Initiation to Witten Laplacians Methods in Statistical Mechanics

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Main goals

The aim of these lectures is to present some basic properties of the Witten Laplacians and to explain how they can be used in statistical mechanics.

We will look at the simplest situations where this method appears to be powerful, that is for the analysis of the properties of a measure on $\mathbb{R}^N$ (or on an open set therein) taking the form

$$\exp -\frac{\Phi}{h}\, dx^N$$

where Φ is a phase (in $C^\infty(\mathbb{R}^N)$) describing the properties of the model, dx^N denotes the Lebesgue measure on $\mathbb{R}^N$ and h is a small parameter.

Two asymptotics will be considered :

h small (this could be the temperature)
called the **semi-classical limit**,

or N large (related to the number of particles),
called the **thermodynamic limit**.

The analysis of simultaneous asymptotics is the most interesting but we will just analyze two examples where only one of the two asymptotics is involved.

Another parameter $\mathcal{J}$ measuring the size of the interaction will also appear permitting the use of perturbative techniques.

Contributors using the method

Witten (but not in this context) (1982)

Helffer-Sjöstrand (1994), Sjöstrand (1994)
(but previous works on Schrödinger operators...1982-...)

Further developments 1995-2005

In alphabetic order

Bach, Bodineau, Helffer, Jecko, Johnsen, Matte, Moeller, Nier, Sjöstrand, W.M. Wang ...

But not disjoint of other methods : Bakry-Emery ... Ledoux

In other domains it has inspired :

Naddaf-Spencer : On homogeneization and scaling limit of some gradient perturbation of a massless free field

Deuschel : The random Walk representation for interacting diffusion processes..

For a extensive presentation of the subject, see Helffer (book World Scientific), Helffer-Nier (LN in Mathematics 1862).

Witten Laplacians approach

We start with some standard mathematical basics.

The De Rham Complex

We denote by $\Omega^0(\mathbb{R}^m)$ the space of the real C^∞ 0-forms corresponding consequently to $C^\infty(\mathbb{R}^m\ ;\ \mathbb{R})$, by $\Omega^1(\mathbb{R}^m)$ the real C^∞ 1-forms :

$$\omega = \sum_{j=1}^{m} \omega_j dx^j \ .$$

More generally we can define p-forms and we recall that the m-forms can be identified with functions through the correspondence $\phi \mapsto \phi dx_1 \wedge dx_2 \cdots \wedge dx_m$. The operation $\wedge$ associates to two forms $\omega \in \Omega^p$ and $\omega' \in \Omega^{p'}$ the exterior product $\omega \wedge \omega'$ in $\Omega^{p+p'}$ and we have the property

$$\omega \wedge \omega' = (-1)^{pp'} \omega' \wedge \omega \ ,$$

with in particular

$$dx_i \wedge dx_i = 0 \ , \ dx_i \wedge dx_j = -dx_j \wedge dx_i \ .$$

The **exterior differential** d is defined on $\oplus_{j=0}^{m} \Omega^j(\mathbb{R}^m)$ with the properties :

- d restricted to $\Omega^0(\mathbb{R}^m)$ is denoted by $d^{(0)}$ and goes from $\Omega^0(\mathbb{R}^m)$ into $\Omega^1(\mathbb{R}^m)$:

  $$d^{(0)} u = \sum_j (\partial_{x_j} u) dx_j \ .$$

  More generally the restriction of d to $\Omega^p(\mathbb{R}^m)$ is denoted by $d^{(p)}$ and goes from $\Omega^p(\mathbb{R}^m)$ into $\Omega^{p+1}(\mathbb{R}^m)$

- $$d \circ d = 0 \ , \tag{1}$$

  that is

  $$d^{(p+1)} \circ d^{(p)} = 0 \ , \ \forall p = 0, \cdots, m-1 \ . \tag{2}$$

- For $\omega \in \Omega^p$ and $\omega' \in \Omega^{p'}$,

$$d^{(p+p')}(\omega^\wedge\omega') = d^{(p)}\omega \wedge \omega' + (-1)^p \omega \wedge d^{(p')}\omega' \, .$$

In particular $d^{(1)}$ on $\Omega^1(\mathbb{R}^m)$ is the operator defined for $\omega = \sum_{j=1}^m \omega_j dx_j$ by

$$\begin{aligned} d^{(1)}\omega \quad &= \textstyle\sum_j (d^{(0)}\omega_j) \wedge dx_j \\ &= \textstyle\sum_{j,k} (\partial_{x_k}\omega_j) dx_k \wedge dx_j \\ &= \textstyle\sum_{j<k} (\partial_{x_j}\omega_k - \partial_{x_k}\omega_j) dx_j \wedge dx_k \, . \end{aligned}$$

- $$d^{(m)} = 0 \, .$$

The De Rham Complex is well defined (see (2)) by

$$0 \mapsto \Omega^0 \overset{d^{(0)}}{\mapsto} \Omega^1 \overset{d^{(1)}}{\mapsto} \Omega^2 \mapsto \cdots \mapsto \Omega^m \mapsto 0 \, . \tag{3}$$

All this can be more generally defined on general C^∞ manifolds M.

We are now interested in the introduction of a prehilbertian structure on these C^∞- forms which are now assumed to be with compact support (in this case, we write Ω_0^p). This will permit to define by completion the corresponding Hilbert spaces of L^2 p-forms. It is sufficient for this to define the natural norms on the tensor products. In particular we have

- For u in $\Omega^0(\mathbb{R}^m)$

$$||u||^2 = \int_{\mathbb{R}^m} u(X)^2 \, dX \ , \tag{4}$$

  where dX corresponds to the usual Lebesgue measure. By completion, we shall get the usual standard $L^2(\mathbb{R}^m)$ space.

- For ω in $\Omega^1(\mathbb{R}^m)$

$$||\omega||^2 = \sum_{j=1}^m \int_{\mathbb{R}^m} \omega_j(X)^2 dX \ .$$

- For σ in $\Omega^2(\mathbb{R}^m)$, we get

$$||\sigma||^2 = \sum_{j<k} \int_{\mathbb{R}^m} \sigma_{jk}(X)^2 dX \ .$$

These norms are of course associated to scalar products denoted by $\langle \cdot \mid \cdot \rangle$. We get of course

$$\langle \omega \mid \omega \rangle = ||\omega||^2 \ .$$

Once we have this prehilbertian structure, it is immediate to associate to d a formal adjoint d^*. More specifically we associate to each $d^{(p)}$ defined from Ω^p into Ω^{p+1} a formal adjoint $d^{*,(p)}$ which maps Ω^{p+1} into Ω^p and is uniquely defined, for $\omega \in \Omega^{p+1}$, by the relation

$$\langle {d^*}^{(p)} \omega \mid \omega' \rangle = \langle \omega \mid d^{(p)} \omega' \rangle \ ,$$

for all $\omega' \in \Omega^p_0$.

${d^*}^{(p)}$ is a differential operator of order 1 mapping explicitely Ω^{p+1} into Ω^p.
We get in particular the following formulas :

- For a one-form ω, we have

$$
{d^*}^{(0)}\omega = -\sum_j \partial_{x_j}\omega_j \;,
$$

- For a 2-form σ, we have

$$
({d^*}^{(1)}\sigma)_k = -\sum_j \partial_{x_j}\sigma_{jk} \;,
$$

  where $x \mapsto \sigma_{jk}(x)$ has been extended as a function with values in the space of the antisymmetric matrices.

It is also easy to verify that d^* is also a complex :

$$
d^* \circ d^* = 0 \text{ or } {d^*}^{(p)} \circ {d^*}^{(p+1)} = 0 \;. \qquad (5)
$$

The corresponding Laplacians $\Delta^{(\cdot)}$ defined in a contracted way as

$$\Delta^{(\cdot)} = (d + d^*)^2 \tag{6}$$

and more explicitely, for any p, whose by

$$\Delta^{(p)} = {d^*}^{(p)} \circ d^{(p)} + d^{(p-1)} \circ {d^*}^{(p-1)} \,. \tag{7}$$

The Laplacian $\Delta^{(p)}$ maps $\Omega^p(\mathbb{R}^m)$ into itself. This is an elliptic operator of order 2 with diagonal principal symbol.
All these constructions are valid when $\mathbb{R}^m$ is replaced by a Riemannian manifold. The operator $\Delta^{(0)}$ is usually called the Laplace-Beltrami operator.
In the case of $\mathbb{R}^m$, the situation is particularly simple because we have the identity

$$\Delta^{(p)} = \Delta^{(0)} \otimes Id \,.$$

When $p = 1$, this gives

$$\Delta^{(1)}\omega = \sum_j (\Delta^{(0)}\omega_j)\, dx_j \ .$$

Note also that

$$\Delta^{(0)} = -\Delta = -\sum_{j=1}^{m} \partial_{x_j}^2 \ .$$

In the case of a compact riemannian manifold M the introduction of these De Rham Complexes and of the corresponding Laplacians leads to the so-called Hodge theory.

A central role is played by these Laplacians for which a Fredholm theory can be developed. In particular, they can be defined as unbounded operators on L^2-forms. These operators will be selfadjoint and under some assumtions with compact resolvent. In particular, their spectrum is a sequence of eigenvalues with finite multiplicity tending to $+\infty$ with a corresponding sequence of eigenvectors forming an orthonormal basis.

Witten Complex

The distorted exterior differential is

$$d_\Phi = \exp -\Phi/2 \; d \; \exp \Phi/2 \; . \tag{8}$$

Note that

$$d_\Phi^{(0)} u = d^{(0)} u + \frac{1}{2} u \; d^{(0)} \Phi \; ,$$

and more generally we have

$$d_\Phi = d + \frac{1}{2} d\Phi \wedge \; . \tag{9}$$

This is a complex :

$$d_\Phi \circ d_\Phi = 0 \; . \tag{10}$$

Keeping the prehilbertian structure given by the initial L^2 norm, we can then define d_Φ^* by duality

$$\langle d_\Phi^* \omega \,|\, \omega' \rangle = \langle \omega \,|\, d_\Phi \omega' \rangle \;.$$

For a 1-form, we get in particular

$${d_\Phi^*}^{(0)} \omega = {d^*}^{(0)} \omega + \frac{1}{2} \omega(\nabla \Phi) \;.$$

Other distortions

More general distortions (sometimes for technical reasons) can be introduced. See later (in the proof of the decay estimates) and also Bach-Jecko-Sjöstrand, Bach-Möller.

Witten Laplacians

We can define now the Witten Laplacians on the forms by
$$\Delta_\Phi^{(\cdot)} = (d_\Phi + d_\Phi^*)^2 \; . \tag{11}$$

All these constructions were initially introduced by E. Witten on a compact manifold and Φ was a Morse function (that is with non degenerate critical points). His idea was to relate some invariants of the manifold M with some indices of the Morse function Φ at the critical points.
These relations are called Morse inequalities.

In the case of $\mathbb{R}^m$, we get more explicitly

$$\Delta_\Phi^{(0)} := \Delta^{(0)} + \frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi \; , \tag{12}$$

and
$$\Delta_\Phi^{(1)} := \Delta_\Phi^{(0)} \otimes Id + \mathrm{Hess}\Phi \; . \tag{13}$$

Note that we have the following important relation

$$d_\Phi^{(0)}\Delta_\Phi^{(0)} = \Delta_\Phi^{(1)} d_\Phi^{(0)}\ , \tag{14}$$

which is only the explicitation on the 0-forms of

$$\begin{aligned} d_\Phi \circ (d_\Phi + d_\Phi^*)^2 \quad &= d_\Phi \circ (d_\Phi d_\Phi^* + d_\Phi^* d_\Phi) \\ &= d_\Phi \circ d_\Phi^* \circ d_\Phi \\ &= (d_\Phi d_\Phi^* + d_\Phi^* d_\Phi) \circ d_\Phi\ . \end{aligned}$$

What is changed ?

At least if $|\nabla\Phi(X)| \to +\infty$ and Φ is with bounded second derivatives, one can show that, although we are on a non compact manifold, the operators $\Delta_\Phi^{(0)}$ and $\Delta_\Phi^{(1)}$ can be extended as positive, selfadjoint operators, with compact resolvent on $L^2(\mathbb{R}^m)$ or $L^2(\mathbb{R}^m;\mathbb{R}^m)$.

For more information on criteria of compactness of the resolvent, see the book of Helffer-Nier.

For the positivity, we observe indeed (at least for the compactly supported forms) the identity

$$\langle \Delta_\Phi \omega \mid \omega \rangle = ||(d_\Phi + d_\Phi^*)\omega||^2 \ .$$

Semi-classical considerations

If we replace Φ by Φ/h, we get $h^2\Delta_{\frac{\Phi}{h}}$, a new family of h-dependent Laplacians considered by E. Witten in the case of the compact manifolds. We get

$$\Delta^{(0)}_{\Phi;h} = -h^2\Delta + \frac{1}{4}|\nabla\Phi(X)|^2 - \frac{h}{2}\Delta\Phi(X)\ . \qquad (15)$$

This has the form of a Schrödinger operator

$$-h^2\Delta + V_h$$

with

$$V_h := V_0 + hV_1$$

and

$$V_0(X) := \frac{1}{4}|\nabla\Phi(X)|^2\ .$$

For the Laplacian on $1-$forms

$$\Delta^{(1)}_{\Phi;h} = (-h^2\Delta+\frac{1}{4}|\nabla\Phi(X)|^2-\frac{h}{2}\Delta\Phi(X))\otimes I+h\mathrm{Hess}\Phi\ , \qquad (16)$$

V_1 becomes a non diagonal matrix !

Conversely

If you start from Schrödinger

$$S_{h,V} := -h^2\Delta + V - \lambda_1(h)$$

so that 0 is the lowest eigenvalue.

Consider

$$\Phi = -2h \log u_0$$

where u_0 is the groundstate.

Then

$$S_{h,V} = \Delta^{(0)}_{\Phi;h} \ .$$

About Witten Idea

The main idea of E. Witten was that the dimension of the kernels of these various Laplacians
– related by the Hodge theory to the Betti numbers
–
can be estimated from above by a rather crude semiclassical analysis of the lowest eigenvalues of these operators as $h \to 0$.

If we observe that the minimum of V_0 is obtained at the critical values of Φ and that the minima are non degenerate if and only if the critical points are non degenerate the harmonic approximation gives a satisfactory answer. The only zeros eigenvalues of the approximating harmonic operators correspond to the minima of Φ.

For $\Delta^{(1)}_{\Phi;h}$, we will meet the critical points of Φ of type 1.

An alternative point of view : Dirichlet forms

The idea is simply that we start from the usual De Rham Complex but take for the definition of the adjoint the prehilbertian structure given by the scalar product associated to the norm in $L^2(\mathbb{R}^m; \exp -\Phi \, dX)$.

This gives a natural adjoint complex $d^{*,\Phi}$ and one can similarly introduce associated Laplacians which are denoted by $A_\Phi^{(\cdot)}$.

In particular :

$$A_\Phi^{(0)} = -\Delta + \nabla\Phi \cdot \nabla \; , \tag{17}$$

and

$$A_\Phi^{(1)} = A_\Phi^{(0)} \otimes Id + \mathrm{Hess}\Phi \; . \tag{18}$$

These operators are unbounded operators on $L^2(\mathbb{R}^m,\mathbb{R};\exp-\Phi\, dX)$ and $L^2(\mathbb{R}^m,\mathbb{R}^m;\exp-\Phi\, dX)$. They are also associated with the quadratic forms

$$u \mapsto q_\Phi^{(0)}(u,u) = \int |\nabla u|^2\ \exp-\Phi\, dX\ ,$$

which is called the Dirichlet form, and

$$\begin{aligned}\omega \mapsto\ & q_\Phi^{(1)}(\omega,\omega)\\ &= \int |d^{(1)}\omega|^2 \exp-\Phi dX + \int |d^{*,(0),\Phi}\omega|^2\ \exp-\Phi\, dX\ .\end{aligned}$$

The relation (on the compactly supported forms) is given by :

$$q_\Phi^{(0)}(u,u) = \langle A_\Phi^{(0)} u \mid u\rangle_{L^2(\mathbb{R}^m;\ \exp-\Phi\, dX)}\ ,$$

and similarly

$$q_\Phi^{(1)}(\omega,\omega) = \langle A_\Phi^{(1)} \omega \mid \omega\rangle_{L^2(\mathbb{R}^m,\mathbb{R}^m;\ \exp-\Phi\, dX)}\ .$$

In this point of view, the techniques based on the Lax-Milgram Lemma and the Friedrichs extension are available for defining these Laplacians associated to the Dirichlet forms.

In the case of a compact riemannian manifold M, we have also a (modified) identity (18)
– In particular the Lebesgue measure is replaced by the Riemannian measure on M and $\mathrm{Hess}\Phi$ is replaced by $\mathrm{Hess}\Phi + Ric(M)$ where $Ric(M)$ is the Ricci curvature on the manifold –
or more precisely a modification of the corresponding "form" version written in the form

$$q_\Phi^{(1)}(\omega,\omega) = \sum_{jk} ||\partial_{x_j}\omega_k||^2 + \int (\mathrm{Hess}\Phi\ \omega)\cdot\omega\ \exp -\Phi\ dX\ ,$$

is known from the geometers as the Bochner-Lichnerowicz-Weitzenbock formula. It is actually usually written only for exact 1-forms.

This point of view is unitary equivalent to the Witten Laplacian point of view. We simply observe the following relations

$$\Delta_{\Phi}^{(\cdot)} = \exp -\frac{\Phi}{2} \circ A_{\Phi}^{(\cdot)} \circ \exp +\frac{\Phi}{2} \, .$$

We note of course that $u \mapsto \exp \frac{\Phi}{2}\, u$ is a unitary map from $L^2(\mathbb{R}^m)$ onto $L^2(\mathbb{R}^m\, ; \exp -\Phi \; dX)$.
Having this unitary equivalence, one can transfer all the identities we have obtained for the Witten Laplacians and we get in particular the identity

$$d^{(0)} A_{\Phi}^{(0)} = A_{\Phi}^{(1)} d^{(0)} \, . \tag{19}$$

When trying to use standard theorems (about Schrödinger or about compact injections) it is usually more easy to come back to the point of view of the Witten Laplacians which are defined on $L^2(\mathbb{R}^m)$.

This intertwining property of d and A_Φ, leads to simple connections between the spectra of the various Laplacians.

Typically, if u is an eigenvector corresponding to a non zero eigenvalue of $A_\Phi^{(0)}$, then $d^{(0)}u$ is an eigenfunction of $A_\Phi^{(1)}$. Conversely, if ω is an eigenform of $A_\Phi^{(1)}$, then $d^{*,\Phi}\omega$ if not zero is an eigenvector of $A_\Phi^{(0)}$.

This will be mainly used for the two first Laplacians but see also Matte-Moeller for a nice application of this supersymmetric trick.

Supersymmetry

In other words (coming back to W.L), we can consider the "super charge operator"

$$Q = d_\Phi + d_\Phi^*$$

The Witten Laplacian corresponds to Q^2 and the Hilbert space $\mathcal{H}$ of the L^2-forms is splitted in the direct sum $\mathcal{H}_- \oplus \mathcal{H}_+$ of the odd and even forms. If γ is defined as the multiplication operator by $\pm$ on $\mathcal{H}_\pm$, we get

$$\Delta_\Phi = Q^2 \; , \; Q\gamma + \gamma Q = 0 \; , \; \gamma^2 = 1 \; ,$$

giving the classical supersymmetric picture (see Witten).

Towards statistical mechanics

We start from a phase Φ representing the physical system. We assume (possibly after renormalization)

$$Z := \int \exp -\Phi(X)\, dX = 1 \,, \tag{20}$$

and $h = 1$.

The integration is over $\mathbb{R}^N$, where N corresponds (is proportional) to the number of particles.

In the context of statistical mechanics, one is interested in the limit $N \to +\infty$, but for us this will mean mainly estimates with uniform control with respect to the dimension.

When temperature is involved, one has to replace Φ by $\beta\Phi$ with $\beta = 1/T$. Semi-classical techniques are efficient when temperature is small.

A nice formula for the covariance

If we denote by $\langle f\rangle$, the mean value of f with respect to the measure $\exp -\Phi(X)\ dX$, the covariance of two functions f and g is defined by

$$\text{Cov }(f,g) = \langle (f-\langle f\rangle)\cdot(g-\langle g\rangle)\rangle\ . \tag{21}$$

The variance is recovered by taking $f=g$.

If we have in mind to write an expression of the covariance in the form

$$\text{Cov }(f,g) = \langle dg \mid \omega\rangle_{L^2(\mathbb{R}^m,\mathbb{R}^m\,;\,\exp -\Phi\ dX)}\ ,$$

for a suitable 1-form ω, we get, observing that $dg = d(g-\langle g\rangle)$,

$$\text{Cov }(f,g) = \int (g-\langle g\rangle)(d^{*,\Phi}\omega)\ \exp -\Phi\ dX\ .$$

This leads to the question of solving

$$f-\langle f\rangle = d^{*,\Phi}\omega\ . \tag{22}$$

We can try to solve

$$f - \langle f \rangle = d^{*,\Phi}\omega \ .$$

with an ω in the form $\omega = du$, and this leads to the following equation for u

$$\left.\begin{array}{l} f - \langle f \rangle = A_\Phi^{(0)} u \\ \langle u \rangle = 0 \end{array}\right\} . \tag{23}$$

As previously mentioned, $A_\Phi^{(0)}$ is an unbounded selfadjoint operator.

This operator $A_\Phi^{(0)}$ is positive (this is a Laplacian) and $A_\Phi^{(0)} + Id$ is an invertible operator which maps $D(A_\Phi^{(0)})$ onto $L^2(\mathbb{R}^m; \exp -\Phi \, dX)$ and its inverse is (under suitable assumptions) a selfadjoint compact operator. It can then be shown that $A_\Phi^{(0)}$ can be diagonalized in the same basis with the property that the eigenvalues $\lambda_n = \lambda_n^{(0)}$.
$A_\Phi^{(0)}$ being a positive operator, we know that $\lambda_1^{(0)} \geq$

0. But it is immediately seen that

$$A_\Phi^{(0)}\, 1 = 0\ .$$

So $X \mapsto u_1(X) = 1$ is a possible first eigenvector (or ground state) of $A_\Phi^{(0)}$ attached to $\lambda_1^{(0)} = 0$.

Now, in order to solve the equation (22)

$$\left.\begin{array}{l} f - \langle f \rangle = A_\Phi^{(0)} u \\ \langle u \rangle = 0 \end{array}\right\} .$$

it is clear that the necessary and sufficient condition for solving $A_\Phi^{(0)} u = \tilde{f}$ is the orthogonality of $\tilde{f}$ with the first eigenvector which is expressed by $\langle \tilde{f} \rangle = 0$. But $\tilde{f} = f - \langle f \rangle$ satisfies this condition, and this explains why we can solve (22).

For f in C^1, such that df is bounded, we have seen that there exists u such that (22)

$$\left.\begin{array}{l} f - \langle f \rangle = A_\Phi^{(0)} u \\ \langle u \rangle = 0 \end{array}\right\} .$$

is satisfied.

We get by differentiation and using the commutation relation (19) with $\omega := du$,

$$df = A_\Phi^{(1)} \omega \ . \tag{24}$$

There is for $A_\Phi^{(1)}$ an analogous Fredholm theory as for $A_\Phi^{(0)}$. If we denote by $\lambda_n^{(1)}$ the increasing sequence of the eigenvalues, we know that

$$0 \leq \lambda_1^{(1)} \leq \lambda_2^{(1)} \leq \cdots \tag{25}$$

In the case when $\lambda_1^{(1)}$ is strictly positive then $A_\Phi^{(1)}$ becomes invertible and we can consequently write that

$$\omega = (A_\Phi^{(1)})^{-1} df \ ,$$

and this leads to the "nice" formula for the covariance of two functions :

$$\text{Cov } (f,g) = \langle (A_\Phi^{(1)})^{-1} df \mid dg \rangle_{\mathcal{H}^{(1)}} \ , \tag{26}$$

where $\mathcal{H}^{(1)}$ is the Hilbert space of the 1-forms with coefficients in the space $L^2(\mathbb{R}^m; \exp -\Phi \, dX)$.
In particular we get for the variance :

$$\text{Var } (f) = \int \Big((A_\Phi^{(1)})^{-1} df \Big) \cdot df \ \exp -\Phi \ dX \ , \tag{27}$$

and the Poincaré inequality

$$\text{Var } (f) \leq (\lambda_1^{(1)})^{-1} ||df||^2_{\mathcal{H}^{(1)}} \ . \tag{28}$$

This inequality is not optimal in general.

Remark

We have the inequality

$$\mathrm{Var}\ (f) \leq (\lambda_2^{(0)})^{-1} ||df||^2_{\mathcal{H}^{(1)}}\ . \tag{29}$$

Hint

Observe that

$$||df||^2_{\mathcal{H}^{(1)}} = \langle A_\Phi^{(0)}(f - \langle f \rangle) \mid (f - \langle f \rangle) \rangle_{\mathcal{H}}\ .$$

Remark

Using (19), one can show

$$\lambda_1^{(1)} \leq \lambda_2^{(0)}\ . \tag{30}$$

We now look at the question of getting explicit lower bounds for $\lambda_1^{(1)}$. This is rather easy in the convex case.

Convex phase

Let us now consider situations when an easy lower bound of $\lambda_1^{(1)}$ is available. In the case when Φ is uniformly strictly convex,

$$\mathrm{Hess}\Phi(X) \geq \sigma > 0 \ , \tag{31}$$

we observe the following inequality between selfadjoint operators

$$A_\Phi^{(1)} = A_\Phi^{(0)} \otimes Id + \mathrm{Hess}\Phi \geq \mathrm{Hess}\Phi \geq \sigma > 0 \ , \tag{32}$$

which is an immediate consequence of (31), the positivity of $A_\Phi^{(0)}$, (18), and, using abstract analysis, we obtain

$$(A_\Phi^{(1)})^{-1} \leq (\mathrm{Hess}\Phi)^{-1} \ . \tag{33}$$

The Brascamp-Lieb inequality is then an immediate consequence of (26) and says

$$\text{Var}\,(f) \leq \langle (\mathrm{Hess}\Phi)^{-1} df \mid df \rangle_{\mathcal{H}^{(1)}} \,. \tag{34}$$

Of course, together with (32), the inequality (34) implies the Poincaré inequality.

Notes

- The Witten Laplacians were introduced in a different context by E. Witten as an elegant way to prove by analytic technics Morse inequalities on a compact manifold (see later).

- The appearance of these Laplacians in the analysis of Poincaré inequalities is rather old, at least in Riemannian geometry. In this context, the Witten Laplacians appear together with Bochner Laplacians.

- The understanding of the role of the Witten Laplacians in the analysis of decay estimates is due to Helffer-Sjöstrand and more explicitly to Sjöstrand, but it is not so far from the approach of Bakry-Emery (see Ledoux) or of other techniques used in Quantum field theory.

Typical models coming from statistical mechanics

Our aim is to analyze the thermodynamic properties (and particularly the decay of correlations) of measures. Here we more precisely consider the case with "boundary", that is measures $\exp -\Phi^{\Lambda,\omega}(X)\, dX$ with some $\Phi^{\Lambda,\omega}$, associated with cubes $\Lambda \subset \mathbb{Z}^d$ and some $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ defining the boundary condition, in the form, for $X \in \mathbb{R}^\Lambda$,

$$\Phi^{\Lambda,\omega}(X) = \sum_{j\in\Lambda} \phi(x_j) + \frac{\mathcal{J}}{2} \sum_{(\{j\}\cup\{k\})\cap\Lambda\neq\emptyset\ ,\ j\sim k} |z_j - z_k|^2\ ,$$

where

- $X = (x_j)_{j\in\Lambda}$,
- ϕ is a one-particle phase on $\mathbb{R}$ with at least quadratic increase[1]

[1] One can in some case consider a weaker assumption but we always assume that $\int \exp -\phi(t)\, dt < +\infty$. $\mathbb{R}$ can be replaced by $\mathbb{R}^k$.

- $$\begin{aligned} z_j &= x_j \quad \text{if } j \in \Lambda\ , \\ z_j &= \omega_j \quad \text{if } j \notin \Lambda\ . \end{aligned}$$

- $j \sim k$ means that j and k are nearest neighbors [2] for the ℓ^1- distance in $\mathbb{Z}^d$.

We shall sometimes use the following decomposition

$$\Phi^{\Lambda,\omega} = \Phi_d^{\Lambda} + \mathcal{J}\ \Phi_i^{\Lambda,\omega}\ ,$$

with

$$\Phi_d^{\Lambda}(X) = \sum_{j\in\Lambda} \phi(x_j)\ ,$$

and

$$\Phi_i^{\Lambda,\omega}(X) = \sum_{(\{j\}\cup\{k\})\cap\Lambda\neq\emptyset\ ,\ j\sim k} |z_j - z_k|^2\ .$$

[2] One can also analyze the case when j and k are nearest neighbors in Λ considered as a discrete torus. The same questions can be considered on trees (Bach-Moeller)

One can also consider the **free boundary condition** corresponding to the phase

$$\Phi^{\Lambda,f} = \Phi_d^{\Lambda} + \mathcal{J}\ \Phi_i^{\Lambda,f}\ , \tag{35}$$

with

$$\Phi_i^{\Lambda,f}(X) = \sum_{j,k\in\Lambda\ ,\ j\sim k} |x_j - x_k|^2\ . \tag{36}$$

If necessary, the dependence on $\mathcal{J}$ will be mentioned by the notation $\Phi^{\Lambda,\omega} = \Phi^{\Lambda,\omega,\mathcal{J}}$ or $\Phi^{\Lambda,f} = \Phi^{\Lambda,f,\mathcal{J}}$.

Single-spin phase

We assume that ϕ is C^∞ on $\mathbb{R}$ and convex at ∞, so there exists $C > 0$ such that

$$\phi''(x) \geq \frac{1}{C} \ , \ \forall x \in \mathbb{R} \text{ s.t. } |x| \geq C \ . \qquad (37)$$

We assume also that there exists $\rho > 0$ and, for all $k \in \mathbb{N}$, C_k such that,

$$|\phi^{(k+1)}(x)| \leq C_k < \phi'(x) >^{(1-\rho k)_+} \ , \ \forall x \in \mathbb{R} \ , \qquad (38)$$

where, for $u \in \mathbb{R}$, $< u >:= (1 + |u|^2)^{\frac{1}{2}}$ and, for $t \in \mathbb{R}$, $(t)_+ := \max(t, 0)$.
The typical example will be ($\lambda > 0$)

$$\phi(x) = \frac{1}{12}\lambda x^4 + \frac{1}{2}\nu x^2 \ .$$

We would like to analyze the possibility of having any sign for ν. That is to analyze non convex cases (but convex at ∞).

Our main problem will be to analyze the properties of the measure

$$d\mu_{\Lambda,\omega} := \exp -\Phi^{\Lambda,\omega}(X)\, dX / \left(\int_{(\mathbb{R}^N)^\Lambda} \exp -\Phi^{\Lambda,\omega}(X)\, dX \right) \;,$$

or of the measure

$$d\mu_{\Lambda} := \exp -\Phi^{\Lambda,f}(X)\, dX / \left(\int_{(\mathbb{R}^N)^\Lambda} \exp -\Phi^{\Lambda,f}(X)\, dX \right) \;.$$

We shall in particular analyze the covariance associating to $f, g \in C^\infty_{temp}(\mathbb{R}^\Lambda)$

$$\text{Cov}_{\;\Lambda,\omega}(f,g) = \langle (f - \langle f \rangle_{\Lambda,\omega})(g - \langle g \rangle_{\Lambda,\omega}) \rangle_{\Lambda,\omega}$$

where $\langle \cdot \rangle_{\Lambda,\omega}$ denotes the mean value with respect to $d\mu_{\Lambda,\omega}$ and $C^\infty_{temp}(\mathbb{R}^\Lambda)$ is the space of C^∞ functions with polynomial growth.
Similarly, we associate with $\Phi^{\Lambda,f}$ and the measure $d\mu_\Lambda$ the mean value $\langle \cdot \rangle_\Lambda$ and the covariance Cov $_\Lambda$.

Main Theorem

Let us consider, for any $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ and $\Lambda \subset \mathbb{Z}^d$, the phase $\Phi = \Phi^{\Lambda,\omega,\mathcal{J}} = \Phi_d^\Lambda + \mathcal{J}\Phi_i^{\Lambda,\omega}$ with ϕ satisfying (37) and (38). Then there exists a constant C and $\mathcal{J}_0 > 0$ such that, for any $\mathcal{J} \in [0, \mathcal{J}_0]$, for any ω, for any $\Lambda \subset \mathbb{Z}^d$ and any tempered functions f and g on $\mathbb{R}^\Lambda$, we have :

$$\begin{array}{l} |\ \mathrm{Cov}\ _{\Lambda,\omega}(f,g)| \leq \\ C\ \exp -\frac{1}{C} d(S_f^\Lambda, S_g^\Lambda)\ ||d_\Lambda f||_{L^2}\ ||d_\Lambda g||_{L^2}\ . \end{array} \tag{39}$$

In this case, we say shortly that we have uniform decay estimates.

Here S_f^Λ (which is called the lattice support of f in Λ) is the smallest subset of Λ such that $f(X) = \tilde{f}(X^{S_f^\Lambda})$ where $\tilde{f}$ is a function on $\mathbb{R}^{S_f^\Lambda}$.

For example the support of $X \mapsto f(X) = x_i$ is $\{i\}$.

When $f = g$, we recover Poincaré inequality.

When $f = x_i$ and $g = x_j$, we get under the same assumptions for the pair correlation

$$\text{Cor } (i,j) = \text{ Cov } (x_i, x_j) \ ,$$

There exists $\mathcal{J}_0 > 0$, C and κ, such that, for any Λ, ω and $\mathcal{J} \in [0, +\mathcal{J}_0]$, the correlation pair function satisfies

$$| \text{ Cor }_{\Lambda,\omega}(i,j)| \leq C \exp -\kappa d(i,j) \ , \ \forall i,j \in \Lambda \ . \tag{40}$$

The proof relies essentially on getting a

lower bound for the spectrum of the Witten Laplacian on 1-forms

Let us recall that the Witten Laplacian on 1-forms attached to the phase $\Phi = \Phi^{\Lambda,\omega}$ is defined as,

$$\Delta_{\Phi}^{(1)} := \left[\sum_{j\in\Lambda}\left(-\frac{\partial}{\partial x_j}+\frac{1}{2}\frac{\partial\Phi}{\partial x_j}\right)\left(\frac{\partial}{\partial x_j}+\frac{1}{2}\frac{\partial\Phi}{\partial x_j}\right)\right]\otimes I + \mathrm{Hess}\Phi\ .$$

defined on the L^2 $1-$forms with respect to the standard Lebesgue measure on $\mathbb{R}^m$, with $m = |\Lambda|$.

This Witten Laplacian $\Delta_{\Phi}^{(1)}$ is essentially selfadjoint from C_0^∞ if Φ is C^2.

The main results are perturbative in nature (small interaction).

The aim of this part is :

Theorem LB1

For any $\Lambda \subset \mathbb{Z}^d$ and $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, let $\Phi := \Phi^{\Lambda,\omega,\mathcal{J}}$ be the phase on $\mathbb{R}^\Lambda$ with ϕ satisfying (37)-(38). There exists $\mathcal{J}_0$ and $\sigma_1 > 0$, such that the lowest eigenvalue $\lambda_1^{\Lambda,\omega,\mathcal{J}}$, of the corresponding Witten Laplacian on 1-forms $\Delta_\Phi^{(1)}$, satisfies, for any cube Λ, $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ and $\mathcal{J} \in [0, \mathcal{J}_0]$,

$$\lambda_1^{\Lambda,\omega,\mathcal{J}} \geq \sigma_1 \ . \tag{41}$$

Similarly, we have also :

Theorem LB2

For any $\Lambda \subset \mathbb{Z}^d$, let $\Phi := \Phi^{\Lambda,f,\mathcal{J}}$ be the phase on $\mathbb{R}^\Lambda$ with ϕ satisfying (37)-(38). There exists $\mathcal{J}_0$ and $\sigma_1 > 0$, such that the lowest eigenvalue $\lambda_1^{\Lambda,f,\mathcal{J}}$, of the corresponding Witten Laplacian on 1-forms $\Delta_\Phi^{(1)}$, satisfies, for any cube Λ and $\mathcal{J} \in [0, \mathcal{J}_0]$,

$$\lambda_1^{\Lambda,f,\mathcal{J}} \geq \sigma_1 \ . \tag{42}$$

The starting point for the proof is the basic identity

$$\langle \Delta_{\Phi}^{(1)} u \mid u \rangle_{L^2} = \sum_{j,k} ||X_k\, u_j||^2 + \sum_{j,k} \int \frac{\partial^2 \Phi}{\partial x_j\, \partial x_k} u_j\, u_k\, dX \ , \tag{43}$$

with

$$X_j = \partial_j + \frac{1}{2} \partial_j \Phi \ , \tag{44}$$

and

$$\partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j} \ . \tag{45}$$

Denote by $w_j^{(0)}$ and $w_j^{(1)}$ the single-spin Witten Laplacians (respectively on 0- and 1- forms) attached to the variable x_j and the phase on $\mathbb{R}$

$$\phi_j(x_j) := \Phi^{\Lambda, \omega, \mathcal{J}}(X) \ . \tag{46}$$

The function ϕ_j' depends actually only on the z_ℓ with $\ell \sim j$ and recall that $z_\ell = x_\ell$ if $\ell \in \Lambda$ and $z_\ell = \omega_\ell$ if $\ell \in \mathbb{Z}^d \setminus \Lambda$. We have indeed

$$\phi_j(t) = \phi(t) + \mathcal{J} \sum_{(\{\ell\} \cup \{j\}) \cap \Lambda \neq \emptyset \ , \ \ell \sim j} |t - z_\ell|^2 + \hat{\phi}(\hat{z}_j) \ . \tag{47}$$

The last term is independent of t and irrelevant in the discussion. The operators $w_j^{(0)}$ and $w_j^{(1)}$ depend only on the z_ℓ with $\ell \sim j$.

It is quite important that the estimates are proved independently of these parameters.
We note the relations

$$w_j^{(0)} = X_j^* \, X_j \ , \tag{48}$$

and

$$w_j^{(1)} = X_j \, X_j^* = X_j^* \, X_j \ + \frac{\partial^2 \Phi}{\partial x_j^2} \ . \tag{49}$$

According to the context, we shall see these identities as identities between differential operators on $L^2(\mathbb{R}^\Lambda)$ or on $L^2(\mathbb{R}_{x_j})$ (the other variables being considered as parameters).

With these conventions, we have

$$\begin{aligned}\langle \Delta_\Phi^{(1)} u \mid v\rangle_{L^2} = &\textstyle\sum_{j,k\in\Lambda\,,\, j\neq k}\langle w_k^{(0)}\, u_j \mid v_j\rangle \\ &+ \textstyle\sum_{j\in\Lambda}\langle w_j^{(1)} u_j \mid v_j\rangle + \mathcal{J}\ \langle \mathrm{Hess}'\Phi_i\, u \mid v\rangle\ ,\end{aligned} \tag{50}$$

where $\Phi_i = \Phi_i^{\Lambda,\omega}$ denotes the interaction phase and Hess' means that we consider only the terms outside of the diagonal of the Hessian, that is such that for $k,\ell\in\Lambda$

$$\begin{aligned}(\mathrm{Hess}'\Phi_i)_{k\ell} \quad &= -1 \quad \text{if } k\sim\ell \\ &= 0 \qquad \text{else}\ .\end{aligned} \tag{51}$$

Here we observe that $\mathrm{Hess}\Phi_i$ is independent of z and that $\mathcal{J}\,\mathrm{Hess}\Phi_i$ corresponds to a perturbation in $\mathcal{O}(\mathcal{J})$, where $\mathcal{O}$ is uniform with respect to Λ, using the discrete Schur's Lemma.
Note that we get from (51) the inequality

$$\begin{aligned}&\langle \Delta_\Phi^{(1)} u \mid u\rangle_{L^2} \geq \\ &\quad \textstyle\sum_{j\in\Lambda}\langle w_j^{(1)} u_j \mid u_j\rangle + \mathcal{J}\ \langle \mathrm{Hess}'\Phi_i\, u \mid u\rangle\ .\end{aligned} \tag{52}$$

We get consequently the following theorem :
Theorem
Let us assume that there exists $\rho_1 > 0$ such that, for any $j \in \mathbb{Z}^d$ and any $z \in \mathbb{R}^{\mathbb{Z}^d \setminus \{j\}}$, the operator[3] $w_j^{(1)}$ satisfies

$$w_j^{(1)} \geq \rho_1 \;, \tag{53}$$

then, for any $\epsilon > 0$, there exists $\mathcal{J}_0 > 0$ such that the Witten Laplacian W_1^Φ, with $\Phi := \Phi^{\Lambda,\omega,\mathcal{J}}$ or with $\Phi := \Phi^{\Lambda,f,\mathcal{J}}$, satisfies for any $\Lambda \subset \mathbb{Z}^d$, $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ and $\mathcal{J} \in [0, +\mathcal{J}_0]$,

$$\Delta_\Phi^{(1)} \geq (\rho_1 - \epsilon) \;. \tag{54}$$

We recall that the positivity of $w_j^{(1)}$ is immediate from the definition. We also observe that it is sufficient to treat the case of a fixed j_0 all the families being unitary equivalent (after a simple change of the names of the parameters).
The other important point is to verify the condition of uniformity. This will be done later.

[3] For a given $j \in \mathbb{Z}^d$, the effective parameters are actually the z_ℓ such that $\ell \sim j$.

Uniform estimates for a family of 1-dimensional Witten Laplacians

We shall discuss various conditions under which these uniform estimates can be obtained. ϕ is at least C^2.

If ψ is a C^2 phase, we shall denote by $w_\psi^{(0)}$ and $w_\psi^{(1)}$ the corresponding Witten Laplacians defined in this simple case by

$$w_\psi^{(0)} = -\frac{d^2}{dt^2} + \frac{1}{4}\psi'(t)^2 - \frac{1}{2}\psi''(t)\ , \tag{55}$$

and

$$w_\psi^{(1)} = -\frac{d^2}{dt^2} + \frac{1}{4}\psi'(t)^2 + \frac{1}{2}\psi''(t)\ . \tag{56}$$

These two operators (being positive) are automatically selfadjoint on $L^2(\mathbb{R})$ starting from C_0^∞.

The first condition is (for reference to the strictly convex situation which was analyzed through the Bakry-Emery argument) the existence of $C > 0$ such that

$$(sc) \quad \phi''(t) \geq \frac{1}{C}\,,\ \forall t \in \mathbb{R}\,. \tag{57}$$

A weaker condition is

$$(sc(\infty)) \quad \phi''(t) \geq \frac{1}{C}\,,\ \forall t \in \mathbb{R}\,,\ |t| \geq C\,. \tag{58}$$

A still weaker assumption is that there exists a bounded function χ in C^2 such that

$$(scm) \quad \phi''(t) + \chi''(t) \geq \frac{1}{C}\,,\ \forall t \in \mathbb{R}\,. \tag{59}$$

These three conditions are ordered :

$$(sc) \Rightarrow (sc)(\infty) \Rightarrow (scm)\,.$$

Another family of conditions corresponds to assumptions on the operator

$$w^{red} := -\frac{d^2}{dx^2} + \frac{1}{2}\phi'' \,, \tag{60}$$

or more precisely to the quadratic form associated to w^{red} :

$$u \mapsto q^{red}(u) = \langle w^{red} u \mid u \rangle \,. \tag{61}$$

We consider a new family of conditions starting with :

$$(qsc) \quad q^{red}(u) \geq \frac{1}{C} \, ||u||^2 \,,\ \forall u \in C_0^\infty(\mathbb{R}) \,. \tag{62}$$

A weaker condition is

$$(qsc(\infty)) \quad q^{red}(u) \geq \frac{1}{2C} ||u||^2 \,,\ \forall u \in C_0^\infty(\mathbb{R} \setminus]-C, +C[) \,. \tag{63}$$

Finally a still weaker assumption is the existence of a bounded function χ in C^2 and of C s.t. $\forall u \in C_0^\infty(\mathbb{R})$

$$(qscm) \quad q^{red}(u) + \frac{1}{2} \int_{\mathbb{R}} \chi''(t) \, |u(t)|^2 \, dt \geq \frac{1}{2C} \, ||u||^2 \,. \tag{64}$$

These three new conditions are again ordered :

$$(qsc) \;\Rightarrow\; (qsc(\infty)) \;\Rightarrow\; (qscm)$$

and it is also clear that (qsc) is weaker than (sc). The condition (qsc) permits to treat roughly speaking functions which are strictly convex in mean value. Explicit criteria exist for verifying (qsc) and are related to sharp versions of the Garding inequality.

We have shown that the proof of a uniform lower bound for the Witten Laplacian $\Delta_\Phi^{(1)}$ can be deduced from the study of one-dimensional Witten Laplacians. We want to analyze

$$w^{(1)}_{\phi_{\mathcal{J},\alpha}} := -\frac{d^2}{dt^2} + \frac{1}{4}(\phi'_{\mathcal{J},\alpha}(t))^2 + \frac{1}{2}(\phi''_{\mathcal{J},\alpha}(t)) \;, \quad (65)$$

with

$$\alpha = -2\mathcal{J} \sum_{0 \sim k} z_k \;. \quad (66)$$

The first theorem is the following :

Theorem ulb1

Let ϕ be a phase satisfying (62). Then, there exists $\mathcal{J}_0$, and $\rho_1 > 0$ such that, for any $(\alpha, \mathcal{J}) \in \mathbb{R} \times [0, +\mathcal{J}_0]$, we have

$$w^{(1)}_{\phi_{\mathcal{J},\alpha}} \geq \rho_1 \; . \tag{67}$$

Proof : Just observe that

$$w^{(1)}_{\phi_{\mathcal{J},\alpha}} \geq w^{red} + \mathcal{J} d \; . \tag{68}$$

Note that in the strictly convex case (sc), we simply write

$$w^{(1)}_{\phi_{\mathcal{J},\alpha}} \geq \phi'' + 2\mathcal{J} d \; . \tag{69}$$

The second theorem is :

Theorem ulb2

Let ϕ be a phase satisfying (64). Then, there exists $\mathcal{J}_0$, and $\rho_1 > 0$ such that, for any $(\alpha, \mathcal{J}) \in \mathbb{R} \times [0, +\mathcal{J}_0]$, we have

$$w^{(1)}_{\phi_{\mathcal{J},\alpha}} \geq \rho_1 \; . \tag{70}$$

The proof is based on the following lemma :

Lemma

If $\lambda_1(\psi)$ is the bottom of the spectrum of the Witten Laplacian $w_\psi^{(1)}$ attached to ψ, then, for χ and ϕ C^2 such that χ is bounded, we have

$$\lambda_1(\phi) \geq (\exp -2||\chi||_{L^\infty}) \;\cdot\; \lambda_1(\phi+\chi)\,. \tag{71}$$

The proof consists in observing :

$$\begin{aligned}\langle w_\phi^{(1)} u | u\rangle \quad &= \langle \exp -\tfrac{\chi}{2}\left(-\tfrac{d}{dt}+\tfrac{1}{2}(\phi'+\chi')\right)\exp\tfrac{\chi}{2}\,u\,|\\ &\qquad \exp -\tfrac{\chi}{2}\left(-\tfrac{d}{dt}+\tfrac{1}{2}(\phi'+\chi')\right)\exp\tfrac{\chi}{2}\,u\rangle\,.\end{aligned} \tag{72}$$

We then deduce, that, for any $u\in C_0^\infty(\mathbb{R})$,

$$\begin{aligned}\langle w_\varphi^{(1)} u|u\rangle \quad &\geq \exp -||\chi||_{L^\infty}\,\langle w^{(1)}_{\varphi+\chi}\exp\tfrac{\chi}{2}u|\exp\tfrac{\chi}{2}u\rangle\\ &\geq \exp -||\chi||_{L^\infty}\,\lambda_1(\varphi+\chi)\;||\exp\tfrac{\chi}{2}u||^2\,,\end{aligned} \tag{73}$$

and consequently

$$\langle w_\varphi^{(1)} u|u\rangle \geq \exp -2||\chi||_{L^\infty}\,\lambda_1(\varphi+\chi)||u||^2\,. \tag{74}$$

The minimax principle and (74) give (71). This ends the proof of Lemma .

A second proof consists in working with the second eigenvalue $\lambda_2^{(0)}$ of $w_\psi^{(0)}$.

For the proof of Theorem ulb2, we can then use the lemma and apply theorem ulb1, with $\phi_{\alpha,\mathcal{J}}$ replaced by $\phi_{\alpha,\mathcal{J}} + \chi$.

A proof of the uniform decay of correlations

We now present the proof of the uniform decay of correlations and show that one has just to use the uniform Poincaré inequality for the Witten Laplacian associated to the single-spin phase $w^{(0)}_{\phi_{\mathcal{J},\alpha}}$ (in other words the existence of a uniform lower bound for the second eigenvalue $\lambda_2^{(0)}(\alpha,\mathcal{J})$ of this Laplacian) instead of the uniform control of the lowest eigenvalue $\lambda_1^{(1)}(\alpha,\mathcal{J})$ of the single-spin Laplacian $w^{(1)}_{\phi_{\mathcal{J},\alpha}}$ on all 1-forms.

This makes no difference in our case because our single particle phase are defined on $\mathbb{R}$ but this remark is useful for extensions to the case when ϕ is defined on $\mathbb{R}^k$.

We would like to control the quantity Cov (f, g).

The initial remark was that we can solve

$$f - \langle f \rangle = A_\Phi^{(0)} u \ , \tag{75}$$

from which we get the identity

$$df = d\, A_\Phi^{(0)}\, u = A_\Phi^{(1)}\, du \ . \tag{76}$$

We have in mind to take $f = x_i$, $g = x_j$ with i and j in Λ. The idea consists in the introduction of weighted spaces on Λ, associated with strictly positive weights satisfying

$$\exp -\kappa \leq \rho(\ell)/\rho(k) \leq \exp \kappa \ , \tag{77}$$

where $\ell \sim k$ (this means that ℓ and k are nearest neighbors in $\mathbb{Z}^d$) and κ will be determined later.

For a given $i \in \Lambda$, the function $\rho(\ell) = \exp -\kappa d(i,\ell)$ where d is a usual distance on $\mathbb{R}^d$ satisfies this condition. We will rather take later $\rho(\ell) = \exp -\kappa d(S_f^\Lambda, \ell)$

For a given $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| = m$, let us now associate with a given weight ρ on Λ the $m \times m$ diagonal matrix $M = M^\Lambda$ defined by

$$M_{k\ell} = \delta_{k\ell}\, \rho(\ell)\ , \quad \text{for } \ell,\, k \in \Lambda\ . \tag{78}$$

We consequently can write

$$\begin{aligned} \text{Cov}_{\,\Lambda,\omega}(f,g) &= \langle du \cdot dg \rangle \\ &= \langle (M^{-1} du) \cdot (M dg) \rangle\ . \end{aligned} \tag{79}$$

We have to control

$$\sigma := M^{-1} du\ . \tag{80}$$

In order to do that, we rewrite (76) in the form

$$\begin{aligned} M^{-1}df &= M^{-1}A_1 M M^{-1} du \\ &= A_1\sigma + (M^{-1}A_1 M - A_1)\sigma \\ &= A_1\sigma + (M^{-1}\mathrm{Hess}\Phi M - \mathrm{Hess}\Phi)\sigma \ . \end{aligned} \tag{81}$$

We now take the scalar product in $\Omega_\Phi^{1,2}$ with σ in the identity (81) and get

$$\langle (M^{-1}df)\cdot\sigma\rangle \geq \langle (A_1\sigma)\cdot\sigma\rangle - C\mathcal{J}||\sigma||^2_{\Omega_\Phi^{1,2}} \ . \tag{82}$$

Here we have used for getting the last term the pointwise estimate of $||M^{-1}\mathrm{Hess}\Phi_i M||$ in $\mathcal{L}(\ell^2(\Lambda\,;\,\mathbb{R}))$.

We observe indeed that, for all $X \in \mathbb{R}^\Lambda$,

$$\begin{aligned} &||\mathrm{Hess}\Phi(X) - M^{-1}\mathrm{Hess}\Phi(X)M||_{\mathcal{L}(\ell^2)} = \\ &|\mathcal{J}| \ ||\mathrm{Hess}\Phi_i(X) - M^{-1}\mathrm{Hess}\Phi_i(X)M||_{\mathcal{L}(\ell^2)} \ . \end{aligned} \tag{83}$$

Observing that the coefficients of

$$\delta_M(\mathrm{Hess}\Phi_i) =: \mathrm{Hess}\Phi_i - M^{-1}\mathrm{Hess}\Phi_i M$$

vanish if $k \not\sim \ell$, it is immediate, using Schur's Lemma, to get

$$\begin{array}{l} ||\delta_M(\mathrm{Hess}\Phi_i)||_{\mathcal{L}(\ell^2)} \\ \quad \leq 2d\, \sup_{\ell\sim k} |(1-\frac{\rho(\ell)}{\rho(k)})| \\ \quad \leq 2d\, \max\left((1-\exp -\kappa), (\exp\kappa - 1)\right) \\ \quad = 2d\,\theta\ , \end{array} \tag{84}$$

with $\theta := (\exp\kappa - 1)$, and this is uniform with respect to the lattice.

Then we rewrite (82) in the form

$$\langle (M^{-1}df)\cdot\sigma\rangle \geq \sum_j \langle (a^{(1)}_{\phi_j}\sigma_j)\cdot\sigma_j\rangle - C\mathcal{J}||\sigma||^2_{\Omega^{1,2}_\Phi}\ , \tag{85}$$

where $a^{(1)}_{\phi_j}$ is unitary equivalent to $w^{(1)}_j$ (through the map $\omega \mapsto \exp -\frac{\Phi}{2}\,\omega$) and C is uniform with respect to all the parameters.

The proof is then like for the uniform Poincaré inequality.
On $L^2_{x_j}(\mathbb{R})$, we first get

$$\lambda_1^{(1);j,z_k}\,||\sigma_j||^2_{\Omega^{1,2}_{\phi_j}} \leq \langle a_j^{(1)}\sigma_j\ ,\sigma_j\rangle^2_{L^2_{\phi_j}}\ . \tag{86}$$

We then multiply by $\exp -(\Phi - \phi_j)$ this inequality, integrate over $(\mathbb{R})^{\Lambda\setminus\{j\}}$ with respect to the other variables $\hat{x}_j$, and obtain

$$\begin{array}{l}\langle (M^{-1}df)\cdot\sigma\rangle \geq \\ \sum_j \left(\inf_{z_k\,;\,k\neq j}\lambda_1^{(1);j,z_k}\right)\ ||\sigma_j||^2_{\Omega^{1,2}_\Phi} - C\mathcal{J}||\sigma||^2_{\Omega^{1,2}_\Phi}\ .\end{array} \tag{87}$$

For a suitable constant C and for $\mathcal{J}\geq 0$ small enough, we get, after use of Cauchy-Schwarz in (87), the inequality

$$||(M^{-1}df)||_{\Omega^{1,2}_\Phi}\cdot||\sigma||_{\Omega^{1,2}_\Phi} \geq \frac{1}{C}||\sigma||^2_{\Omega^{1,2}_\Phi}\ , \tag{88}$$

and finally

$$||M^{-1}du||_{\Omega^{1,2}_{\Phi}} \leq C||M^{-1}df||_{\Omega^{1,2}_{\Phi}} . \qquad (89)$$

The end of the proof is the consequence of

$$|\text{ Cov }_{\Lambda,\omega}(f,g)| \leq ||M^{-1}du||_{\Omega^{1,2}_{\Phi}} \cdot ||Mdg||_{\Omega^{1,2}_{\Phi}} , \qquad (90)$$

and (89).
We choose now the weight

$$\rho = \exp \kappa d(S_f^{\Lambda}, \cdot) ,$$

for implementation in the matrix M. This ends the proof of the theorem.

Remark 1

As observed by J. Moeller, the proof gives that the results is true with $\kappa \sim \eta(-\log J)$ with $\eta > 0$ small enough.
This leads to the improve statement :

Improved Main Theorem

Let us consider, for any $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ and $\Lambda \subset \mathbb{Z}^d$, the phase $\Phi = \Phi^{\Lambda,\omega,\mathcal{J}} = \Phi_d^\Lambda + \mathcal{J}\Phi_i^{\Lambda,\omega}$ with ϕ satisfying (37) and (38). Then there exists a constant C and $\mathcal{J}_0 > 0$ such that, for any $\mathcal{J} \in [0, \mathcal{J}_0]$, for any ω, for any $\Lambda \subset \mathbb{Z}^d$ and any tempered functions f and g on $\mathbb{R}^\Lambda$, we have :

$$
\begin{array}{l}
| \text{ Cov }_{\Lambda,\omega}(f,g)| \leq \\
C \;\exp -\frac{1}{C}|\log J| d(S_f^\Lambda, S_g^\Lambda)\; ||d_\Lambda f||_{L^2}\; ||d_\Lambda g||_{L^2}\; .
\end{array}
$$

Remark 2

The quadratic interaction was just proposed for giving an example. The proof is actually much more general.

Extension : Weighted decay estimates.

One can improve (see Helffer-Bodineau, Bach-Moeller..) the decay estimates by taking account of the possible growth of ϕ'' at ∞. This could be quite useful for considering more general interactions. For example one has:

New improvment of Main Theorem

Under the assumptions of the previous Corollary, there exists $C > 0$, $\mathcal{J}_0 > 0$ such that, for any $\Lambda \subset \mathbb{Z}^d$, any $\mathcal{J} \in [0, \mathcal{J}_0]$, any $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, and any tempered functions f and g on $\mathbb{R}^\Lambda$,

$$\begin{aligned} &|\ \mathrm{Cov}\ _{\Lambda,\omega}(f,g)| \leq \\ &\quad C\ \left(\exp -\frac{1}{C} d(S_f^\Lambda, S_g^\Lambda)\right)\ ||\Theta \cdot d_\Lambda f||_{L^2}\ ||\Theta \cdot d_\Lambda g||_{L^2}\ , \end{aligned}$$

with

$$(\Theta(X))_{jk} = (\phi''(x_j) + C)^{-\frac{1}{2}} \delta_{jk}\ .$$

This should permit to control more general interactions.

Notes

- Most of the material presented is taken from the series of papers by Helffer, which were some elaboration of Helffer-Sjöstrand to a non-convex situation. In the same spirit, let us also mention the papers by Bach-Jecko-Sjöstrand, Bach-Moeller, Matte... .

- The analysis of the decay of correlations has a long story and we mention in particular Dobrushin, Gross, Foellmer, Kuensch, Sokal and Deuschel. May be one originality is the Hilbertian character of the right hand side.

- Theorem on Uniform decay is proved in Helffer. Without the uniform control with respect to ω, it was also obtained by Bach-Jecko-Sjöstrand. Further much finer results on the correlations are obtained by J. Sjöstrand in the semi-classical context.

- The strict positivity of the Witten Laplacian on 1-forms is a consequence of general arguments (see J. Sjöstrand and J. Johnsen). But this proof does not give automatically the uniformity of the lower bound with respect to parameters.

- The uniform decay when $d = 1$ has been proved by B. Zegarlinski.

- We do not consider in these lectures the problem of analyzing the existence and the uniqueness of a limit measure as $\Lambda \to \mathbb{Z}^d$. This leads in particular to the notion of Gibbs measure.

  See Dobrushin, Bellissard-Hoeghkrohn and references therein, Antoniouk, Deuschel, Albeverio, Kondratev, Roeckner and coauthors, Matte (and references therein) for a discussion.

Uniform log-Sobolev inequalities

We recall that our aim is to analyze the thermodynamic properties of measures $\exp -\Phi^{\Lambda,\omega}(X)\, dX$ in the case when $\Phi^{\Lambda,\omega}$, which is associated with subsets $\Lambda \subset \mathbb{Z}^d$ and some $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ defining the boundary condition.

Our main assumption is an assumption of convexity at ∞ of the single spin phase ϕ. We assume that there exists a bounded C^∞ function s such that $\tilde{\phi} := \phi + s$ is strictly convex, that is that there exists $\rho > 0$ such that

$$(\phi + s)''(t) \geq \rho > 0 \;,\; \forall t \in \mathbb{R} \;.$$

Our main problem will be to analyze the properties of the renormalized measure

$$dE^{\Lambda,\omega} := Z_{\Lambda,\omega}^{-1} \cdot \exp -\Phi^{\Lambda,\omega}(X)\, dX \;, \tag{91}$$

with

$$Z_{\Lambda,\omega} := \int_{\mathbb{R}^\Lambda} \exp -\Phi^{\Lambda,\omega}(X)\, dX \;. \tag{92}$$

More specifically we would like to analyze under which conditions we can prove the existence of an uniform log-Sobolev inequality attached to this family of probability measures. The following theorem is mainly due to Zegarlinski with additional arguments of N. Yoshida and Bodineau-Helffer.

Theorem (Uniform LogSobolev)
Let us assume that () is satisfied. Then there exists J_0 and a constant c in $]0,+\infty[$ such that for any $\Lambda \subset \mathbb{Z}^d$, any $J \in [0, J_0]$ and any $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, we have

$$\langle f \ln f \rangle_{\Lambda,\omega} \leq 2c\, \langle |\nabla f^{\frac{1}{2}}|^2 \rangle_{\Lambda,\omega} + \langle f \rangle_{\Lambda,\omega}\; \ln \langle f \rangle_{\Lambda,\omega} \;,$$

for all nonnegative functions f for which the right-hand side is finite.

The proof depends heavily,
– according to a criterion of Strook-Zegarlinski (see also Dobrushin-Shlosman)–
on

- the uniform decay estimates and
- some “p-spin”(p bounded) uniform LogSobolev inequality,

and an argument due to Lu-Yau.

Zegarlinski and Yosida were using Superconvexity at ∞.

Our proof gives the same result under Uniform Convexity at ∞.
See also a nice paper by Ledoux.

There was some hope to use more directly the Witten Laplacian method for a Bakry-Emery like proof (which work under a strictly convex assumption of the phase). But to my knowledge, this has not been a success.

These results can be completed :

Theorem (Yoshida-Bodineau-Helffer).
Under the assumption that the single-spin phase is superconvex at ∞, then the following conditions are equivalent :

1. Correlations decay exponentially fast uniformly with respect to Λ, ω in the sense of (39).

2. The Poincaré inequalities hold uniformly with respect to Λ, ω.

3. The log-Sobolev inequalities hold uniformly with respect to Λ, ω.

Remark

Note that this last theorem does not say if the log-Sobolev inequality holds uniformly or not. This is actually true when $d = 1$ and for $\mathcal{J}$ small enough for a general d. It just establishes that this property is equivalent to simpler properties.

Decay of correlation in the semi-classical limit

Here we mention : Helffer-Sjöstrand, Sjöstrand, Bach-Jecko-Sjöstrand, Sjöstrand, Bach-Matte-Moeller ....

The first results have been obtained under assumptions of convexity. More generally, one hope to treat the case where the one-particule phase create a unique well (say at 0) and we have a, say nearest neighbour interaction, controlled by the main potential.

Although it is natural to think that everything should be localized, in the semi-classical limit, near the bottom of the potential, it was quite difficult to prove it technically : the localization should be well controlled with respect to the dimension !

On the other hand, one can relate the decay of the correlations more directly to the germ of Φ at 0, and more precisely to $\mathrm{Hess}\Phi(0)$ which will be assumed to be definite positive.

Actually, it will be useful (see Bach-Jecko-Sjöstrand, Bach-Möller) to use the approximation

$$\Delta^{(1)}_{\Phi;h} \sim \Delta^{(0)}_{\Phi;h} \otimes I + h\mathrm{Hess}\Phi(0) ,$$

instead of the Harmonic approximation.

Feschbach-Grushin Method

Under different names this method of reduction appears to be powerful in many contexts. So it is may be useful to explain it here. As we have seen, the important point is to have a good information on the W.L on one-forms.

One introduces the auxiliary operator

$$R_+ \ : \ \Omega^{1,2}(\mathbb{R}^\Lambda) \Longrightarrow \ell^2(\Lambda)$$

associating to a 1-form $u = \sum_j u_j dx_j$ the sequence $\langle u_j, \exp -\Phi/2h\rangle$.

We consider instead of $\Delta^{(1)}_{\Phi;h}$ the z-dependent family

$$\mathcal{P}(z) = \begin{pmatrix} h^{-1}\Delta^{(1)}_{\Phi;h} - z & R_+^* \\ R_+ & 0 \end{pmatrix}$$

for $z \in \mathbb{C}$.

For z in the neighborhood of 0, this operator can be shown to be uniformly invertible, and the inverse takes the form

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} .$$

Moreover, one has a good semi-classical approximation by

$$E_+(z) = R_- + \mathcal{O}(h^{\frac{1}{2}}) \;,\; E_- = R_+ + \mathcal{O}(h^{\frac{1}{2}})$$
$$E_{-+} = z - \mathrm{Hess}\Phi(0) + \mathcal{O}(h^{\frac{1}{2}}) \;.$$

At least formally, it is standard that $z \in$ Spect $h^{-1}\Delta^{(1)}_{\Phi;h}$ iff $0 \in$ Spect $E_{-+}(z)$. But this correspondance is more explicit, because we get the following expression for the inverse

$$h(\Delta^{(1)}_{\Phi;h})^{(-1)} = E(0) - E_+(0)(E_{-+})^{-1}(0)E_-(0) \;.$$

According to the covariance formula, applied with $f = x_i$ and $g = x_j$, this leads immediately to an expression for the correlation Cor $(i, j) =$ Cov (x_i, x_j).

This is the second term on the right hand side which gives the main term and relate it to the decay properties of $\mathrm{Hess}\Phi(0)^{-1}$ far from the diagonal.

Fokker-Planck operators and Witten Laplacians

The analysis of spectral problems for the Fokker-Planck Operators initiated by Hérau-Nier leads to new questions on the Witten Laplacians.

Questions about the return to equilibrium are indeed strongly related to estimates on the gap between the two first eigenvalues (or on the lowest positive eigenvalue) of a Witten Laplacian. (See the book by Helffer-Nier)

The Fokker-Planck operator (or Kramer's operator) is defined as an unbounded, maximal accretive operator on $L^2(\mathbb{R}^{2n}_{x,y}$:

$$K := -\partial_y^2 + y^2 - n + \lambda\left(\nabla\Phi(x)\partial_y - y\partial_x\right) \ .$$

Note that $\exp -(\Phi(x) + \frac{y^2}{2})$ is in the kernel of K and K^*.

The strong links between these two operators is clear if you think of the restriction of K to the space $\exp -(\frac{y^2}{2}) \otimes L^2(\mathbb{R}_x)$.

Note that

$$K(f(x)\exp -y^2/2) = -\lambda\, y{\cdot}(\nabla\Phi f - \nabla f)\ \exp -y^2/2\ .$$

and consider then $||Kf||^2$.

K can be also extended on p-forms and there exists a supersymmetric structure (J. Kurchan and coauthors).

Exponentially small eigenvalues in the semi-classical context

We now forget "large dimension problems" and are interested in the exponentially small eigenvalues of the Dirichlet realization in an open set Ω or on a manifold of the semiclassical Witten Laplacian on 0-forms

$$\Delta_{f,h}^{(0)} = -h^2\Delta + |\nabla f(x)|^2 - h\Delta f(x)\ .$$

SORRY THERE IS A SMALL CHANGE IN THE CONVENTIONS

f was $\frac{f}{2}$ in the previous slides.

Our aim is to describe the recent results which have been recently obtained in three cases :

- The case of $\mathbb{R}^n$ (Bovier-Gayrard-Klein,Helffer-Klein-Nier),
- The case of a compact riemannian manifold (HelfferKleinNier),
- The case of a bounded set Ω with regular boundary (HelfferNier) (in this case, we consider the Dirichlet realization of this operator).

Main assumptions

Assumption 1
The function f is assumed to be a C^∞- function on $\overline{\Omega}$ and a Morse function on Ω.

In the case when $\Omega = \mathbb{R}^n$,
Assumption 2

$$\liminf_{|x|\to+\infty} |\nabla f(x)|^2 > 0 \ ,$$

and

$$|D_x^\alpha f| \leq C_\alpha(|\nabla f|^2 + 1) \ ,$$

for $|\alpha| = 2$.

In the case with boundary,
Assumption 3
The function f has no critical points at the boundary and the function $f_{/\partial\Omega}$ is a Morse function on $\partial\Omega$.

Initiated by E. Witten for compact manifolds

It is known (see Simon, Witten, Helffer-Sjostrand and more recently Chang-Liu, Helffer-Nier2) that the Witten Laplacians on functions $\Delta_{f,h}^{(0)}$ admits exactly m_0 eigenvalues in some interval $[0, Ch^{\frac{6}{5}}]$ for $h > 0$ small enough, where m_0 is the number of local minima in Ω .

This is easy to guess :
Consider, near each of the local minima $U_j^{(0)}$, the function $\chi_j(x) \exp -\frac{f}{h}$, where χ_j is a suitable cut-off function localizing near $U_j^{(0)}$ as suitable quasimode.

This shows that these small eigenvalues are actually exponentially small as $h \to 0$.

Note that we consider the Dirichlet problem.

So

Assumption 3
The function f has no critical points at the boundary and the function $f_{/\partial\Omega}$ is a Morse function on $\partial\Omega$.

implies that the eigenfunctions corresponding to low lying eigenvalues are localized far from the boundary.

Morse Inequalities

In the compact case, this was the main point of the semi-classical proof suggested by Witten of the Morse inequalities.

Each of the W-Laplacians is essentially selfadjoint and an analysis based on the harmonic approximation shows that the dimension of the eigenspace corresponding to $]0, h^{\frac{6}{5}}]$ is, for h small enough, equal to m_p the number of critical points of index p (the index at a critical point U being defined as the number of negative eigenvalues of the Hessian of f at U.

Note that the dimension of the kernel of $\Delta_{h,f}^{(p)}$ being equal to the Betti number b_p, this gives immediately the so called “weak Morse Inequalities” :

$$b_p \leq m_p \ , \quad \text{for all } p \in \{0, \dots, n\} \ .$$

Questions in the case with boundary

In the case with boundary, two natural questions appear :

What is the interesting selfadjoint realization to consider (in order for example to show a Morse inequality) ?

How do we define the notion of critical point and of index for a point at the boundary?

We mainly concentrate here on the analysis of the Witten Laplacians on 0-forms and 1-forms.

Our aim is to get the optimal accuracy asymptotic formulas for the m_0 first eigenvalues of the Dirichlet realization of $\Delta^{(0)}_{f,h}$.

Previously via a probabilistic approach : Freidlin-Wentcel, Holley-Kusuoka-Strook, Miclo, Kolokoltsov, Bovier-Eckhoff-Gayrard-Klein and Bovier-Gayrard-Klein, but the proof of optimal accuracy (except may be for the case of dimension 1) was open.

The Witten Laplacian is, in the case of an open set Ω, associated to the Dirichlet form

$$C_0^\infty(\Omega) \ni u \mapsto \int_\Omega |(h\nabla + \nabla f)u(x)|^2 \; dx \;.$$

As already mentioned the probabilists look equivalently at :

$$C_0^\infty(\Omega) \ni v \mapsto h^2 \int_\Omega |\nabla v(x)|^2 \; e^{-2f(x)/h} \, dx \;.$$

The case of $\mathbb{R}^n$

In the case of $\mathbb{R}^n$ and under assumptions 1 and 2 (together with a generic assumption), one gets :

Theorem

The first eigenvalues $\lambda_k(h)$, $k \in \{2, \ldots, m_0\}$, of $\Delta^{(0)}_{f,h}$ have the following expansions :

$$\lambda_k(h) = \frac{h}{\pi}|\widehat{\lambda}_1(U^{(1)}_{j(k)})| \sqrt{\frac{\left|\det(\mathrm{Hess} f(U^{(0)}_k))\right|}{\left|\det(\mathrm{Hess} f(U^{(1)}_{j(k)}))\right|}} \times \exp -\frac{2}{h}\left(f(U^{(1)}_{j(k)}) - f(U^{(0)}_k)\right) \times (1 + r_1(h)) \,,$$

with $r_1(h) = o(1)$.

Here the $U^{(0)}_k$ denote the local minima of f ordered in some specific way, the $U^{(1)}_{j(k)}$ are "saddle points" (critical points of index 1) attached in a specific way to the $U^{(0)}_k$ and $\widehat{\lambda}_1(U^{(1)}_{j(k)})$ is the negative eigenvalue of $\mathrm{Hess} f(U^{(1)}_{j(k)})$.

Actually, the estimate

$$r_1(h) = \mathcal{O}(h^{\frac{1}{2}}|\log h|) \ ,$$

is obtained in Bovier-Gayrard-Klein (under weaker assumptions on f) and a complete asymptotics

$$r_1(h) \sim \sum_{j\geq 1} r_{1j} h^j \ ,$$

is obtained in Helffer-Klein-Nier.
Here we have left out the case $k=1$, which leads to a specific assumption (see Assumption 2) in the case of $\mathbb{R}^n$ for f at ∞. This implies that $\Delta^{(0)}_{f,h}$ is essentially selfadjoint and that the bottom of the essential spectrum is bounded below by some $\epsilon_0 > 0$ (independently of $h \in]0, h_0]$, h_0 small enough). If the function $\exp -\frac{f}{h}$ is in L^2, then

$$\lambda_1(h) = 0 \ .$$

But other examples like $f(x) = -(x^2 - 1)^2$ (with $n = 1$) are interesting and an asymptotic of $\lambda_1(h)$ can be given for this example.

The approach given in HKN intensively uses, together with the techniques of HelSj4, that

- the Witten Laplacian is associated to a cohomology complex
- $\exp -\frac{f(x)}{h}$ is in the kernel of the Witten Laplacian on $0-$forms

This permits to construct – and this is specific of the case of $\Delta^{(0)}_{f,h}$– very efficiently quasimodes.

Witten complex, Reduced Witten complex

It is more convenient to consider the singular values of the restricted differential $d_{f,h} : F^{(0)} \to F^{(1)}$. The space $F^{(\ell)}$ is the m_ℓ-dimensional spectral subspace of $\Delta^{(\ell)}_{f,h}$, $\ell \in \{0,1\}$,

$$F^{(\ell)} = \operatorname{Ran} 1_{I(h)}(\Delta^{(\ell)}_{f,h}) \ ,$$

with $I(h) = [0, h^{\frac{6}{5}}]$ and the property

$$1_{I(h)}(\Delta^{(1)}_{f,h}) d_{f,h} = d_{f,h} 1_{I(h)}(\Delta^{(0)}_{f,h}) \ .$$

We will analyze :

$$\beta^{(\ell)}_{f,h} := (d^{(\ell)}_{f,h})_{/F^{(\ell)}} \ .$$

We will mainly concentrate on the case $\ell = 0$.

Singular values

In order to exploit all the information which can be extracted from well chosen quasimodes, working with singular values of $\beta^{(0)}_{f,h}$ happens to be more efficient than considering their squares, the eigenvalues of $\Delta^{(0)}_{f,h}$. The main point is probably that the errors appear "multiplicatively" when computing the matrix of $\beta^{(0)}_{f,h}$ in approximate well localized "almost" orthogonal basis of $F^{(0)}$ and $F^{(1)}$.
By this we mean :

$$\lambda = \lambda^{app}(1 + error) \;,$$

instead of additively

$$\lambda = \lambda^{app} + error \;,$$

as for example in HeSj4. Here λ_{app} will be explicitly obtained from the WKB analysis.

The main result in the case with boundary

In the case with boundary, the function $\exp -\frac{f}{h}$ does not satisfy the Dirichlet condition, so the smallest eigenvalue can not be 0.

For this case, a starting reference Freidlin-Wentzel, which says (in particular) that, if f has no critical points except a unique non degenerate local minimum x_{min}, then the lowest eigenvalue $\lambda_1(h)$ of the Dirichlet realization $\Delta_{f,h}^{(0)}$ in Ω satisfies :

$$\lim_{h \to 0} -h \log \lambda_1(h) = 2 \inf_{x \in \partial\Omega} (f(x) - f(x_{min})) \; .$$

Other results are given in the case of many local minima but they are again limited to the determination of logarithmic equivalents.

We can show (Helffer-Nier) that, under a suitable generic assumption "Assgeneric", one can

- label the m_0 local minima
- introduce an injective map j from the set of the local minima into the set of the m_1 (generalized) saddle points of the Morse functions in $\overline{\Omega}$ of index 1.

We recall that ∇f does not vanish at the boundary. Our problem leads us to define a point of index 1 at the boundary as a point U which is a local minimum of $f_{/\partial\Omega}$ and for which the external normal derivative of f is strictly positive.

At a generalized critical point U with index 1, we can associate the Hessians $\mathrm{Hess} f(U)$, if $U \in \Omega$, or $(\mathrm{Hess} f\big|_{\partial\Omega})(U)$, if $U \in \partial\Omega$. When $U \in \Omega$, $\widehat{\lambda}_1(U)$ denotes the negative eigenvalue of $\mathrm{Hess} f(U)$.

Theorem

Under Assumptions 1, 3 and "Assgeneric", there exists h_0 such that, for $h \in (0, h_0]$, the spectrum in $[0, h^{\frac{3}{2}})$ of the Dirichlet realization of $\Delta^{(0)}_{f,h}$ in Ω, consists of m_0 eigenvalues $\lambda_1(h) < \ldots < \lambda_{m_0}(h)$ of multiplicity 1, which are exponentially small and admit the following asymptotic expansions :

$$\lambda_k(h) = \tfrac{h}{\pi}|\widehat{\lambda}_1(U^{(1)}_{j(k)})| \sqrt{\frac{\left|\det(\mathrm{Hess} f(U^{(0)}_k))\right|}{\left|\det(\mathrm{Hess} f(U^{(1)}_{j(k)}))\right|}} \left(1 + hc^1_k(h)\right) \times$$
$$\times \exp -\tfrac{2}{h}\left(f(U^{(1)}_{j(k)}) - f(U^{(0)}_k)\right) \;, \quad \text{if } U^{(1)}_{j(k)} \in \Omega \;,$$

and

$$\lambda_k(h) = \frac{2h^{1/2}|\nabla f(U^{(1)}_{j(k)})|}{\pi^{1/2}} \sqrt{\frac{\left|\det(\mathrm{Hess} f(U^{(0)}_k))\right|}{\left|\det(\mathrm{Hess} f\big|_{\partial\Omega}(U^{(1)}_{j(k)}))\right|}} \left(1 + hc^1_k(h)\right) \times$$
$$\times \exp -\tfrac{2}{h}\left(f(U^{(1)}_{j(k)}) - f(U^{(0)}_k)\right) \;, \quad \text{if } U^{(1)}_{j(k)} \in \partial\Omega \;.$$

Here $c_k^1(h)$ admits a complete expansion :

$$c_k^1(h) \sim \sum_{m=0}^{\infty} h^m c_{k,m} \,.$$

This theorem extends to the case with boundary the previous results of BovGayKl and HKN (see also the books FrWe and Ko and references therein).

About the proof in the case with boundary

As in HelSj4, the proof is deeply connected with the analysis of the small eigenvalues of a suitable realization (which is **not** the Dirichlet realization) of the Laplacian on the 1-forms. In order to understand the strategy, three main points have to be explained.

First point: define the Witten complex and the associate Laplacian.

The case of a compact manifold was treated in the foundational paper of Witten.
The case of $\mathbb{R}^n$ requires some care (See Johnsen, Helffer or Helffer-Nier1).

The case with boundary creates specific new problems.

Our starting problem being the analysis of the Dirichlet realization of the Witten Laplacian, we were led to find the right realization of the Witten Laplacian on 1-forms in the case with boundary in order to extend the commutation relation

$$\Delta^{(1)}_{f,h} d^{(0)}_{f,h} = d^{(0)}_{f,h} \Delta^{(0)}_{f,h}.$$

in a suitable "strong" sense (at the level of the selfadjoint realizations).

Towards the boundary conditions

The answer was present in the literature Chang-Liu in connection with the analysis of the relative cohomology.
Let us explain how we can find the right condition by looking at the eigenvectors.

If u is eigenvector of the Dirichlet realization of $\Delta_{f,h}^{(0)}$, then by commutation relation, $d_{f,h}^{(0)}u$ (which can not vanish) should be an eigenvector in the domain of the realization of $\Delta_{f,h}^{(1)}$. But $d_{f,h}^{(0)}u$ does not satisfy the Dirichlet condition in all its components, but only in its tangential components.

This is the natural condition in the definition of the variational domain to take for the quadratic form $\omega \mapsto ||d_{f,h}^{(1)}\omega||^2 + ||d_{f,h}^{(0)\,*}\omega||^2$.

The selfadjoint realization $\Delta_{f,h}^{(1)\,DT}$ obtained as the Friedrichs extension associated to the quadratic form gives the right answer.

Observing also that $d_{f,h}^{(0),*}(d_{f,h}^{(0)}u) = \lambda u$ (with $\lambda \neq 0$), we get the second natural (Neumann type)-boundary condition saying that a one form ω in the domain of the operator $\Delta_{f,h}^{(1)\,DT}$ should satisfy

$$d_{f,h}^{(0),*}\omega_{/\partial\Omega} = 0\ .$$

Second point : "rough" localization of the spectrum of this Laplacian on 1-forms.

The analysis was performed in Chang-Liu, in the spirit of Witten's idea, extending the so called Harmonic approximation. But these authors, because they were interested in the Morse theory, used the possibility to add symplifying assumptions on f and the metric near the boundary. We emphasize that HelNi2 treats the general case.

Third point : construction of WKB solutions for the critical points.

This was one in HelSj4 for the case without boundary, as an extension of previous constructions of HelSj1.

The new point is the construction of WKB near critical points of the restriction of the Morse function at the boundary is done in Helffer-Nier for 1-forms.

The analysis of a point of index 1 at the boundary of a WKB solution is done in HelNi2. Let us explain the main lines of the construction.

The construction is done locally around a local minimum U_0 of $f\Big|_{\partial\Omega}$ with $\partial_n f(U_0) > 0$. The function Φ is a local solution of the eikonal equation

$$|\nabla\Phi|^2 = |\nabla f|^2 \ ,$$

which also satisfies

$$\Phi = f \text{ on } \partial\Omega$$

and

$$\partial_n\Phi = -\partial_n f \text{ on } \partial\Omega$$

and we normalize f so that $f(U_0) = f(0) = 0$.
We first consider a local solution u_0^{wkb} near the point $x = 0$ of

$$e^{\frac{\Phi}{h}}\Delta_{f,h}^{(0)} u_0^{wkb} = \mathcal{O}(h^\infty) \ ,$$

with u_0^{wkb} in the form

$$u_0^{wkb} = a(x,h) e^{-\frac{\Phi}{h}} \ ,$$

$$a(x,h) \sim \sum_j a_j(x) h^j \ ,$$

and the condition at the boundary

$$a(x,h) e^{-\frac{\Phi}{h}} = e^{-\frac{f}{h}} \quad \text{on } \partial\Omega \ ,$$

which leads to the condition

$$a(x,h)\big|_{\partial\Omega} = 1 \ .$$

In order to verify locally the boundary condition for our future u_1^{wkb}, we substract $e^{-\frac{f}{h}}$ and still obtain

$$e^{\frac{\Phi}{h}} \, \Delta_f^{(0)} (u_0^{wkb} - e^{-\frac{f}{h}}) = \mathcal{O}(h^\infty) \ .$$

We now define the WKB solution u_1^{wkb} by considering :

$$u_1^{wkb} := d_{f,h} u_0^{wkb} = d_{f,h}(u_0^{wkb} - e^{-\frac{f}{h}}) \ .$$

The 1-form $u_1^{wkb} = d_{f,h} u_0^{wkb}$ satisfies locally the Dirichlet tangential condition on the boundary and u_1^{wkb} gives a good approximation for a ground state of a suitable realization of $\Delta_{f,h}^{(1)}$ in a neighborhood of this boundary critical point.

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