explain here "by hand " how it can be constructed for our specific choice of privileged coordinates. On $\mathbb{R}^n$ we have the Lebesgue measure

$$du = \prod_i du_i \,,$$

and in these local coordinates the measure $d\mu$ is in the form

$$d\mu = a(x,u)du \,,$$

where $(x,u) \mapsto a(x,u)$ is C^∞ in both variables x and u.
In a small nhbd of 0, the nilpotentized measure at x can be defined by

$$d\widehat{\mu}_x := a(x,0)du \,. \tag{6}$$

Then we denote by $Y_{i,x}$, the image of Y_i by θ_x, which is simply Y_i written in the local canonical coordinates around x.
It can be shown that θ_x is also C^∞ with respect to x.

On $\mathbb{R}^n$, with coordinates $u = (u_i)$, we introduce the family of dilations given by

$$\delta_t(u_i) = (t^{w_i} u_i)\,,$$

where w_i is defined as follows: there exists a unique $j \in \{1, \ldots, n\}$ such that $n_{j-1} + 1 \leq i \leq n_j$, and we set $w_i = j$.
We then define the homogeneous dimension by

$$Q := \sum_i w_i\,. \tag{7}$$

G. Métivier [48] (Theorem 3.1) proves (in addition with the regularity of θ_x already mentioned above) the following theorem.

Métivier Approximation Theorem

For any x, $X_{j,x}$ is of order ≤ 1. Furthermore,

- $$X_{k,x} = \widehat{X}_{k,x} + R_{k,x}\,,$$

  where $\widehat{X}_{k,x}$ is homogeneous of order 1 and $R_{k,x}$ is (for a suitable natural definition) of order ≤ 0.
- The $\widehat{X}_{k,x}$ generate a nilpotent Lie algebra $\mathcal{G}_x$ of dimension n and rank r.
- The mapping $x \mapsto \widehat{X}_{j,x}$ is smooth.

Although, the lemma is established for a specific θ_x, there are other posssible choices permitting to get the same conclusion.

By Métivier's theorem (see also Colin de Verdière–Hillairet-Trélat) we have the following Weyl formula, giving the structure of the Weyl constant.

Spectral Theorem of G. Métivier

There exists a continuous, positive function $x \mapsto c_x^{\mathrm{Weyl}}$ such that the counting function, of the selfadjoint realization of $-\Delta = -\Delta_{\mathbf{X}}^{M,\mu}$ satisfies, as $\lambda \to +\infty$,

$$N_{-\Delta}(\lambda) := \#\{j : \ \lambda_j(-\Delta) \le \lambda\} \sim \Big(\int_M c_x^{\mathrm{Weyl}}\, d\mu(x)\Big)\lambda^{\frac{Q}{2}}\,. \qquad (8)$$

Note that, when $r = 2$, there were important contributions on the subject starting from the end of the seventies. One has to give a more explicit way to determine c_x^{Weyl}.

A basic example

To make the general theory more concrete, let us consider examples.
We denote coordinates on $\mathbb{R}^3$ by (x, y, z).
In $\Omega \subset \mathbb{R}^3$, we consider

$$X_1 = \frac{\partial}{\partial x} + K_1(x, y)\frac{\partial}{\partial z}\,, \qquad X_2 = \frac{\partial}{\partial y} + K_2(x, y)\frac{\partial}{\partial z}\,,$$

with $\mathrm{curl}\ \vec{K} = \frac{\partial}{\partial x}K_2 - \frac{\partial}{\partial y}K_1 > 0$.
The measure μ is simply the Lebesgue measure $dxdydz$.
Our aim is to give an explicit criterion for getting Pleijel's theorem for the Dirichlet realization of the subLaplacian in Ω

$$-\Delta_{\mathbf{X}}^{\Omega} = -X_1^2 - X_2^2\,.$$

Let $(x_0, y_0, z_0) \in \Omega$. Then in the construction G_{x_0,y_0,z_0} is the Heisenberg group $\mathbb{H}_1$ and the privileged coordinates at (x_0, y_0, z_0) are given (modulo higher order term if we use the canonical privileged coordinates) in the form

$$u_1 = x - x_0\,, \qquad u_2 = y - y_0\,, \qquad u_3 = \widehat{\delta}(z - z_0) + P(x - x_0, y - y_0)\,, \tag{9}$$

where P is a polynomial of order 2 and

$$\widehat{\delta} = \frac{1}{\mathrm{curl}\ \vec{K}(x_0, y_0)}\,.$$

Hence the candidate to be the nilpotentized measure at (x_0, y_0, z_0) is

$$\mathrm{curl}\ \vec{K}(x_0, y_0)\, du_1 du_2 du_3\,. \tag{10}$$

In these coordinates we have

$$\widehat{X}_1 = \frac{\partial}{\partial u_1} - \frac{1}{2} u_2 \frac{\partial}{\partial u_3}\,, \qquad \widehat{X}_2 = \frac{\partial}{\partial u_2} + \frac{1}{2} u_1 \frac{\partial}{\partial u_3}\,, \qquad \widehat{X}_3 = \frac{\partial}{\partial u_3}\,. \tag{11}$$

Conclusion in the setting of the example

We denote by $c^{\mathrm{FK}}(\mathbb{H}_1)$ the Faber–Krahn constant on the Heisenberg group $\mathbb{H}_1$.

We now apply the main Theorem or, more precisely, its version for the sub-Laplacian on an open set with Dirichlet boundary conditions. The condition (15) reads

$$\left(\left(c^{\mathrm{FK}}(\mathbb{H}_1)\right)^2 \hat{W}(\mathbb{H}_1)\right) > 1\,.$$

Under this condition, Pleijel's theorem holds.

Currently we have no proof that

$$\left(c^{\mathrm{FK}}(\mathbb{H}_1)\right)^2 \hat{W}(\mathbb{H}_1) > 1\,.$$

Although this holds provided a well-known conjecture by Pansu concerning the isoperimetric inequality on the Heisenberg group is true.

Other examples

To have a positive example, we could repeat the above analysis with

$$X_1^2 + X_2^2 + \Delta_w$$

on $\Omega \subset \mathbb{R}^3 \times \mathbb{R}^k$ with $k \geq 3$.

We can also consider for $n \in \mathbb{N}$, $j = 1, \cdots, n$

$$\begin{aligned} X_j' &= \partial_{x_j} - K_1^j(x_j, y_j)\partial_z \,, \\ X_j'' &= \partial_{y_j} - K_2^j(x_j, y_j)\partial_z \,, \end{aligned}$$

with $\mathrm{Curl}\, K^j > 0$.

Main result for sub-Laplacians in the equiregular case

By the nilpotent approximation (as also explicited in the previous examples), we can associate to each point $x \in M$ a nilpotent group G_x (identified with the algebra $\mathcal{G}_x$ in the exponential coordinates) and a corresponding sub-Laplacian

$$\widehat{\Delta}_x = \sum_{i=1}^{p} \widehat{X}_{i,x}^2$$

in $\mathcal{U}_2(\mathcal{G}_x)$ (the elements in the enveloping algebra which are homogeneous of degree 2) .
According to Varopoulos (see also Folland and Rothschild), we have always, when $Q > 2$ with Q introduced in (7), a Sobolev inequality $L^2 - L^q$ with

$$q = 2Q/(Q-2)\,. \tag{12}$$

By our assumption of equiregularity, Q is independent of $x \in M$.

In addition, for any $x \in M$ we also have, for all $\Omega \subset G_x$ open, for all $v \in C_0^\infty(\Omega)$, a Faber–Krahn inequality in the form

$$\langle -\widehat{\Delta}_x v, v\rangle_{L^2(G_x,\widehat{\mu}_x)} \geq c\ \widehat{\mu}_x(\Omega)^{-\frac{2}{Q}} \|v\|^2_{L^2(G_x,\widehat{\mu}_x)}\,, \qquad (13)$$

with $c > 0$.
By definition c_x^{FK} is the largest constant such that (13) holds.
When $Q > 2$, a lower bound for c_x^{FK} can be deduced from the Sobolev inequality (see Varopoulos or Rothschild).

Our main statement (Frank-Helffer) is the following theorem:

Main Theorem

Let $-\Delta = \sum_\ell X_\ell^\star X_\ell$ be an equiregular sub-Riemannian Laplacian on a closed connected manifold M. Then

$$\limsup_{k \to +\infty} \frac{\nu(k)}{k} \leq \left(\int_M (c_x^{\mathrm{FK}})^{-\frac{Q}{2}} d\mu(x) \right) \cdot \left(\int_M c_x^{\mathrm{Weyl}} d\mu(x) \right)^{-1}, \quad (14)$$

where $\nu(k)$ denotes the maximal number of nodal domains of an eigenfunction of $-\Delta$ associated with the eigenvalue λ_k,

Corollary

If

$$(c_x^{\mathrm{FK}})^{\frac{Q}{2}}\, c_x^{\mathrm{Weyl}} > 1\,, \tag{15}$$

then Pleijel's theorem holds.

We can come back to the examples.

When M is replaced by an open set Ω in a fixed graded group G and $-\Delta^{\Omega}$ is the Dirichlet realization of the sub-Laplacian (associated with this group) in Ω, examples where the condition given in the corollary holds can be given. We can also consider a Dirichlet realization $-\Delta^{\Omega}$ of an equiregular sub-Riemannian Laplacian in $\Omega \subset M$. Other examples where the theorem can be applied are in the form $M_3 \times \mathbb{T}^k$ with k large enough and with M_3 a 3-dimensional contact manifold.

Second Part

In the first part, we have seen how we can hope to get Pleijel's theorem by analyzing the case of nilpotent groups. The formulas involved indeed some c_x^{FK} relative to a group G_x with a specific Haar measure $\hat{\mu}_x$ and some c_x^{Weyl} relative to the spectral density of $-\hat{\Delta}_x$ in G_x.

The Pleijel argument for $\mathbb{H}_n \times \mathbb{R}^k$

We work on $\mathbb{H}_n \times \mathbb{R}^k$, where $n \in \mathbb{N}$, $k \in \mathbb{N}$. The case $k = 0$ corresponds to the Heisenberg group $\mathbb{H}_n$.Typically, we will denote coordinates in $\mathbb{H}_n$ by (x, y, z) with $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$, and we will denote coordinates in $\mathbb{R}^k$ by w. The measure $dx\,dy\,dz\,dw$ is the Lebesgue measure on $\mathbb{R}^{2n+1+k}$.
For the vector fields we use the following normalization,

$$X_j = \partial_{x_j} + 2y_j\partial_z\,, \qquad Y_j = \partial_{y_j} - 2x_j\partial_z\,, \qquad W_j = \partial_{w_j}\,.$$

The sub-Laplacian is

$$\Delta^{\mathbb{H}_n \times \mathbb{R}^k} = \sum_{j=1}^{n}(X_j^2 + Y_j^2) + \sum_{i=1}^{k} W_i^2\,.$$

If $\Omega \subset \mathbb{H}_n \times \mathbb{R}^k$ is an open set of finite measure, then the spectrum of the Dirichlet realization of $-\Delta_\Omega^{\mathbb{H}_n \times \mathbb{R}^k}$ is discrete and we can denote its eigenvalues, in nondecreasing order and repeated according to multiplicities, by $\lambda_\ell(\Omega)$.
We denote by $\nu_\ell(\Omega)$ the maximum number of nodal domains of eigenfunctions corresponding to eigenvalue $\lambda_\ell(\Omega)$. We are interested in an upper bound on

$$\limsup_{\ell \to \infty} \frac{\nu_\ell(\Omega)}{\ell}$$

that depends only on n and k. Just as Pleijel's bound, our bound depends on two constants. The Weyl asymptotics states that, for any open set $\Omega \subset \mathbb{H}_n \times \mathbb{R}^k$ of finite measure,

$$\mu^{-\frac{2n+2+k}{2}} \#\{\ell : \ \lambda_\ell(\Omega) < \mu\} \to \mathcal{W}(\mathbb{H}_n \times \mathbb{R}^k)|\Omega| \qquad \text{as } \mu \to \infty \,.$$

We will give a (relatively) explicit expression for the constant $\mathcal{W}(\mathbb{H}_n \times \mathbb{R}^k)$.

The Faber–Krahn constant $C^{\mathrm{FK}}(\mathbb{H}_n \times \mathbb{R}^k)$ is the largest constant such that for any open $\Omega \subset \mathbb{H}_n \times \mathbb{R}^k$ of finite measure and for any $u \in S_0^1(\Omega)$ one has

$$\begin{aligned} &\int_\Omega (\sum_{j=1}^n ((X_j u)^2 + (Y_j u)^2) + \sum_{i=1}^k (W_i u)^2)\, dx\, dy\, dz\, dw \\ &\quad \geq C^{\mathrm{FK}}(\mathbb{H}_n \times \mathbb{R}^k)|\Omega|^{-\frac{2}{2n+2+k}} \int_\Omega u^2\, dx\, dy\, dz\, dw\,. \end{aligned} \tag{16}$$

Here $S_0^1(\Omega)$ denotes the form domain of the Dirichlet realization of $-\Delta$ on Ω. The defining inequality for the Faber–Krahn constant can also be stated as

$$\lambda_1(\Omega) \geq C^{\mathrm{FK}}(\mathbb{H}_n \times \mathbb{R}^k)|\Omega|^{-\frac{2}{2n+2+k}}$$

for all open $\Omega \subset \mathbb{H}_n \times \mathbb{R}^k$ of finite measure.

Let us set

$$\gamma(\mathbb{H}_n \times \mathbb{R}^k) := \left(C^{\mathrm{FK}}(\mathbb{H}_n \times \mathbb{R}^k)\right)^{-\frac{2n+2+k}{2}} \left(\mathcal{W}(\mathbb{H}_n \times \mathbb{R}^k)\right)^{-1} . \quad (17)$$

Here is our Pleijel-type bound.

Theorem (Frank–Helffer [20])

For any open $\Omega \subset \mathbb{H}_n \times \mathbb{R}^k$ of finite measure,

$$\limsup_{\ell \to \infty} \frac{\nu_\ell(\Omega)}{\ell} \leq \gamma(\mathbb{H}_n \times \mathbb{R}^k) .$$

Proof of Pleijel's Theorem

Let u be an eigenfunction corresponding to the eigenvalue $\lambda_\ell(\Omega)$. Let $(\omega_\alpha)_\alpha$ be its nodal domains and let $\nu_\ell(u)$ be their number. W know that $\lambda_\ell(\Omega) = \lambda_1(\omega_\alpha)$ and that $u|_{\omega_\alpha}$ is the ground state of the Dirichlet realization on ω_α. Thus,

$$\begin{aligned}
\frac{\nu_\ell(u)}{\ell} &= \frac{\lambda_\ell(\Omega)^{\frac{2n+2+k}{2}}}{\ell} \sum_\alpha \lambda_1(\omega_\alpha)^{-\frac{2n+2+k}{2}} \\
&\leq \frac{\lambda_\ell(\Omega)^{\frac{2n+2+k}{2}}}{\ell} \left(C^{\mathrm{FK}}(\mathbb{H}_n \times \mathbb{R}^k)\right)^{-\frac{2}{2n+2+k}} \sum_\alpha |\omega_\alpha| \\
&\leq \frac{\lambda_\ell(\Omega)^{\frac{2n+2+k}{2}}}{\ell} \left(C^{\mathrm{FK}}(\mathbb{H}_n \times \mathbb{R}^k)\right)^{-\frac{2}{2n+2+k}} |\Omega| \,.
\end{aligned}$$

Since this holds for any eigenfunction, we get

$$\frac{\nu_\ell(\Omega)}{\ell} \leq \frac{\lambda_\ell(\Omega)^{\frac{2n+2+k}{2}}}{\ell} \left(C^{\mathrm{FK}}(\mathbb{H}_n \times \mathbb{R}^k)\right)^{-\frac{2}{2n+2+k}} |\Omega| \,.$$

Taking the limsup as $\ell \to \infty$ and with in mind the Weyl asymptotics, we arrive at the claimed bound.

It remains to give conditions on n and k for which $\gamma(\mathbb{H}_n \times \mathbb{R}^k) < 1$. We recall that for $n = 0$ this was shown to be the case for $k = 2$ by Pleijel [55] and for general k by Bérard and Meyer [5]. Moreover, Helffer and Persson Sundqvist [35] showed that, for $n = 0$, $k \mapsto \gamma(\mathbb{R}^k)$ is decreasing and actually exponentially decreasing. At the moment we have the following.

Theorem Frank–Helffer

If $k = 0$, then for all $n \geq 4$ one has $\gamma(\mathbb{H}_n) < 1$.
If $k = 1$, then for all $n \geq 3$ one has $\gamma(\mathbb{H}_n \times \mathbb{R}) < 1$.
If $n = 1$, then for all $k \geq 2$ one has $\gamma(\mathbb{H}_1 \times \mathbb{R}^k) < 1$.
If $n = 2$, then for all $k \geq 1$ one has $\gamma(\mathbb{H}_2 \times \mathbb{R}^k) < 1$.

The cases that are still open are when both $n \geq 3$ and $k \geq 2$. And $k = 0, n = 1, 2, 3,\ k = 1, n = 1$.

The constant in the Weyl asymptotics in the case of $\mathbb{H}_n$

Here we follow more explicit computations of Hansson–Laptev [29], which yields the Weyl asymptotics under the sole assumption that Ω is an open set of finite measure.

$$\mathcal{W}(\mathbb{H}_n) = \frac{1}{2(n+1)} \frac{1}{(2\pi)^{n+1}} \sum_{m \in \mathbb{N}} \binom{m+n-1}{m} \frac{1}{(2m+n)^{n+1}} . \tag{18}$$

Faber-Krahn or Sobolev for $\mathbb{H}_n$

We obtain a bound on the Faber–Krahn constant in terms of the (critical) Sobolev inequality on $\mathbb{H}_n \times \mathbb{R}^k$. By definition, $S^{\mathrm{Sob}}(\mathbb{H}_n \times \mathbb{R}^k)$ is the largest constant such that for all $u \in S_0^1(\mathbb{H}_n \times \mathbb{R}^k)$

$$\int_{\mathbb{H}_n \times \mathbb{R}^k} \left(\sum_{j=1}^{n} ((X_j u)^2 + (Y_j u)^2) + \sum_{i=1}^{k} (W_i u)^2 \right) dx\, dy\, dz\, dw$$

$$\geq C^{\mathrm{Sob}}(\mathbb{H}_n \times \mathbb{R}^k) \left(\int_{\mathbb{H}_n \times \mathbb{R}^k} |u|^{\frac{2(2n+2+k)}{2n+k}}\, dx\, dy\, dz\, dw \right)^{\frac{2n+k}{2n+2+k}} .$$

By application of Hölder, we have

Lemma Sob-to-FK

$$C^{\mathrm{FK}}(\mathbb{H}_n \times \mathbb{R}^k) \geq C^{\mathrm{Sob}}(\mathbb{H}_n \times \mathbb{R}^k).$$

An explicit expression for $C^{\mathrm{Sob}}(\mathbb{H}_n)$ was found by Jerison and Lee [39]; for an alternative proof see [22]. We have

$$C^{\mathrm{Sob}}(\mathbb{H}_n) = \frac{4\pi n^2}{(2^{2n} n!)^{\frac{1}{n+1}}}. \tag{19}$$

The Pleijel constant $\gamma(\mathbb{H}_n)$

Our goal in this subsection is to prove the part of our Theorem for $k = 0$, that is, we are going to prove that $\gamma(\mathbb{H}_n) < 1$ for $n \geq 4$. To bound $\gamma(\mathbb{H}_n)$ we use our previous bounds to get

$$\gamma(\mathbb{H}_n) \leq \left(C^{\mathrm{Sob}}(\mathbb{H}_n)\right)^{-n-1} \mathcal{W}(\mathbb{H}_n)^{-1} = \frac{2^n(n+1)!}{n^{2(n+1)}} \frac{1}{c_n} =: \tilde{\gamma}_n \,, \tag{20}$$

where c_n is defined by

$$c_n := \sum_{m \in \mathbb{N}} \binom{m+n-1}{m} \frac{1}{(2m+n)^{n+1}} \,.$$

Numerics treats the case $n \leq 13$. The second step is to show that $\tilde{\gamma}_n / \tilde{\gamma}_{n-1}$ becomes < 1. This is done first asymptotically and in a second step, the estimate of the remainder in the asymptotics shows that this holds for $n \geq 13$.

[1] V. Arnaiz and G. Rivière.
Quantum limits of perturbed subRiemannian contact Laplacians in dimension 3.
arXiv:2306.10757v1 (2023).

[2] H. Bahouri.
Sur la propriété de prolongement unique pour les opérateurs de Hörmander.
Journées équations aux dérivées partielles (1983), p. 1-7 and AIF (1984).

[3] J.M. Bony.
Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour des opérateurs elliptiques dégénérés.
Annales de l'institut Fourier, 1969, Vol. 19, no 1, p. 277-304.

[4] A. Bellaïche and J.J. Risler.
sub-Riemannian Geometry. Birkhäuser (1996).

[5] P. Bérard and D. Meyer.
Inégalités isopérimétriques et applications.

Annales scientifiques de l'École Normale Supérieure, Série 4, Tome 15, no 3, p. 513–541 (1982).

[6] T. P. Branson, L. Fontana, C. Morpurgo.
Moser–Trudinger and Beckner–Onofri's inequalities on the CR sphere.
Ann. of Math. (2) 177 (2013), no. 1, 1–52.

[7] L. Capogna, D. Danielli, S. D. Pauls, J. T. Tyson.
An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem
Progr. Math., 259. Birkhäuser Verlag, Basel, 2007, xvi+223 pp.

[8] Y. Chitour, D. Prandi, and L. Rizzi.
Weyl's law for singular Riemannian manifolds
ArXiv.

[9] W. S. Cohn, G. Lu.
Best constants for Moser-Trudinger inequalities on the Heisenberg group.
Indiana Univ. Math. J. 50 (2001), no. 4, 1567–1591.

[10]Yves Colin de Verdière, Luc Hillairet and Emmanuel Trélat.

Spectral asymptotics for sub-Riemannian Laplacians, I: Quantum ergodicity and quantum limits in the 3-dimensional contact case.
Duke Math. Journal, 167 (1): 109-174, 2018.

[11]Yves Colin de Verdière, Luc Hillairet and Emmanuel Trélat.

Small-time asymptotics of hypoelliptic heat kernels near the diagonal, nilpotentization and related results.
Annales Henri Lebesgue 4 (2021): 897-971.

[12] Yves Colin de Verdière, Luc Hillairet and Emmanuel Trélat.
Spectral asymptotics for sub-Riemannian Laplacians.
ArXiv 2022.

[13] R. Courant and D. Hilbert.
Methods of Mathematical Physics: Partial Differential Equations, Vol. 2. John Wiley and Sons, (2008).

[14] D. Danielli, N. Garofalo, and D.-M. Nhyeu.
Non-doubling Ahlfors measures, perimeter measures, and the characterization of the trace spaces of Sobolev functions in Carnot-Carathéodory spaces.
Memoirs of the AMS, Vol. 182, No 537 (2006).

[15] M. Derridj.
Un problème aux limites pour une classe d'opérateurs du second ordre hypoelliptiques.
Ann. Inst. Fourier, Grenoble, 21, 4 (1971), 99-148.

[16] N. De Ponti, S. Farinelli, I.Y. Violo.
Pleijel nodal domain theorem in non-smooth setting
ArXiv 2023.

[17] S. Eswarathasan and C. Letrouit.
Nodal sets of Eigenfunctions of sub-Laplacians.
ArXiv January 2023. International Mathematics Research Notices (2023).

[18] G. Folland.

Subelliptic estimates and function spaces on nilpotent groups.
Ark. Math. 13 (1975), 161-207.

[19] G.B. Folland and E.M. Stein.
Estimates for the $\overline{\partial}_b$-complex and analysis on the Heisenberg group.
Comm. Pure Appl. Math. 27 (1974), p. 429-522.

[20] R.L. Frank and B. Helffer.
ArXiv v3 (2024).

[21] R.L. Frank, A. Laptev, and T. Weidl.
Schrödinger operators: eigenvalues and Lieb–Thirring inequalities.
Cambridge studies in advanced mathematics 200.

[22] Rupert L. Frank and Elliott H. Lieb. Sharp constants in several inequalities on the Heisenberg group.
Annals of mathematics (2012). 349–381.

[23] R.L. Frank, D. Gontier, and M. Lewin.

The nonlinear Schrödinger equation for orthonormal functions II: application to Lieb-Thirring inequalities.
Commun. Math. Phys. 384, 1783–1828 (2021).

[24] N. Garofalo.
Unique continuation for a class of elliptic operators which degenerate on a manifold of arbitrary codimension,
J. Differential Equations 104 (1993), no. 1.

[25] B. Helffer, T. Hoffmann-Ostenhof, and S. Terracini.
Nodal domains and spectral minimal partitions. Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009).

[26] N. Garofalo, D.-M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces*. Comm. Pure Appl. Math. **49** (1996), no. 10, 1081–1144.

[27] N. Garofalo, D.-M. Nhieu, *Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot–Carathodory spaces*. J. Anal. Math. **74** (1998), 67–97.

[28]R.W. Goodman.
Nilpotent Lie groups.
Lecture Notes in Mathematics. No 562. Springer (1976).

[29] A. M. Hansson and A. Laptev.
Sharp spectral inequalities for the Heisenberg Laplacian.
London Math. Soc. Lecture Note Ser., 354, pages 100–115, 2008.

[30] Asma Hassannezhad, David Sher.
On Pleijel's nodal domain theorem for the Robin problem
arXiv

[31] P. Hajłasz, P. Koskela, *Sobolev met Poincaré*. Mem. Amer. Math. Soc. **145** (2000), no. 688, x+101 pp.

[32] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear potential theory of degenerate elliptic equations*. Unabridged republication of the 1993 original. Dover Publications, Inc., Mineola, NY, 2006. xii+404 pp.

[33] B. Helffer and J. Nourrigat.

Approximation d'un système de champs de vecteurs et applications à l'hypoellipticité.
Ark. Mat. 17(1-2): 237-254 (1979).

[34] B. Helffer and J. Nourrigat.
Hypoellipticité Maximale pour des Opérateurs Polynômes de Champs de Vecteurs.
Progress in Mathematics. Birkhäuser. 1985

[35] B. Helffer and M. Persson Sundqvist.
On nodal domains in Euclidean balls.
Proc. Amer.Math. Soc. 144 (11): 4777–4791 (2017).

[36] L. Hörmander.
Hypoelliptic second order differential equations
Acta Math. 119: 147-171 (1967).

[37] F. Jean.
Control of nonholonomic systems: from sub-Riemannian geometry to motion planning.
Monograph. Springer (2014).

[38] D. Jerison.
The Dirichlet problem for the Kohn Laplacian on the Heisenberg group.
J. of Functional Analysis 43 (1981), Part I, 97–141, Part II, 224–257.

[39] D. Jerison and J.M. Lee.
Extremal for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem.
Journal of the AMS, Vol. 1, Number 1, January 1988.

[40]J. Kinnunen, J. Lehrbäck, A. Vähäkangas, *Maximal function methods for Sobolev spaces*. Mathematical Surveys and Monographs, 257. American Mathematical Society, Providence, RI, 2021. xii+338 pp.

[41] C. Léna.
Pleijel's nodal domain theorem for Neumann and Robin eigenfunctions.
ArXiv:1609.02331 (2016) and Annales de l'Institut Fourier 69 (1), 283-301, 2019.

[42] G. Leoni.
A first course in Sobolev spaces.
Grad. Stud. Math., 181, American Mathematical Society, Providence, RI, 2017, xxii+734 pp.

[43] M. Levitin, D. Mangoubi, and I. Polterovich.
Topics in spectral Geometry.
AMS Graduate Studies in Mathematics series, volume 237 (2023).

[44] E. H. Lieb, M. Loss.
Analysis
Grad. Stud. Math., 14, American Mathematical Society, Providence, RI, 2001, xxii+346 pp.

[45] W. Magnus, F. Oberhettinger, and R.P. Soni.
Formulas and Theorems for the special functions of mathematical Physics.
Die Grundlehre der mathematischen Wissenschaften. Band 52. Third edition. Springer (1966).

[46] A. Menikoff and J. Sjöstrand.
On the eigenvalues of a class of hypoelliptic operators.
Math. Ann. 235 (1978), 55–85.

[47] A. Menikoff and J. Sjöstrand, On the eigenvalues of a class of hypoelliptic operators II,
Global analysis (Proc. Biennial Sem. Canad. Math. Congr., Univ. Calgary, Calgary, Alta., 1978), pp. 201–247, Lecture Notes in Math., 755, Springer, Berlin, 1979.

[48] G. Métivier.
Fonction spectrale et valeurs propres d'une classe d'opérateurs non elliptiques.
Publications des séminaires de mathématiques et informatique de Rennes, 1976, fascicule 1 Séminaires d'analyse fonctionnelle p. 1-5. and CPDE 1 (1976), 467–519.

[49] G. Métivier.
Journées Equations aux Dérivées Partielles (1976).

[50] A. Mohamed.

Étude spectrale d'opérateurs hypoelliptiques caractéristiques multiples
Journées équations aux dérivées partielles (1981) and AIF.

[51] R. Monti, D. Morbidelli.
Non-tangentially accessible domains for vector fields.
Indiana University Mathematics Journal, Vol. 54, No. 2 (2005).

[52]B.D.S. Nagy.
Über Integralungleichungen zwischen einer Funktion und ihrer Ableitung.
Acta Sci. Math. 10, 64–74 (1941).

[53]P. Pansu.
An isoperimetric inequality for the Heisenberg group. CRAS Math. 295.2 (1982), 127–130.

[54]P. Pansu.
An isoperimetric inequality on the Heisenberg group.
Conference on differential geometry on homogeneous spaces

(Torino, 1983). Rend. Sem. Mat. Univ. Politec. Torino 1983, Special Issue, 159–174 (1984).

[55] A. Pleijel.
Remarks on Courant's nodal theorem.
Comm. Pure. Appl. Math., 9: 543–550, 1956.

[56] L.P. Rothschild.
A criterion for hypoellipticity of operators constructed of vector fields.
Comm. in PDE 4 (6) (1979), 248–315.

[57] L.P. Rothschild and E.M. Stein.
Hypoelliptic differential operators and nilpotent groups.
Acta Mathematica 137, 248–315.

[58] G. Talenti.
The standard isoperimetric theorem.
North-Holland Publishing Co., Amsterdam, 1993, 73–123.

[59] N. Th. Varopoulos.
Analysis on nilpotent groups.

Journal of Functional Analysis 66, 406–431 (1986).

[60] K. Watanabe
Sur l'unicité du prolongement des solutions des équations elliptiques dégénérées.
Tohoku Math. Journ. 34(1982), 239-249.