Global stability of the normal state of superconductors in the presence of a strong electric current
(Le cours dans l'isle de Berder, last part)

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Abstract

We (Y. Almog+ B. Helffer) consider the time-dependent Ginzburg-Landau model of superconductivity in the presence of an electric current flowing through a two-dimensional wire. We show that when the current is sufficiently strong the solution converges in the long-time limit to the normal state. We provide two types of upper bounds for the critical current where such global stability is achieved: by using the principal eigenvalue of the magnetic Laplacian associated with the normal magnetic field, and through the norm of the resolvent of the linearized steady-state operator. In the latter case we estimate the resolvent norm in large domains by the norms of approximate operators defined on the plane and the half-plane. We also obtain a lower bound, in large domains, for the above critical current by obtaining the current for which the normal state looses its local stability.

The starting point on the mathematical side was a paper of Yaniv Almog at Siam J. Math. Appl. . This work was continued in collaboration with Y. Almog and X. Pan by the analysis of specific toy models. Here, in collaboration with Y. Almog, we treat a rather general situation and show how the toy models are involved in the question. Applications to control theory (collaboration with K. Beauchard, R. Henry and L. Robbiano) and results by R. Henry are also described.

Time Dependent Ginzburg-Landau equation.

Consider a superconductor placed at a temperature lower than the critical one. It is well-understood from numerous experimental observations, that a sufficiently strong current, applied through the sample, will force the superconductor to arrive at the normal state. To explain this phenomenon mathematically, we use the time-dependent Ginzburg-Landau model which is defined by the following system of equations, and will be referred to as (TDGL1) (Time Dependent Ginzburg-Landau equation).

(TDGL1)

$$\frac{\partial\psi}{\partial t} + i\phi\psi = (\nabla - iA)^2\,\psi + \psi\left(1 - |\psi|^2\right)\,, \qquad \text{in } \mathbb{R}_+ \times \Omega\,, \tag{1a}$$

$$\kappa^2 \;\text{curl}\;{}^2 A + \sigma\left(\frac{\partial A}{\partial t} + \nabla\phi\right) = \mathrm{Im}\,(\bar{\psi}\cdot(\nabla - iA)\psi)\,, \qquad \text{in } \mathbb{R}_+ \times \Omega\,, \tag{1b}$$

$$\psi = 0\,, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_c\,, \tag{1c}$$

$$(\nabla - iA)\psi\cdot\nu = 0\,, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_i\,, \tag{1d}$$

$$\sigma\left(\frac{\partial A}{\partial t} + \nabla\phi\right)\cdot\nu = J\,, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_c\,, \tag{1e}$$

$$\sigma\left(\frac{\partial A}{\partial t} + \nabla\phi\right)\cdot\nu = 0\,, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_i\,. \tag{1f}$$

$$\frac{1}{|\partial\Omega|}\int_{\partial\Omega} \operatorname{curl} A(t,x)\, ds = h_{ex}\,, \text{ on } \mathbb{R}_+\,, \quad (1g)$$

$$\psi(0,x) = \psi_0(x)\,, \qquad \text{in } \Omega\,, \quad (1h)$$

$$A(0,x) = A_0(x)\,, \qquad \text{in } \Omega\,, \quad (1i)\,.$$

In the above ψ denotes the order parameter, A is the magnetic potential, ϕ is the electric potential, κ denotes the Ginzburg-Landau parameter, which is a material property, and the normal conductivity of the sample is denoted by σ. ds denotes the induced measure on $\partial\Omega$. The domain $\Omega \subset\subset \mathbb{R}^2$, occupied by the superconducting sample, has a smooth interface, denoted by $\partial\Omega_c$, with a conducting metal which is at the normal state.

We require that ψ would vanish on $\partial\Omega_c$ in (1c), and allow for a smooth current

$$J = hJ_r\,,$$

satisfying

$$(J1) \quad J_r \in C^2(\overline{\partial\Omega_c}), \tag{3}$$

to enter the sample in (1e).
We further require that

$$(J2) \quad \int_{\partial\Omega_c} J_r\,ds = 0\,, \tag{4}$$

and

$$(J3) \quad \text{the sign of } J_r \text{ is constant on each connected component of } \partial\Omega_c\,. \tag{5}$$

We allow for $J_r \neq 0$ at the corners. (By convention, $J_r = 0$ on $\partial\Omega \setminus \partial\Omega_c$).

The rest of the boundary, denoted by $\partial\Omega_i$ is adjacent to an insulator. To simplify some of our arguments (or simply have a proof) we introduce the following geometrical assumption on $\partial\Omega$:

$$(R1)\left\{\begin{array}{l}(a)\ \partial\Omega_i \text{ and } \partial\Omega_c \text{ are of class } C^3\,;\\ (b)\quad \text{Near each edge, } \partial\Omega_i \text{ and } \partial\Omega_c \text{ are flat}\\ \qquad \text{and meet with an angle of } \frac{\pi}{2}\,.\end{array}\right. \tag{6}$$

We also require:

$$(R2) \qquad \text{Both } \partial\Omega_c \text{ and } \partial\Omega_i \text{ have two components.} \tag{7}$$

Figure 1 presents a typical sample with properties (R1) and (R2). Most wires would fall into the above class of domains.

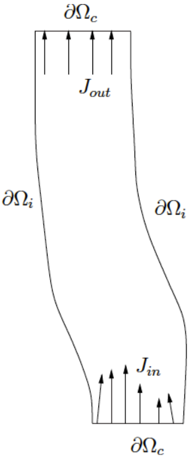

FIGURE 1. Typical superconducting sample. The arrows denote the direction of the current flow (J_{in} for the inlet, and J_{out} for the outlet).

We assume, for the initial conditions (1h,i), that

$$\psi_0 \in H^1(\Omega, \mathbb{C}) \text{ and } A_0 \in H^1(\Omega, \mathbb{R}^2)\,, \tag{8}$$

and:

$$\|\psi_0\|_\infty \leq 1\,. \tag{9}$$

We mainly consider Coulomb gauge solutions of (1):

$$\text{div } A = 0 \; \text{ in } \Omega,\; A \cdot \nu = 0 \text{ on } \partial\Omega\,. \tag{10}$$

Note that for the proof of existence of solutions it is better to consider first solutions in the Lorentz gauge:

$$\phi = \omega \text{ div } A\,.$$

Equivalent boundary conditions.

Instead of considering the boundary conditions (1e,f,g), it is possible to use an equivalent boundary condition where we prescribe instead the magnetic field. By (1b,e,f), on each point on $\partial\Omega$, except for the corners, we have

$$\frac{\partial}{\partial\tau} \operatorname{curl} A(t,\cdot) = \frac{1}{\kappa^2} J(\cdot)\,, \tag{11}$$

where $\partial/\partial\tau$ denotes the tangential derivative along $\partial\Omega$ in the positive direction. For convenience we set

$$J_r(x) \equiv 0 \text{ on } \partial\Omega_i\,. \tag{12}$$

Thus, if we introduce on the boundary the function B_r by

$$\operatorname{curl} A(t,x) = h\,B_r(t,x) \quad \text{on } \partial\Omega\,, \tag{13}$$

where h denotes a parameter measuring the intensity of the magnetic field.

One can recover the magnetic field $B_r(t,\cdot)$

$$B_r(t,x) = h_r - \frac{1}{\kappa^2\,|\partial\Omega|}\int_{\partial\Omega} |\Gamma(\tilde{x},x)|\, J_r(\tilde{x}) ds(\tilde{x}) \ \text{ for } x \in \partial\Omega\,. \quad (14)$$

where $h_{ex} = hh_r$, $J = hJ_r$ and $|\Gamma(\tilde{x},x)|$ is the length inside the boundary between x and $\tilde{x}$.

This shows that $B_r(t,x) = B_r(x)$ on the boundary, hence independent of t.

Note also that

The magnetic field B_r is constant along each component of $\partial\Omega_i$. (15)

Hence the system (TGDL1) is equivalent to the system (TGDL2) (same equations except (1e-1g) replaced by)

$$\text{curl } A(t,x) = hB_r(x)\,, \text{ on } \mathbb{R}_+ \times \partial\Omega\,, \tag{16}$$

where B_r is given by (14).
Of course functional spaces should be introduced to give a precise mathematical sense to this statement of equivalence.

(TDGL2)

$$\frac{\partial \psi}{\partial t} + i\phi\psi = (\nabla - iA)^2\,\psi + \psi\,(1 - |\psi|^2)\ , \qquad \text{in } \mathbb{R}_+ \times \Omega\,, \tag{17a}$$

$$\kappa^2 \text{ curl }^2 A + \sigma \left(\frac{\partial A}{\partial t} + \nabla \phi \right) = \mathrm{Im}\,(\bar{\psi}\,\cdot\,(\nabla - iA)\psi)\,, \qquad \text{in } \mathbb{R}_+ \times \Omega\,, \tag{17b}$$

$$\psi = 0\,, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_c\,, \tag{17c}$$

$$(\nabla - iA)\psi \cdot \nu = 0\,, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_i\,, \tag{17d}$$

$$\text{curl } A(t,x) = h\,B_r(x) \qquad \text{on } \mathbb{R}_+\,,\ (1g) \tag{17e}$$

$$\psi(0,x) = \psi_0(x)\,, \qquad \text{in } \Omega\,,\ (1h) \tag{17f}$$

$$A(0,x) = A_0(x)\,, \qquad \text{in } \Omega\,,\ (1i)\,. \tag{17g}$$

Conversely, a solution of (TGDL2) must satisfy (TGDL1) with

$$J_r = \kappa^2 \frac{\partial B_r}{\partial \tau} \text{on } \partial\Omega \,,$$

and

$$h_r = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} B_r(x) ds \,.$$

Stationary normal solutions.

If we assume time independence and a solution of (TDGL1) $(0, A_n, \phi_n)$, we get for the magnetic and electric normal potentials A_n and ϕ_n. These equations are obtained by setting $\psi \equiv 0$ in (1b), yielding

$$\begin{cases} -c \ \text{curl}\ ^2 A_n + \nabla \phi_n = 0 & \text{in } \Omega\,, \\ -\sigma \frac{\partial \phi_n}{\partial \nu} = J_r & \text{on } \partial\Omega\,, \\ \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \ \text{curl}\ A_n \, ds = h_r\,, & \end{cases}$$

in which $c = \kappa^2/\sigma$, J_r and h_r respectively denote some reference electric current and magnetic field.

If we fix the Coulomb gauge for A_n, we can prove the existence, uniqueness, and regularity of solutions to the above problem.
Note that ϕ_n is a solution of

$$\Delta\phi_n = 0$$

$$\int_\Omega \phi_n dx = 0\,,$$

and

$$-\sigma\frac{\partial\phi_n}{\partial\nu} = J_r\,.$$

This is Neumann but for a problem with corners ! H^2-regularity is OK when the angles are $\frac{\pi}{2}$.
See Kondratev, Grisvard, Dauge for these questions of regularity.

The next assumption (which can be expressed in term of J_r and h_r), is

$$(B) \quad B_n \neq 0 \text{ at the corners}\,, \tag{18}$$

where $B_n = \text{ curl } A_n$.

For some of the results, we assume for technical reasons

$$(C) \quad \nabla\phi_n \perp \partial\Omega \;\text{ on } B_n^{-1}(0)\cap\partial\Omega\,. \tag{19}$$

One possible mechanism which contributes to the breakdown of superconductivity by a strong current is the magnetic field induced by the current. In the absence of electric current, it was proved by Giorgi-Phillips in [16] that, when a sufficiently strong magnetic field is applied on the sample's boundary (or when h is sufficiently large), the normal state, for which $\psi \equiv 0$, becomes the unique solution for the steady-state version of (1) (cf. also Fournais-Helffer [15] and the references therein).
For the time-dependent Ginzburg-Landau equations it was proved in Feireisl-Takac [13] that every solution must reach an equilibrium in the long-time limit. When combined with the results in [16] it follows that when the applied magnetic field is sufficiently large the normal state becomes globally stable.

No such result was available in the presence of electric currents. The results in [13] are based on the fact that, in the absence of currents, the Ginzburg-Landau energy functional serves as a Lyapunov functional. In the presence of a current one has to take account of the work it produces, which does not necessarily decrease the energy (cf. [30] for instance).
Moreover, the magnetic field is not the only mechanism which forces the sample into the normal state when the electric current is sufficiently large.

Consider the reduced model where one neglects the induced magnetic field and set $A \equiv 0$ in (1). It has been proved in [22, 31, 1] that the normal state is at least locally stable when the current is sufficiently strong. In a recent contribution [2], together with Pan, we show that the critical current where the normal state looses its local stability tends to the critical value for the reduced model [22] in the small conductivity limit, or when $c \to \infty$. This result suggests that stability is being forced not only by the magnetic field that the current induces, but also by the potential term in (1a).

In the present contribution we prove global stability of the normal state, as a solution of (1), for sufficiently large currents. We begin by proving global existence and uniqueness of solutions for (1) and obtain their regularity. While these questions have previously addressed (cf. [6], [14], and [9] to name just a few references) the fact that the boundary is not smooth at the corners requires some additional attention.

A non self-adjoint operator.

Let

$$\mathcal{L}_h = -\nabla^2_{hA_n} + i\, h\phi_n \,,$$

be defined over the domain

$$D(\mathcal{L}_h) = \{u \in H^2(\Omega) \,|\, u|_{\partial\Omega_c} = 0 \,;\; \nabla u \cdot \nu|_{\partial\Omega_i} = 0\,\}\,.$$

We prove that a proper bound on the resolvent of $\mathcal{L}_h$, which is the elliptic operator in (1a) linearized near $(0, hA_n, h\phi_n)$ gives the stability.

In the present contribution we prove global stability of the normal state, as a solution of (1), for sufficiently large currents. We begin by proving global existence and uniqueness of solutions for (1) and obtain their regularity. While these questions have previously addressed (cf. [6], [14], and [9] to name just a few references) the fact that the boundary is not smooth at the corners requires some additional attention.

We prove that if the current is strong enough, the magnetic field induced by this current forces the semigroup associated with (1) to become asymptotically a contraction. Let

$$\mu(h) = \inf_{\substack{u \in H^1(\Omega,\mathbb{C}) \\ u|_{\partial\Omega_c}=0\ ;\ \|u\|_2=1}} \|\nabla_{hA_n} u\|_2^2 .$$

This is simply the ground state energy of the magnetic Laplacian (selfadjoint part of $\mathcal{L}_h$).

Analysis of the linearized problem

Consider first the linearized version of (1):

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}u = 0\,, & \text{in } \mathbb{R}_+ \times \Omega\,, \\ (i\nabla + hA_n)u \cdot \nu = 0\,, & \text{on } \mathbb{R}_+ \times \partial\Omega_i\,, \\ u = 0\,, & \text{on } \mathbb{R}_+ \times \partial\Omega_c\,, \\ u(0,\cdot) = u_0(\cdot)\,, & \text{in } \Omega\,. \end{cases} \qquad (20)$$

In the above

$$-\mathcal{L} = (\nabla - ihA_n)^2 + ih\phi_n + 1\,.$$

It is easy to show using integration by parts that for any $v \in D(\mathcal{L})$ we have

$$\langle v, \mathcal{L}v \rangle = \|\nabla_{hA_n} v\|_2^2 - \|v\|_2^2\,.$$

Hence

$$\langle v, \mathcal{L}v \rangle \geq (\mu - 1)\|v\|_2^2\,.$$

Note that if v is a ground state of $\mathcal{L}$ the above inequality becomes an identity. Hence, it follows that the operator $\mathcal{L}$ is accretive if and only if $\mu \geq 1$. Consequently, it is easy to show from the Lumer-Phillips Theorem (Theorem 8.3.5 in [7]) that the semigroup associated with (20) is a contraction semigroup if and only if $\mu \geq 1$. If $\mu > 1$ one can easily show that any solution of (20) decays exponentially fast (with a decay like $\exp -(\mu - 1)t$) and hence, that $u \equiv 0$ is asymptotically stable.

If we now consider the linearized part of (1b), (after taking its curl), we get the equation for the first variation w of curl A

$$\begin{cases} \sigma\partial_t w - \kappa^2\Delta w = 0 \text{ in } \mathbb{R}^+ \times \Omega\,, \\ w(t,\cdot) = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega\,, \\ w(0,\cdot) = w_0(\cdot) \text{ on } \Omega\,. \end{cases}$$

From the above we can conclude an $\mathcal{O}(e^{-\lambda_D ct})$-decay for $w(t,\cdot)$, where λ^D is the Dirichlet Laplacian and $c = \frac{\kappa^2}{\sigma}$.

The first result is the following

Theorem 1

Let (ψ, A, ϕ) denote a solution of (1) and (10) satisfying (9). Then, there exists $\gamma > 0$ for which whenever

$$\mu(h) > 1 + \frac{\gamma}{\kappa^2} + \frac{\gamma^2}{\kappa^4}\,, \tag{21}$$

there exist $C = C(\Omega, \kappa, c, \|\psi_0\|_2, \|A_0\|_2, h) > 0$ and $\lambda_m = \lambda_m(c, \kappa, \mu(h), \Omega) > 0$ such that, for all $t > 0$, we have:

$$\|\psi\|_2 + \|A - hA_n\|_2 + \|\phi - h\phi_n\|_2 \leq Ce^{-\lambda_m t}\,. \tag{22}$$

Furthermore, there exists $t^*(\kappa, c, \|A_0\|_2, \Omega)$ such that $[t^* + 1, +\infty) \ni t \mapsto \|\psi(t,\cdot)\|_2$ is monotone decreasing.

Note that (22) means that the semigroup associated with the linearized version of (1) is a contraction. Precise values of γ, λ_m, and t^* can be established in the large κ limit.

Theorem 2

Let $\nu \geq 0$. There exists $\kappa_0 > 0$ and $C_1 > 0$ such that, if for some $\kappa > \kappa_0$ we have

$$\sup_{\gamma \in \mathbb{R}} \|(\mathcal{L}_h - i\gamma - \nu)^{-1}\| < 1 - \frac{C_1}{\kappa^2}, \tag{23}$$

then, any solution of (1) must satisfy

$$\int_0^\infty e^{2\nu t} \|\psi(t,\cdot)\|_2^2 \, dt < \infty. \tag{24}$$

The above stability is proved in the large κ limit both for (1) and we treat the same system, scaled with respect to the penetration depth, which is obtained by applying the transformation $x \to x/\kappa$ in (1).
As the resolvent of $\mathcal{L}_h$ in an arbitrary domain is difficult to control, we provide an estimate of its norm for large values of h, which can be applied for either large domains, or large κ values.

Large domains Ω_R

Our aim is to show that the norm of the resolvent can be controlled from two approximated problems, with constant current defined either in $\mathbb{R}^2$ or in $\mathbb{R}^2_+$ with Dirichlet boundary conditions. From resolvent estimates, together with the results of Almog-Helffer-Pan in [4, 2, 3] we deduce that the critical current, for which the normal state looses its local stability, can be approximated by the same critical current obtained for the above $\mathbb{R}^2_+$ problem. Before to state the result let us describe the toy models.

Two toy models

We now give the definitions of these model operators in $\mathbb{R}^2$ and $\mathbb{R}^2_+ = \{y > 0\}$.
These models depend on two real parameters $c \neq 0$ and j.
The first one is

$$\mathcal{A}(\mathrm{j}, c) = D_x^2 + (D_y - \mathrm{j}x^2)^2 + ic\mathrm{j}y\,, \tag{25}$$

defined on

$$D(\mathcal{A}) = \{u \in L^2(\mathbb{R}^2)\,|\,\mathcal{A}u \in L^2(\mathbb{R}^2)\}\,. \tag{26}$$

It has empty spectrum and we have a good control of the resolvent depending only of the real part of the spectral parameter.

The second one is $\mathcal{A}_+(\mathrm{j}, c)$, which is defined (via de Lax-Milgram theorem) by the same differential formula of $\mathcal{A}$ but on the domain

$$D(\mathcal{A}_+) = \{u \in \tilde{V} \,:\, \mathcal{A}_+ u \in L^2(\mathbb{R}^2_+, \mathbb{C})\}, \tag{27}$$

where

$$\tilde{V} = H_0^{1,\mathrm{mag}}(\mathbb{R}^2_+, \mathbb{C}) \cap L^2(\mathbb{R}^2_+, \mathbb{C}; y\, dxdy)\,. \tag{28}$$

Here the analysis of the spectrum is more difficult. The guess is that it is non-empty. This is only proven for $|c|$ large enough or small enough.

Towards the last theorem

Set, for $z \in \bar{\Omega}$,

$$\mathfrak{j}(z) := h|\nabla B_n(z)| = \frac{h}{c}|\nabla \phi_n(z)|\,, \qquad (29)$$

and then define,

$$\mathcal{A}(z) = \mathcal{A}(\mathfrak{j}(z), c) \quad ; \quad \mathcal{A}_+(z) = \mathcal{A}_+(\mathfrak{j}(z), c) \qquad (30)$$

Under all of the above assumptions $B_n^{-1}(0)$ is either empty, or else consists of a single curve Γ connecting between the two connected components of $\partial\Omega_c$.

We treat the second case. We denote the two points of intersection by z_1 and z_2 and then set

$$\nu_m(z_1, z_2, c) = \min_{i=1,2} \inf_{\lambda \in \sigma(\mathcal{A}_+(z_i))} \mathrm{Re}\lambda \,. \tag{31}$$

Large domain limit

Let then $R > 0$. We denote by Ω_R the image of Ω under the dilation $x \to R\,x$ We assume that the domain Ω has the property (R1)-(R2) and that assumptions (J1)-(J3), (B) and (C) are met. Denote the transformed electric field by ϕ_R. It satisfies the problem

$$\begin{cases} \Delta\phi_R = 0 & \text{in } \Omega_R\,, \\ \frac{\partial\phi_R}{\partial\nu} = -\frac{J_R(x)}{\sigma} & \text{on } \partial\Omega_R\,, \end{cases}$$

where

$$J_R(x) = J_r(x/R)\,.$$

Note that

$$\phi_R(x) = R\,\phi_n(x/R)\,.$$

The transformed magnetic potential, which we denote by A_R then satisfies

$$A_R(x) = R^2\,A_n(x/R)\,.$$

Let then

$$\mathcal{L}_h^R = -\nabla_{hA_R}^2 + ih\phi_R\,, \tag{32}$$

and let

$$\mu(R) = \inf_{\lambda\in\sigma(\mathcal{L}_h^R)} \mathbb{R}e\lambda \quad \text{and} \quad \mu_\infty = \liminf_{R\to\infty} \mu(R)\,. \tag{33}$$

We can now state

Theorem 3

Under the previous assumptions,

$$\mu_\infty = \nu_m \,.$$

Furthermore, let $\nu < \mu_\infty$. Then, $\exists\ R_0,\ C,$ such that, for $R \geq R_0$,

$$\sup_{\gamma\in\mathbb{R}} \|(\mathcal{L}_h^R - \nu - i\gamma)^{-1}\| \leq$$

$$\max\left(\sup_{z_0\in\Gamma}\|(\mathcal{A}(z_0)-\nu)^{-1}\|, \sup_{\substack{\gamma\in\mathbb{R}\\ i=1,2}}\|(\mathcal{A}_+(z_i)-\nu-i\gamma)^{-1}\|\right)\left(1+\frac{C}{R^{1/4}}\right)$$

$$+\frac{C}{R^{1/4}}\,. \quad (34)$$

One can deduce from (34) an upper bound for the critical current where the normal state $(0, hA_n, h\phi_n)$ becomes globally stable. Let

$$\mathrm{j}_m = \inf_{z\in\Gamma} \mathrm{j}(z)\,, \tag{35a}$$

and

$$\mathrm{j}_+ = \inf_{i=1,2} \mathrm{j}(z_i)\,. \tag{35b}$$

When the domain size is multiplied by R, the resolvent norm of $\mathcal{L}_h$ is given by the left-hand-side of (34). By (23) it then follows that if R and κ are sufficiently large, and if

$$\mathrm{j}_m > \|\mathcal{A}^{-1}(1,c)\|^{3/2} \tag{36a}$$

and

$$\mathrm{j}_+ > \sup_{\gamma\in\mathbb{R}} \|(\mathcal{A}_+(1,c) - i\gamma)^{-1}\|^{3/2}\,, \tag{36b}$$

then the normal state must be globally stable. The above conditions serve as an upper bound for the critical current where the normal state becomes globally stable.

On the semiclassical side

This corresponds to the spectral analysis of

$$\sum_j (\hbar D_{x_j} - A_j)^2 + i\hbar\phi(x)\,,$$

in the limit $\hbar \to 0$. With $\phi = 0$, this analysis plays an important role in the analysis of the superconductivity. In the above questions, we have $\nabla\phi \cdot \nabla$ curl $A = 0$ and the zeros of curl A consists in a curve Γ joining two points of the boundary where the Dirichlet condition is assumed.

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