Improving semigroups bounds with resolvent estimates

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(after Helffer-Sjöstrand)

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Abstract

The purpose of this talk is to revisit the proof of the Gearhart-Prüss-Huang-Greiner theorem for a semigroup

$$S(t) = e^{tA}$$

following the general idea of the proofs that we have seen in the literature and to get an explicit estimate on $\|S(t)\|$ in terms of bounds on the resolvent of the generator.

A first version of this paper was presented by the two authors in ArXiv (2010) together with applications in semi-classical analysis and a part of these results has been published later in two books written by the authors.

Our aim is to present new improvements, partially motivated by a paper of D. Wei.

On the way we discuss optimization problems confirming the optimality of our results.

The paper appears in Integral Equations and Operator Theory (2021).

Finally we discuss (following Helffer-Sjöstrand-Viola) more recent results about the optimality of Wei's bound.

Let $\mathcal{H}$ be a complex Hilbert space and let $[0,+\infty[\ni t\mapsto S(t)\in\mathcal{L}(\mathcal{H},\mathcal{H})$ be a strongly continuous semigroup with $S(0)=I$.

Recall that there exist $M\geq 1$ and $\omega_0\in\mathbb{R}$ such that $S(t)$ has the property

$$P(M,\omega_0):\quad \|S(t)\|\leq Me^{\omega_0 t},\ t\geq 0. \tag{1}$$

If A is the generator of the semigroup (we write $S(t)=e^{tA}$) we have

$$(z-A)^{-1}=\int_0^\infty S(t)e^{-tz}dt,\quad \|(z-A)^{-1}\|\leq\frac{M}{\operatorname{Re}z-\omega_0}\,, \tag{2}$$

when $P(M,\omega_0)$ holds and z belongs to the open half-plane $\operatorname{Re}z>\omega_0$.

We recall the Gearhart-Prüss-Huang-Greiner theorem, see Engel-Nagel :

GPHG-Theorem

(a) Assume that $\|(z-A)^{-1}\|$ is uniformly bounded in the half-plane $\mathrm{Re}\, z \geq \omega$. Then there exists a constant $M > 0$ such that $P(M, \omega)$ holds.

(b) If $P(M, \omega)$ holds, then for every $\alpha > \omega$, $\|(z-A)^{-1}\|$ is uniformly bounded in the half-plane $\mathrm{Re}\, z \geq \alpha$.

Our purpose is to revisit the proof of (a), following the general idea of the proofs that we have seen in the literature and to get explicit t-dependent estimate on $e^{-\omega t}\|S(t)\|$, implying explicit bounds on M.

This idea is essentially to use that the resolvent and the inhomogeneous equation $(\partial_t - A)u = w$ in exponentially weighted spaces are related via Fourier-Laplace transform and we can use Plancherel's formula.

We will obtain general results of the form :

If $\|S(t)\| \leq m(t)$ *for some positive function* m*, and if we have a certain bound on the resolvent of* A*, then* $\|S(t)\| \leq \widetilde{m}(t)$ *and hence* $\|S(t)\| \leq \min(m(t), \widetilde{m}(t))$ *for a new function* $\widetilde{m}$ *that can be explicitly described.*

The next question would be to see what we get by iterating the procedure (we have preliminary results on this subject together with J. Sjöstrand and J. Viola).

Let

$$\omega_1 = \inf\{\omega \in \mathbb{R};\ \{z \in \mathbb{C}; \mathrm{Re}\, z > \omega\} \subset \rho(A) \text{ and } \sup_{\mathrm{Re}\, z > \omega} \|(z-A)^{-1}\| < \infty\}.$$

For $\omega > \omega_1$, we may define $r(\omega)$ by

$$\frac{1}{r(\omega)} = \sup_{\mathrm{Re}\, z > \omega} \|(z - A)^{-1}\|. \tag{3}$$

The main result which was obtained in 2010 was :

HS-Theorem (2010)

Under the assumptions of GPHG-Theorem, (a) and for some $\omega \in \mathbb{R}$ $r(\omega) > 0$ be as in (3).
Let $m(t)$ be a continuous positive function such that

$$\|S(t)\| \leq m(t)\,.$$

Then for all $t, a, b > 0$, such that $t \geq a + b$, we have

$$\|S(t)\| \leq \frac{e^{\omega t}}{r(\omega)\|\frac{1}{m}\|_{e^{-\omega\cdot}L^2(0,a)}\|\frac{1}{m}\|_{e^{-\omega\cdot}L^2(0,b)}}\,. \tag{4}$$

Here the norms are always the natural ones obtained from $\mathcal{H}$, L^2, In [18], Dongyi Wei, motivated by our first version [8] has proved the following theorem :

W-theorem

Let $H = -A$ be an accretive operator in a Hilbert space $\mathcal{H}$. Then

$$||S(t)|| \leq e^{-r(0)t+\frac{\pi}{2}}\,,\ \forall t \geq 0\,. \tag{5}$$

This is trivial for $tr(0) < \frac{\pi}{2}$.

Our aim is to deduce and improve these two theorems as a consequence of a unique basic estimate.
Let Φ satisfy

$$0 \leq \Phi \in C^{1,pw}([0,+\infty[) \text{ with } \Phi(0) = 0 \text{ and } \Phi(t) > 0 \text{ for } t > 0\,, \qquad (6)$$

and assume that Ψ has the same properties.
For $t > 0$, let ι_t be the reflection with respect to $t/2$.

Theorem HS1

Under the assumptions of GPHG-Theorem, for any Φ and Ψ we have

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq e^{\omega t} \frac{\|(r(\omega)^2\Phi^2 - \Phi'^2)_-^{\frac{1}{2}}\, m\|_{e^{\omega\cdot}L^2([0,t[)}\, \|(r(\omega)^2\Psi^2 - \Psi'^2)_-^{\frac{1}{2}}\, m\|_{e^{\omega\cdot}L^2([0,t[)}}{\int_0^t (r(\omega)^2\Phi^2 - \Phi'^2)_+^{\frac{1}{2}} (r(\omega)^2\iota_t\Psi^2 - \iota_t\Psi'^2)_-^{\frac{1}{2}}\, ds}\,. \qquad (7)$$

Here for $a \in \mathbb{R}$, $a_+ = \max(a,0)$ and $a_- = \max(-a,0)$.

We now discuss the consequences of this theorem that can be obtained with suitable choices of Φ, Ψ.
The first one is a Wei like version of our (2010)-Theorem.

Theorem HS2

For positive a and b, we have, for $t > a + b$,

$$||S(t)|| \leq \frac{e^{\omega t - r(\omega)(t-a-b)}}{r(\omega)} \frac{1}{\|\frac{1}{m}\|_{e^{-\omega \cdot} L^2(0,a)} \|\frac{1}{m}\|_{e^{-\omega \cdot} L^2(0,b)}} . \tag{8}$$

Note the multiplicative gain $e^{-r(\omega)(t-a-b)}$ in comparison with [HS]-Theorem.

In the case of Wei's theorem we have $\omega = 0$, $m = 1$. With $b = a$ we first get

$$||S(t)|| \leq \frac{1}{ar(0)} \exp -r(0)(t-2a)\,, \quad t > 2a.$$

Minimization with respect to a leads to

$$||S(t)|| \leq 2e \exp -r(0)t\,, \quad t > \frac{1}{r(0)}\,,$$

which is not quite as sharp as (5), since $e^{\pi/2} \approx 4.81$, $2e \approx 5.44$.

We will show that a finer approach will permit to recover (5) and generalize it to more general m.

An important step will be to prove (we assume $\omega = 0$, $r(0) = 1$) as a consequence of Theorem HS1, the following key proposition

Key proposition

Assume that $\omega = 0$, $r(\omega) = 1$. Let a, b positive. Then for $t \geq a + b$,

$$||S(t)|| \leq \exp -(t-a-b) \frac{\left(\inf_u \int_0^a m(s)^2 (u'^2(s) - u^2(s))_+ ds\right)^{1/2}}{\left(\sup_\theta \int_0^b \frac{1}{m^2} (\theta(s)^2 - \theta'(s)^2)\, ds\right)^{1/2}}, \quad (9)$$

where

- $u \in H^1(]0, a[)$ satisfies $u(0) = 0$, $u(a) = 1$;
- $\theta \in H^1((]0, b[)$ satisfies $\theta(b) = 1$ and $|\theta'| \leq \theta$.

This proposition implies rather directly Theorem HS2 but we want better !

To refine the analysis of the right hand side of (9), we have to analyze quantities

$$I_{\rm inf}(a) := \inf_u \int_0^a m(s)^2 (u'(s)^2 - u^2(s))_+ \, ds$$

and

$$J_{\rm max}(b) := \sup_\theta \int_0^b \frac{1}{m^2} (\theta(s)^2 - \theta'(s)^2) \, ds \,,$$

where u and θ satisfy the above conditions.

The optimization leads to consider the Dirichlet-Robin realization $K_{m,a}^{DR}$ of the operator in $]0,a[$

$$K_m := -\frac{1}{m^2}\partial_s \circ m^2 \partial_s - 1\,, \tag{10}$$

with Dirichlet-Robin condition

$$u(0) = 0\,,\ u'(a) = u(a)\,, \tag{11}$$

If $\lambda^{DR}(a, m)$ denote the lowest eigenvalue of $K^{DR}_{m,a}$. we define

$$a^* = a^*(m) = \sup\{\widetilde{a} \in]0, \infty[; \lambda^{DR}(a, m) > 0 \text{ for } 0 < a < \widetilde{a}\}, \tag{12}$$

so that $a^*(m) \in]0, +\infty]$.
By continuity we have in the case $a^* < \infty$

$$\lambda^{DR}(a^*, m) = 0, \quad \lambda^{DR}(a, m) > 0 \text{ for } 0 < a < a^*.$$

Next we define on $]0, a^*[$

$$\psi_0(s; m) = \psi_0(s) := u_0'(s)/u_0(s)\,, \quad 0 < s < a^*, \tag{13}$$

where u_0 is the first eigenfunction of the *DR*-problem in $]0, a^*[$ and observe that ψ_0 is a solution of a Riccati equation,

$$\psi' = -(\psi^2 + 2\mu\psi + 1)\,, \tag{14}$$

with $\mu = m'/m$.
This plays an important role in the analysis of the optimality of the results.

Then we have

Theorem HS3

Let $\omega = 0$, $r(\omega) = 1$. When $a, b \in]0, +\infty[\cap]0, a^*]$ and $t > a + b$, we have

$$||e^t S(t)|| \leq \exp(a+b) m(a) m(b) \psi_0(a)^{\frac{1}{2}} \psi_0(b)^{\frac{1}{2}} \,. \tag{15}$$

In particular, when $a^* < +\infty$, we have (taking $a = b = a^*$)

$$||e^t S(t)|| \leq \exp(2a^*)\, m(a^*)^2 \,, \quad t > 2a^* \,. \tag{16}$$

This theorem is the analog of Wei's theorem for general weights m.

By a general procedure we have actually a more general statement for a pair $(\omega, r(\omega))$.

We now assume $\omega = 0$ and $r(0) = 1$.

We introduce

$$\mathfrak{H} := \mathcal{H}_{0,a}^{0,1} = \{u \in H^1(]0, a[), u(0) = 0, u(a) = 1; 0 \leq u \leq u'\}. \quad (17)$$

We also introduce

$$\mathfrak{G} = \{\theta \in H^1(]0, b[);\ |\theta'| \leq \theta,\ \theta(b) = 1\}\,.$$

Given some $t > a + b$, we now give the conditions satisfied by Φ :

Property $P_{a,b}$

1. $\Phi = e^a u$ on $]0, a]$ and $u \in \mathfrak{H}$.
2. On $[a, t-b]$, we take $\Phi(s) = e^s$, so $\Phi'^2(s) - \Phi(s)^2 = 0$.
3. On $[t-b, t]$ we take $\Phi(s) = e^{t-b}\theta(t-s)$ with $\theta \in \mathfrak{G}$.

Hence, we have

$$\mathrm{Supp}(\Phi^2 - \Phi'^2)_+ \subset [t-b, t]\,.$$

Similar constructions can be made for Ψ.

Analysis of the differentiation operator on an interval.

The starting point is some toy model in the book of Embree-Trefethen [17] (Chapter 15).

The goal is to prove that in this case the Wei constant $e^{\pi/2}$ in (5) is optimal.

We consider the operator A defined on $L^2(]0,1[)$ by

$$D(A) = \{u \in H^1(]0,1[)\,,\ u(1) = 0\}\,, \tag{18a}$$

and

$$Au = u'\,,\ \forall u \in D(A)\,. \tag{18b}$$

One easily verifies that, for any $z \in \mathbb{C}$, $(z - A)$ is inversible and that the inverse is given by

$$[(z-A)^{-1}f](x) = \int_x^1 \exp z(x-s)\, f(s)\, ds\,. \tag{19}$$

Observing that, for any $\alpha \in \mathbb{R}$, the conjugation by the map $u \mapsto U_\alpha u := \exp i\alpha x\, u$ gives $U_\alpha^{-1} A U_\alpha = A + i\alpha$, we deduce that $||(A-z)^{-1}||$ depends only on $\mathrm{Re}\, z$.
For $u \in D(A)$, we have

$$-\mathrm{Re}\,\langle (A-z)u\,,\, u\rangle = \mathrm{Re}\, z\|u\|^2 + \frac{1}{2}|u(0)|^2 \geq z\|u\|^2\,.$$

In particular $-A$ is accretive and satisfies the assumption of Wei [18].

In order to apply Wei's theorem, we have to compute $r(0) = 1/\psi(0)$. Hence we have to compute $||A^{-1}||$. In our case, we get that $r(0)$ is the square root of the smallest eigenvalue of A^*A, computed directly as $\pi^2/4$. So $r(0) = \pi/2$, and Wei's theorem gives

$$|| \exp tA|| \leq \exp \frac{\pi}{2} \exp \left(- \frac{\pi}{2} t \right) . \tag{20}$$

One question for the optimality is to ask if the constant $\exp \frac{\pi}{2}$ can be improved.
We prove that this constant is indeed optimal in the case of our toy model.

One can indeed directly compute the norm of $\exp tA$.
We have indeed for $u \in L^2(0,1)$:

$$(\exp tAu)(x) = \tilde{u}(x+t)\,,$$

where $\tilde{u}$ is the extension of u by 0 on $(1,+\infty)$.
For $t > 1$, one immediately sees that

$$\exp(tA) = 0\,.$$

For $t < 1$, one gets

$$||\exp tA|| = 1\,.$$

This shows that Wei's constant is optimal.

An interesting theorem is mentioned in the book of Embree-Trefethen (chapter 15, Theorem 15.6).

Theorem ET

Let A be a closed linear operator generating a C_0 semigroup. For any $\tau > 0$, the following properties are equivalent

1. $e^{\tau A} = 0$.
2. $\sigma(A) = \emptyset$ and there exists $C > 0$ and $\omega_0 < 0$ s. t., for $\omega \in (-\infty, \omega_0]$

$$\frac{1}{r(\omega)} \leq C e^{-\tau\omega}\,. \tag{21}$$

This can be recovered by Theorem [HS] by considering a sequence $\omega_n \to -\infty$.

Questions

Can we improve by iteration ?

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MORE

proof

The proof that (1) implies (2) is a consequence of the formula

$$(A - z)^{-1} = \int_0^\tau e^{-tz} S(t) dt \,,$$

together with the Banach-Steinhaus theorem.
The proof that (2) implies (1) is an easy application of Theorem [HS]. By the semi-group theory we can take, for some $M > 0$ and $\omega_0 \geq 0$

$$m(t) = M \exp \omega_0 t \,.$$

(The accretive case corresponds to $M = 1$ and $\omega_0 = 0$.)

We apply the theorem in the limiting case when $a = b = t/2$ and (4) gives when $\omega < \omega_0$

$$||S(t)|| \leq 2M^2(\omega_0 - \omega)\frac{e^{\omega t}}{r(\omega)}\,(1 - e^{(\omega-\omega_0)t})^{-1}\,. \tag{22}$$

By (21), for all $\omega \leq \omega_0$

$$\frac{e^{\omega t}}{r(\omega)} \leq Ce^{\omega(t-\tau)}.$$

When $t > \tau$, in the limit $\omega \to -\infty$ the estimate (22) gives $S(t) = 0$ as claimed.

When applied to the differential operator A introduced in (18), the estimate (21) is proven with $\tau = 1$ (see [17] or [7] (Chapter 14, (14.1.3))). We propose below an alternative approach to the control of $||(A - z)^{-1}||$ for our differential operator A. As above in the case $z = 0$, it is based on the property that, for $z \in \mathbb{R}$, $1/||(A - z)^{-1}||$ is the square root of the smallest eigenvalue of the operator

$$B(z) := (A^* - z)(A - z) \tag{23a}$$

whose domain reads

$$D(B(z)) = \{u \in H^2(0,1), u(1) = 0, u'(0) - zu(0) = 0\} . \tag{23b}$$

and is a realization of $-\frac{d^2}{dx^2} + z^2$ on this domain.
At the end we will be interested in the square root of the lowest eigenvalue of $B(z)$.

We first analyze the spectrum of

$$C(z) := B(z) - z^2\,.$$

It is rather standard to determine the lowest eigenvalue in function of $z \in \mathbb{R}$. We can first try an eigenfunction of the form

$$\phi_\mu(x) = \sin \mu(1-x)\,.$$

Here μ is determined by the Robin condition at 0 :

$$\mu \cos \mu = -z \sin \mu\,.$$

If $\mu(z)$ is a solution of this equation the corresponding eigenvalue will be $\mu(z)^2$. We choose $\mu(z)$ such that this eigenvalue is minimal, corresponding by Sturm-Liouville property with the condition that $\phi_\mu(x)$ does not vanish in $(0,1)$. When $z=0$, we recover $\mu(0)=\pi/2$.
If we look at the continuous branch of solution such that $\mu(0)=\pi/2$ we obtain that this branch could be defined on $[-1,+\infty)$.
When $z<-1$, we should instead look at

$$\phi_\mu(x) = \sinh(\mu(1-x))\,.$$

Here μ is determined as above by the Robin condition at 0 :

$$\mu \cosh \mu = -z \sinh \mu\,.$$

As $z \to -\infty$ we get

$$r(z) \sim 2|z|e^{-|z|}\,,$$

as stated in Theorem 14.3 in [7].

Other elements in the proof

Consider $(A - \partial_t)u = 0$ on $[0, +\infty[$ with $u \in L^2_{\omega\cdot}([0, +\infty[)$.
Let Φ satisfy (6) and add temporarily the assumption that $\Phi(s)$ is constant for $s \gg 0$. Then Φu, $\Phi' u$ can be viewed as elements of $L^2_{\omega\cdot}(\mathbb{R})$ and from

$$(A - \partial_t)\Phi u = -\Phi' u\,,$$

we get, by the definition of $r(\omega)$,

$$\|\Phi u\|_{\omega\cdot} \leq \frac{1}{r(\omega)}\|\Phi' u\|_{\omega\cdot},$$

or, taking the square,

$$((r(\omega)^2\Phi^2 - \Phi'^2)u|u)_{\omega\cdot} \leq 0\,.$$

This can be rewritten as

$$((r(\omega)^2\Phi^2 - \Phi'^2)_+ u|u)_{\omega\cdot} \leq ((r(\omega)^2\Phi^2 - \Phi'^2)_- u|u)_{\omega\cdot}, \tag{24}$$

or

$$\|(r(\omega)^2\Phi^2 - \Phi'^2)_+^{1/2} u\|_{\omega\cdot} \leq \|(r(\omega)^2\Phi^2 - \Phi'^2)_-^{1/2} u\|_{\omega\cdot}. \tag{25}$$

Writing $\Phi = e^{\phi}$, $\phi \in C^1(]0,+\infty[)$, $\phi(t) \to -\infty$ when $t \to 0$, we have

$$r(\omega)^2\Phi^2 - \Phi'^2 = (r(\omega)^2 - \phi'^2)e^{2\phi}\,,$$

and (24), (25) become

$$((r(\omega)^2 - \phi'^2)_+ u|u)_{\omega\cdot -\phi} \leq ((r(\omega)^2 - \phi'^2)_- u|u)_{\omega\cdot -\phi}\,, \qquad (26)$$

$$\|(r(\omega)^2 - \phi'^2)_+^{1/2} u\|_{\omega\cdot -\phi} \leq \|(r(\omega)^2 - \phi'^2)_-^{1/2} u\|_{\omega\cdot -\phi}\,. \qquad (27)$$

We have in mind the case when $r(\omega)^2 - (\phi')^2 > 0$ away from a bounded neighborhood of $t = 0$.

Let $S(t) = e^{tA}$, $t \geq 0$ and let $m(t) > 0$ be a continuous function such that

$$\|S(t)\| \leq m(t),\ \ t \geq 0. \tag{28}$$

Then we get

$$\|(r(\omega)^2 - \phi'^2)_+^{1/2} u\|_{\omega\cdot-\phi} \leq \|(r(\omega)^2 - \phi'^2)_-^{1/2} m\|_{\omega\cdot-\phi} |u(0)|_{\mathcal{H}}. \tag{29}$$

Note that we have also trivially

$$\|(r(\omega)^2 - \phi'^2)_-^{1/2} u\|_{\omega\cdot-\phi} \leq \|(r(\omega)^2 - \phi'^2)_-^{1/2} m\|_{\omega\cdot-\phi} |u(0)|_{\mathcal{H}}. \tag{30}$$

This is only one part of the proof !