On the Courant sharp property : examples and methods

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Introduction

Given a bounded open set Ω in $\mathbb{R}^n$ (or in a Riemannian manifold) and a partition $\mathcal{D}$ of Ω by k open sets D_j, we can consider the quantity $\Lambda(\mathcal{D}) := max_j \lambda(D_j)$ where $\lambda(D_j)$ is the ground state energy of the Dirichlet realization of the Laplacian in D_j. If we denote by $\mathfrak{L}_k(\Omega)$ the infimum over all the k-partitions of $\Lambda(\mathcal{D})$ a minimal k-partition is then a partition which realizes the infimum. Although the analysis is rather standard when $k = 2$ (we find the nodal domains of a second eigenfunction), the analysis of higher k's becomes non trivial and quite interesting.

We recall that the Courant nodal domain theorem (1923) says that, for $k \geq 1$, and if λ_k denotes the k-th eigenvalue and $E(\lambda_k)$ the eigenspace of $H(\Omega)$ associated with λ_k, then, for all real-valued $u \in E(\lambda_k) \setminus \{0\}$, $\mu(u) \leq k$.

A theorem due to Pleijel [31] (1956) says that this cannot be true when the dimension (here we consider the $2D$-case) is larger than one.

Together with T. Hoffmann-Ostenhof and S. Terracini [23], we have proved that the only minimal partitions which are nodal correspond to the Courant sharp case. It is consequently interesting to determine when we are in this case.

If we look at the square, it is immediate that the first, second and fourth eigenvalues are Courant sharp. We will analyze the statement by Pleijel saying that there are no other cases. The completion of the proof was done in collaboration with P. Bérard (2013).

There are other cases which have been analyzed (I put in blue when the results are complete):

1. the disk. Helffer–Hoffmann-Ostenhof–Terracini (2006)
2. the irrational rectangle. Helffer–Hoffmann-Ostenhof–Terracini (2006)
3. the square (Neumann). Helffer–Persson-Sundqvist (2014)
4. the annulus (Neumann). Helffer–Hoffmann-Ostenhof (2011)
5. the sphere. Leydold (1992) , Helffer-Hoffmann-Ostenhof–Terracini (2010)
6. the irrational torus. Helffer–Hoffmann-Ostenhof (2011).
7. the square torus. Léna (2014)
8. the hexagonal torus. Bérard (2014)
9. the triangle. Bérard-Helffer (2015)
10. the isotropic harmonic oscillator. Leydold (1989) , Bérard-Helffer (2014), Charron (2014).

The techniques are related to another question considered by Antonie Stern in 1925: the construction of family of eigenfunctions (corresponding to a family of eigenvalues tending to $+\infty$) with only two nodal domains. Her contribution seems to have been forgotten (except one reference in footnote in Courant-Hilbert).

The guess for the answer can also be deduced from the analysis of the problem of minimal partitions. Hence numerics can give good indications for the result. See contributions of V. Bonnaillie-Noël, B. Helffer, C. Léna, G. Vial starting form 2007.

Minimal spectral partitions

We now introduce for $k \in \mathbb{N}$ ($k \geq 1$), the notion of k-partition. We will call k-**partition** of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of mutually disjoint open connected sets in Ω. We denote by $\mathfrak{O}_k(\Omega)$ this set. A spectral minimal partition sequence is defined by

Definition

For any integer $k \geq 1$, and for $\mathcal{D}$ in $\mathfrak{O}_k(\Omega)$, we set

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i). \tag{1}$$

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D} \in \mathfrak{O}_k} \Lambda(\mathcal{D}). \tag{2}$$

and call $\mathcal{D} \in \mathfrak{O}_k$ a minimal k-partition if $\mathfrak{L}_k = \Lambda(\mathcal{D})$.

One can show (Conti-Terracini-Verzini [13, 14, 15] and Helffer–Hoffmann-Ostenhof–Terracini [23]) that minimal spectral partitions always exist, are actually as regular[1] as the nodal sets of an eigenfunction.
In particular, minimal partitions have the equal angle property.
For any integer $k \geq 1$, we denote by $L_k(\Omega)$ the smallest eigenvalue whose eigenspace contains an eigenfunction of $H(\Omega)$ with k nodal domains. We set $L_k(\Omega) = \infty$, if there are no eigenfunctions with k nodal domains.

[1] up to set of capacity 0

Characterization of nodal minimal partitions

In general, one can show, that

$$\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega) \leq L_k(\Omega) \,. \tag{3}$$

Theorem 3

Suppose $\Omega \subset \mathbb{R}^2$ is regular.
If $\mathfrak{L}_k = L_k$ or $\mathfrak{L}_k = \lambda_k$ then

$$\lambda_k = \mathfrak{L}_k = L_k \,.$$

In addition, one can find a Courant-sharp pair (u, λ_k).

This is therefore interesting to determine for a given open set all the Courant sharp cases. This is what we will discuss in this talk.

Sharp Pleijel's theorem

Pleijel's theorem as stated in the introduction is the consequence of a more precise theorem which gives a link between Pleijel's theorem and minimal partitions. The classical proof is indeed going through the proposition

Proposition Pleijel 1

$$\limsup_{n\to+\infty} \frac{\mu(\phi_n)}{n} \leq \frac{4\pi}{A(\Omega)\liminf_{k\to+\infty}\frac{\mathfrak{L}_k(\Omega)}{k}}\,, \tag{4}$$

where $\mu(\phi_n)$ is the cardinal of the nodal components of $\Omega \setminus N(\phi_n)$ and then to establishing a lower bound for $A(\Omega)\liminf_{k\to+\infty}\frac{\mathfrak{L}_k(\Omega)}{k}$. This proposition is a consequence of Weyl's theorem on the counting function of eigenvalues.

Hint

Having in mind Weyl's formula

$$N(\lambda) \sim \frac{A(\Omega)}{4\pi}\,\lambda\,,$$

the first step in Pleijel's formula can also be written in the form:

$$\liminf_{\lambda\to+\infty} \frac{N(\lambda)}{\lambda}\ \limsup_{n\to+\infty} \frac{\mu(\phi_n)}{n}\ \liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \leq 1\,. \tag{5}$$

Faber-Krahn

Hence any progress relative to the lower bound of $\liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k}$ leads to an improvement for the sharp Pleijel theorem.
Faber-Krahn says:

Faber-Krahn's inequality

$$A(\omega)\,\lambda(\omega) \geq \pi j_0^2\,.$$

Here πj_0^2 is the ground state energy of the Dirichlet Laplacian in the disk of area 1.

Applying to each element of an equicontinuous partition and summing we get as the second step for sharp Pleijel's theorem:

Proposition Pleijel 2

$$A(\Omega)\,\frac{\mathfrak{L}_k(\Omega)}{k} \geq \pi j_0^2\,.$$

Recent improvments (with application to Pleijel's theorem) have been obtained by J. Bourgain and Steinerberger.

In any case, we get Pleijel's refined theorem

Pleijel's Theorem

$$\limsup_{n\to+\infty} \frac{\mu(\phi_n)}{n} \leq \frac{4}{j_0^2} < 1\,. \tag{6}$$

How to determine Courant sharpness

1. Explicit computations in the case of one dimensional eigenspaces.
2. Twisting trick.
3. Courant sharp with symmetries
4. Pleijel with control of the constants together with Faber-Krahn.
5. Direct analysis of the nodal domains of one parameter families of eigenfunctions.

The case of the irrational rectangle

Consider the rectangle $\mathcal{R}(a,b) =]0, a\pi[\times]0, b\pi[$. The eigenvalues are given by

$$\hat{\lambda}_{m,n} = (\frac{m^2}{a^2} + \frac{n^2}{b^2})\,, \;\; m,n \geq 1,$$

with a corresponding basis of eigenfunctions given by

$$\phi_{m,n}(x,y) = \sin\frac{mx}{a}\,\sin\frac{ny}{b}\,.$$

It is easy to determine the Courant sharp eigenvalues when b^2/a^2 is irrational, because each eigenspace is one dimensional and one can explicitly compute the number of nodal domains mn. One has then to proceed to the ordering of the eigenvalues.

In the long rectangle $]0, a[\times]0, 1[$ the eigenfunction $\sin(k\pi x/a)\sin\pi y$ is Courant-sharp for $a \geq \sqrt{(k^2-1)/3}$.

The case of the disk

Although the spectrum is explicitly computable, we are mainly interested in the ordering of the eigenvalues corresponding to different angular momenta.

Consider the Dirichlet realization in the unit disk $B_1 \subset \mathbb{R}^2$. We have in polar coordinates :

$$-\Delta = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,,$$

and the Dirichlet boundary conditions require that any eigenfunction u satisfies $u(r,\theta) = 0$ for $r = 1$.

We analyze for any $\ell \in \mathbb{N}$ the eigenvalues $\lambda_{\ell,j}$ of

$$(-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{\ell^2}{r^2}) f_{\ell,j} = \lambda_{\ell,j} f_{\ell,j} \ , \quad \text{in } (0,1) \ .$$

We observe that the operator is self adjoint for the scalar product in $L^2((0,1), r\,dr)$.

The corresponding eigenfunctions of the eigenvalue problem take the form

$$u(r,\theta) = f_{\ell,j}(r)\,(a\cos \ell\theta \;+\; b\sin \ell\theta\,)\,, \quad \text{with } a^2 + b^2 > 0\,, \qquad (7)$$

where the $f_{\ell,j}(r)$ are suitable Bessel functions. For the corresponding $\lambda_{\ell,j}$'s, we find the following ordering.

$$\begin{aligned} \lambda_1 = \lambda_{0,1} < \lambda_2 = \lambda_3 = \lambda_{1,1} < \lambda_4 = \lambda_5 & \\ = \lambda_{2,1} < \lambda_6 = \lambda_{0,2} < \lambda_7 = \lambda_8 = \lambda_{3,1} < \ldots & \\ \cdots < \lambda_9 = \lambda_{10} = \lambda_{1,2} < \lambda_{11} = \lambda_{12} = \lambda_{4,1} < \ldots & \end{aligned} \qquad (8)$$

We recall that the zeros of the Bessel functions are related to the eigenvalues by the relation

$$\lambda_{\ell,k} = (j_{\ell,k})^2 \,. \tag{9}$$

Moreover all the $j_{\ell,k}$ are distinct (Watson).
The multiplicity is either 1 (if $\ell = 0$) or 2 if $\ell > 0$.
We hence have

$$\begin{array}{l} \mu(u_1) = 1, \\ \mu(u) = 2, \text{ for any eigenfunction u associated to } \lambda_2 = \lambda_3, \\ \mu(u) = 4, \text{ for any eigenfunction u associated to } \lambda_4 = \lambda_5, \\ \mu(u_6) = 2, \\ \mu(u) = 6, \text{ for any eigenfunction u associated to } \lambda_7 = \lambda_8, \\ \mu(u) = 4, \text{ for any eigenfunction u associated to } \lambda_9 = \lambda_{10}, \end{array} \tag{10}$$

Hence

$$L_1 = \lambda_1 \, , \; L_2 = \lambda_2 \, , \; L_3 = \lambda_{15} \, , \; L_4 = \lambda_4 \, . \tag{11}$$

In the case of the disk, we have

Proposition

Except the cases $k = 1$, 2 and 4, minimal partitions never correspond to nodal domains.

Proof

One can use the twisting trick (Helffer–Hoffmann-Ostenhof) for eliminating all the eigenvalues $\lambda_{\ell,m}$, for which $m \geq 2$ and $\ell > 0$. This trick goes roughly as follows. When $\ell > 0$, we can divide the disk as the union of a smaller disk and of its complementary, each of these sets being the union of at least two nodal domains. Then by small rotation of the small disk, we get a new partition which has the same energy. If the initial one was minimal, the new one should be also minimal, but it is easy to show that the new one has not the "equal angle meeting" property of a regular partition. This gives the contradiction.

So we have finally to analyze the eigenvalues $\lambda_{0,k}$ and the family $\lambda_{\ell,1}$.

For the first family, we observe that $\lambda_{0,k}$ can neither be the k-th eigenvalue as soon as $k \geq 2$.

For the second family, which occurs only for $k = 2\ell$ even, inspection of the tables leads to the condition $k \leq 4$, we observe indeed that $\lambda_{0,2} < \lambda_{3,1}$.

We also observe that when k is odd, we get that necessarily $L_k = \lambda_{0,k}$.

The case of the sphere

Here we can use the argument of Courant sharpness with symmetry. The argument first appears in the PHD of J. Leydold (see also Helffer–Hoffmann-Ostenhof–Terracini).
The involved symmetry is the antipodal map σ and its representation Σ on $L^2(\mathbb{S}^2)$ defined by

$$(\Sigma u)(x) = u \circ \sigma(x)\,.$$

Associated with Σ, we can decompose $L^2(\mathbb{S}^2)$ as the direct sum

$$L^2(\mathbb{S}^2) = L^{2,sym}(\mathbb{S}^2) \oplus L^{2,asym}(\mathbb{S}^2)\,.$$

A particular feature is then that for each eigenvalue of the Laplacian on the sphere, the corresponding eigenspace is either in $L^{2,sym}(\mathbb{S}^2)$ or in $L^{2,asym}(\mathbb{S}^2)$.

We can then apply separately a Courant theorem for the "symmetric" eigenvalues and the "antisymmetric" eigenvalues and obtain

$$\mu(u_\ell) \leq \ell(\ell - 1) + 2\,,$$

where u_ℓ denotes a spherical harmonic of degree ℓ.
Note that the usual Courant theorem gives only

$$\mu(u_\ell) \leq \ell^2 + 1\,.$$

Note also that the twisting trick can be used to eliminate many cases.

Leydold stated the following conjecture on the maximal cardinal of nodal sets of a spherical harmonic.

Conjecture

$$\max_{u \in \mathcal{H}_\ell} \mu(u) = \begin{cases} \frac{1}{2}(\ell+1)^2 & \text{if } \ell \text{ is odd,} \\ \frac{1}{2}\ell(\ell+2) & \text{if } \ell \text{ is even.} \end{cases}$$

This conjecture is proved by Leydold (1993) for $\ell \leq 6$. Leydold's Conjecture implies

Theorem

In the case of the Laplacian on the sphere, the unique Courant sharp situation correspond to the first and second eigenvalues.

This last statement is true as a consequence of Courant sharp with symmetry or by using the following estimate by Karpushkin (1989).

$$\max_{u \in \mathcal{H}_\ell} \mu(u) \leq \begin{cases} \ell(\ell-2)+5 & \text{if } \ell \text{ is odd,} \\ \ell(\ell-2)+4 & \text{if } \ell \text{ is even.} \end{cases}$$

Remark

This idea is also used for the isotropic harmonic oscillator (Leydold (1989)) and in the analysis of the Neumann problem for the square (Helffer–Persson-Sundqvist (2014)).

Pleijel's analysis in the case of the square

In [31], Pleijel claims that in the case of the square, the Dirichlet eigenvalue λ_k is Courant sharp if and only if $k = 1, 2, 4$. His proof involves the reduction to the analysis of the cases $k = 5, 7, 9$, and does not seem well justified for this last point (see below).

Let us consider the general question of analyzing the zero set of the Dirichlet eigenfunctions for the square $\mathcal{S}$. We have:

$$\phi_{m,n}(x,y) = \phi_m(x)\phi_n(y)\,, \text{ with } \phi_m(t) = \sin(m\pi t)\,.$$

Due to multiplicities, we have (at least) to consider the family of eigenfunctions,

$$(x,y) \mapsto \Phi_{m,n}(x,y,\theta) := \cos\theta\,\phi_{m,n}(x,y) + \sin\theta\,\phi_{n,m}(x,y)\,,$$

with $m, n \geq 1$, and $\theta \in [0,\pi[$.

In Pleijel's analysis [31] of the Courant sharp property for $\mathcal{S}$, it is shown that it is enough to consider the eigenvalues λ_5, λ_7 and λ_9 with correspond respectively to the pairs $(m,n) = (1,3)$, $(m,n) = (2,3)$ and $(m,n) = (1,4)$.

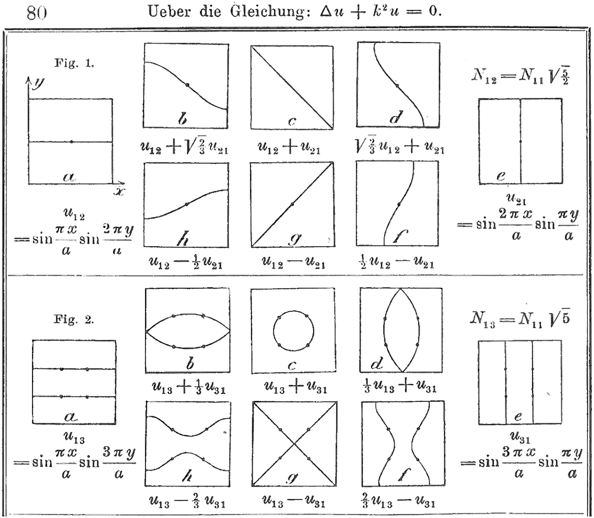

Figure: Nodal sets, Dirichlet eigenvalues λ_2 and λ_5 (Pockels, [32]).

Pleijel's reduction argument

Let us briefly recall Pleijel's argument. Let $N(\lambda) := \#\{n \mid \lambda_n < \lambda\}$ be the counting function. Using a covering of $\mathbb{R}^2$ by the squares $]k, k+1[\times]\ell, \ell+1[$, he first establishes the estimate

$$N(\lambda) > \frac{\pi}{4}\lambda - 2\sqrt{\lambda} - 1\,. \tag{12}$$

For any n such that $\lambda_{n-1} < \lambda_n$, we have $N(\lambda_n) = n - 1$, and

$$n > \frac{\pi}{4}\lambda_n - 2\sqrt{\lambda_n}\,. \tag{13}$$

On the other hand, if λ_n is Courant sharp, the Faber-Krahn inequality gives the necessary condition

$$\frac{\lambda_n}{n} \geq \frac{\mathbf{j}^2}{\pi}$$

or

$$\frac{n}{\lambda_n} \leq \pi \mathbf{j}^{-2} \sim 0.545\,. \tag{14}$$

Recall that $\pi \mathbf{j}^2$ is the ground state energy of the disk of area 1.

Combining (13) and (14), leads to the inequality

$$\lambda_n < 68\,. \tag{15}$$

After re-ordering the values $m^2 + n^2$, we get the spectral sequence for $\lambda_n \leq 68$,

It remains to analyze, among the eigenvalues which are less than 68, the ones which could be Courant sharp, and hence satisfy (14). Computing the quotients $\frac{n}{\lambda_n}$ in the list, this leaves us with the eigenvalues λ_5, λ_7 and λ_9.
For these last three cases, Pleijel refers to pictures in Courant-Hilbert [17], actually reproduced from Pockel [32], see above. Although the choice of pictures suggests that some theoretical analysis is involved, one cannot see any systematic analysis, the difficulty being that we have to analyze the nodal sets of eigenfunctions living in two-dimentional eigenspaces. Hence one has to give a detailed proof that eigenvalues λ_5, λ_7 and λ_9 are not Courant sharp.

Of course we know that $\Phi_{m,n}$ has mn nodal components (this corresponds to the "product" situation with $\theta = 0$ or $\theta = \frac{\pi}{2}$). However, we have already mentioned that the number of nodal domains for a linear combination of two given independent eigenfunctions can be smaller or larger than the number of nodal domains of the given eigenfunctions.

The three cases left by Pleijel

Behind all the computations we have the property that, for $x \in]0, \pi[$,

$$\sin mx = \sqrt{1 - u^2}\, U_{m-1}(u)\,, \tag{16}$$

where U_{m-1} is the Chebyshev polynomial of second type and $u = \cos x$.

First case : eigenvalue λ_5 $((m,n)=(1,3))$.

We look at the zeroes of $\Phi_{1,3}(x,y,\theta)$. Let,

$$\cos x = u\,,\ \cos y = v\,. \tag{17}$$

This is a C^∞ change of variables from the square $]0,\pi[\times]0,\pi[$ onto $]-1,+1[\times]-1,+1[$. In these coordinates, the zero set of $\Phi_{1,3}(x,y,\theta)$ inside the square is given by

$$\cos\theta\,(4v^2-1)+\sin\theta\,(4u^2-1)=0\,. \tag{18}$$

Except the two easy cases when $\cos\theta = 0$ or $\sin\theta = 0$, which can be analyzed directly (product situation), we immediately get that the only possible critical point is $(u, v) = (0, 0)$, i.e. $(x, y) = (\frac{\pi}{2}, \frac{\pi}{2})$, and that this can only occur for $\cos\theta + \sin\theta = 0$, i.e. for $\theta = \frac{\pi}{4}$.

This analysis shows rigorously that the number of nodal domains is 2, 3 or 4 as claimed in [31]. As a matter of fact, we have a complete description of the situation.

Two other cases: eigenvalue λ_7 $((m,n)=(2,3))$ and λ_9 $((m,n)=(1,4))$.

As observed by Mikael Persson Sundqvist, we can use an argument à la Leydold.
We observe that

$$\phi_{m,n}(\pi - x, \pi - y, \theta) = (-1)^{m+n}(x, y, \theta)\,.$$

Hence, when $m+n$ is odd, any eigenfunction corresponding to m^2+n^2 has necessarily an even number of nodal domains. Hence λ_7 and λ_9 cannot be Courant sharp.
This achieves the analysis of the Courant sharp cases for the square (Dirichlet).

On Stern's Thesis in 1925

One can actually observe that for some values of θ the function $\Phi_{1,4}(x, y, \theta)$, has exactly two nodal domains. This phenomenon was studied by Antonie Stern who claims that for all $k \geq 2$, there are eigenfunctions associated with the Dirichlet eigenvalue $1 + 4k^2$ of the square $[0, \pi]^2$, with exactly two nodal domains. This has been proved rigorously by Bérard-Helffer (2014).

Although not necessary for the analysis of the case $(1,3)$, we mention a few arguments which have to be used for more general cases.

Property P1= Checkerboard argument

Let ϕ and ψ be two linearly independent eigenfunctions associated with the same eigenvalue for the square $\mathcal{S}$. Let μ be a real parameter, and consider the family of eigenfunctions $\phi_\mu = \psi + \mu\phi$. Let $N(\phi)$ denote the nodal set of the eigenfunction ϕ.

1. Consider the domains in $\mathcal{S} \setminus N(\phi) \cup N(\psi)$ in which $\mu\,\phi\,\psi > 0$ and hatch them ('schraffieren'). Then the nodal set $N(\phi_\mu)$ avoids the hatched domains,
2. The points in $N(\phi) \cap N(\psi)$ belong to the nodal set $N(\phi_\mu)$ for all μ,

These properties are trivial but very powerful (see for the sphere, the Neumann case, the triangle case).
A. Stern also mentions the following.

Property P2

The nodal set $N(\phi_\mu)$ depends continuously on μ.

which is rather clear near regular point, but not so clear near multiple points (see however Leydold in 1989).

All in all, the arguments given by A. Stern seem very sketchy and it is necessary to write the details in the same spirit as for Pleijel's statement. The complete proof for the case $(1, 2k)$ is based on:

1. Complete determination of the multiple points of $N(\Phi^{\frac{\pi}{4}})$;
2. Absence of multiple points in $N(\Phi^\theta)$, when θ is different from $\frac{\pi}{4}$, and close to $\frac{\pi}{4}$;
3. Connectedness of the nodal set $N(\Phi^\theta)$, or why there are no other components, e.g. closed inner components, in the nodal set.
4. Separation lemmas.

About Neumann

In the case of Neumann, one difficulty is that we cannot apply Faber-Krahn for the nodal domains touching the boundary along segments. Hence, we should estimate the cardinal of these nodal domains.
A first rough estimate was performed for the square by Pleijel, leading to Pleijel's theorem for the square.

Pleijel's theorem for Neumann was proved by I. Polterovich for domains with piecewise analytic domains, using a result by Toth-Zelditch.

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