# Nodal domains and spectral minimal partitions.

B. Helffer (Université Paris-Sud)

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(After B. Helffer,
T. Hoffmann-Ostenhof and S. Terracini)

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# Introduction and main results

We consider mainly two-dimensional Laplacians operators in bounded domains. We would like to analyze the relations between the nodal domains of the eigenfunctions of the Dirichlet Laplacians and the partitions by $k$ open sets $D_i$ which are minimal in the sense that the maximum over the $D_i$'s of the ground state energy of the Dirichlet realization of the Laplacian in $D_i$ is minimal.

Most of the results can be extended to Schrödinger operators with an $L^\infty$-potential but we will omit this here.

Let $\Omega$ be a regular bounded domain ($C^{(1,+)}$ i.e. $C^{(1,\alpha)}$ for some $\alpha > 0$)

Let us consider the Laplacian $H(\Omega)$ on a bounded regular domain $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary condition. We denote by $\lambda_j(\Omega)$ the increasing sequence of its eigenvalues and by $u_j$ some associated orthonormal basis of eigenfunctions. We know that the groundstate $u_1$ can be chosen to be strictly positive in $\Omega$, but the other eigenfunctions $u_k$ must have zerosets. We define for any function $u \in C_0^0(\overline{\Omega})$

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}} \tag{1}$$

and call the components of $\Omega \setminus N(u)$ the nodal domains of $u$. The number of nodal domains of such a function will be called $\mu(u)$.

# Partitions

We now introduce the notions of partition and minimal partition.

**Definition 1**

*Let $1 \leq k \in \mathbb{N}$. We will call* **partition** *(or $k$-partition for indicating the cardinal of the partition) of $\Omega$ a family $\mathcal{D} = \{D_i\}_{i=1}^{k}$ of mutually disjoint sets such that*

$$\cup_{i=1}^{k} D_i \subset \Omega \ . \tag{2}$$

*We call it* **open** *if the $D_i$ are open sets of $\Omega$,* **connected** *if the $D_i$ are connected. We denote by $\mathfrak{O}_k$ the set of open connected partitions.*

Sometimes (at least for the proof) we have to relax this definition by considering measurable sets for the partitions.

# Minimal partitions

We now introduce the notion of spectral minimal partition sequence.

## Definition 2

*For any integer $k \geq 1$, and for $\mathcal{D}$ in $\mathfrak{D}_k$, we introduce*

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i). \tag{3}$$

*Then we define*

$$\mathfrak{L}_k = \inf_{\mathcal{D} \in \mathfrak{D}_k} \Lambda(\mathcal{D}). \tag{4}$$

*and call $\mathcal{D} \in \mathfrak{D}_k$ minimal if $\mathfrak{L}_k = \Lambda(\mathcal{D})$.*

## Remark 3

*If $k = 2$, it is rather well known (see for example [HeHO1] or [CTV3]) that $\mathfrak{L}_2$ is the second eigenvalue and the associated minimal 2-partition is a nodal partition, i.e. a partition whose elements are the nodal domains of some eigenfunction corresponding to $\lambda_2$.*

We discuss roughly the notion of regular and strong partition.

**Definition 4**

*A partition $\mathcal{D} = \{D_i\}_{i=1}^k$ of $\Omega$ in $\mathfrak{D}_k$ is called strong if*

$$\mathrm{Int}\left(\overline{\cup_i D_i}\right) \setminus \partial\Omega = \Omega \ . \tag{5}$$

Attached to a strong partition, we can naturally associate a closed set in $\overline{\Omega}$ defined by

$$N(\mathcal{D}) = \overline{\cup_i \left(\partial D_i \cap \Omega\right)} \ . \tag{6}$$

$N(\mathcal{D})$ plays the role of the nodal set (in the case of a nodal partition).

# Regular partitions

This leads us to introduce the set $\mathcal{R}(\Omega)$ of regular partitions (or nodal like) through the properties of the associated closed set.

**Definition 5**

**(i)** *There are finitely many distinct $x_i \in \Omega \cap N$ and associated positive integers $\nu_i$ with $\nu_i \geq 2$ such that, in a sufficiently small neighborhood of each of the $x_i$, $N$ is the union of $\nu_i(x_i)$ smooth curves with one end at $x_i$ and such that in the complement of these points in $\Omega$, $N$ is locally diffeomorphic to a regular curve.*

**(ii)** *$\partial\Omega \cap N$ consists of a (possibly empty) finite set of points $z_i$, s.t. at each $z_i$, $\rho_i$, with $\rho_i \geq 1$ lines hit the boundary. Moreover, $\forall z_i \in \partial\Omega$, then $N$ is near $z_i$ the union of $\rho_i$ distinct smooth half-curves which hit $z_i$.*

**(iii)** *Moreover, $N$ has the equal angle meeting property i.e. the half curves cross with equal angle at each critical point of $N$ and also at the boundary together with the boundary.*
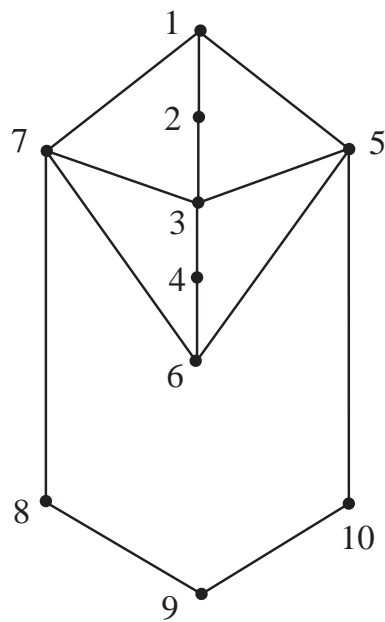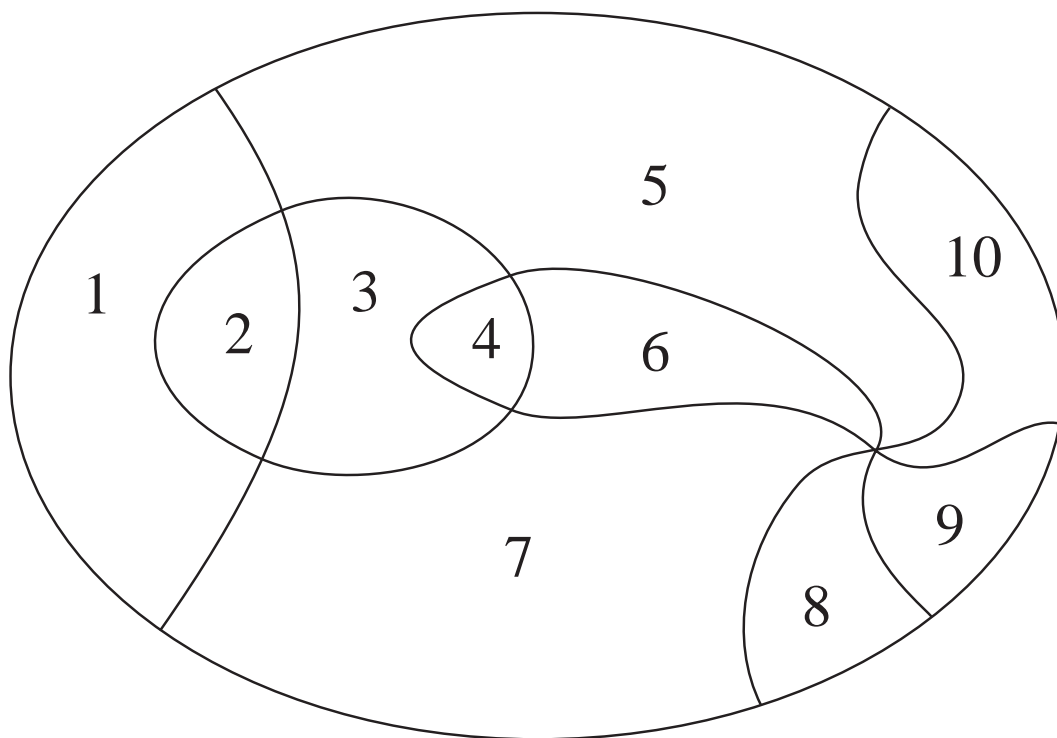
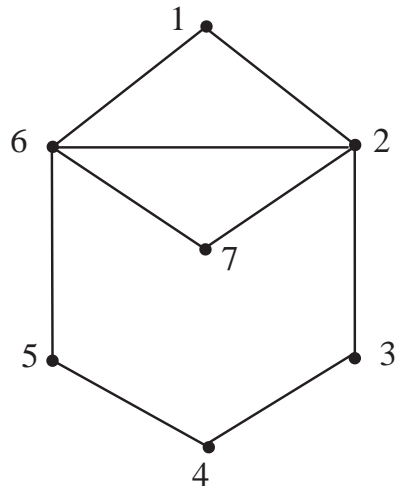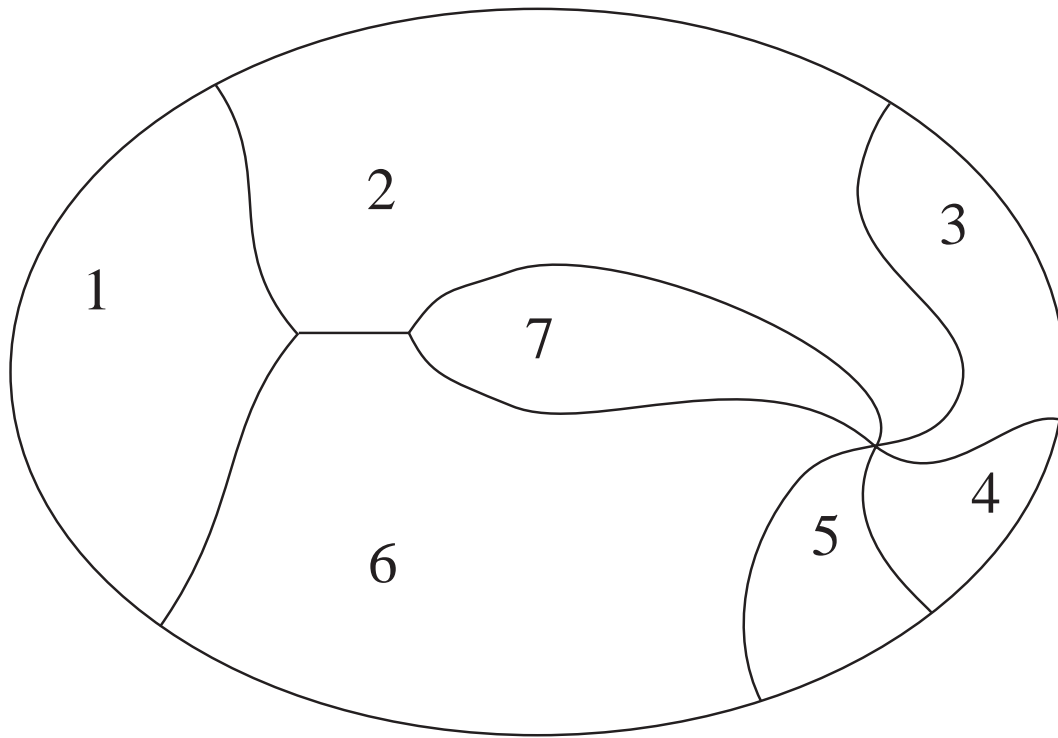Figure 1: An example of regular strong bipartite partition with associated graph.

Figure 2: An example of regular strong nonbipartite partition with associated graph.

# Main results

It has been proved by Conti-Terracini-Verzini [CTV1, CTV2, CTV3] that

**Theorem 6**

*For any $k$, there exists a minimal regular $k$-partition.*

This result is completed by (see Helffer–Hoffmann-Ostenhof–Terracini [HeHOTe]) :

**Theorem 7**

*Any minimal $k$-partition has a regular representative.*

A natural question is whether a minimal partition is the nodal partition induced by an eigenfunction. Next theorem will give a simple criterion for a partition to be associated to a nodal set.

For this we need some additional definitions.

We say that $D_i, D_j$ are neighbors or $D_i \sim D_j$, if $D_{i,j} := \mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected.

We associate to each $\mathcal{D}$ a graph $G(\mathcal{D})$ by associating to each $D_i$ a vertex and to each pair $D_i \sim D_j$ an edge.

We will say that the graph is bipartite if it can be colored by two colors (two neighbours having two different colors).

We recall that the graph associated to a collection of nodal domains of an eigenfunction is always bipartite.

We have now the following converse theorem :

## Theorem 8
*If the graph of the minimal partition is bipartite this is a nodal partition.*

A natural question is now to determine how general is the situation described in the previous theorem. The surprise is that this will only occur in the so called Courant-Sharp situation.

Let us recall some old results and notations. The Courant nodal theorem says :

**Theorem 9**

*Let $k \geq 1$, $\lambda_k$ be the $k$-th eigenvalue and $E(\lambda_k)$ the eigenspace of $H(\Omega)$ associated to $\lambda_k$. Then,*

$$\forall u \in E(\lambda_k) \setminus \{0\} \ , \ \mu(u) \leq k \ .$$

Then we say, as in [AnHeHO], that $u$ is Courant-sharp if

$$u \in E(\lambda_k) \setminus \{0\} \quad \text{and} \quad \mu(u) = k \ .$$

For any integer $k \geq 1$, we denote by $L_k$ the smallest eigenvalue whose eigenspace contains an eigenfunction with $k$ nodal domains. We set $L_k = \infty$, if there are no eigenfunctions with $k$ nodal domains.

In general, one can show, that

$$\lambda_k \leq \mathfrak{L}_k \leq L_k \ . \tag{7}$$

The last goal consists in giving the full picture of the equality cases :

**Theorem 10**
*Suppose $\Omega \subset \mathbb{R}^2$ is regular. Then, if $\mathfrak{L}_k = L_k$ or $\mathfrak{L}_k = \lambda_k$ then*

$$\lambda_k = \mathfrak{L}_k = L_k \ .$$

*In addition, one can find in $E(\lambda_k)$ a Courant-sharp eigenfunction.*

In other words, the only case when the $k$- nodal domains of an eigenfunction of $H(\Omega)$ form a minimal partition is the case when this eigenfunction is Courant-Sharp.

## Remarks.

(i) *For the one dimensional case the standard Sturm-Liouville theory leads easily to the following*

$$L_k = \mathfrak{L}_k = \lambda_k \ , \ \forall k \geq 1 \ . \tag{8}$$

(ii) *It is easy to show, that for a given $H$*

$$\mathfrak{L}_1 = L_1 = \lambda_1, \tag{9}$$

*(by the property of the ground state) and (we recall) that*

$$\mathfrak{L}_2 = L_2 = \lambda_2, \tag{10}$$

*by the orthogonality of $u_2$ to the ground state combined with Courant's nodal Theorem.*

(iii) *Once it is shown that $\mathcal{L}_k$ is obtained for (at least) a partition, it is easy to see that the sequence $(\mathfrak{L}_k)_{k \in \mathbb{N}}$ is strictly increasing.*

# Example : the case of the rectangle

Using Theorem 10 it is now easier to analyze the situation for the disk or for rectangles (at least in the irrational case), since we have just to check for which eigenvalues one can find associated Courant-sharp eigenfunctions.

For a rectangle of sizes $a$ and $b$, the spectrum is given by $\pi^2(m^2/a^2 + n^2/b^2)$ $((m,n) \in (\mathbb{N}^*)^2)$. The first remark is that all the eigenvalues are simple if $\frac{a^2}{b^2}$ is irrational. We assume

$$(a/b)^2 \text{ is irrational.}$$

So we can associate to each eigenvalue $\lambda_{m,n}$, an (essentially) unique eigenfunction $u_{m,n}$ such that $\mu(u_{m,n}) = nm$. Given $k \in \mathbb{N}^*$, the lowest eigenvalue corresponding to $k$ nodal domains is given by

$$L_k = \pi^2 \inf_{mn=k} \left(m^2/a^2 + n^2/b^2\right).$$

In the irrational case, $\lambda_{m,n}$ cannot lead to a Courant-sharp situation if $\inf(m,n) \geq 3$ or if $\inf(m,n) \geq 2$ and $m$ or $n$ larger than $4$.

So there are only very few cases to analyze by hand, for which the answer can depend on $\frac{a}{b}$.

In the case of the square, it is not to difficult to see that $\mathcal{L}_3$ is strictly less than $L_3$. We observe indeed that $\lambda_4$ is Courant-sharp, so $\mathcal{L}_4 = \lambda_4$, and there is no eigenfunction corresponding to $\lambda_2 = \lambda_3$ with three nodal domains (by Courant's Theorem).

Restricting to the half-rectangle and assuming that there is a minimal partition which is symmetric with one of the symmetry axes of the square perpendicular to two opposite sides, one is reduced to analyze a family of Dirichlet-Neumann problems.

Numerical computations performed by V. Bonnaillie-Noël and G. Vial lead to a natural candidate for a symmetric minimal partition.
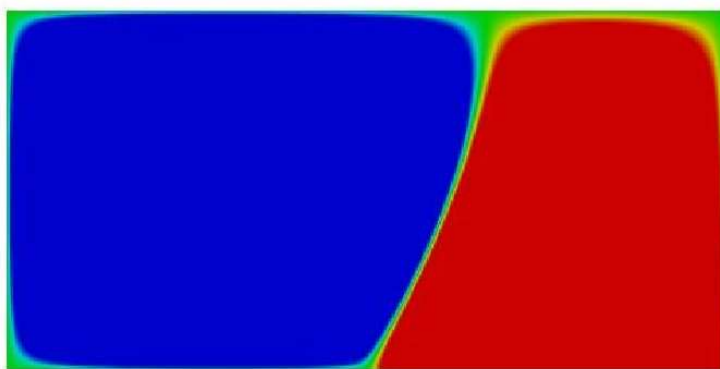See http://www.bretagne.ens-cachan.fr /math/Simulations/MinimalPartitions/



Figure 3: Trace on the half-square of the candidate for the 3-partition of the square. The complete structure is obtained from the half square by symmetry with respect to the horizontal axis.

# About minimal $3$-partitions of a simply connected domain.

Here I discuss the starting point of recent results obtained with Thomas Hoffmann-Ostenhof [HeHO4].

**Proposition 11**

*Let $\Omega$ be simply-connected and consider a minimal $3$-partition $\mathcal{D} = (D_1, D_2, D_3)$ associated to $\mathfrak{L}_3$ and suppose that $\lambda_3 < \mathfrak{L}_3$. Let $X(\mathcal{D})$ the singular points of $N(\mathcal{D}) \cap \Omega$ and $Y(\mathcal{D}) = N(\mathcal{D}) \cap \partial\Omega$. Then there are three cases.*

**(a)** $X(\mathcal{D})$ *consists of one point $x$ with a meeting of three half-lines ($\nu(x) = 3$) and $Y(\mathcal{D})$ consists of*

- *either three $y_1, y_2, y_3$ points with $\rho(y_1) = \rho(y_2) = \rho(y_3) = 1$,*

- *or two points $y_1, y_2$ with $\rho(y_1) = 2$, $\rho(y_2) = 1$,*

- *or one point $y$ with $\rho(y) = 3$.*

*Here, for $y \in \partial\Omega$, $\rho(y)$ is the number of half-lines ending at $y$.*

**(b)** $X(\mathcal{D})$ *consists of two distinct points* $x_1, x_2$
*so that* $\nu(x_1) = \nu(x_2) = 3$.
$Y(\mathcal{D})$ *consists*
*either of two points* $y_1, y_2$ *such that*

$$\rho(y_1) + \rho(y_2) = 2$$

*or of one point* $y$ *with*

$$\rho(y) = 2 \ .$$

**(c)** $X(\mathcal{D})$ *consists again of two distinct points*
$x_1, x_2$ *with* $\nu(x_1) = \nu(x_2) = 3$,
*but* $Y(\mathcal{D}) = \emptyset$.

The proof of Proposition 11 relies essentially on Euler formula.

This leads (with some success) to analyze the minimal partition with some topological type. If in addition, we introduce some symmetries, this leads to guess some candidates for minimal partitions.

# About the proof of the regularity of a minimal partition

First we have to relax the assumptions for the elements of the partition.

**Definition.**
*For any measurable $D \subset \Omega$, let $\lambda_1(D)$ denotes the first eigenvalue of the Dirichlet realization of the Schrödinger operator in the following generalized sense. We define*

$$\lambda_1(D) = +\infty \, ,$$

*if $\left\{ u \in W_0^1(\Omega) \, , u \equiv 0 \text{ a.e. on } \Omega \setminus D \right\} = \{0\}$, and*

$$\lambda_1(D) = \\ \min \left\{ \frac{\int_\Omega \left( |\nabla u(x)|^2 \right) dx}{\int_\Omega |u(x)|^2 dx} \; : \right. \\ \left. u \in W_0^1(\Omega) \setminus \{0\} \, , u \equiv 0 \text{ a.e. on } \Omega \setminus D \right\} \, ,$$

*otherwise.*
*We call groundstate any function $\phi$ achieving the above minimum.*

We also extend the minimal partition problem, by considering :

$$\mathcal{L}_{k,p} := \inf_{\mathcal{B}_k} \left( \frac{1}{k} \sum_{i=1}^{k} \left( \lambda_1(D_i) \right)^p \right)^{1/p} ,$$

where the minimization is taken over the class of partitions in $k$ "disjoint" measurable subsets of $\Omega$

$$\mathcal{B}_k :=$$
$$\left\{ \mathcal{D} = (D_1, \ldots, D_k) : \ \bigcup_{i=1}^{k} D_i \subset \Omega, \right.$$
$$\left. |D_i \cap D_j| = 0 \ if \ i \neq j \right\} .$$

Our previous problem corresponds to $p = +\infty$ and we were only minimizing on open partitions.

These problems have their own interest (with application to biomathematics).

The main result is the following

**Theorem 12**
*Let $\mathcal{D} = (\widetilde{D}_1, ..., \widetilde{D}_k) \in \mathcal{B}_k$ be any minimal partition associated with $\mathcal{L}_k$ and let $(\widetilde{\phi}_i)_i$ be any set of positive eigenfunctions normalized in $L^2$ corresponding to $(\lambda_1(\widetilde{D}_i))_i$. Then, there exist $a_i \geq 0$, not all vanishing, s. t. the functions $\widetilde{u}_i = a_i \widetilde{\phi}_i$ verify in $\Omega$ the differential inequalities*

*(I1)* $-\Delta \widetilde{u}_i \leq \mathcal{L}_k \widetilde{u}_i$, $\forall i = 1, \ldots, k$,

*(I2)* $-\Delta \left( \widetilde{u}_i - \sum_{j \neq i} \widetilde{u}_j \right) \geq \mathcal{L}_k \left( \widetilde{u}_i - \sum_{j \neq i} \widetilde{u}_j \right)$, $\forall i = 1, \ldots, k$.

Note that at this stage we do not know if the $\widetilde{D}_i$'s are connected and consequently if the $\widetilde{\phi}_i$'s are unique. But these properties are true in two dimensions.

The following results was proved by Conti-Terracini-Verzini in [CTV3]:

**Theorem.**
*Let $p \in [1, +\infty)$ and let $\mathcal{D} = (D_1, ..., D_k) \in \mathcal{B}_k$ be a minimal partition associated with $\mathcal{L}_{k,p}$ and let $(\phi_i)_i$ be any set of positive eigenfunctions normalized in $L^2$ corresponding to $(\lambda_1(D_i))_i$. Then, $\exists a_i > 0$, such that the functions $u_i = a_i \phi_i$ verify in $\Omega$*

*(I1) $-\Delta u_i \leq \lambda_1(D_i) u_i$,*

*(I2) $-\Delta \left( u_i - \sum_{j \neq i} u_j \right) \geq \lambda_1(D_i) u_i - \sum_{j \neq i} \lambda_1(D_j) u_j$.*

These inequalities imply that $U = (u_1, ..., u_k)$ is in the class $\mathcal{S}^*$ as defined in [CTV2]. Hence Theorem 8.3 in [CTV2] ensures the Lipschitz continuity of the $u_i$'s in $\Omega$. Therefore we can choose a partition made of open representatives $D_i = \{u_i > 0\}$.

The following result was shown in [CTV3]:

**Theorem.**
*There holds, for every $p \in [1, +\infty)$,*

$$\frac{1}{k^{1/p}} \mathcal{L}_k \leq \mathcal{L}_{k,p} \leq \mathcal{M}_{k,p} \leq \mathcal{L}_k .$$

*In particular*

$$\lim_{p \to +\infty} \mathcal{L}_{k,p} = \mathcal{L}_k .$$

*Moreover, there exists a minimizer of $\mathcal{L}_k$ such that (I1)- (I2) hold for suitable non negative multiples $u_i = a_i \phi_i$ of an appropriate set of associated eigenfunctions.*

One can go further in dimension two (many problems remain open in larger dimension).

**Theorem.**
*If $N = 2$ and $\Omega$ is bounded, connected with a piecewise $\mathcal{C}^{1,+}$ boundary, then the assertion of Theorem 12 holds with $a_i > 0$.*
*Moreover any minimizing partition $\mathcal{D}$ admits an open connected representative, having a piecewise $\mathcal{C}^{1,+}$ boundary.*

# Let us sketch the proof of Theorem 12.

Let $(\widetilde{D}_1, ..., \widetilde{D}_k) \in \mathcal{B}_k$ be a particular minimal partition associated with $\mathcal{L}_k$ and let $(\widetilde{\phi}_1, ..., \widetilde{\phi}_k)$ be any choice of associated eigenfunctions. We wish to prove that (I1)-(I2) hold for a suitable set of multiples of the $\widetilde{\phi}_j$'s. We consider, for a given

$$q \in (1, (N/(N-2))) , \tag{11}$$

(or $q \in (1, +\infty)$ when $N = 2$), the penalized Rayleigh quotient:

$$\mathcal{F}_{k,p}(u_1, ..., u_k)$$
$$= \left( \frac{1}{k} \sum_{i=1}^{k} \left( \frac{\int_\Omega |\nabla u_i(x)|^2 dx}{\int_\Omega |u_i(x)|^2 dx} \right)^p \right)^{1/p}$$
$$+ \sum_{i=1}^{k} \left( 1 - \frac{\int_\Omega u_i(x)^q \widetilde{\phi}_i(x)^q \, dx}{\left( \int_\Omega u_i(x)^{2q} \, dx \int_\Omega \widetilde{\phi}_i(x)^{2q} \, dx \right)^{1/2}} \right) .$$

We consider the minimization problem

$$\mathcal{M}_{k,p} = \inf \left\{ \mathcal{F}_{k,p}(u_1, ..., u_k) \ : \ (u_1, ..., u_k) \in \mathcal{U} \right\} ,$$
$$(12)$$

where

$$\mathcal{U} =$$
$$\left\{ (u_1, ..., u_k) \in (W_0^1(\Omega))^k \ : \ u_i \cdot u_j = 0 \ , \ \text{for} \ i \neq j \ , \right.$$
$$\left. u_i \geq 0 \ , u_i \not\equiv 0 \ , \ \forall i = 1, ..., k \right\} \ .$$
$$(13)$$

We note that the condition on $q$ permits us to have (weak and strong) continuity and differentiability in $W_0^1(\Omega)$ of the penalization term, which involves integrals of powers of $u_i$. This permits us to apply the direct method of the Calculus of Variations and to differentiate $\mathcal{F}_{k,p}$ at the minimum.

This permits to show the differential inequalities.

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B. HELFFER: DÉPARTEMENT DE MATHÉMATIQUES, BAT. 425, UNIVERSITÉ PARIS-SUD, 91 405 ORSAY CEDEX, FRANCE.

EMAIL: BERNARD.HELFFER@MATH.U-PSUD.FR